Quasibosons composed of two q-fermions: realization by deformed oscillators

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Abstract. Composite bosons or *quasibosons* (e.g. mesons, excitons, etc.) occur in various physical situations. Quasibosons differ from bosons or fermions as their creation and annihilation operators obey non-standard commutation relations, even for the "fermion+fermion" composites. Our aim is to realize the operator algebra of quasibosons composed from two fermions or two q-fermions (q-deformed fermions) by the respective operators of deformed oscillators, the widely studied objects. For this, the restrictions on quasiboson creation/annihilation operators and on the deformed oscillator (= deformed boson) algebra are obtained. Their resolving proves uniqueness of the family of deformations and gives explicitly the deformation structure function (DSF) which provides the desired realization. In case of two fermions as constituents, such realization is achieved when the DSF is quadratic polynomial in the number operator. In the case of two q-fermions, $q \neq 1$, the obtained DSF inherits the parameter q and does not reduce, when $q \to 1$, to DSF of the first case.

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1. Introduction

Theoretical treatment of many-particle systems is connected with a number of complications. Some of them can be resolved by introducing of the concept of quasiparticle or "composite particle", if this is possible. However, on this way we generally encounter various factors of internal structure, which cannot be completely encapsulated into internal degrees of freedom of a composite particle. These are the nontrivial commutation relations, or the interaction of the constituents between themselves and with other particles, etc. It is desirable to have an equivalent description of many-(composite-)particle systems, almost as simple as the description of ideal/pointlike particle system, but taking into account the mentioned factors. Deformed bosons or deformed oscillators, see e.g. the review [1], provide possible means for the realization of such an intention. In such case all the factors connected with the internal structure should be encoded in one or more deformation parameters.

A particular realization of the mentioned idea to describe quasibosons [2] (boson-like composite particles) in terms of deformed Heisenberg algebra was demonstrated by Avancini and Krein in [3] who utilized the quonic [4] version of deformed boson algebra. Note that if two or more copies (modes) are involved, different modes of quons do not commute [3, 4]. Unlike quons, the deformed oscillators of Arik-Coon type are independent [5, 6] (that is, the operators corresponding to different copies, mutually commute).

Regardless of their intrinsic origin, diverse models of deformed oscillators have received much attention during the nineties and till now. Among the best known and extensively studied deformed oscillators one encounters the q-deformed Arik-Coon (AC) [5] or Biedenharn-Macfarlane (BM) [7, 8] ones, the q-deformed Tamm-Dancoff oscillator [9, 10, 11], and also the two-parameter p,q-deformed oscillator [12, 13]. On the other hand, the so-called μ-deformed oscillator is studied much less. Introduced in [14] almost two decades ago, this deformed oscillator essentially differs from the models we have already mentioned and exhibits rather unusual properties [15, 16]. Note that there exists general approach to description of deformed oscillators based on the concept of DSF given in [17, 1]. As the extension of the standard quantum harmonic oscillator, deformed oscillators find diverse applications in description of miscellaneous physical systems involving essential nonlinearities, from say quantum optics and Landau problem to high energy particle phenomenology and modern quantum field theory, see e.g. [18, 19, 20, 21, 22, 23, 24, 25].

Although a great variety of models of deformed oscillators exists, as mentioned above, the detailed analysis of possible realizations, on their base, of composite particles along with interpretation of deformation parameters in terms of internal structure, as far as we know, is lacking. To fill this gap, in our preceding paper [26] some steps in this direction were undertaken, and first results were obtained. Namely, we carried out the detailed analysis for quasibosons consisting from two ordinary fermions, so that the creation operator $A_\alpha^\dagger = \Phi_\alpha^{\mu\nu} a_\mu^\dagger b_\nu^\dagger$ of quasiboson in some α-th mode is bilinear combination of the constituents' creation operators of general form. The analysis implies the realization of quasibosons by deformed oscillators characterized by most general DSF $\phi(N)$ which unambiguously determines [17, 1] the deformed algebra within one mode. Our present study further extends the results obtained in [26] by using, instead of the usual fermions, *their deformed analogs* for the constituents' operators.

