Determinant formula for the partition function of the six-vertex model with a non-diagonal reflecting end

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Abstract

With the help of the F-basis provided by the Drinfeld twist or factorizing F-matrix for the open XXZ spin chain with non-diagonal boundary terms, we obtain the determinant representation of the partition function of the six-vertex model with a non-diagonal reflecting end under domain wall boundary condition.

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1 Introduction

The domain wall (DW) boundary condition of a statistical model on a finite two-dimensional lattice was introduced in [1] for the six-vertex model. The partition function of the model (or DW partition function) was then given in terms of some determinant [2, 3]. Such a determinant representation of the partition function has played an important role in constructing norms of Bethe states, correction functions [4, 5, 6] and thermodynamical properties of the six-vertex model [7], and also in the Toda theories [8]. Moreover, it has been proven to be very useful in solving some pure mathematical problems, such as the problem of alternating sign matrices [9]. Recently, the determinant representations of the DW partition function have been obtained for various models [10, 11, 12].

Since the DW partition function is calculated [3] as an inner product of pseudo-vacuum and some Bethe state which is generated by pseudo-particle creation operators (given by one-row monodromy matrix related to closed spin chain [5]) on the pseudo-vacuum state, the computation of the function is simplified dramatically and can be directly calculated in the so-called F-basis [13] provided by the Drinfeld twist [14] or factorizing F-matrix. Such a magic F-matrix has been studied extensively for other models [15, 16, 17, 18, 19, 20].

For a two-dimensional statistical model with a reflection end [21], in addition to the local interaction vertex, a reflecting matrix or K-matrix which describes the boundary interactions needs to be introduced at the reflection end of the lattice (see figure 4 below). This problem is closely related to that of open spin chain [22]. The DW partition function of the six-vertex model with a diagonal reflection end was exactly calculated and expressed in terms some determinant [21]. Then such an explicit determinant representation was re-derived [23, 24] by using F-basis of the closed XXZ chain. However, it is well known that to obtain exact solution of open spin chain with non-diagonal boundary terms is very *non-trivial* [25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41] comparing with that of the one with simple diagonal boundary terms. In this paper, we will investigate the determinant representation of the DW partition function of the six-vertex model with a non-diagonal reflection end which is specified by a generic non-diagonal K-matrix found in [42, 43]. The result will be essential to construct the determinant representations of scalar products of Bethe states of the open XXZ chain with non-diagonal boundary terms [44].

The paper is organized as follows. In section 2, after introducing our notation and some basic ingredients, we construct the four boundary states which specify the DW boundary condition of the six-vertex model with a non-diagonal reflection end. In section 3, using the vertex-face correspondence relation we express the DW boundary partition function in terms of the particular matrix element of the product of face type pseudo-particle creation operators. In section 4, we present the F-matrix of the open XXZ chain with non-diagonal boundary terms and give the completely symmetric and polarization free representations of the pseudo-particle creation operators in the F-basis. In section 5, with the help of the F-basis provided by the F-matrix we obtain the determinant representation of the DW partition function. In section 6, we summarize our results and give some discussions. Some detailed technical proof is given in Appendix A.

2 Six-vertex model with a reflecting end

In this section, we briefly review the DW boundary condition for the six-vertex model with non-diagonal reflecting end on an $N \times 2N$ rectangular lattice.

2.1 The six-vertex R-matrix and associated K-matrix

Throughout, V denotes a two-dimensional linear space. The well-known six-vertex model R-matrix $\overline{R}(u) \in \mathrm{End}(V \otimes V)$ [5] is given by

$$\overline{R}(u) = \begin{pmatrix} a(u) & & & \\ & b(u) & c(u) & \\ & c(u) & b(u) & \\ & & & a(u) \end{pmatrix}. \tag{2.1}$$

The coefficient functions read: $a(u) = 1$, $b(u) = \frac{\sin u}{\sin(u+\eta)}$, $c(u) = \frac{\sin \eta}{\sin(u+\eta)}$. Here we assume η is a generic complex number. The R-matrix satisfies the quantum Yang-Baxter equation (QYBE),

$$R_{1,2}(u_1 - u_2)R_{1,3}(u_1 - u_3)R_{2,3}(u_2 - u_3) = R_{2,3}(u_2 - u_3)R_{1,3}(u_1 - u_3)R_{1,2}(u_1 - u_2), \tag{2.2}$$

and the unitarity, crossing-unitarity and quasi-classical properties [28]. We adopt the standard notations: for any matrix $A \in \mathrm{End}(V)$, A_j (or A^j) is an embedding operator in the tensor space $V \otimes V \otimes \cdots$, which acts as A on the j-th space and as identity on the other factor spaces; $R_{i,j}(u)$ is an embedding operator of R-matrix in the tensor space, which acts as identity on the factor spaces except for the i-th and j-th ones.

For a model with reflection end or open spin chain [22], in addition to the R-matrix, one needs to introduce K-matrix $K(u)$ which satisfies the reflection equation (RE)

$$\overline{R}_{1,2}(u_1-u_2)K_1(u_1)\overline{R}_{2,1}(u_1+u_2)K_2(u_2) \\ = K_2(u_2)\overline{R}_{1,2}(u_1+u_2)K_1(u_1)\overline{R}_{2,1}(u_1-u_2). \tag{2.3}$$

In this paper, we consider the K-matrix $K(u)$ which is a generic solution to the RE (2.3) associated the six-vertex model R-matrix [42, 43]

$$K(u) = \begin{pmatrix} k_1^1(u) & k_2^1(u) \\ k_1^2(u) & k_2^2(u) \end{pmatrix}. \tag{2.4}$$

The coefficient functions are

$$\begin{aligned} k_1^1(u) &= \frac{2\cos(\lambda_1-\lambda_2)-\cos(\lambda_1+\lambda_2+2\zeta)e^{-2iu}}{4\sin(\lambda_1+\zeta+u)\sin(\lambda_2+\zeta+u)}, \\ k_2^1(u) &= \frac{-i\sin(2u)e^{-i(\lambda_1+\lambda_2)}e^{-iu}}{2\sin(\lambda_1+\zeta+u)\sin(\lambda_2+\zeta+u)}, \\ k_1^2(u) &= \frac{i\sin(2u)e^{i(\lambda_1+\lambda_2)}e^{-iu}}{2\sin(\lambda_1+\zeta+u)\sin(\lambda_2+\zeta+u)}, \\ k_2^2(u) &= \frac{2\cos(\lambda_1-\lambda_2)e^{-2iu}-\cos(\lambda_1+\lambda_2+2\zeta)}{4\sin(\lambda_1+\zeta+u)\sin(\lambda_2+\zeta+u)}. \end{aligned} \tag{2.5}$$

It is very convenient to introduce a vector $\lambda \in V$ associated with the boundary parameters $\{\lambda_i\}$,

$$\lambda = \sum_{k=1}^{2} \lambda_k \epsilon_k, \tag{2.6}$$

where $\{\epsilon_i,\, i=1,2\}$ form the orthonormal basis of V such that $\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}$.

2.2 The model

The partition function of a statistical model on a two-dimensional lattice is defined by the following:

$$Z = \sum \exp\{-\frac{E}{kT}\},$$

where E is the energy of the system, k is the Boltzmann constant, T is the temperature of the system, and the summation is taken over all possible configurations under the particular boundary condition such as the DW boundary condition. The model we consider here has six allowed local bulk vertex configurations

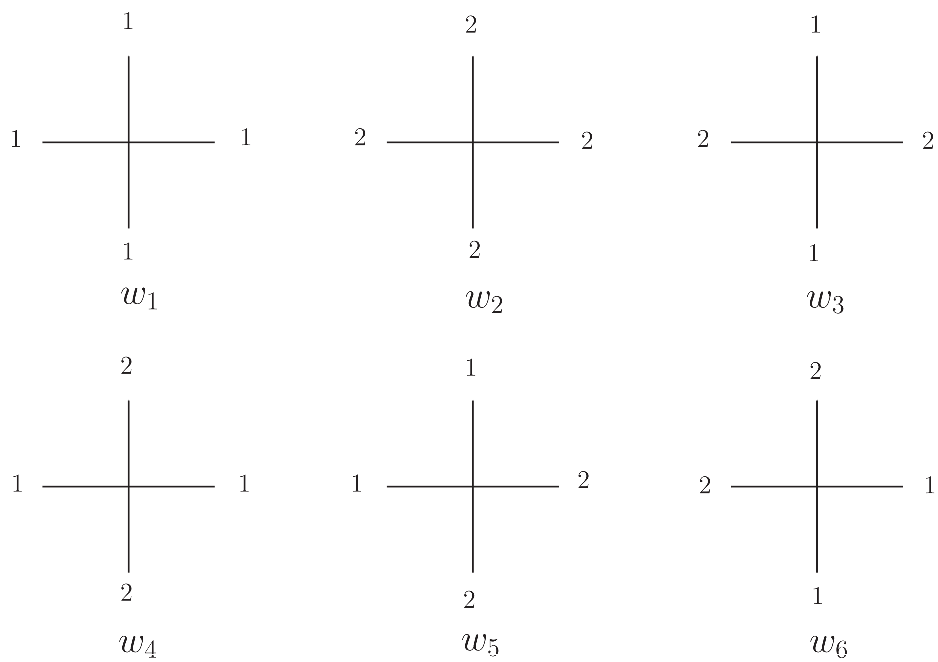

Figure 1. Vertex configurations and their associated Boltzmann weights.

