arXiv:1107.5533v1 [math-ph] 27 Jul 2011

GEOMETRICALLY RELATING MOMENTUM CUT-OFF AND DIMENSIONAL REGULARIZATION

SUSAMA AGARWALA

ABSTRACT. The β function for a scalar field theory describes the dependence of the coupling constant on the renormalization mass scale. This dependence is affected by the choice of regularization scheme. I explicitly relate the β-functions of momentum cut-off regularization and dimensional regularization on scalar field theories by a gauge transformation using the Hopf algebras of the Feynman diagrams of the theories.

Perturbative quantum field theories (QFTs), when naively calculated, lead to divergent integrals and undefined quantities. To address this problem, physicists have developed many regularization and renormalization schemes to extract finite values from divergent integrals. Introducing a regularization parameter forces the quantities in the Lagrangian of the field theory to be dependent on the energy scale of the calculation. This scale dependence captured by a new parameter called the renormalization mass. The theory's dependence on the renormalization mass is described by a set of differential equations, called the renormalization group equations, or RGEs. The simplest of these solves for the dependence of the coupling constant on the renormalization mass, and gives the β function of the theory. The RGEs depend on the regularization scheme. Different regularization schemes give rise to different RGEs. Very little is understood about the relationship between different regularization schemes.

In this paper, I compare regularization schemes with logarithmic singularities and finite poles to those with only finite poles. Specifically, I study the relationship between sharp momentum cut-off regularization and dimensional regularization, and the associated β functions.

Recently, a literature has emerged geometrically describing this process of renormalization and regularization for a QFT, in which the β-function is defined by a connection on a renormalization bundle with sections representing different regularization schemes [4, 5, 6, 7, 8, 1]. I extend the analysis in these papers to include the logarithmic singularities found in momentum cut-off and related regularization schemes, and express the corresponding β function in terms of connection on this new regularization bundle. As a result, β functions for dimensional regularization and momentum cut-off, which are known to be different for gauge theories, even at the one loop level, can be related in terms of a gauge transformation.

Section 1 recalls some useful facts about Feynman integrals, dimensional regularization and cut-off regularization. Section 2 constructs the new renormalization bundle and defines the relevant β functions in terms of connections on it.

1. Momentum Cutoff and Dimensional Regularization

In this section I consider Feynman integrals of a massive ϕ^4 theory in $\mathbb{R}^4$

$$\mathcal{L} = \frac{1}{2}\phi(\Delta + m^2)\phi + g\phi(x)^4 \ .$$

The same arguments can be made for other renormalizable theories. For a graph, Γ with l loops, I internal edges, and J external edges with assigned momenta $e_1 \ldots e_J$, the Feynman integral, is of the form

$$\int_{\mathbb{R}^{4l}} \prod_{k=1}^{I} \frac{1}{f_k(p_i, e_j)^2 + m^2} \prod_{i=1}^{l} d^4 p_i \ , \tag{1}$$

where the p_i are the loop momentum assigned to each loop, $f(p_i, e_j)$ is a linear combination of the loop and external momenta representing the momenta assigned to each internal leg, and the square refers to a dot product of the vectors. All calculations in this paper are done in Euclidean space, all integrals

2010 *Mathematics Subject Classification.* 81R50,16T05.

Key words and phrases. β function, Hopf Algebra, momentum cut-off regularization, dimensional regularization.

have been Wick rotated. These integrals are generally divergent as written. The process of regularization and renormalization extracts physical, finite values from these divergent integrals. In this section, I recall properties of dimensional regularization and momentum cut-off regularization. Both regularization schemes can then be renormalized using minimal subtraction.

For dimensional regularization, write the integral in (1) in spherical coordinates,

$$A(4)^l \int_0^\infty \prod_{k=1}^{I} \frac{1}{f_k(p_i, e_j)^2 + m^2} \prod_{i=1}^{l} p_i^3 dp_i \ ,$$

where $A(d) = \frac{\Gamma(d)}{(4\pi)^{d/2}}$ is the volume of S^{d-1}, the sphere in $d-1$ dimensions. Dimensional regularization exploits the fact that the integral above is convergent if taken over $d = 4 + z$, dimensions, with z a complex parameter. Notice that $A(d)$ is holomorphic in z, and does not contribute to the polar structure of the graph. The dimensionally regularized integral is

$$\varphi_{dr}(z)(\Gamma) = A(d)^l \int_0^\infty \prod_{k=1}^{I} \frac{1}{f_k(p_i, e_j)^2 + m^2} \prod_{i=1}^{l} p_i^{d-1} dp_i \ ,$$

. Put another way, dimensional regularization assigns a holomorphic function, $A(d)$, times the Mellin transform of the each loop integral in the Feynman integral. If the original integral is divergent, this expression has a pole at $d = 4$.

Momentum cut-off regularization multiplies the integrand of the Feynman integral in polar coordinates by a cutoff function. The simplest is to impose a sharp cut off function

$$\chi(p) = \begin{cases} 1 & \text{if } p \leq \Lambda, \\ 0 & \text{if } p > \Lambda. \end{cases} \ .$$

However, this destroys some nice analytic properties, and sometimes it is better to examine a smooth cutoff function. The calculations in the paper are done using sharp cut-off, but the analysis generalizes to the smooth case. The philosophy behind cut-off regularization is that physical theories are only valid in a certain domain. Once the energy scale is large enough, one doesn't expect the theory to hold. Therefore, one should only consider energy scales at which the theory is valid. The integral in (1) becomes

$$\varphi_{mc}(\Lambda)(\Gamma) = \int_{-\Lambda}^{\Lambda} \prod_{k=1}^{I} \frac{1}{f_k(p_i, e_j)^2 + m^2} \prod_{i=1}^{l} d^4 p_i \ .$$

In order to compare momentum cut-off with dimensional regularization, one needs to consider what occurs if Λ is a complex regulator. Consider the integral

$$\varphi_{mc}(\Lambda e^{i\theta}) = \int_C \prod_{k=1}^{I} \frac{1}{f_k(p_i, e_j) \cdot \overline{f_k(p_i, e_j)} + m^2} \prod_{i=1}^{l} d^4 p_i$$

taken along the contour $C = te^{i\theta}$ for $t \in [-\Lambda, \Lambda]$. The symmetries of the integrand gives $\varphi_{mc}(\Lambda)(\Gamma) = \varphi_{mc}(\Lambda e^{i\theta})(\Gamma)$.