The paper is organized as follows. Section 2, which serves as a base for our analysis, concerns the case of quasibosons whose constituents are ordinary fermions (the particular $q = 1$ case of q-fermions). Here, after introducing the creation and annihilation operators for composite quasibosons, we recapitulate main facts and results from [26] (note that some of these results, only sketched in [26], here are presented in full detail: in particular, that concerns the extended treatment given in subsection 2.3). We establish important relations for quasibosons' operators. These include necessary conditions for the representation of quasibosons in terms of deformed bosons to hold. Those conditions are partially solved in subsection 2.1 yielding the DSFs of the effective deformation $\phi(N)$ and completely solved in subsection 2.3. There we obtain explicitly all possible internal structures for quasibosons in terms of the matrices $\Phi_\alpha^{\mu\nu}$. Section 3 presents further development of the ideas and results of [26]. Now for the constituents' operators, instead of usual fermions, we take their q-deformed analogs (that means the nontrivial fermionic deformation: $q \neq 1$). We investigate the quasibosons represented by deformed oscillators of general kind – those characterized by most general DSF $\phi(N)$ defining the deformed oscillator (or deformed boson) algebra [17, 1], within one mode. The paper ends with concluding remarks and some outlook.

2. System of quasibosons composed of two fermions

The general task of representing the quasibosons consisting of q-fermions divides in two particular situations: i) the constituents are pure fermions ($q = 1$); ii) the constituents are essentially deformed q-fermions ($q \neq 1$). This section is devoted to the first case: similarly to [26] we deal with the system of composite boson-like particles (*quasibosons* [2]) such that each copy (mode) of them is built from two fermions. We study the realization of quasibosons in terms of the set of *independent* identical copies of deformed oscillators of general form (for some examples of mode-independent systems see [6]).

Let us denote the creation and annihilation operators of the two (mutually anticommuting) sets of usual fermions by $a^\dagger_\mu$, $b^\dagger_\nu$, a_μ, b_ν respectively with their standard anticommutation relations, namely

$$\begin{aligned} &\{a_\mu, a^\dagger_{\mu'}\} \equiv a_\mu a^\dagger_{\mu'} + a^\dagger_{\mu'} a_\mu = \delta_{\mu\mu'}, \qquad \{a_\mu, a_\nu\} = 0, \\ &\{b_\nu, b^\dagger_{\nu'}\} \equiv b_\nu b^\dagger_{\nu'} + b^\dagger_{\nu'} b_\nu = \delta_{\nu\nu'}, \qquad \{b_\mu, b_\nu\} = 0. \end{aligned} \tag{1}$$

Besides, each of $a^\dagger_\mu$, a_μ anticommutes with each of $b^\dagger_\nu$, b_ν. So, we use these fermions to construct quasibosons. Then, the corresponding quasibosonic creation and destruction operators $A^\dagger_\alpha$, A_α (where α labels particular quasiboson and denotes the whole set of its quantum numbers) are given as

$$A^\dagger_\alpha = \sum_{\mu\nu} \Phi^{\mu\nu}_\alpha a^\dagger_\mu b^\dagger_\nu, \quad A_\alpha = \sum_{\mu\nu} \overline{\Phi}^{\mu\nu}_\alpha b_\nu a_\mu \,. \tag{2}$$

For the matrices Φ_α we assume the following normalization condition:

$$\sum_{\mu\nu} \Phi^{\mu\nu}_\alpha \overline{\Phi}^{\mu\nu}_\beta \equiv \mathrm{Tr}\, \Phi_\alpha \Phi^\dagger_\beta = \delta_{\alpha\beta}$$

One can easily check that

$$[A_\alpha, A_\beta] = [A^\dagger_\alpha, A^\dagger_\beta] = 0. \tag{3}$$

For the remaining commutator one finds [3]

$$[A_\alpha, A^\dagger_\beta] = \sum_{\mu\nu\mu'\nu'} \overline{\Phi}^{\mu\nu}_\alpha \Phi^{\mu'\nu'}_\beta \Big([a_\mu, a^\dagger_{\mu'}] b_\nu b^\dagger_{\nu'} + a^\dagger_{\mu'} a_\mu [b_\nu, b^\dagger_{\nu'}] \Big) = \delta_{\alpha\beta} - \Delta_{\alpha\beta} \tag{4}$$

where

$$\Delta_{\alpha\beta} \equiv \sum_{\mu\nu\mu'} \overline{\Phi}^{\mu\nu}_\alpha \Phi^{\mu'\nu}_\beta a^\dagger_{\mu'} a_\mu + \sum_{\mu\nu\nu'} \overline{\Phi}^{\mu\nu}_\alpha \Phi^{\mu\nu'}_\beta b^\dagger_{\nu'} b_\nu.$$

The entity $\Delta_{\alpha\beta}$ in (4) shows deviation from pure bosonic canonical relation. Note that if $\Delta_{\alpha\beta} = 0$ we have $\Phi^{\mu\nu}_\alpha = 0$.