and four allowed configurations at each reflection end

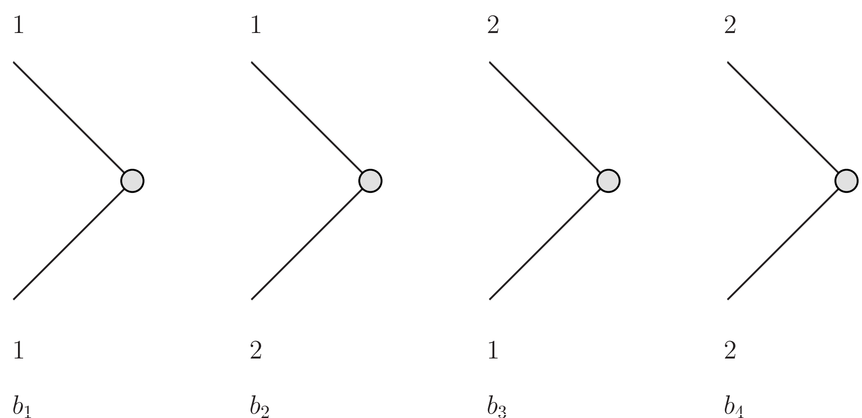

Figure 2. Reflection ends and the associated Boltzmann weights.

where 1 and 2 respectively denote the spin up and down states. Each of the six bulk configurations is assigned a statistical weight (or Boltzmann weight) w_i, while each of the four reflection configurations is assigned a weight b_i (see Figs. 1 and 2). Then the partition function of the model with a reflection end can be rewritten as

$$Z = \sum w_1{}^{n_1}\, w_2{}^{n_2}\, w_3{}^{n_3}\, w_4{}^{n_4}\, w_5{}^{n_5}\, w_6{}^{n_6}\, b_1^{l_1}\, b_2^{l_2}\, b_3^{l_3}\, b_4^{l_4},$$

where the summation is over all possible configurations with n_i and l_j being the number of the vertices of type i and the number of the reflection end of type j respectively. The bulk Boltzmann weights which we consider here have Z_2-symmetry, i.e.,

$$a \equiv w_1 = w_2, \quad b \equiv w_3 = w_4, \quad c \equiv w_5 = w_6, \tag{2.7}$$

and the variables a, b, c satisfy a function relation, or equivalently, the local Boltzmann weights $\{w_i\}$ can be parameterized by the matrix elements of the six-vertex R-matrix R (2.1) as in figure 3. At the same time, the weights $\{b_i\}$ corresponding to the reflection end can be parameterized by the matrix elements of the K-matrix K (2.4) as in figure 3.

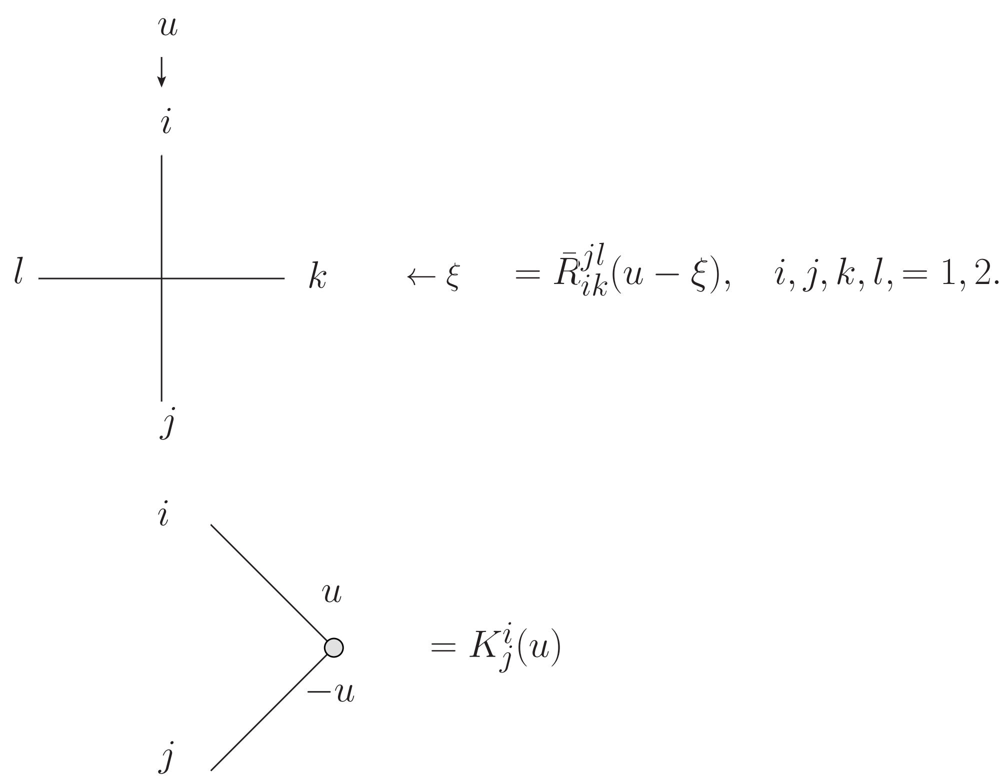

Figure 3. The Boltzmann weights and elements of the six-vertex R-matrix and K-matrix.

then the corresponding model is called the six-vertex model with a reflection end. Therefore the partition function of the model is give by

$$Z = \sum a^{n_1+n_2}\, b^{n_3+n_4}\, c^{n_5+n_6} b_1^{l_1}\, b_2^{l_2}\, b_3^{l_3}\, b_4^{l_4}.$$

In order to parameterize the local bulk Boltzmann weights in terms of the elements of the R-matrix, one needs to assign spectral parameters u and ξ respectively to the vertical line and horizontal line of each vertex of the lattice, as shown in figure 3. In an inhomogeneous model, the statistical weights are site-dependent. Hence two sets of spectral parameters $\{u_\alpha\}$ and $\{\xi_i\}$ are needed, see figure 4. The horizontal lines are enumerated by indices $1, \ldots, N$ with spectral parameters $\{\xi_i\}$, while the vertical lines are enumerated by indices $\bar{1}, \ldots, \bar{N}$ with spectral parameters $\{\bar{u}_\alpha\}$ (The $2N$ parameters $\{\bar{u}_\alpha\}$ are assigned as follow: $\bar{u}_{2i} = u_i$ and $\bar{u}_{2i+1} = -u_i$, as shown in figure 4.). The DW boundary condition is specified by four boundary states $|\Omega^{(2)}(\lambda)\rangle$, $|\bar{\Omega}^{(1)}(\lambda)\rangle$, $\langle\Omega^{(1)}(\lambda)|$ and $\langle\bar{\Omega}^{(2)}(\lambda)|$ (the definitions of the boundary states will be given later, see (2.32)-(2.35) below). These four states correspond to the particular choices of spin states on the four boundaries of the lattice .

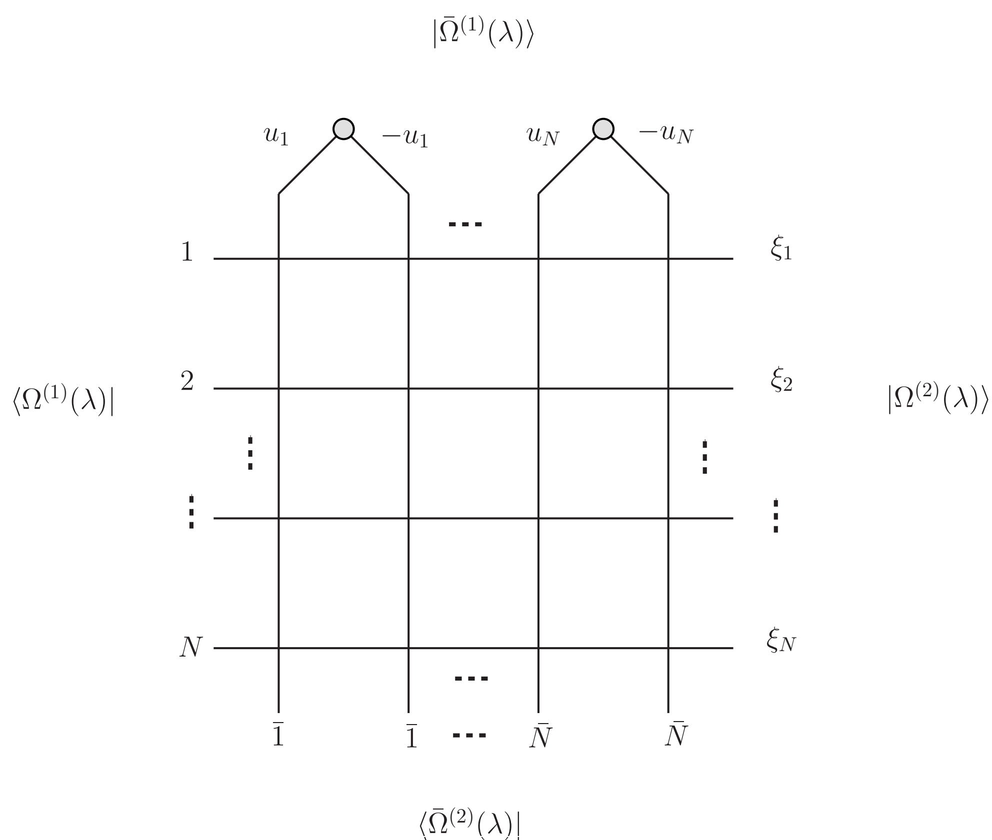

Figure 4. The six-vertex model with a non-diagonal reflection end under the DW condition.