Definition 1. A one particle irreducible graph, 1PI graph, is a connected graph that is still connected after the removal of any single (internal) edge.

Dimensional analysis and power counting arguments show that the only divergent integrals of renormalizable ϕ^4 theory in $\mathbb{R}^4$ are those associated to 1PI graphs with either 2 or 4 external legs ($J \in \{2, 4\}$).

Definition 2. For ϕ^4 in $\mathbb{R}^4$, the superficial degree of divergence of a 1PI graph, Γ is $\omega(\Gamma) = 4l - 2I$.

If $\omega(\Gamma) < 0$ then the integral is convergent. If $\omega(\Gamma) \geq 0$, the integral is divergent, [9] §8.1.3. Dimensional regularization of integrals in renormalizable theories give holomorphic functions with finite poles at $z = 0$. Momentum cutoff regularization for these integrals have logarithmic and polynomial singularities at $\Lambda \to \infty$, loc. cit. §8.2.1. One can impose different cut off function to maintain smoothness or other analytic properties. Then the regulator depends on the cutoff function.

1.1. **Renormalization group action.** The renormalization group is the torsor $M \simeq \mathbb{R}_{>0}$. The renormalization group action on a field theory the measures how changing the energy scale effects the field theory. The regularized integrals, $\varphi_{dr}(z)$ and $\varphi_{mc}(\Lambda)$ also depend on the mass, m, the external momenta, e_j, and the scale, t, at which the integral is calculated. The action of the renormalization group takes the Feynman integral with respect to tp_i instead of taking the integral with respect to dp_i. Writing these dependencies explicitly, let $\varphi_{dr}(m, e_j, t, z)$ be the integral taken at tp_i, the change of variables $p_i \to tp_i$ changes the Feynman integrals as

$$\begin{aligned} t^{zl}\varphi_{dr}(m, e_j, z)(\Gamma) &\to t^{-\omega(\Gamma)}\varphi_{dr}(tm, te_j, z)(\Gamma) \\ \varphi_{mc}(m, e_j, \Lambda)(\Gamma) &\to t^{-\omega(\Gamma)}\varphi_{mc}(tm, te_j, t\Lambda)(\Gamma)\ . \end{aligned} \tag{2}$$

The variable t is the energy scale of the field theory. The extra factor of t^{zl} in the case of dimensional regularization is introduced to keep certain quantities dimensionless. It is called the t'Hooft mass. The introduction of an energy scale t changes the mass of the theory, $m \to tm$. Changing the energy scale changes the external momenta from $e_j \to te_j$. In the case of momentum cutoff regularization, it changes the regulator $\Lambda \to t\Lambda$. On the level of the Lagrangian density, introducing the energy scale also affects the coupling constant g and the field ϕ.

The effect of the action on the Lagrangian defining the theory is calculated by writing the regularized Lagrangian in terms of renormalized and counterterm components. The bare, or unrenormalized Lagrangian is

$$\mathcal{L}_B = \frac{1}{2}(|d\phi_B|^2 - m_B^2\phi_B^2) + g_B\phi_B^3\ .$$

A renormalized theory gives Greens functions of a renormalized field, $\phi_B = \sqrt{Z(g_B, m_B, z)}\phi_r$, where $\lim_{z\to 0} Z - 1 = \infty$. Then the bare Lagrangian can be written

$$\begin{aligned} \mathcal{L}_B &= \frac{1}{2}Z|d\phi_r|^2 - m_r^2 Z\phi_r^2) + g_r Z^{3/2}\phi_r^3 \\ &= \frac{1}{2}(|d\phi_r|^2 - m_r^2\phi_r^2) + g_r\phi_r^3 \\ &\quad + \frac{1}{2}((Z-1)|d\phi_r|^2 - (Z-1)m_r^2\phi_r^2) + (Z^{3/2}-1)g_r\phi_r^3\ . \end{aligned}$$

The second line is called the renormalized Lagrangian, consisting of finite quantities $\mathcal{L}_{fp}$, and last line is the counterterm $\mathcal{L}_{ct}$. Writing the Lagrangian as the sum

$$\mathcal{L}_B = \mathcal{L}_{ct} + \mathcal{L}_{fp}$$

shows the components that lead to counterterm and finite parts of the Feynman integrals. For more details on this process see [12], chapters 21 and 10.

The quantities ϕ_r, m_r and g_r depend on the scale of the theory, while, for a good choice of regularization and renormalization schemes, the coefficients of $\mathcal{L}_{ct}$ do not. The differential equation

$$\beta(g_r) = \frac{1}{t}\frac{\partial g_r}{\partial t}$$

gives the dependence of the coupling constant on the scale. The β function is useful in solving the other dependencies. It is solved for as an asymptotic expansion by loop number of the theory.

The β function for a theory can change depending on which regularization method is employed. For a scalar field theory, as in the example computed above, the β function for dimensional regularization and cut off regularization are the same up to 3 loop orders [9]. To one loop order, the β function is

$$\beta(g) = \frac{3g^2}{16\pi^2}\ .$$

This is not the case for QED. The β function to one loop order for QED under dimensional regularization is

$$\beta(e) = \frac{e^3}{12\pi^2} + O(e^5)$$

where e is the dimensionless electric charge [12]. However, the first loop order calculation of the β function for QED under a cut-off regularization scheme is [9]

$$\beta(\alpha) = \frac{2\alpha^2}{3\pi} \ .$$

where α is the fine structure constant given by $\alpha = e^2/4\pi$, the fine structure constant.