Remark that unlike the realization of quasibosonic operators using quonic version of deformed oscillator algebra, as it was done in [3], in all our analysis we consider (the set of) completely independent copies of deformed oscillators. That is, we assume validity of (3) and also require $[A_\alpha, A^\dagger_\beta] = 0$ for $\alpha \neq \beta$.

The most simple type of deformed oscillator is the Arik-Coon q-deformation [5]. So it is of interest, first, to try to use this set of q-deformed bosons for representing the system of independent quasibosons. However, as it was shown in [26], the representation of quasibosons with the *independent* system of q-deformed bosons of the Arik-Coon type leads to inconsistency. For that reason we set the goal to examine other deformed oscillators in general form given by their structure function $\phi(N)$.

Necessary conventions. Our goal is to operate with A_α, $A^\dagger_\alpha$ and N_α constructed from $a^\dagger_\mu, a_\mu, b^\dagger_\nu, b_\nu$ (N_α is some effective number operator for composite particles) as with the elements (operators) of some deformed oscillator algebra, "forgetting" about their internal structure. It means that we are looking for subalgebras of the enveloping algebra $\mathfrak{A}\{A_\alpha, A^\dagger_\alpha, N_\alpha\}$, generated by A_α, $A^\dagger_\alpha$, N_α, isomorphic to some deformed oscillator algebras $\mathfrak{A}\{\mathcal{A}_\alpha, \mathcal{A}^\dagger_\alpha, \mathcal{N}_\alpha\}$, generated by $\mathcal{A}_\alpha$, $\mathcal{A}^\dagger_\alpha$, $\mathcal{N}_\alpha$:

$$\mathfrak{A}\{A_\alpha, A^\dagger_\alpha, N_\alpha\} \simeq \mathfrak{A}\{\mathcal{A}_\alpha, \mathcal{A}^\dagger_\alpha, \mathcal{N}_\alpha\}.$$

We are going to establish necessary and sufficient conditions for the existence of such isomorphism. We also require the isomorphism of representation spaces of mentioned algebras:

$$L\{(a^\dagger_\mu)^r (b^\dagger_\nu)^s \ldots |O\rangle\} \supset H \simeq \mathcal{H} = L\{\mathcal{A}^\dagger_{\gamma_1} \ldots \mathcal{A}^\dagger_{\gamma_n}|O\rangle\}, \tag{5}$$

where $L\{...\}$ denotes a linear span. Thus, if the algebra of deformed oscillator operators is given by the relations

$$G_i(\mathcal{A}_\alpha, \mathcal{A}^\dagger_\alpha, \mathcal{N}_\alpha) = 0 \quad \Leftrightarrow \quad G_i(\mathcal{A}_\alpha, \mathcal{A}^\dagger_\alpha, \mathcal{N}_\alpha)\mathcal{A}^\dagger_{\gamma_1} \ldots \mathcal{A}^\dagger_{\gamma_n}|O\rangle = 0, \quad n = 0, 1, 2, ... \tag{6}$$

then necessary and sufficient conditions for the isomorphism to exist can be written as

$$G_i(A_\alpha, A^\dagger_\alpha, N_\alpha) \cong 0 \quad \overset{def}{\Longleftrightarrow} \quad G_i(A_\alpha, A^\dagger_\alpha, N_\alpha)A^\dagger_{\gamma_1} \ldots A^\dagger_{\gamma_n}|O\rangle = 0. \tag{7}$$