Some remarks are in order. The boundary states not only depend on the spectral parameters ($|\Omega^{(2)}(\lambda)\rangle$ and $\langle\Omega^{(1)}(\lambda)|$ depend on $\{\xi_i\}$, while $|\bar{\Omega}^{(1)}(\lambda)\rangle$ and $\langle\bar{\Omega}^{(2)}(\lambda)|$ depend on $\{u_\alpha\}$) but also on two continuous parameters λ_1 and λ_2. However, after a diagonal similarity transformation generated by $\text{Diag}(1, e^{-i(\lambda_1+\lambda_2)})$ and then taking $\lambda_1 \to +i\infty$, the corresponding boundary states $|\Omega^{(2)}(\lambda)\rangle$ and $\langle\bar{\Omega}^{(2)}(\lambda)|$ (or $|\bar{\Omega}^{(1)}(\lambda)\rangle$ and $\langle\Omega^{(1)}(\lambda)|$) become the state of all spin down and its dual (or the state of all spin up and its dual) up to some over-all scalar factors. Moreover after the same similarity transformation and taking the limit the resulting K-matrix becomes the diagonal matrix solution of the reflection equation (2.3). Hence the partition function in the limit reduces to that of the six-vertex model [21].

The partition function of the six-vertex model with a non-diagonal reflection end specified by the generic K-matrix $K(u)$ (2.4) under the DW boundary condition is a function of $2N+3$ variables $\{u_\alpha\}$, $\{\xi_i\}$, λ_1, λ_2 and ζ, which is denoted by $Z_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)$. Due to the fact that the local Boltzmann weights of each vertex and reflection end of the lattice are given by the matrix elements of the six-vertex R-matrix and the associated K-matrix (see figure 3), the partition function can be expressed in terms of the product of the R-matrices, the K-matrix and the four boundary states

$$\begin{aligned} Z_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta) = \; & \langle\Omega^{(1)}(\lambda)|\langle\bar{\Omega}^{(2)}(\lambda)| \\ & \times\overline{R}_{\bar{1},N}(u_1-\xi_N)\ldots\overline{R}_{\bar{1},1}(u_1-\xi_1)K_{\bar{1}}(u_1)\overline{R}_{1,\bar{1}}(u_1+\xi_1)\ldots\overline{R}_{N,\bar{1}}(u_1+\xi_N) \\ & \qquad\vdots \\ & \times\overline{R}_{\bar{N},N}(u_N-\xi_N)\ldots\overline{R}_{\bar{N},1}(u_N-\xi_1)K_{\bar{N}}(u_N)\overline{R}_{1,\bar{N}}(u_N+\xi_1)\ldots\overline{R}_{N,\bar{N}}(u_N+\xi_N) \\ & \times|\bar{\Omega}^{(1)}(\lambda)\rangle|\Omega^{(2)}(\lambda)\rangle. \end{aligned} \tag{2.8}$$

One can rearrange the product of the R-matrices in (2.8) in terms of a product of the so-called double-row monodromy matrices

$$Z_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)=\langle\Omega^{(1)}(\lambda)|\langle\bar{\Omega}^{(2)}(\lambda)|\,\mathbb{T}_{\bar{1}}(u_1)\ldots\mathbb{T}_{\bar{N}}(u_N)\,|\bar{\Omega}^{(1)}(\lambda)\rangle|\Omega^{(2)}(\lambda)\rangle, \tag{2.9}$$

where the monodromy matrix $\mathbb{T}_{\bar{i}}(u)$ is given by

$$\begin{aligned} \mathbb{T}_{\bar{i}}(u) \equiv \mathbb{T}_{\bar{i}}(u;\xi_1,\ldots,\xi_N;\zeta) \;=\; & \overline{R}_{\bar{i},N}(u_i-\xi_N)\ldots\overline{R}_{\bar{i},1}(u_i-\xi_1) \\ & \times K_{\bar{i}}(u_i)\,\overline{R}_{1,\bar{i}}(u_i+\xi_1)\ldots\overline{R}_{N,\bar{i}}(u_i+\xi_N). \end{aligned} \tag{2.10}$$

The double-row matrix $\mathbb{T}(u)$ has played an important role to construct the transfer matrix for an open spin chain [22]. The QYBE (2.2) of the R-matrix and the reflection equation (2.3) of the K-matrix give rise to that the monodromy matrix $\mathbb{T}_{\bar{i}}(u)$ satisfy the following exchange relation

$$\overline{R}_{\bar{i},\bar{j}}(u_i-u_j)\,\mathbb{T}_{\bar{i}}(u_i)\,\overline{R}_{\bar{j},\bar{i}}(u_i+u_j)\,\mathbb{T}_{\bar{j}}(u_j) = \mathbb{T}_{\bar{j}}(u_j)\,\overline{R}_{\bar{i},\bar{j}}(u_i+u_j)\,\mathbb{T}_{\bar{i}}(u_i)\,\overline{R}_{\bar{j},\bar{i}}(u_i-u_j). \tag{2.11}$$

2.3 The boundary states

From the orthonormal basis $\{\epsilon_i\}$ of V, we define

$$\hat{\imath} = \epsilon_i - \overline{\epsilon}, \quad \overline{\epsilon} = \frac{1}{2}\sum_{k=1}^{2}\epsilon_k, \quad i = 1, 2, \qquad \text{then} \sum_{i=1}^{2}\hat{\imath} = 0. \tag{2.12}$$

Let $\mathfrak{h}$ be the Cartan subalgebra of A_1 and $\mathfrak{h}^*$ be its dual. A finite dimensional diagonalizable $\mathfrak{h}$-module is a complex finite dimensional vector space W with a weight decomposition $W = \oplus_{\mu\in\mathfrak{h}^*}W[\mu]$, so that $\mathfrak{h}$ acts on $W[\mu]$ by $x\,v = \mu(x)\,v$, $(x \in \mathfrak{h},\ v \in\ W[\mu])$. For example, the non-zero weight spaces of the fundamental representation $V_{\Lambda_1} = \mathbb{C}^2 = V$ are

$$W[\hat{\imath}] = \mathbb{C}\epsilon_i, \ \ i = 1, 2. \tag{2.13}$$

For a generic $m \in V$, define

$$m_i = \langle m, \epsilon_i\rangle, \quad m_{ij} = m_i - m_j = \langle m, \epsilon_i - \epsilon_j\rangle, \quad i, j = 1, 2. \tag{2.14}$$

Let $R(u, m) \in \mathrm{End}(V \otimes V)$ be the R-matrix of the six-vertex SOS model, which is trigonometric limit of the eight-vertex SOS model [45] given by

$$R(u;m)=\sum_{i=1}^{2} R(u;m)^{ii}_{ii}E_{ii}\otimes E_{ii}+\sum_{i\neq j}^{2}\left\{R(u;m)^{ij}_{ij}E_{ii}\otimes E_{jj}+R(u;m)^{ji}_{ij}E_{ji}\otimes E_{ij}\right\}, \tag{2.15}$$

where E_{ij} is the matrix with elements $(E_{ij})^l_k = \delta_{jk}\delta_{il}$. The coefficient functions are

$$R(u;m)^{ii}_{ii} = 1, \quad R(u;m)^{ij}_{ij} = \frac{\sin u \sin(m_{ij} - \eta)}{\sin(u + \eta)\sin(m_{ij})}, \quad i \neq j, \tag{2.16}$$

$$R(u;m)^{ji}_{ij} = \frac{\sin\eta \sin(u + m_{ij})}{\sin(u + \eta)\sin(m_{ij})}, \quad i \neq j, \tag{2.17}$$

and m_{ij} is defined in (2.14). The R-matrix satisfies the dynamical (modified) quantum Yang-Baxter equation (or the star-triangle relation) [45]

$$\begin{aligned} &R_{1,2}(u_1 - u_2; m - \eta h^{(3)})R_{1,3}(u_1 - u_3; m)R_{2,3}(u_2 - u_3; m - \eta h^{(1)}) \\ &\quad = R_{2,3}(u_2 - u_3; m)R_{1,3}(u_1 - u_3; m - \eta h^{(2)})R_{1,2}(u_1 - u_2; m). \end{aligned} \tag{2.18}$$