Remark 1. This difference is explained by the fact that cut-off regularization does not preserved the gauge symmetries of the theory, and therefore cutoff regularization is not the appropriate regularization scheme to use in this context. Nonetheless I show that the β functions of the two regularization schemes can be geometrically related to each other.

1.2. **BPHZ on Cut off renormalization.** The divergences in momentum cut-off regularization can be subtracted off by BPHZ renormalization [9]. The BPHZ algorithm calculates the counterterms, or divergent quantity, associated to Γ as the Taylor series of $\varphi(\Lambda)(e_i)(\Gamma)$ in its external momenta, calculated up to $\omega(\Gamma)$,

$$T(\varphi_{mc}(tm, te_j, t\Lambda))(\Gamma) = \sum_{i=0}^{\omega(\Gamma)} \frac{\vec{e}^i}{i!} D_e^i \varphi_{mc}(tm, te_j, t\Lambda)(\Gamma) \ ,$$

where D_e^i is the multi dimensional matrix of i^{th} derivatives in the variables $\{e_1 \dots e_{J-1}\}$.

Definition 3. Let T be the Taylor series operator described above,

$$T(f(e_j, \Lambda)(\Gamma)) = \sum_{i=0}^{\omega(\Gamma)} \frac{\vec{e}^i}{i!} D_e^i f(e_j, \Lambda)(\Gamma) \ .$$

Write the unrenormalized quantity $U(\Gamma) = \varphi_{mc}(\Lambda, m, e_i)(\Gamma)$. The counterterm is then

$$C(\Gamma) = T\left(U(\Gamma) + \sum_{\substack{\gamma\subset\Gamma \\ \text{divergent}}} C(\gamma)U(\Gamma/\gamma)\right),$$

where the sum is over all sub-graphs of Γ that are divergent. The graph Γ/γ is obtained by contracting the edges in each connected component of γ to a vertex. The renormalized part is the difference, $R(\Gamma) = U(\Gamma) - C(\Gamma)$. The counterterms thus derived are independent of the scaling factor t.

Theorem 1.1. *The BPHZ algorithm yields scale invariant counterterms if the Taylor series operator, T, is replaced by the minimal subtraction operator, π, that subtracts only the singular part of ϕ_{dr}.*

Proof. Write

$$\varphi_{mc}(tm, te_j, t\Lambda)(\Gamma) = \varphi^f_{mc}(tm, te_j, t\Lambda)(\Gamma) + \varphi^d_{mc}(tm, te_j, t\Lambda)(\Gamma),$$

where the superscripts f and d denote the terms of $\varphi_{mc}(\Lambda)(\Gamma)$ that are finite and divergent as $\Lambda \to \infty$. That is, $\pi(\varphi_{mc}(tm, te_j, t\Lambda)) = \varphi^d_{mc}(tm, te_j, t\Lambda)$.

For any graph Γ, $\varphi^d_{mc}(tm, te_j, t\Lambda)(\Gamma)$ is a homogeneous polynomial in e_i and m of weight less than or equal to $\omega(\Gamma)$ [3]. Therefore,

$$T(\varphi_{mc}(tm, te_j, t\Lambda))(\Gamma) = \varphi^d_{mc}(tm, te_j, t\Lambda)(\Gamma) + \text{finite terms} \ .$$

This quantity is not dependent on the energy scale t. The t dependence of the finite terms cannot cancel the time dependence of the divergent parts. Therefore

$$\pi(U(\Gamma)) = \varphi^d_{mc}(tm, te_j, t\Lambda)(\Gamma)$$

is independent of the energy scale.

If Γ has only one divergent subdiagram, γ, then Γ/γ is not divergent, and

$$\begin{aligned} C(\Gamma) = T\left(U(\Gamma) + C(\gamma)U(\Gamma/\gamma)\right) = \\ T\big((\varphi^d_{mc}(tm, te_j, t\Lambda) + \varphi^f_{mc}(tm, te_j, t\Lambda))(\Gamma/\gamma) + \\ (\varphi^d_{mc}(tm, te_j, t\Lambda) + \text{finite terms})(\gamma)(\varphi^d_{mc}(tm, te_j, t\Lambda) + \varphi^f_{mc}(tm, te_j, t\Lambda)(\Gamma/\gamma)\big) \ . \end{aligned} \tag{3}$$

Since $\varphi^d_{mc}(tm, te_j, t\Lambda)(\gamma)$, $\varphi^d_{mc}(tm, te_j, t\Lambda)(\Gamma)$ and $\varphi^d_{mc}(tm, te_j, t\Lambda)(\Gamma/\gamma)$ are all polynomials, energy scale independence of $\varphi^d_{mc}(tm, te_j, t\Lambda)(\Gamma)$ results from the energy scale independence of (3). The general case is proved by induction. □

Connes and Kreimer use BPHZ renormalization on dimensional regularization with the minimal subtraction operator instead of the Taylor series operator in their work [4, 5]. In this paper, I extend their work to include cutoff regularization. The substitution of the minimal subtraction operator for the Taylor series operator in BPHZ renormalization of different regularization schemes is well established. For example, Collins [2] does so for dimensional regularization, and Speer [11] for analytic regularization.

Writing $z = 1/\Lambda$, $\varphi_{mc}(\Gamma) \in \mathbb{R}[z^{-1}, \log(z/m)][[z]]$ for any Feynman diagram Γ.

Definition 4. Define $\mathscr{A} := \mathbb{C}[z^{-1}, y][[z]][[y]]/(e^y - z/m)$ to be the target algebra for dimensional regularization and momentum cut off regularization.

If one allows for a complex cut-off, the logarithmic poles are no longer well defined. The quotient by the ideal $(e^y - z/m)$ settles this ambiguity. For any Feynman diagram Γ, $\varphi_{dr}(\Gamma), \varphi_{mc}(\Gamma) \in \mathscr{A}$. Any element $f \in \mathscr{A}$ can be written $f = \sum_{j=0}^{\infty}\sum_{i=-n}^{\infty} a_{ij} z^i y^j$. The minimal subtraction operator is a projection onto the subalgebra of $\mathscr{A}$ that contains only the term that are singular at $z = 0$.