Here the symbol of weak equality $\cong$ is introduced which means equality on all the n-(quasi)particle states. Next we notice that

$$G_i A^\dagger_{\gamma_1}|O\rangle = 0 \quad \Leftrightarrow \quad [G_i, A^\dagger_{\gamma_1}]|O\rangle = 0 \tag{8}$$

and, by induction,

$$G_i A^\dagger_{\gamma_1} ... A^\dagger_{\gamma_n}|O\rangle = 0 \quad \Leftrightarrow \quad [...[G_i, A^\dagger_{\gamma_1}]..., A^\dagger_{\gamma_n}]|O\rangle = 0. \tag{9}$$

For general deformed oscillator defined using the structure function $\phi(N)$, see e.g. [1], the relation (7) takes the form

$$\begin{cases} \mathcal{A}^\dagger_\alpha \mathcal{A}_\alpha = \phi(\mathcal{N}_\alpha), \\ [\mathcal{A}_\alpha, \mathcal{A}^\dagger_\alpha] = \phi(\mathcal{N}_\alpha + 1) - \phi(\mathcal{N}_\alpha), \\ [\mathcal{N}_\alpha, \mathcal{A}^\dagger_\alpha] = \mathcal{A}^\dagger_\alpha, \quad [\mathcal{N}_\alpha, \mathcal{A}_\alpha] = -\mathcal{A}_\alpha. \end{cases} \tag{10}$$

Here the expressions for $[\mathcal{A}_\alpha, \mathcal{A}^\dagger_\beta]$, $\alpha \neq \beta$, if any, may be added. Thus, the set of functions G_i applicable in this case reads as follows:

$$G_0(A_\alpha, A^\dagger_\alpha, N_\alpha) = A^\dagger_\alpha A_\alpha - \phi(N_\alpha),$$
$$G_1(A_\alpha, A^\dagger_\alpha, N_\alpha) = [A_\alpha, A^\dagger_\alpha] - (\phi(N_\alpha+1) - \phi(N_\alpha)),$$
$$G_2(A^\dagger_\alpha, N_\alpha) = [N_\alpha, A^\dagger_\alpha] - A^\dagger_\alpha, \quad \text{and possibly some others.}$$

Such functions G_i are determined by the structure function of deformation $\phi(N_\alpha)$. So, the relations (7) can be used for deducing the connection between matrices $\Phi^{\mu\nu}_\alpha$, which determine the operators $A^\dagger_\alpha$, and the DSF $\phi(N_\alpha)$.

2.1. Necessary conditions on $\Phi^{\mu\nu}_\alpha$ and $\phi(n)$

In the subsequent analysis we study the independent quasibosons system realized by deformed oscillators without indication of the particular model of deformation. The aim of this section is to obtain necessary conditions for such realization in terms of the matrices Φ_α. Note that the results of this section are not sensitive to the form of the definition of $N_\alpha(\cdot)$ as a function of A_α, $A^\dagger_\alpha$, etc.

Using the relations (7)-(10) and taking into account independence of modes, we come to the following weak equalities for the commutators:

$$\begin{cases} [A_\alpha, A^\dagger_\beta] \cong 0 \quad \text{for} \quad \alpha \neq \beta, \\ [N_\alpha, A^\dagger_\alpha] \cong A^\dagger_\alpha, \quad [N_\alpha, A_\alpha] \cong -A_\alpha, \\ [A_\alpha, A^\dagger_\alpha] \cong \phi(N_\alpha+1) - \phi(N_\alpha). \end{cases} \tag{11}$$

Treatment of modes independence. From the first relation in (11) we derive the equivalent requirements of independence in terms of matrices Φ:

$$\sum_{\mu'\nu'} \left(\Phi^{\mu\nu'}_\beta \overline{\Phi}^{\mu'\nu'}_\alpha \Phi^{\mu'\nu}_\gamma + \Phi^{\mu\nu'}_\gamma \overline{\Phi}^{\mu'\nu'}_\alpha \Phi^{\mu'\nu}_\beta \right) = 0, \quad \alpha \neq \beta, \tag{12}$$

which can be rewritten in the matrix form

$$\Phi_\beta \Phi^\dagger_\alpha \Phi_\gamma + \Phi_\gamma \Phi^\dagger_\alpha \Phi_\beta = 0, \quad \alpha \neq \beta. \tag{13}$$