Here we have adopted the convention

$$R_{1,2}(u, m - \eta h^{(3)})\, v_1 \otimes v_2 \otimes v_3 = (R(u, m - \eta\mu) \otimes \mathrm{id})\, v_1 \otimes v_2 \otimes v_3, \quad \text{if } v_3 \in W[\mu]. \tag{2.19}$$

Moreover, one may check that the R-matrix satisfies weight conservation condition,

$$\left[h^{(1)} + h^{(2)},\, R_{1,2}(u;m)\right] = 0, \tag{2.20}$$

unitary condition,

$$R_{1,2}(u;m)\, R_{2,1}(-u;m) = \mathrm{id} \otimes \mathrm{id}, \tag{2.21}$$

and crossing relation

$$R(u;m)^{kl}_{ij} = \varepsilon_l\, \varepsilon_j \frac{\sin(u)\sin((m-\eta\hat{\imath})_{21})}{\sin(u+\eta)\sin(m_{21})} R(-u-\eta; m-\eta\hat{\imath})^{\bar{j}\,k}_{\bar{l}\,i}, \tag{2.22}$$

where

$$\varepsilon_1 = 1,\, \varepsilon_2 = -1, \quad \text{and } \bar{1} = 2,\, \bar{2} = 1. \tag{2.23}$$

Define the following functions: $\theta^{(1)}(u) = e^{-iu}$, $\theta^{(2)}(u) = 1$. Let us introduce two intertwiners which are 2-component column vectors $\phi_{m,m-\eta\hat{\jmath}}(u)$ labelled by $\hat{1}$, $\hat{2}$. The k-th element of $\phi_{m,m-\eta\hat{\jmath}}(u)$ is given by

$$\phi^{(k)}_{m,m-\eta\hat{\jmath}}(u) = \theta^{(k)}(u + 2m_j). \tag{2.24}$$

Explicitly,

$$\phi_{m,m-\eta\hat{1}}(u) = \begin{pmatrix} e^{-i(u+2m_1)} \\ 1 \end{pmatrix}, \qquad \phi_{m,m-\eta\hat{2}}(u) = \begin{pmatrix} e^{-i(u+2m_2)} \\ 1 \end{pmatrix}. \tag{2.25}$$

Obviously, the two intertwiner vectors $\phi_{m,m-\eta\hat{\imath}}(u)$ are linearly *independent* for a generic $m \in V$.

Using the intertwiner vectors, one can derive the following face-vertex correspondence relation [27]

$$\begin{aligned} &\overline{R}_{1,2}(u_1-u_2)\phi^1_{m,m-\eta\hat{\imath}}(u_1)\phi^2_{m-\eta\hat{\imath},m-\eta(\hat{\imath}+\hat{\jmath})}(u_2) \\ &\quad = \sum_{k,l} R(u_1-u_2;m)^{kl}_{ij}\phi^1_{m-\eta\hat{l},m-\eta(\hat{l}+\hat{k})}(u_1)\phi^2_{m,m-\eta\hat{l}}(u_2). \end{aligned} \tag{2.26}$$

Then the QYBE (2.2) of the vertex-type R-matrix $\overline{R}(u)$ is equivalent to the dynamical Yang-Baxter equation (2.18) of the SOS R-matrix $R(u,m)$. For a generic m, we can introduce other types of intertwiners $\bar{\phi}$, $\tilde{\phi}$ which are both row vectors and satisfy the following conditions,

$$\bar{\phi}_{m,m-\eta\hat{\mu}}(u)\,\phi_{m,m-\eta\hat{\nu}}(u) = \delta_{\mu\nu}, \quad \tilde{\phi}_{m+\eta\hat{\mu},m}(u)\,\phi_{m+\eta\hat{\nu},m}(u) = \delta_{\mu\nu}, \tag{2.27}$$

from which one can derive the relations,

$$\sum_{\mu=1}^{2}\phi_{m,m-\eta\hat{\mu}}(u)\,\bar{\phi}_{m,m-\eta\hat{\mu}}(u)=\mathrm{id},\tag{2.28}$$

$$\sum_{\mu=1}^{2}\phi_{m+\eta\hat{\mu},m}(u)\,\tilde{\phi}_{m+\eta\hat{\mu},m}(u)=\mathrm{id}.\tag{2.29}$$

One may verify that the K-matrices $K(u)$ given by (2.4) can be expressed in terms of the intertwiners and *diagonal* matrices $\mathcal{K}(\lambda|u)$ as follows

$$K(u)^s_t=\sum_{i,j}\phi^{(s)}_{\lambda-\eta(\hat{\imath}-\hat{\jmath}),\ \lambda-\eta\hat{\imath}}(u)\mathcal{K}(\lambda|u)^j_i\bar{\phi}^{(t)}_{\lambda,\ \lambda-\eta\hat{\imath}}(-u).\tag{2.30}$$

Here the *diagonal* matrix $\mathcal{K}(\lambda|u)$ is given by

$$\mathcal{K}(\lambda|u)\equiv\mathrm{Diag}(k(\lambda|u)_1,\,k(\lambda|u)_2)=\mathrm{Diag}(\frac{\sin(\lambda_1+\zeta-u)}{\sin(\lambda_1+\zeta+u)},\,\frac{\sin(\lambda_2+\zeta-u)}{\sin(\lambda_2+\zeta+u)}).\tag{2.31}$$

Although the vertex type K-matrix $K^-(u)$ given by (2.4) is generally non-diagonal, after the face-vertex transformation (2.30), the face type counterpart $\mathcal{K}(\lambda|u)$ becomes diagonal. This fact enabled the authors to apply the generalized algebraic Bethe ansatz method developed in [30] for SOS type integrable models to diagonalize the transfer matrix of the open XXZ chain with non-diagonal terms [28, 40].

Now we are in the position to construct the boundary states to specify the DW boundary condition of the six-vertex model with a non-diagonal reflection end, see figure 4. For any vector $m\in V$, we introduce four states which live in the two N-tensor spaces of V (one is indexed by $1,\ldots,N$ and the other is indexed by $\bar{1},\ldots,\bar{N}$) or their dual spaces as follows:

$$|\Omega^{(2)}(m)\rangle = \phi^1_{m,m-\eta\hat{2}}(\xi_1)\,\phi^2_{m-\eta\hat{2},m-2\eta\hat{2}}(\xi_2)\ldots\phi^N_{m-\eta(N-1)\hat{2},m-\eta N\hat{2}}(\xi_N),\tag{2.32}$$

$$\begin{aligned}|\bar{\Omega}^{(1)}(m)\rangle &= \phi^{\bar{1}}_{m-(N-2)\eta\hat{1},m-(N-2)\eta\hat{1}-\eta\hat{1}}(-u_1)\,\phi^{\bar{1}}_{m-(N-4)\eta\hat{1},m-(N-4)\eta\hat{1}-\eta\hat{1}}(-u_2)\\ &\quad\times\ldots\phi^{\bar{N}}_{m+N\eta\hat{1},m+N\eta\hat{1}-\eta\hat{1}}(-u_N),\end{aligned}\tag{2.33}$$

$$\langle\Omega^{(1)}(m)| = \tilde{\phi}^1_{m,m-\eta\hat{1}}(\xi_1)\,\tilde{\phi}^2_{m-\eta\hat{1},m-2\eta\hat{1}}(\xi_2)\ldots\tilde{\phi}^N_{m-\eta(N-1)\hat{1},m-\eta N\hat{1}}(\xi_N),\tag{2.34}$$

$$\begin{aligned}\langle\bar{\Omega}^{(2)}(m)| &= \tilde{\phi}^{\bar{1}}_{m-N\eta\hat{1},m-N\eta\hat{1}-\eta\hat{2}}(u_1)\,\tilde{\phi}^{\bar{1}}_{m-(N-2)\eta\hat{1},m-(N-2)\eta\hat{1}-\eta\hat{2}}(u_2)\\ &\quad\times\ldots\tilde{\phi}^{\bar{N}}_{m+\eta(N-2)\hat{1},m-\eta(N-2)\hat{1}-\eta\hat{2}}(u_N).\end{aligned}\tag{2.35}$$

The boundary states which have been used to define the DW boundary condition can be obtained through the above states by special choices of m and i (for example, m is specified to