Definition 5. Let π be the minimal subtraction operator on $\mathscr{A}$. The operator $\mathbb{I} - \pi$ is the projection map

$$\begin{aligned} \mathbb{I} - \pi : \mathscr{A} &\rightarrow \mathscr{A}_+ := \mathbb{C}[z, yz][[z]][[y]]/(e^y - z/m) \\ \sum_{j=0}^{\infty}\sum_{i=-n}^{\infty} a_{ij} z^i y^j &\mapsto \sum_{j \le i} a_{ij} z^i y^j \end{aligned}$$

that maps to the subalgebra of $\mathscr{A}$ that is finite at $z = 0$. Define $\mathscr{A}_-$ such that $\pi(\mathscr{A}) = \mathscr{A}_-$ and $\mathscr{A} = \mathscr{A}_- \oplus \mathscr{A}_+$.

In the rest of the paper, I apply the methods of [4], [5], and [6] to build a Hopf algebra of Feynman diagrams, define the counterterms using Birkhoff decomposition, and define the β function for cut-off regularization on the corresponding renormalization bundle.

2. The renormalization bundle

In [4], Connes and Kreimer build a Hopf algebra, $\mathcal{H}$, out of the divergence structure of the Feynman diagrams for a scalar field theory under dimensional regularization. They use the BPHZ algorithm to renormalize the theory, replacing Taylor subtraction around 0 external momenta with the minimal subtraction operator. The key to constructing this Hopf algebra is the sub-divergence structure of the graphs as defined by power counting arguments. The co-product of the Hopf algebra is defined to express the same sub-divergence data as in Zimmermann's subtraction formula for BPHZ renormalization [9]. Replacing the Taylor series operator in BPHZ for the minimal subtraction operator does not change the divergence structure of the diagrams. Therefore, I use the same Hopf algebra to study cut-off regularization. In [13], van Suijlekom constructs a Hopf algebra that captures the renormalization structure of QED under dimensional regularization. This is the same Hopf algebra that is needed to study QED under cut-off regularization. The arguments in this paper apply to scalar ϕ^4 and QED, even though cut-off regularization does not preserved the gauge symmetries of QED.

To briefly recall notation, let

$$\mathcal{H} = \mathbb{C}[\{1PI \text{ graphs with 2 or 4 external edges}\}]$$

to be the Hopf algebra of Feynman diagrams, with multiplication defined by disjoint union. It is graded by loop number, with Y the grading operator. If $\Gamma \in \mathcal{H}_n$, $Y(\Gamma) = n\Gamma$. The co-unit z is 0 on $\mathcal{H}_{\geq 1}$, and is the identity map on $\mathcal{H}_0$. An admissible sub-graph of a 1PI graph, Γ is a graph, γ, or product of graphs, that can be embedded into Γ with 2 or 3 external edges on each connected component. The graph $\Gamma//\gamma$ is the graph obtained by contracting each connected component of γ to a point. The admissible sub-graphs correspond to the divergences subtracted by Zimmermann's subtraction algorithm. Using Sweedler notation, the co-product on $\mathcal{H}$ is given by the sum

$$\Delta\Gamma = 1 \otimes \Gamma + \Gamma \otimes 1 + \sum_{\gamma \text{ admis}} \gamma \otimes \Gamma//\gamma \, .$$

Let ϵ and η denote the co-unit and unit of this Hopf algebra.

The Hopf algebra is connected and each graded component $\mathcal{H}_n$ is finitely generated as an algebra. Write the graded dual of this Hopf algebra $\mathcal{H}^* = \oplus_n \mathcal{H}_n^*$. The product on $\mathcal{H}^*$ is the convolution product $f \star g(\Gamma) = m(f \otimes g)\Delta(\Gamma)$. The antipode, S, on the restricted dual defines the inverse of a map under this convolution product, $f^{\star -1} = S(f)$. By the Milnor-Moore theorem, $\mathcal{H}^* \simeq \mathcal{U}(\mathfrak{g})$ is isomorphic to the universal enveloping algebra of the Lie algebra $\mathfrak{g}$, generated by the infinitesimal derivatives

$$\delta_\Gamma(\Gamma') = \begin{cases} 1 & \Gamma = \Gamma' \text{ 1PI} \\ 0 & \Gamma \neq \Gamma' \end{cases} .$$

The generators of the Lie algebra are infinitesimal characters

$$\delta_\Gamma(\gamma\Gamma') = \epsilon(\gamma)\delta_\Gamma(\Gamma') + \epsilon(\Gamma')\delta_\Gamma(\gamma) .$$

The Lie bracket is given by $[f, g] = f \star g - g \star f$. The corresponding Lie group $G = e + \mathfrak{g}$ is the group of algebra homomorphisms $\mathrm{Hom}_{alg}(\mathcal{H}, \mathbb{C}) = \mathrm{Spec}\ \mathcal{H}$. See [8] for more discussion of this Lie group and Lie algebra.

In this paper, I study regularization procedures that induce maps from from the Hopf algebra $\mathcal{H}$ to the algebra generated by the regulation parameter, $\mathscr{A}$. In dimensional regularization, z corresponds to the complex "dimension" regulator. In momentum cut-off regulation, z corresponds to the complexification of the inverse of the cut-off, $z = 1/\Lambda$, and polynomials in y correspond to polynomials in $\log(z/m)$. Minimal subtraction on both these regulation schemes is encoded by considering the direct sum decomposition $\mathscr{A} = \mathscr{A}_- \oplus \mathscr{A}_+$, where $\mathscr{A}_+ = \mathbb{C}[z, yz][[z]][[y]]/(e^y - z/m)$. The projection map

$$\pi : \mathscr{A} \to \mathscr{A}_-$$

is the subtraction map used in minimal subtraction. This projection map is a Rota-Baxter operator on $\mathscr{A}$. The algebra $\mathscr{A}$ and this Rota-Baxter operator are discussed in detail in [10] in the context of cut-off regularization, and other applications.