Conditions on $\Phi^{\mu\nu}_\alpha$ within one mode α. Since $A^\dagger_\alpha A_\alpha \cong \phi(N_\alpha)$ and $A_\alpha A^\dagger_\alpha \cong \phi(N_\alpha+1)$, we have

$$[A^\dagger_\alpha A_\alpha, A_\alpha A^\dagger_\alpha] \cong 0 \qquad \text{and} \qquad [\Delta_{\alpha\alpha}, N_\alpha] \cong 0. \tag{14}$$

The first equality can equivalently be rewritten as

$$[A^\dagger_\alpha A_\alpha, \Delta_{\alpha\alpha}] = [A^\dagger_\alpha A_\alpha, \sum_{\mu\nu\mu'} \overline{\Phi}^{\mu\nu}_\alpha \Phi^{\mu'\nu}_\alpha a^\dagger_{\mu'} a_\mu + \sum_{\mu\nu\nu'} \overline{\Phi}^{\mu\nu}_\alpha \Phi^{\mu\nu'}_\alpha b^\dagger_{\nu'} b_\nu] \cong 0.$$

Calculation of this commutator gives

$$[A^\dagger_\alpha A_\alpha, \Delta_{\alpha\alpha}] = 2A^\dagger_\alpha \sum_{\mu\nu} \left(\Psi^\dagger_\alpha\right)^{\nu\mu} b_\nu a_\mu - 2\sum_{\mu'\nu'} \Psi^{\mu'\nu'}_\alpha a^\dagger_{\mu'} b^\dagger_{\nu'} A_\alpha \cong 0, \tag{15}$$

$$\Psi_\alpha \equiv \Phi_\alpha \Phi^\dagger_\alpha \Phi_\alpha.$$

With the account of (2) one can see: the validity of (15) on the one-quasiboson state requires that the commutator with the creation operator on the vacuum should be

$$\Big[(\overline{\Psi}_\alpha^{\mu\nu}\Phi_\alpha^{\mu'\nu'} - \overline{\Phi}_\alpha^{\mu\nu}\Psi_\alpha^{\mu'\nu'})a^\dagger_{\mu'}b^\dagger_{\nu'}b_\nu a_\mu, \Phi_\alpha^{\lambda\rho}a^\dagger_\lambda b^\dagger_\rho\Big]\,|O\rangle =$$
$$= \Big(\Phi_\alpha^{\mu'\nu'}a^\dagger_{\mu'}b^\dagger_{\nu'}\cdot\overline{\Psi}_\alpha^{\mu\nu}\Phi_\alpha^{\mu\nu} - \Phi_\alpha^{\mu'\nu'}a^\dagger_{\mu'}b^\dagger_{\nu'}\Delta[\Psi,\Phi] - \Psi_\alpha^{\mu'\nu'}a^\dagger_{\mu'}b^\dagger_{\nu'} + \Psi_\alpha^{\mu'\nu'}a^\dagger_{\mu'}b^\dagger_{\nu'}\cdot\Delta_{\alpha\alpha}\Big)|O\rangle$$
$$= \Big(\Phi_\alpha^{\mu'\nu'}\cdot\mathrm{Tr}(\Psi_\alpha^\dagger\Phi_\alpha) - \Psi_\alpha^{\mu'\nu'}\Big)\,a^\dagger_{\mu'}b^\dagger_{\nu'}|O\rangle = 0$$

(the summation over repeated indices is meant). From this we obtain the requirement

$$\Phi_\alpha\Phi_\alpha^\dagger\Phi_\alpha = \mathrm{Tr}(\Phi_\alpha^\dagger\Phi_\alpha\Phi_\alpha^\dagger\Phi_\alpha)\cdot\Phi_\alpha\,, \tag{16}$$

which is also sufficient one. This requirement guarantees not only weak equality as in (15) but also the corresponding strong (operator) equality.

Thus, we have two independent requirements (13) and (16) for the matrices Φ_α.

