A subset of $\{1,2,3,\ldots\}^n$ whose non-computability leads to the existence of a Diophantine equation whose solvability is logically undecidable

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Abstract. Let $B(n) = \Big\{(x_1,\ldots,x_n) \in \{1,2,3,\ldots\}^n$: for each positive integers $y_1,\ldots,y_n$ the conjunction

$$\Big(\forall i,j,k \in \{1,\ldots,n\}\ (x_i + x_j = x_k \Longrightarrow y_i + y_j = y_k)\Big) \wedge$$
$$\forall i,j,k \in \{1,\ldots,n\}\ (x_i \cdot x_j = x_k \Longrightarrow y_i \cdot y_j = y_k)$$

implies that $x_1 = y_1\Big\}$. We conjecture that the sets $B(n)$ are not computable for sufficiently large values of n. We prove: if the set $B(n)$ is not computable for some n, then there exists a Diophantine equation whose solvability in positive integers (non-negative integers, integers) is logically undecidable.

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Let $B(n) = \Big\{(x_1,\ldots,x_n) \in \{1,2,3,\ldots\}^n$: for each positive integers $y_1,\ldots,y_n$ the conjunction

$$\Big(\forall i,j,k \in \{1,\ldots,n\}\ (x_i + x_j = x_k \Longrightarrow y_i + y_j = y_k)\Big) \wedge$$
$$\forall i,j,k \in \{1,\ldots,n\}\ (x_i \cdot x_j = x_k \Longrightarrow y_i \cdot y_j = y_k)$$

implies that $x_1 = y_1\Big\}$.

The tuple $A = (3,\ 9,\ 27,\ 26,\ 25,\ 5,\ 1)$ belongs to $B(7)$ if and only if in the domain of positive integers the system

$$\Big\{y_i + y_j = y_k :\ (i, j, k \in \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7\}) \wedge (A[i] + A[j] = A[k])\Big\} \cup$$

$$\Big\{y_i \cdot y_j = y_k :\ (i, j, k \in \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7\}) \wedge (A[i] \cdot A[j] = A[k])\Big\}$$

implies that $y_1 = 3$. It means that the system

$$\begin{cases} y_4 + y_7 &= y_3 \\ y_5 + y_7 &= y_4 \\ y_1 \cdot y_1 &= y_2 \\ y_1 \cdot y_2 &= y_3 \\ y_6 \cdot y_6 &= y_5 \\ y_1 \cdot y_7 &= y_1 \\ y_2 \cdot y_7 &= y_2 \\ y_3 \cdot y_7 &= y_3 \\ y_4 \cdot y_7 &= y_4 \\ y_5 \cdot y_7 &= y_5 \\ y_6 \cdot y_7 &= y_6 \\ y_7 \cdot y_7 &= y_7 \end{cases}$$

implies that $y_1 = 3$. In the domain of positive integers, the last seven equations say that $y_7 = 1$. Therefore, in the domain of positive integers the system equivalently expresses that $y_6^2 + 2 = y_1^3$. Hence, $(3,\ 9,\ 27,\ 26,\ 25,\ 5,\ 1) \in B(7)$ if and only if in the domain of positive integers only the pair $(5, 3)$ solves the equation $x^2 + 2 = y^3$. The last claim is true, see [6, pp. 398–399].

The statement

$$\Big(238,\ 239,\ 239^2,\ 239^2 + 1,\ 13,\ 13^2,\ 13^4,\ 1\Big) \in B(8)$$

equivalently expresses that in the domain of positive integers only the pair $(238, 13)$ solves the equation $(x + 1)^2 + 1 = 2y^4$. The last claim is true, see [3], [5] and [1].

The statement

$$\big(164,\ 165,\ 164\cdot 165,\ (164\cdot 165)^2,$$

$$132,\ 133,\ 132\cdot 133,\ (132\cdot 133)^2,$$

$$143,\ 144,\ 143\cdot 144,\ (143\cdot 144)^2,\ 1\big) \in B(13)$$

equivalently expresses that in the domain of positive integers only the triples $(132, 143, 164)$ and $(143, 132, 164)$ solve the equation

$$x^2(x+1)^2 + y^2(y+1)^2 = z^2(z+1)^2$$

The last claim is still not proved, see [4, p. 53].

Conjecture. *The sets $B(n)$ are not computable for sufficiently large values of n.*

Lemma. *For each integers a_1, y_1, we have $a_1 \neq y_1$ if and only if there exists a positive integer x such that $(a_1 - y_1 - x)(y_1 - a_1 - x) = 0$.*

The conclusion of the following Theorem is unconditionally true and well-known as the corollary of the negative solution to Hilbert's Tenth Problem, see [2, p. 231].

Theorem. *If the set $B(n)$ is not computable for some n, then there exists a Diophantine equation whose solvability in positive integers (non-negative integers, integers) is logically undecidable.*

Proof. To a tuple $(a_1, \ldots, a_n)$ of positive integers we assign the equation

$$D_{(a_1,\ldots,a_n)}(x, y_1, \ldots, y_n) = (a_1 - y_1 - x)^2(y_1 - a_1 - x)^2 +$$

$$\sum_{\substack{(i,j,k)\in\{1,\ldots,n\}^3 \\ a_i + a_j = a_k}} (y_i + y_j - y_k)^2 + \sum_{\substack{(i,j,k)\in\{1,\ldots,n\}^3 \\ a_i \cdot a_j = a_k}} (y_i \cdot y_j - y_k)^2 = 0$$

By the Lemma, for each positive integers $a_1, \ldots, a_n$, the tuple $(a_1, \ldots, a_n)$ does not belong to $B(n)$ if and only if the equation $D_{(a_1,\ldots,a_n)}(x, y_1, \ldots, y_n) = 0$ has a solution in positive integers $x, y_1, \ldots, y_n$. We prove that there exist positive integers $a_1, \ldots, a_n$ for which the solvability of the equation $D_{(a_1,\ldots,a_n)}(x, y_1, \ldots, y_n) = 0$ in positive integers $x, y_1, \ldots, y_n$ is logically undecidable. Suppose, on the contrary, that for each positive integers $a_1, \ldots, a_n$ the solvability of the equation $D_{(a_1,\ldots,a_n)}(x, y_1, \ldots, y_n) = 0$ can be either proved or disproved. This would yield the following algorithm for deciding whether for positive integers $a_1, \ldots, a_n$ the tuple $(a_1, \ldots, a_n)$ belongs to $B(n)$: examine all proofs (in order of length) until for the equation $D_{(a_1,\ldots,a_n)}(x, y_1, \ldots, y_n) = 0$ a proof that resolves the solvability question one way or the other is found.

If a Diophantine equation $W(x_1, \ldots, x_n) = 0$ is logically undecidable in positive integers, then the equation $W(x_1 + 1, \ldots, x_n + 1) = 0$ is logically undecidable in non-negative integers. By Lagrange's four-square theorem, if a Diophantine equation $W(x_1, \ldots, x_n) = 0$ is logically undecidable in positive integers, then the equation

$$W^2(x_1, \ldots, x_n) +$$
$$\left(x_1 - 1 - t_{1,1}^2 - t_{1,2}^2 - t_{1,3}^2 - t_{1,4}^2\right)^2 + \ldots + \left(x_n - 1 - t_{n,1}^2 - t_{n,2}^2 - t_{n,3}^2 - t_{n,4}^2\right)^2 = 0$$

is logically undecidable in integers. □

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