2.1. **Generalization of Birkhoff Decomposition.** Let $\varphi_{mc}, \varphi_{dr} \in \mathrm{Hom}_{alg}(\mathcal{H}, \mathscr{A})$ be the algebra homomorphisms from $\mathcal{H}$, the Hopf algebra of Feynman graphs, to $\mathscr{A}$ the algebra spanned by the regulating parameters corresponding to momentum cut-off regularization and dimensional regularization respectively. Paralleling the work of Connes and Kreimer in [4], I write the counterterm and the renormalized part of cut-off regularization and dimensional regularization under minimal subtraction as a Birkhoff-type decomposition of φ_{mc} and φ_{dr}.

Before proceeding, I introduce Rota-Baxter operators.

Definition 6. A Rota-Baxter operator, R, of weight θ on an algebra A is a linear map

$$R : A \to A$$

that satisfies the relationship

$$R(x)R(y) + \theta R(xy) = R(xR(y)) + R(R(x)y) .$$

The pair (A, R) is called a Rota-Baxter algebra.

Ebrahimi-Fard, Guo and Kreimer show that, if the algebra $\mathscr{A}$ is endowed with a Rota-Baxter subtraction operator, R, there is an unique expression for each $\varphi \in \mathrm{Hom}_{alg}(\mathcal{H}, \mathscr{A})$ as $\varphi_- \star \varphi_+$ such that φ_- lies in the image of R, if $x \in \mathcal{H}$, and φ_-, $\varphi_+ \in \mathrm{Hom}_{alg}(\mathcal{H}, \mathscr{A})$. If R corresponds to a subtraction operator for BPHZ, $\varphi_-(x)$ corresponds to the counterterm of x and $\varphi_+(x)$ the renormalized part [7]. The following theorem follows directly from this result.

Theorem 2.1. *Let $\varphi \in Hom_{alg}(\mathcal{H}, \mathscr{A})$. Define the projection map $\pi : \mathscr{A} \to \mathscr{A}_-$. There is a unique decomposition of each $\varphi = \varphi_-^{\star -1} \star \varphi_+$ with $\varphi_-(\Gamma) \in \mathscr{A}_-$ for $\Gamma \in \ker \epsilon$, $\phi_-(1) = 1$ and $\varphi(\Gamma) \in G(\mathscr{A}_+)$.*

Proof. Notice that $\pi : \mathscr{A} \to \mathscr{A}_-$ is a Rota-Baxter operator of weight 1. Let $\mathrm{Hom}(\mathcal{H}, \mathscr{A})$ be the algebra of linear maps from $\mathcal{H}$ to $\mathscr{A}$, with point-wise multiplication and unit $e = \eta_{\mathscr{A}} \circ \epsilon$. For $\varphi \in \mathrm{Hom}(\mathcal{H}, \mathscr{A})$, let $R = \pi \circ \varphi$. Then R is a Rota-Baxter operator on $\mathrm{Hom}(\mathcal{H}, \mathscr{A})$. By extending the convolution product on $\mathcal{H}^*$

to $\mathrm{Hom}(\mathcal{H}, \mathscr{A})$, each algebra homomorphism $\varphi \in \mathrm{Hom}_{alg}(\mathcal{H}, \mathscr{A})$ can be uniquely decomposed according to π. For all $\Gamma \in \ker(\epsilon)$,

$$\varphi_-(\Gamma) = -\pi(\varphi(\Gamma) + \sum_{\gamma \text{ admis.}} \varphi_-(\gamma)\varphi(\Gamma//\gamma)$$

$$\varphi_+(\Gamma) = (e - \pi)(\varphi(\Gamma) + \sum_{\gamma \text{ admis.}} \varphi_-(\gamma)\varphi(\Gamma//\gamma) \ .$$

The maps φ, φ_- and φ_+ are algebra homomorphisms from $\mathcal{H}$ to $\mathscr{A}$, $\mathbb{C} \oplus \mathscr{A}_-$ and $\mathscr{A}_+$ respectively. That is,

$$\varphi(1) = \varphi_-(1) = \varphi_+(1) = 1_{\mathscr{A}} \ .$$

However, for $\Gamma \in \ker(\epsilon)$, $\varphi_-(\Gamma) \in \mathscr{A}_-$. □

This is a generalization of the Birkhoff decomposition theorem, which says that any simple closed curve, C, in $\mathbb{CP}^1$ that does not pass through 0 or ∞, and a map

$$\varphi : C \to G \ ,$$

for a complex Lie group G, there is a function φ_- that is holomorphic on the connected component of $\mathbb{CP}^1 \setminus C$ that contains ∞ and a function φ_+ that is holomorphic on the connected component of $\mathbb{CP}^1 \setminus C$ that contains 0, such that $\varphi = \varphi_- \varphi_+$. In the setting of dimensional regularization, $\varphi_{dr} \in \mathrm{Hom}_{alg}(\mathcal{H}, \mathbb{C}[z^{-1}][[z]]) = G(\mathbb{C}[z^{-1}][[z]])$, is viewed as a map from a loop in Spec $\mathbb{C}[z^{-1}][[z]] \subset \mathbb{C}$ to $G =$ Spec $\mathcal{H}$. The Birkhoff decomposition theorem on loops directly gives the existence of such a decomposition. The Rota-Baxter algebra argument in loc. cit. generalizes the Birkhoff decomposition setting to other algebras.

2.2. **The renormalization bundle.** So far, I have considered φ_{mc} and $\varphi_{dr} \in \mathrm{Hom}_{alg}(\mathcal{H}, \mathscr{A})$ to be sections of a (trivial) G principal bundle over Spec $\mathscr{A}$. Call this bundle $K \simeq G \times \text{Spec } \mathscr{A} \to \text{Spec } \mathscr{A}$. Sections of this bundle correspond to algebra homomorphisms from $\mathcal{H}$ to $\mathscr{A}$. The maps $\varphi_{dr}(z)$ and $\varphi_{mc}(z, y)$ are two such maps. In this geometric context, consider the renormalization group $\mathbb{C}^\times$.