λ which is related to the parameters of the K-matrix $K(u)$). Then the DW partition function $Z_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)$ of the six-vertex model with a non-diagonal reflection end given by (2.8) becomes

$$\begin{aligned}
&Z_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta) = \\
&\quad \tilde{\phi}^1_{\lambda,\lambda-\eta\hat{1}}(\xi_1)\ldots\tilde{\phi}^N_{\lambda-(N-1)\eta\hat{1},\lambda-N\eta\hat{1}}(\xi_N)\,\tilde{\phi}^{\bar{1}}_{\lambda-N\eta\hat{1},\lambda-N\eta\hat{1}-\eta\hat{2}}(u_1)\ldots\tilde{\phi}^{\bar{N}}_{\lambda+(N-2)\eta\hat{1},\lambda+(N-2)\eta\hat{1}-\eta\hat{2}}(u_N)\\
&\qquad \times R_{\bar{1},N}(u_1-\xi_N)\ldots R_{\bar{1},1}(u_1-\xi_1)\,K_{\bar{1}}(u_1)\,R_{1,\bar{1}}(u_1+\xi_1)\ldots R_{N,\bar{1}}(u_1+\xi_N)\\
&\qquad\qquad \vdots\\
&\qquad \times R_{\bar{N},N}(u_N-\xi_N)\ldots R_{\bar{N},1}(u_N-\xi_1)\,K_{\bar{N}}(u_N)\,R_{1,\bar{N}}(u_N+\xi_1)\ldots R_{N,\bar{N}}(u_N+\xi_N)\\
&\qquad \times \phi^1_{\lambda,\lambda-\eta\hat{2}}(\xi_1)\ldots\phi^N_{\lambda-(N-1)\eta\hat{2},\lambda-N\eta\hat{2}}(\xi_N)\,\phi^{\bar{1}}_{\lambda-(N-2)\eta\hat{1},\lambda-(N-1)\eta\hat{1}}(-u_1)\ldots\phi^{\bar{N}}_{\lambda+N\eta\hat{1},\lambda+(N-1)\eta\hat{1}}(-u_N).
\end{aligned} \tag{2.36}$$

3 Partition function in terms of the face type monodromy matrix

Let us introduce the face type one-row monodromy matrix

$$\begin{aligned}
T_F(l|u) &\equiv T^F_{0,1\ldots N}(l|u)\\
&= R_{0,N}(u-\xi_N;l-\eta\sum_{i=1}^{N-1}h^{(i)})\ldots R_{0,2}(u-\xi_2;l-\eta h^{(1)})R_{0,1}(u-\xi_1;l),\\
&= \begin{pmatrix} T_F(l|u)^1_1 & T_F(l|u)^1_2\\ T_F(l|u)^2_1 & T_F(l|u)^2_2\end{pmatrix}
\end{aligned} \tag{3.1}$$

where l is a generic vector in V. The monodromy matrix satisfies the face type quadratic exchange relation [46, 47]. Applying $T_F(l|u)^i_j$ to an arbitrary vector $|i_1,\ldots,i_N\rangle$ in the N-tensor product space $V^{\otimes N}$ given by

$$|i_1,\ldots,i_N\rangle = \epsilon^1_{i_1}\ldots\epsilon^N_{i_N}, \tag{3.2}$$

we have

$$\begin{aligned}
T_F(l|u)^i_j|i_1,\ldots,i_N\rangle &\equiv T_F(m;l|u)^i_j|i_1,\ldots,i_N\rangle\\
&= \sum_{\alpha_{N-1}\ldots\alpha_1}\sum_{i'_N\ldots i'_1} R(u-\xi_N;l-\eta\sum_{k=1}^{N-1}\hat{i}'_k)^{i\ \ i'_N}_{\alpha_{N-1}\ i_N}\ldots\\
&\qquad \times R(u-\xi_2;l-\eta\hat{i}'_1)^{\alpha_2\ i'_2}_{\alpha_1\ i_2}R(u-\xi_1;l)^{\alpha_1\ i'_1}_{j\ \ i_1}\ |i'_1,\ldots,i'_N\rangle,
\end{aligned} \tag{3.3}$$

where $m = l - \eta \sum_{k=1}^{N} \hat{\imath}_k$.

Now we compute the partition function $Z_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)$ defined by (2.8). The expression (2.36) implies that

$$
\begin{aligned}
Z_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta) =& \langle\Omega^{(1)}(\lambda)|\, \tilde{\phi}^{\bar{1}}_{\lambda-(N-1)\eta\hat{1}+\eta\hat{2},\lambda-(N-1)\eta\hat{1}}(u_1)\, \mathbb{T}_{\bar{1}}(u_1)\, \phi^{\bar{1}}_{\lambda-(N-2)\eta\hat{1},\lambda-(N-1)\eta\hat{1}}(-u_1) \\
&\vdots \\
&\times \tilde{\phi}^{\bar{N}}_{\lambda+(N-1)\eta\hat{1}+\eta\hat{2},\lambda+(N-1)\eta\hat{1}}(u_N)\, \mathbb{T}_{\bar{N}}(u_N)\, \phi^{\bar{N}}_{\lambda+N\eta\hat{1},\lambda+(N-1)\eta\hat{1}}(-u_N)\, |\Omega^{(2)}(\lambda)\rangle.
\end{aligned}
\tag{3.4}
$$

With the help of the crossing relation (2.22), the face-vertex correspondence relation (2.26) and the relations (2.27), following the method developed in [30, 48], we find that the partition function $Z_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)$ can be expressed in terms of the face-type double-row monodromy operators as follows:

$$
\begin{aligned}
Z_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta) = \langle 1,\ldots,1|\mathcal{T}^-_F(\lambda-2(M-1)\eta\hat{1},\lambda|u_1)^2_1 \ldots \mathcal{T}^-_F(\lambda,\lambda|u_M)^2_1 \\
\times \mathcal{T}^-_F(\lambda+2\eta\hat{1},\lambda|u_{M+1})^2_1 \ldots \mathcal{T}^-_F(\lambda+N\eta\hat{1},\lambda|u_N)^2_1|2,\ldots,2\rangle.
\end{aligned}
\tag{3.5}
$$

The above double-row monodromy matrix operator $\mathcal{T}^-_F(m,\lambda|u)^2_1$ is given in terms of the one-row monodromy matrix operator $T_F(m;l|u)^i_j$ [48] as follow:

$$
\begin{aligned}
\mathcal{T}^-_F(m,\lambda|u)^2_1 =& \frac{\sin(m_{21})}{\sin(\lambda_{21})} \prod_{k=1}^{N} \frac{\sin(u+\xi_k)}{\sin(u+\xi_k+\eta)} \\
&\times \left\{ \frac{\sin(\lambda_1+\zeta-u)}{\sin(\lambda_1+\zeta+u)} T_F(m,\lambda|u)^2_1 T_F(m+\eta\hat{2},\lambda+\eta\hat{2}|-u-\eta)^2_2 \right. \\
&\left. - \frac{\sin(\lambda_2+\zeta-u)}{\sin(\lambda_2+\zeta+u)} T_F(m+2\eta\hat{2},\lambda|u)^2_2 T_F(m+\eta\hat{1},\lambda+\eta\hat{1}|-u-\eta)^2_1 \right\}.
\end{aligned}
\tag{3.6}
$$

In the next section we construct the Drinfeld twist (or factorizing F-matrix) in the face picture for the six-vertex model with a non-diagonal reflection end. In the resulting F-basis, the pseudo-particle creation operator $\mathcal{T}^-_F$ given by (3.6) takes completely symmetric and polarization free form. This polarization free form allows us to construct the explicit expressions of the partition function $Z_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)$.

4 F-basis

In this section, we give the Drinfeld twist [14] (factorizing F-matrix) on the N-fold tensor product space $V^{\otimes N}$ and the associated representations of the pseudo-particle creation/annihilation operators in this basis.

4.1 Factorizing Drinfeld twist F

Let $\mathcal{S}_N$ be the permutation group over indices $1,\ldots,N$ and $\{\sigma_i|i=1,\ldots,N-1\}$ be the set of elementary permutations in $\mathcal{S}_N$. For each elementary permutation σ_i, we introduce the associated operator $R^{\sigma_i}_{1\ldots N}$ on the quantum space

$$R^{\sigma_i}_{1\ldots N}(l) \equiv R^{\sigma_i}(l) = R_{i,i+1}(\xi_i - \xi_{i+1}|l - \eta\sum_{k=1}^{i-1} h^{(k)}), \tag{4.1}$$

where l is a generic vector in V. For any σ, $\sigma' \in \mathcal{S}_N$, operator $R^{\sigma\sigma'}_{1\ldots N}$ associated with $\sigma\sigma'$ satisfies the following composition law [19](and references therein):

$$R^{\sigma\sigma'}_{1\ldots N}(l) = R^{\sigma'}_{\sigma(1\ldots N)}(l)\, R^{\sigma}_{1\ldots N}(l). \tag{4.2}$$

Let σ be decomposed in a minimal way in terms of elementary permutations,

$$\sigma = \sigma_{\beta_1}\ldots\sigma_{\beta_p}, \tag{4.3}$$

where $\beta_i = 1,\ldots,N-1$ and the positive integer p is the length of σ. The composition law (4.2) enables one to obtain operator $R^{\sigma}_{1\ldots N}$ associated with each $\sigma \in \mathcal{S}_N$. The dynamical quantum Yang-Baxter equation (2.18), weight conservation condition (2.20) and unitary condition (2.21) guarantee the uniqueness of $R^{\sigma}_{1\ldots N}$. Moreover, one may check that $R^{\sigma}_{1\ldots N}$ satisfies the following exchange relation with the face type one-row monodromy matrix (3.1)

$$R^{\sigma}_{1\ldots N}(l)T^F_{0,1\ldots N}(l|u) = T^F_{0,\sigma(1\ldots N)}(l|u)R^{\sigma}_{1\ldots N}(l-\eta h^{(0)}), \qquad \forall\sigma\in\mathcal{S}_N. \tag{4.4}$$