Regularization breaks the scale invariance of the Lagrangian defining the field theory. The action of the renormalization group on sections of K describes the scale dependence. In the following, I consider a more general renormalization group action that the one developed in [6].

Definition 7. Define the renormalization group as $\mathbb{C}^\times$. Parametrize it by $t = e^s$ for $s \in \mathbb{C}$.

Instead of the bundle $K \to \text{Spec } \mathscr{A}$, consider the bundle

$$P \simeq K \times \mathbb{C}^\times \to B \simeq \text{Spec } \mathscr{A} \times \mathbb{C}^\times \ .$$

Sections of this bundle correspond to the group $G(\mathscr{A} \otimes \mathbb{C}[t^{-1}, t])$. For a fixed t_0, $\varphi(t_0, z, y) \in G(\mathscr{A})$. The action of the renormalization group evaluates the character $\phi(t_0, z, y)$ at different values of t

$$\begin{array}{rcl} \mathbb{C}^\times \times G(\mathscr{A} \otimes \mathbb{C}[t, t^{-1}]) & \to & G(\mathscr{A} \otimes \mathbb{C}[t, t^{-1}]) \\ (t, \varphi(u, z, y)) & \mapsto & \varphi(tu, z(t), y) \ . \end{array}$$

From (2), we see that the action of the renormalization group gives

$$\varphi_{dr}(t, z) = t^{zY}\phi(1, z) \quad \text{ and } \quad \varphi_{mc}(t, z, y) = \varphi_{mc}(1, tz, y) \ .$$

The renormalization group in a momentum cut-off regularized theory scales the cut-off parameter $\frac{1}{z} = \Lambda$. In dimensional regularization, it manifests as a multiplicative factor of t^{zY}.

The β function of a theory is given by

$$\beta(\varphi) = t\frac{\partial}{\partial t}\left(\lim_{z\to 0} \varphi^{\star -1}(1, z, y) \star \varphi(t, z, y))\right)|_{t=1} \ .$$

For φ_{dr}, and $\sigma_t(z) = t^z$, the definition of this paper reduces to

$$\beta(\varphi_{dr}) := t\frac{\partial}{\partial t}\left(\lim_{z\to 0} \varphi_{dr}^{\star -1}(z) \star t^{Yz}\varphi_{dr}(1, z)\right)|_{t=1}$$

as defined in [8].

To understand this expression for the $\beta(\varphi)$ recursively, define an operator

$$R(\phi(t,z,y)) := \phi(t,z,y)^{\star -1} \star t\frac{\partial}{\partial t}\phi(t,z,y) \ .$$

The operator R defines a vector field in $\mathfrak{g}(\mathscr{A})$ parametrized by t. To evaluate this at a point $t = t_0$ write

$$R(\phi(t_0,z,y)) := \phi(t_0,z,y)^{\star -1} \star t\frac{\partial}{\partial t}\phi(t,z,y)|_{t=t_0} \ .$$

Theorem 2.2. *The map R is a map from $G(\mathscr{A}\otimes\mathbb{C}[t^{-1},t])$ to $\mathfrak{g}(\mathscr{A}\otimes\mathbb{C}[t^{-1},t])$.*

Proof. To check that $R(\phi) \in \mathfrak{g}(\mathscr{A}\otimes\mathbb{C}[t^{-1},t])$, notice that for any element $\alpha(t,z,y) \in \mathfrak{g}(\mathscr{A}\otimes\mathbb{C}[t^{-1},t])$ and $\phi(t,z,y) \in G(\mathscr{A}\otimes\mathbb{C}[t^{-1},t])$, $\phi\star\alpha$ is an infinitesimal character of $\mathcal{H}$, and thus in $\mathfrak{g}(\mathscr{A}\otimes\mathbb{C}[t^{-1},t])$. For $\rho,\gamma\in\mathcal{H}$,

$$\phi\star\alpha(\gamma\rho) = \phi\otimes\alpha(\Delta(\gamma\rho)) \ .$$

Using Sweedler notation, write $\Delta(\gamma\rho) = \sum \gamma'\rho'\otimes\gamma''\rho''$ and

$$\phi\star\alpha(\gamma\rho) = \sum \phi(x')\phi(y')\otimes[\alpha(x'')\varepsilon(y'') + \alpha(y'')\varepsilon(x'')]) = \phi\star\alpha(\gamma)\varepsilon(\rho) + \phi\star\alpha(\rho)\varepsilon(\gamma) \ .$$

The term $t\frac{\partial}{\partial t}(\varphi(t,z,y)) \in \mathfrak{g}(\mathscr{A}\otimes\mathbb{C}[t,t^{-1}])$ since it is a derivative of a one parameter family of elements in $G(\mathscr{A})$. □

Remark 2. When R is acting on $\varphi_{dr}(t,z)$

$$R(\varphi_{dr}(1,z)) = z\tilde{R}(\varphi_{dr}(1,z)) = \varphi_{dr}(1,z)^{-1}\star zY\varphi_{dr}(1,z) \ .$$

This is a bijection between $G(\mathbb{C}[z^{-1},z]])$ and $\mathfrak{g}(\mathbb{C}[z^{-1},z]])$.

The function $\beta(\phi)$ can be defined as $\lim_{z\to 0} R(\phi(1,z,y))$, if the limit exists.

Definition 8. A section $\varphi(t,z,y)$ has local counterterms if $t\frac{\partial}{\partial t}\varphi_-(t,z,y) = 0$.

Theorem 2.3. *The limit $\lim_{z\to 0} R(\varphi(1,z,y))$, and thus $\beta(\varphi)$ exists if φ has local counterterms.*

This proof is a generalization of arguments in [8].