Now, we construct the face-type Drinfeld twist $F_{1\ldots N}(l) \equiv F_{1\ldots N}(l;\xi_1,\ldots,\xi_N)$ [1] on the N-fold tensor product space $V^{\otimes N}$, which satisfies the following three properties:

$$\text{I. lower} - \text{triangularity}; \tag{4.5}$$

$$\text{II. non} - \text{degeneracy}; \tag{4.6}$$

$$\text{III. factorizing property}: \ R^{\sigma}_{1\ldots N}(l) = F^{-1}_{\sigma(1\ldots N)}(l)F_{1\ldots N}(l),\ \forall\sigma\in\mathcal{S}_N. \tag{4.7}$$

Substituting (4.7) into the exchange relation (4.4) yields the following relation

$$F^{-1}_{\sigma(1\ldots N)}(l)F_{1\ldots N}(l)T^F_{0,1\ldots N}(l|u) = T^F_{0,\sigma(1\ldots N)}(l|u)F^{-1}_{\sigma(1\ldots N)}(l-\eta h^{(0)})F_{1\ldots N}(l-\eta h^{(0)}). \tag{4.8}$$

[1] In this paper, we adopt the convention: $F_{\sigma(1\ldots N)}(l) \equiv F_{\sigma(1\ldots N)}(l;\xi_{\sigma(1)},\ldots,\xi_{\sigma(N)})$.

Equivalently,

$$F_{1\ldots N}(l)T^F_{0,1\ldots N}(l|u)F^{-1}_{1\ldots N}(l-\eta h^{(0)}) = F_{\sigma(1\ldots N)}(l)T^F_{0,\sigma(1\ldots N)}(l|u)F^{-1}_{\sigma(1\ldots N)}(l-\eta h^{(0)}). \quad (4.9)$$

Let us introduce the twisted monodromy matrix $\tilde{T}^F_{0,1\ldots N}(l|u)$ by

$$\begin{aligned} \tilde{T}^F_{0,1\ldots N}(l|u) &= F_{1\ldots N}(l)T^F_{0,1\ldots N}(l|u)F^{-1}_{1\ldots N}(l-\eta h^{(0)}) \\ &= \begin{pmatrix} \tilde{T}_F(l|u)^1_1 & \tilde{T}_F(l|u)^1_2 \\ \tilde{T}_F(l|u)^2_1 & \tilde{T}_F(l|u)^2_2 \end{pmatrix}. \end{aligned} \quad (4.10)$$

Then (4.9) implies that the twisted monodromy matrix is symmetric under $\mathcal{S}_N$, namely,

$$\tilde{T}^F_{0,1\ldots N}(l|u) = \tilde{T}^F_{0,\sigma(1\ldots N)}(l|u), \quad \forall\sigma\in\mathcal{S}_N. \quad (4.11)$$

Define the F-matrix:

$$F_{1\ldots N}(l) = \sum_{\sigma\in\mathcal{S}_N}\sum_{\{\alpha_j\}=1}^{2}{}^{*}\prod_{j=1}^{N} P^{\sigma(j)}_{\alpha_{\sigma(j)}}\, R^{\sigma}_{1\ldots N}(l), \quad (4.12)$$

where P^i_α is the embedding of the project operator P_α in the i^{th} space with matrix elements $(P_\alpha)_{kl} = \delta_{kl}\delta_{k\alpha}$. The sum $\sum^*$ in (4.12) is over all non-decreasing sequences of the labels $\alpha_{\sigma(i)}$:

$$\begin{aligned} &\alpha_{\sigma(i+1)} \geq \alpha_{\sigma(i)} \quad \text{if} \quad \sigma(i+1)>\sigma(i), \\ &\alpha_{\sigma(i+1)} > \alpha_{\sigma(i)} \quad \text{if} \quad \sigma(i+1)<\sigma(i). \end{aligned} \quad (4.13)$$

From (4.13), $F_{1\ldots N}(l)$ obviously is a lower-triangular matrix. Moreover, the F-matrix is non-degenerate because all its diagonal elements are non-zero. It was shown [48] that the F-matrix also satisfies the factorizing property (4.7). Hence, the F-matrix $F_{1\ldots N}(l)$ given by (4.12) is the desirable Drinfeld twist.

4.2 Completely symmetric representations

Direct calculation shows [48] that the twisted operators $\tilde{T}_F(l|u)^j_i$ defined by (4.10) indeed simultaneously have the following polarization free forms. Here we present the results for the relevant operators for our purpose

$$\tilde{T}_F(l|u)^2_2 = \frac{\sin(l_{21}-\eta)}{\sin(l_{21}-\eta+\eta\langle H,\epsilon_1\rangle)}\otimes_i \begin{pmatrix} \frac{\sin(u-\xi_i)}{\sin(u-\xi_i+\eta)} & \\ & 1 \end{pmatrix}_{(i)}, \quad (4.14)$$

$$\tilde{T}_F(l|u)^2_1 = \sum_{i=1}^{N}\frac{\sin\eta\sin(u-\xi_i+l_{12})}{\sin(u-\xi_i+\eta)\sin l_{12}}E^i_{12}\otimes_{j\neq i}\begin{pmatrix} \frac{\sin(u-\xi_j)\sin(\xi_i-\xi_j+\eta)}{\sin(u-\xi_j+\eta)\sin(\xi_i-\xi_j)} & \\ & 1 \end{pmatrix}_{(j)}. \quad (4.15)$$

Applying the above operators to the arbitrary state $|i_1, \ldots, i_N\rangle$ given by (3.2) leads to

$$\tilde{T}_F(m,l|u)^2_2 = \frac{\sin(l_{21}-\eta)}{\sin(l_2-m_1-\eta)} \otimes_i \begin{pmatrix} \frac{\sin(u-\xi_i)}{\sin(u-\xi_i+\eta)} & \\ & 1 \end{pmatrix}_{(i)}, \tag{4.16}$$

$$\begin{aligned}\tilde{T}_F(m,l|u)^2_1 = & \sum_{i=1}^{N} \frac{\sin\eta\sin(u-\xi_i+l_{12})}{\sin(u-\xi_i+\eta)\sin l_{12}} \\ & \times E^i_{12} \otimes_{j\neq i} \begin{pmatrix} \frac{\sin(u-\xi_j)\sin(\xi_i-\xi_j+\eta)}{\sin(u-\xi_j+\eta)\sin(\xi_i-\xi_j)} & \\ & 1 \end{pmatrix}_{(j)}. \end{aligned} \tag{4.17}$$

It then follows that the pseudo-particle creation operator (3.6) in the F-basis has the following completely symmetric polarization free form:

$$\begin{aligned}\tilde{\mathcal{T}}^-_F(m,\lambda|u)^2_1 = & \frac{\sin m_{12}}{\sin(m_1-\lambda_2)} \prod_{k=1}^{N} \frac{\sin(u+\xi_k)}{\sin(u+\xi_k+\eta)} \\ \times & \sum_{i=1}^{N} \frac{\sin(\lambda_1+\zeta-\xi_i)\sin(\lambda_2+\zeta+\xi_i)\sin 2u\sin\eta}{\sin(\lambda_1+\zeta+u)\sin(\lambda_2+\zeta+u)\sin(u-\xi_i+\eta)\sin(u+\xi_i)} \\ & \times E^i_{12} \otimes_{j\neq i} \begin{pmatrix} \frac{\sin(u-\xi_j)\sin(u+\xi_j+\eta)\sin(\xi_i-\xi_j+\eta)}{\sin(u-\xi_j+\eta)\sin(u+\xi_j)\sin(\xi_i-\xi_j)} & \\ & 1 \end{pmatrix}_{(j)}. \end{aligned} \tag{4.18}$$

5 Determinant representation of the partition function

In this section we compute the DW partition function $Z_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)$ from its expression (3.5) and the expansion of the twisted operator $\tilde{\mathcal{T}}^-_F(m,\lambda|u)^2_1$ (4.18) given in the previous section.

5.1 Symmetric expression of the partition function

From the definitions of the F-matrix $F_{1\ldots N}(l)$ given by (4.12), we can show that the state $|2,\ldots,2\rangle$ and the dual state $\langle 1,\ldots,1|$ are invariant under the action of $F_{1\ldots N}(l)$, namely,

$$F_{1\ldots N}(l)|2,\ldots,2\rangle = |2,\ldots,2\rangle, \tag{5.1}$$

$$\langle 1,\ldots,1|F_{1\ldots N}(l) = \langle 1,\ldots,1|. \tag{5.2}$$

Hence the DW partition function $Z_N(\{u_\alpha\};\{\xi_j\};\lambda;\zeta)$ can be expressed in terms of the twisted operator $\tilde{\mathcal{T}}^-_F(m,\lambda|u)^2_1$ as follow

$$Z_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta) = \langle 1,\ldots,1|\tilde{\mathcal{T}}^-_F(\lambda-(N-2)\eta\hat{1},\lambda|u_1)^2_1 \ldots \tilde{\mathcal{T}}^-_F(\lambda+N\eta\hat{1},\lambda|u_N)^2_1|2,\ldots,2\rangle$$

$$
\begin{aligned}
=&\langle 1,\ldots,1|\, F_{1\ldots N}(\lambda-N\eta\hat{1})\,\mathcal{T}_F^-(\lambda-(N-2)\eta\hat{1},\lambda|u_1)_1^2\ldots\\
&\times\mathcal{T}_F^-(\lambda+N\eta\hat{1},\lambda|u_N)_1^2\,F^{-1}_{1\ldots N}(\lambda+N\eta\hat{1})\,|2,\ldots,2\rangle\\
=&\langle 1,\ldots,1|\tilde{\mathcal{T}}_F^-(\lambda-(N-2)\eta\hat{1},\lambda|u_1)_1^2\ldots\tilde{\mathcal{T}}_F^-(\lambda+N\eta\hat{1},\lambda|u_N)_1^2|2,\ldots,2\rangle.
\end{aligned}
$$

Substituting the polarization free expression (4.18) of the twisted operator $\tilde{\mathcal{T}}_F^-(m,\lambda|u)_1^2$ into the above equation, we have

$$
\begin{aligned}
Z_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)=&\prod_{k=1}^{M}\frac{\sin(\lambda_{12}+2k\eta)\sin(\lambda_{12}-2k\eta+\eta)}{\sin(\lambda_{12}+k\eta)\sin(\lambda_{12}-k\eta+\eta)}\prod_{l=1}^{N}\prod_{i=1}^{N}\frac{\sin(u_i+\xi_l)}{\sin(u_i+\xi_l+\eta)}\langle 1,\ldots,1|\\
&\times\sum_{i=1}^{N}\frac{\sin(\lambda_1+\zeta-\xi_i)\sin(\lambda_2+\zeta+\xi_i)\sin 2u_1\sin\eta}{\sin(\lambda_1+\zeta+u_1)\sin(\lambda_2+\zeta+u_1)\sin(u_1-\xi_i+\eta)\sin(u_1+\xi_i)}E_{12}^i\otimes_{j\neq i}\begin{pmatrix}\frac{\sin(u_1-\xi_j)\sin(u_1+\xi_j+\eta)\sin(\xi_i-\xi_j+\eta)}{\sin(u_1-\xi_j+\eta)\sin(u_1+\xi_j)\sin(\xi_i-\xi_j)} & \\ & 1\end{pmatrix}_{(j)}\\
&\quad\vdots\\
&\times\sum_{i=1}^{N}\frac{\sin(\lambda_1+\zeta-\xi_i)\sin(\lambda_2+\zeta+\xi_i)\sin 2u_N\sin\eta}{\sin(\lambda_1+\zeta+u_N)\sin(\lambda_2+\zeta+u_N)\sin(u_N-\xi_i+\eta)\sin(u_N+\xi_i)}E_{12}^i\otimes_{j\neq i}\begin{pmatrix}\frac{\sin(u_N-\xi_j)\sin(u_N+\xi_j+\eta)\sin(\xi_i-\xi_j+\eta)}{\sin(u_N-\xi_j+\eta)\sin(u_N+\xi_j)\sin(\xi_i-\xi_j)} & \\ & 1\end{pmatrix}_{(j)}\\
&\times|2,\ldots,2\rangle.
\end{aligned}
$$

Expanding the last sum term of the above equation which corresponds to the contribution associated with the spectral parameter u_N yields

$$
\begin{aligned}
Z_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)=&\prod_{k=1}^{M}\frac{\sin(\lambda_{12}+2k\eta)\sin(\lambda_{12}-2k\eta+\eta)}{\sin(\lambda_{12}+k\eta)\sin(\lambda_{12}-k\eta+\eta)}\prod_{l=1}^{N}\prod_{i=1}^{N}\frac{\sin(u_i+\xi_l)}{\sin(u_i+\xi_l+\eta)}\\
&\times\sum_{i=1}^{N}\frac{\sin(\lambda_1+\zeta-\xi_i)\sin(\lambda_2+\zeta+\xi_i)\sin 2u_N\sin\eta}{\sin(\lambda_1+\zeta+u_N)\sin(\lambda_2+\zeta+u_N)\sin(u_N-\xi_i+\eta)\sin(u_N+\xi_i)}\prod_{l=1}^{N-1}\frac{\sin(u_l-\xi_i)\sin(u_l+\xi_i+\eta)}{\sin(u_l-\xi_i+\eta)\sin(u_l+\xi_i)}\\
&\times\prod_{j\neq i}\frac{\sin(\xi_j-\xi_i+\eta)}{\sin(\xi_j-\xi_i)}\langle 1,\ldots,1|\\
&\times\sum_{l\neq i}^{N}\frac{\sin(\lambda_1+\zeta-\xi_l)\sin(\lambda_2+\zeta+\xi_l)\sin 2u_1\sin\eta}{\sin(\lambda_1+\zeta+u_1)\sin(\lambda_2+\zeta+u_1)\sin(u_1-\xi_l+\eta)\sin(u_1+\xi_l)}E_{12}^i\otimes_{j\neq l,i}\begin{pmatrix}\frac{\sin(u_1-\xi_j)\sin(u_1+\xi_j+\eta)\sin(\xi_l-\xi_j+\eta)}{\sin(u_1-\xi_j+\eta)\sin(u_1+\xi_j)\sin(\xi_l-\xi_j)} & \\ & 1\end{pmatrix}_{(j)}\\
&\quad\vdots\\
&\times|2,\ldots,2\rangle.
\end{aligned}
$$

Iterating the above procedure, we obtain the complete symmetric expression of the partition function $Z_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)$

$$
\begin{aligned}
Z_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)\;=\;&\prod_{k=1}^{M}\frac{\sin(\lambda_{12}+2k\eta)\sin(\lambda_{12}-2k\eta+\eta)}{\sin(\lambda_{12}+k\eta)\sin(\lambda_{12}-k\eta+\eta)}\prod_{l=1}^{N}\prod_{i=1}^{N}\frac{\sin(u_i+\xi_l)}{\sin(u_i+\xi_l+\eta)}\\
&\times\mathcal{Z}_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta),
\end{aligned}
\tag{5.3}
$$

where the normalized partition function $\mathcal{Z}_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)$ is

$$\mathcal{Z}_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)=\sum_{\sigma\in\mathcal{S}_N}\prod_{n=1}^{N}\left\{\frac{\sin(\lambda_1+\zeta-\xi_{i_{\sigma(n)}})\sin(\lambda_2+\zeta+\xi_{i_{\sigma(n)}})\sin(2u_n)\sin\eta}{\sin(\lambda_1+\zeta+u_n)\sin(\lambda_2+\zeta+u_n)\sin(u_n-\xi_{i_{\sigma(n)}}+\eta)\sin(u_n+\xi_{i_{\sigma(n)}})}\right.$$
$$\left.\times\prod_{k>n}^{N}\frac{\sin(u_n-\xi_{i_{\sigma(k)}})\sin(u_n+\xi_{i_{\sigma(k)}}+\eta)\sin(\xi_{i_{\sigma(n)}}-\xi_{i_{\sigma(k)}}+\eta)}{\sin(u_n-\xi_{i_{\sigma(k)}}+\eta)\sin(u_n+\xi_{i_{\sigma(k)}})\sin(\xi_{i_{\sigma(n)}}-\xi_{i_{\sigma(k)}})}\right\}. \quad (5.4)$$

5.2 Recursive relation and the determinant representation

From the expression (5.4) of the partition function $\mathcal{Z}_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)$, it is easy to check that the partition function is a symmetric function of $\{u_\alpha\}$ and $\{\xi_i\}$ separatively. Moreover, we can derive that the partition function $\mathcal{Z}_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)$ satisfy the following recursive relation

$$\begin{aligned}\mathcal{Z}_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta) &= \sum_{i=1}^{N}\frac{\sin(\lambda_1+\zeta-\xi_i)\sin(\lambda_2+\zeta+\xi_i)\sin(2u_N)\sin\eta}{\sin(\lambda_1+\zeta+u_N)\sin(\lambda_2+\zeta+u_N)\sin(u_N-\xi_i+\eta)\sin(u_N+\xi_i)}\\ &\quad\times\prod_{l=1}^{N-1}\frac{\sin(u_l-\xi_i)\sin(u_l+\xi_i+\eta)}{\sin(u_l-\xi_i+\eta)\sin(u_l+\xi_i)}\prod_{j\neq i}\frac{\sin(\xi_j-\xi_i+\eta)}{\sin(\xi_j-\xi_i)}\\ &\quad\times\mathcal{Z}_{N-1}(\{u_\alpha\}_{\alpha\neq N};\{\xi_j\}_{j\neq i};\lambda;\zeta). \end{aligned}\quad (5.5)$$