Proof. The limit exists if and only if $R(\varphi(1,z,y)) \in \mathfrak{g}(\mathscr{A}_+)$. By construction, for a general $\varphi \in G(\mathscr{A}_+)$, $R(\varphi(t,z,y)) \in \mathfrak{g}(\mathscr{A}_+)$.

Define

$$h(t)(\varphi) = \varphi(1,z,y)^{\star -1}\star\varphi(t,z,y) \ .$$

Then one can write

$$R(\varphi(t,z,y)) = R(\varphi(1,z,y)\star h(t)(\varphi) =$$
$$h^{\star -1}(t)(\varphi)\star\varphi^{\star -1}(1,z,y)\star\varphi(1,z,y)\star t\frac{\partial}{\partial t}h(t)(\varphi) = R(h(t)(\phi)) \ .$$

If $t\frac{\partial}{\partial t}\varphi_-(t,z,y) = 0$ then

$$\begin{aligned} h(t)(\varphi) &= \varphi(1,z,y)_+^{\star -1}\star\varphi(1,z,y)_-\star\varphi(t,z,y)_-^{\star -1}\star\varphi(t,z,y)_+ \\ &= \varphi(1,z,y)_+^{\star -1}\star\varphi(t,z,y)_+ \in G(\mathscr{A}_+) \ , \end{aligned}$$

and $R(\varphi(1)) \in \mathfrak{g}(\mathscr{A}_+)$. □

In the next section, we show that $R(\varphi)$ is a $\mathfrak{g}(\mathscr{A})$ valued function on sections of $P\to B$, and is uniquely defined by a connection on the bundle. Thus, $\beta(\varphi)$ is a $\mathfrak{g}(\mathscr{A}_+)$ valued function on sections of $P\to B$ with local counterterms, which is defined by pullbacks of the connection along these sections.

2.3. **A connection on the renormalization bundle.** The functions φ_{dr} and φ_{mc} both have local counterterms. Thus the β functions $\beta(\varphi_{dr})$ and $\beta(\varphi_{mc})$ are well defined. They are both uniquely defined by a global connection on $P \to B$. This brings me to the main theorem of the paper, which I prove later.

Main Theorem. *The two β functions $\beta(\varphi_{dr})$ and $\beta(\varphi_{mc})$ can be related by a gauge transformation on the bundle $P \to B$.*

There is a flat connection on the bundle $P \to B$, defined by $R(\varphi)$. It can be constructed on sections of the bundle by logarithmic derivatives, as shown below. The following construction is generalization of those presented in [1].

Let ω be the connection on P defined by $R(\varphi(t, z, y))$, defined on pullbacks along sections by the logarithmic differential operator, as in [6].

Definition 9. Let D be a differential operator.

$$\begin{aligned} D : G(\mathscr{A} \otimes \mathbb{C}[t^{-1}, t]) &\to \Omega^1(\mathfrak{g}(\mathscr{A} \otimes \mathbb{C}[t^{-1}, t])) \\ \varphi(t, z, y) &\mapsto \varphi(t, z, y)^{\star -1} \star d(\varphi(t, z, y)) \ . \end{aligned}$$

Lemma 2.4. *For $f,\ g \in G(\mathscr{A} \otimes \mathbb{C}[t^{-1}, t])$, the differential $D(f) = f^\star\omega$ defines a connection on section f of $P \to B$.*

Proof. If D defines a connection, it must satisfy equation

$$(f^{\star -1} \star g)^*\omega = g^{-1}dg + g^{\star -1}(f^*\omega)g \ , \tag{4}$$

for $f,\ g \in G(\mathscr{A} \otimes \mathbb{C}[t^-1, t])$. Since $df^{-1} = -f^{-1}dff^{-1}$,

$$D(f^{-1}g) = Dg - g^{-1}ff^{-1}dff^{-1}g \ ,$$

or

$$Dg = D(f^{-1}g) + (f^{-1}g)^{-1}Df(f^{-1}g) \ .$$

which satisfies equation (4). □

Proposition 2.5. *For any section φ,*

$$D(\varphi) = \varphi^*\omega = \left((1 + z)\varphi(t, z)^{\star -1} \star \frac{\partial}{\partial z}\varphi(t, z)\right) dz + R(\varphi(t, z))dt \ .$$

Proof. I use the relation $mz = e^y$ to write $\frac{\partial}{\partial y} = z\frac{\partial}{\partial z}$, and to write the sections only as a function of t and z, $\varphi(t, z)$. This notation convention gives the factor of $1 + \frac{1}{z}$ in the coefficient for dz.

The coefficient of dt comes from the definition of $R(\varphi)$. □

Write $\varphi^*\omega = a_\varphi dz + b_\varphi dt$, with a_φ and b_φ are defined as

$$a_\varphi(t, z) = (1 + z)\varphi(t, z)^{\star -1} \star \frac{\partial}{\partial z}\varphi(t, z) \tag{5}$$

$$b_\varphi(t, z) = R(\varphi(t, z)) \ . \tag{6}$$

Proposition 2.6. *The connection ω is flat.*

Proof. It is sufficient to check that each pullback is flat. That is, that all the pullbacks satisfy

$$[a_\varphi(t, z), b_\varphi(t, z)] = \partial_t(a_\varphi(t, z)) - \partial_z(b_\varphi(t, z)) \ .$$

□

Proposition 2.7. *The global connection ω is defined by the map R.*

Proof. The coefficient $a_\varphi(t, z)$ in the expression for $\varphi^*\omega$ can be expressed in terms of $b_\varphi(t, z)$,

$$a_\varphi(t, z) = (1 + \frac{1}{z})(R^{-1}(b_\varphi(t, z)) \star \frac{\partial}{\partial z}R^{-1}(b_\varphi(t, z)) \ .$$

Therefore $\varphi^*\omega$ can be written in terms of element of the Lie algebra $R(\varphi)$,and ω can be written in terms of the bijection tR. □

The logarithmic differential operator defining the connection ω has a symmetry under right multiplication by sections of the form $\varphi(t,z) \in G(\mathscr{A} \otimes \mathbb{C}[t^{-1},t])$.