One can show that the initial condition: $\mathcal{Z}_0(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)=1$ and the recursive relation (5.5) *uniquely* determinate the partition function $\mathcal{Z}_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)$ for any positive integer N. This fact allows us to obtain the following determinant representation of the normalized partition function $\mathcal{Z}_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)$:

$$\mathcal{Z}_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)=\frac{\prod_{\alpha=1}^{N}\prod_{i=1}^{N}\sin(u_\alpha-\xi_i)\sin(u_\alpha+\xi_i+\eta)\det\mathcal{N}(\{u_\alpha\};\{\xi_i\})}{\prod_{\alpha>\beta}\sin(u_\alpha-u_\beta)\sin(u_\alpha+u_\beta+\eta)\prod_{k<l}\sin(\xi_k-\xi_l)\sin(\xi_k+\xi_l)}, \quad (5.6)$$

where the $N\times N$ matrix $\mathcal{N}(\{u_\alpha\};\{\xi_i\})$ is given by

$$\begin{aligned}\mathcal{N}(\{u_\alpha\};\{\xi_i\})_{\alpha,j} &= \frac{\sin\eta\sin(\lambda_1+\zeta-\xi_j)}{\sin(u_\alpha-\xi_j)\sin(u_\alpha+\xi_j+\eta)\sin(\lambda_1+\zeta+u_\alpha)}\\ &\quad\times\frac{\sin(\lambda_2+\zeta+\xi_j)\sin(2u_\alpha)}{\sin(\lambda_2+\zeta+u_\alpha)\sin(u_\alpha-\xi_j+\eta)\sin(u_\alpha+\xi_j)}.\end{aligned}\quad (5.7)$$

The proof of this representation is relegated to Appendix A.

Finally, we obtain the determinant representation of the partition function $Z_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)$ defined in (2.8) of the six-vertex model with a non-diagonal reflection end under the DW boundary condition from the expression (5.3)

$$
\begin{aligned}
Z_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta) \;=\; & \prod_{k=1}^{M}\frac{\sin(\lambda_{12}+2k\eta)\sin(\lambda_{12}-2k\eta+\eta)}{\sin(\lambda_{12}+k\eta)\sin(\lambda_{12}-k\eta+\eta)}\prod_{l=1}^{N}\prod_{i=1}^{N}\frac{\sin(u_i+\xi_l)}{\sin(u_i+\xi_l+\eta)} \\
& \times\frac{\prod_{\alpha=1}^{N}\prod_{i=1}^{N}\sin(u_\alpha-\xi_i)\sin(u_\alpha+\xi_i+\eta)\,\det\mathcal{N}(\{u_\alpha\};\{\xi_i\})}{\prod_{\alpha>\beta}\sin(u_\alpha-u_\beta)\sin(u_\alpha+u_\beta+\eta)\prod_{k<l}\sin(\xi_k-\xi_l)\sin(\xi_k+\xi_l)},
\end{aligned}
\tag{5.8}
$$

where the $N\times N$ matrix $\mathcal{N}(\{u_\alpha\};\{\xi_i\})$ is given by (5.7).

6 Conclusions

We have studied the partition function $Z_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)$ of the six-vertex model with a non-diagonal reflection end, where the corresponding K-matrix $K(u)$ given by (2.4) is a generic non-diagonal solution of the RE, under the DW boundary condition. The DW boundary condition is specified by four boundary states (2.32)-(2.35) which are two-parameter generalization of the all-spin-down and all-spin-up states and their dual states. With the help of the F-basis provided by the Drinfeld twist for the open XXZ spin chain with non-diagonal boundary terms, we obtain the the complete symmetric expression (5.3)-(5.4) of the partition function. Such an explicit expression allows us to derive its the recursive relation (5.5). Solving the recursive relation, we obtain the determinant representation (5.8) of the partition function $Z_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)$. The determinant representation of the partition function will play an important role to construct determinant representations of scalar products between an on-shell Bethe state and a general state (or an off-shell Bethe state) of the open XXZ chain with non-diagonal boundary terms [44].

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Appendix A: Proof the determinant representation (5.6)

In this appendix, we prove the determinant representation (5.6) of the normalized partition function $\mathcal{Z}_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)$ defined in (5.3). Let us introduce two series functions $\{B_I(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)\,|\,I=1,\ldots,N\}$ and $\{F_I(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)\,|\,I=1,\ldots,N\}$ which are given respectively by

$$B_I(\{u_\alpha\};\{\xi_i\};\lambda;\zeta) = \prod_{l=1}^{I}\frac{\sin(\lambda_1+\zeta+u_l)\sin(\lambda_2+\zeta+u_l)}{\sin(\lambda_1+\zeta-\xi_l)\sin(\lambda_2+\zeta+\xi_l)\sin 2u_l}\,\mathcal{Z}_I(\{u_\alpha\};\{\xi_i\};\lambda;\zeta), \tag{A.1}$$

$$\begin{aligned} F_I(\{u_\alpha\};\{\xi_i\};\lambda;\zeta) &= \frac{\prod_{\alpha=1}^{I}\prod_{j=1}^{I}\sin(u_\alpha-\xi_j)\sin(u_\alpha+\xi_j+\eta)}{\prod_{\alpha>\beta}\sin(u_\alpha-u_\beta)\sin(u_\alpha+u_\beta+\eta)\prod_{k<l}\sin(\xi_k-\xi_l)\sin(\xi_k+\xi_l)} \\ &\quad\times\det\left|\frac{\sin\eta}{\sin(u_\alpha-\xi_j)\sin(u_\alpha+\xi_j+\eta)\sin(u_\alpha-\xi_j+\eta)\sin(u_\alpha+\xi_j)}\right|. \end{aligned} \tag{A.2}$$

Then the proof of (5.6) is equivalent to the following identification

$$B_I(\{u_\alpha\};\{\xi_i\};\lambda;\zeta) = F_I(\{u_\alpha\};\{\xi_i\};\lambda;\zeta), \quad \text{for any positive integer I.} \tag{A.3}$$

We shall prove the above equation by induction.

- From direct calculation, we can show that (A.3) holds for the case of $N=1$, namely,

  $$B_1(u_1;\xi_1;\lambda;\zeta) = F_1(u_1;\xi_1;\lambda;\zeta) = \frac{\sin\eta}{\sin(u_1-\xi_1+\eta)\sin(u_1+\xi_1)}.$$

- Suppose that (A.3) holds for the case of $I \le N-1$. We are to prove that it is satisfied also for the case of N as follows. It is easy to check that both $B_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)$ and $F_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)$ are symmetric functions of $\{u_\alpha\}$. Hence it is sufficient to prove that as function of u_N they are equal to each other. The recursive relation (5.5) of $\mathcal{Z}_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)$ implies that $B_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)$ satisfies the following relation

  $$\begin{aligned} B_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta) &= \sum_{i=1}^{N}\frac{\sin\eta}{\sin(u_N-\xi_i+\eta)\sin(u_N+\xi_i)}\prod_{l=1}^{N-1}\frac{\sin(u_l-\xi_i)\sin(u_l+\xi_i+\eta)}{\sin(u_l-\xi_i+\eta)\sin(u_l+\xi_i)} \\ &\quad\times\prod_{j\neq i}\frac{\sin(\xi_j-\xi_i+\eta)}{\sin(\xi_j-\xi_i)}\,B_{N-1}(\{u_\alpha\}_{\alpha\neq N};\{\xi_j\}_{j\neq i};\lambda;\zeta). \end{aligned} \tag{A.4}$$

  The determinant representation of the function $F_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)$ implies that it satisfies the following recursive relation

  $$F_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta) = \sum_{i=1}^{N}\frac{\sin\eta}{\sin(u_N-\xi_i+\eta)\sin(u_N+\xi_i)}\prod_{l=1}^{N-1}\frac{\sin(u_l-\xi_i)\sin(u_l+\xi_i+\eta)}{\sin(u_N-u_l)\sin(u_N+u_l+\eta)}$$

$$\times \prod_{j\neq i} \frac{\sin(u_N-\xi_j)\sin(u_N+\xi_j+\eta)}{\sin(\xi_j-\xi_i)\sin(\xi_j+\xi_i)} F_{N-1}(\{u_\alpha\}_{\alpha\neq N};\{\xi_j\}_{j\neq i};\lambda;\zeta). \quad (A.5)$$

The determinant representation (A.2) and its recursive relation (A.5) of the function $F_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)$ and the recursive relation (A.4) of $B_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)$ imply that these two functions, as function of u_N, have the same simple poles located at [2]:

$$\xi_i - \eta,\ -\xi_i \quad \mathrm{mod}(\sqrt{-1}\pi), \qquad i = 1, \ldots, N. \quad (A.6)$$

Direct calculation shows that the residues of the two functions at each simple pole (A.6) are indeed the same. Moreover we can show that

$$B_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)\,|_{u_N\to\infty} = 0 = F_N(\{u_\alpha\};\{\xi_i\};\lambda;\zeta)\,|_{u_N\to\infty}$$

Thanks to the Liouville theorem, we can conclude that (A.3) actually holds for the case of N.

Finally we have completed the proof of (5.6).

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