Definition 10. Two pullbacks of the connection $\varphi^*\omega$ and $\varphi'^*\omega$ are equivalent if and only if one can be written in terms of the action of $G(\mathscr{A}_+ \otimes \mathbb{C}[t^{-1},t])$ on the other

$$\varphi'^*\omega = D\psi + \psi^{\star -1} \star \gamma^*\omega \star \psi$$

for $\psi \in G(\mathscr{A}_+ \otimes \mathbb{C}[t^{-1},t])$.

Remark 3. This gauge equivalence is the same as the statement $\varphi'(t,z) = \varphi(t,z) \star \psi(t,z)$, and specifically, $\varphi'_- = \varphi_-$. The gauge equivalence on the connection classifies pullbacks by the counterterms of the corresponding sections.

The renormalization group action on this bundle is more complicated than the action on the renormalization bundle for dimensional regularization. Therefore, the flat global connection defined by the logarithmic differential is not $\mathbb{C}^\times$ equivariant. However, if a section φ has local counterterms, then $\varphi_-^*\omega$ is $\mathbb{C}^\times$ equivariant. This gives a definition of an equisingular connection on this bundle.

Definition 11. The connection $\varphi^*\omega$ along on $P \to B$ is equisingular when pulled back to the bundle $P \to \operatorname{Spec} \mathscr{A}$ if and only if, for every pair of sections σ, σ' of the $B \to \Delta^*$ bundle, $\sigma(0) = \sigma'(0)$, the corresponding pull backs of the connection ω, $\sigma^*(\varphi^*\omega)$ and $\sigma'^*(\varphi^*\omega)$ are equivalent under the action of $G(\mathscr{A}_+ \otimes \mathbb{C}[t^{-1},t])$.

In other words, $\varphi^*\omega$ is equisingular if and only if φ has local counterterms. This gives the generalization of the main theorem in [6] to this renormalization bundle:

Theorem 2.8. *The flat equisingular connection defined $\varphi^*\omega(z,t) = \varphi^{-1} \star D(\varphi)(t,z)$, on $P \to B$ is uniquely defined by $\beta(\varphi)$.*

This brings me to the main theorem of this paper, which can be restated as

Theorem 2.9. *The β functions $\beta(\varphi_{dr})$ and $\beta(\varphi_{mc})$ can be geometrically related by the connections they define on the bundle $P \to B$ by the gauge transformation*

$$D(\varphi_{dr} \star \varphi_{mc}) = D(\varphi_{mc}) + \varphi_{mc}^{\star -1} \star D(\varphi_{dr}) \star \varphi_{mc} \,.$$

Proof. The relation in the theorem comes directly from the definition of gauge transformations. Since both φ_{dr} and φ_{mc} have local counterterms, the connections are defined by $\beta(\varphi_{dr})$ and $\beta(\varphi_{mc})$. □

The significance of this theorem is a geometric relationship between the $\beta(\varphi_{dr})$ and $\beta(\varphi_{mc})$. Loop-wise calculations for the β functions for dimensionally regularized and cut-off regularized quantum electrodynamics give different values, even at the 1-loop approximation. This gives a geometric structure for understanding the relation between the two. For a scalar field theory, the β functions for one loop graphs agree at the first three loop levels [9]

While this paper has specifically examined a sharp momentum cut-off regulator, there are other related regulation scheme, such as smooth cut-off or Paul-Villars regularization, that also have a structure of logarithmic singularities and finite order poles. Theories under these regularization schemes, and their β functions, can also be expressed in terms of sections and connections of this renormalization bundle.

References

1. Susama Agarwala, *A perspective on renormalization*, Letters in Mathematical Physics **93** (2010), 187–201, arXiv:0909.4117.
2. John C. Collins, *Normal products in dimensional regularization*, Nuclear Physics B **92** (1975), no. 4, 477 – 506.
3. ———, *Renormalization: An introduction to renormalization, the renormalization group, and the operator product expansion*, Cambridge University Press, New York, 1984.
4. Alain Connes and Dirk Kreimer, *Renormalization in quantum field theory and the Riemann-Hilbert problem I: The Hopf algebra structure of graphs and the main theorem*, Communications in Mathematical Physics **210** (2001), 249–273, arXiv:hep-th/9912092v1.
5. ———, *Renormalization in Quantum Field Theory and the Riemann-Hilbert problem II: The β function, diffeomorphisms and renormalization group*, Communications in Mathematical Physics **216** (2001), 215–241, arXiv:hep-th/0003188v1.
6. Alain Connes and Matilde Marcolli, *Quantum fields and motives*, Journal of Geometry and Physics **56** (2006), 55–85.

7. Kurusch Ebrahimi-Fard, Li Guo, and Dirk Kreimer, *Spitzer's identity and the algebraic Birkhoff decomposition in pQFT*, Journal of Physics A **37** (2004), 11037–11052, arXiv:hep-th/0407082v1.
8. Kurusch Ebrahimi-Fard and Dominique Manchon, *On matrix differential equations in the Hopf algebra of renormalization*, Advances in Theoretical and Mathematical Physics **10** (2006), 879–913, arXiv:math-ph/0606039v2.
9. Claude Itzykson and Jean-Bernard Zuber, *Quantum field theory*, Dover ed., Dover Publications, Inc., Mineola, New York, 2005.
10. Dominique Manchon and Sylvie Paycha, *Nested sums of symbols and renormalized multiple zeta values*, International Mathematics Reseach Notices (2010).
11. Eugene R. Speer, *The convergence of BPH renormalization*, Communications in mathematical physics **35** (1974), 151–154.
12. Robin Ticciati, *Quantum field theory for mathematicians*, Cambridge University Press, New York, 1999.
13. Walter van Suijlekom, *The Hopf algebra of Feynman graphs in QED*, Letters in Mathematical Physics **77** (2006), 265–281, arXiv:hep-th/0602126v2.

California Institute of Technology, Mathematics Department, MC 253-37, 1200 E California Ave., Pasadena, CA 91106