Relating Φ_α to the structure function $\phi(n)$. Let us derive the relations that involve the DSF ϕ. Directly from the system (11) we obtain the initial values for the DSF ϕ:

$$\phi(N_\alpha) \cong A_\alpha^\dagger A_\alpha \quad \Rightarrow \quad \phi(0) = 0,$$
$$\phi(N_\alpha+1) \cong A_\alpha A_\alpha^\dagger \quad \Rightarrow \quad \phi(1) = 1.$$

From (4) and the third relation in (11) we have

$$[A_\alpha, A_\alpha^\dagger] = 1 - \Delta_{\alpha\alpha} \cong \phi(N_\alpha+1) - \phi(N_\alpha), \tag{17}$$

or, equivalently,

$$F_{\alpha\alpha} \equiv \Delta_{\alpha\alpha} - 1 + \phi(N_\alpha+1) - \phi(N_\alpha) \cong 0. \tag{18}$$

If the conditions (see (11))

$$[N_\alpha, A_\alpha^\dagger] \cong A_\alpha^\dagger, \quad [N_\alpha, A_\alpha] \cong -A_\alpha \tag{19}$$

do hold (it means that for these relations a subsequent verification is needed), then

$$\phi(N_\alpha)A_\alpha^\dagger \cong A_\alpha^\dagger\phi(N_\alpha+1) \quad \Rightarrow \quad [\phi(N_\alpha), A_\alpha^\dagger] \cong A_\alpha^\dagger(\phi(N_\alpha+1) - \phi(N_\alpha)). \tag{20}$$

As a result, we come to

$$[F_{\alpha\alpha}, A_\alpha^\dagger] \cong 2(\Phi_\alpha\Phi_\alpha^\dagger\Phi_\alpha)^{\mu\nu}a^\dagger_\mu b^\dagger_\nu + A_\alpha^\dagger\Big(\phi(N_\alpha+2) - 2\phi(N_\alpha+1) + \phi(N_\alpha)\Big). \tag{21}$$

Requiring that this commutator vanishes on the vacuum and taking into account that $\phi(0) = 0$, $\phi(1) = 1$ we obtain

$$\Phi_\alpha\Phi_\alpha^\dagger\Phi_\alpha = \Big(1 - \frac{1}{2}\phi(2)\Big)\Phi_\alpha = \frac{f}{2}\Phi_\alpha$$

where the deformation parameter f does appear:

$$\frac{f}{2} \equiv 1 - \frac{1}{2}\phi(2) = \mathrm{Tr}(\Phi_\alpha^\dagger\Phi_\alpha\Phi_\alpha^\dagger\Phi_\alpha) \text{ for } \forall\alpha.$$

Finding admissible $\phi(n)$ explicitly. The equality (21) can be rewritten as

$$[F_{\alpha\alpha}, A_\alpha^\dagger] \cong (2-\phi(2))A_\alpha^\dagger + A_\alpha^\dagger(\phi(N_\alpha+2) - 2\phi(N_\alpha+1) + \phi(N_\alpha)). \quad (22)$$

By induction the equality for n-th commutator can be proven:

$$[\ldots[F_{\alpha\alpha}, A_\alpha^\dagger]\ldots A_\alpha^\dagger] \cong (A_\alpha^\dagger)^n \Big\{\sum_{k=0}^{n+1}(-1)^{n+1-k}C_{n+1}^k\phi(N_\alpha+k)\Big\} \quad (23)$$

(here C_n^k denotes binomial coefficients). The requirement that n-th commutator vanishes on the vacuum leads to the recurrence relation

$$\phi(n+1) = \sum_{k=0}^{n}(-1)^{n-k}C_{n+1}^k\phi(k), \quad n \geq 2. \quad (24)$$

As seen, all the values $\phi(n)$ for $n \geq 3$ are determined unambiguously by the two values $\phi(1)$ and $\phi(2)$, which may in general depend on one or more deformation parameters. Taking into account the equality [29]

$$\sum_{k=0}^{n}(-1)^{n-k}k^mC_n^k = \begin{cases} 0, & m < n, \\ n!, & m = n, \end{cases} \quad (25)$$

we find: the only independent solutions of eq. (24) are n and n^2, as well as their linear combination

$$\phi(n) = \left(1+\frac{f}{2}\right)n - \frac{f}{2}n^2. \quad (26)$$

This structure function satisfies both the initial conditions and the recurrence relations in (24).

Remark 1. In view of uniqueness of the solution with fixed initial conditions, the formula (26) gives general solution of the relation (24).

Remark 2. If we take the Hamiltonian in the form $H = \frac{1}{2}(\phi(N)+\phi(N+1))$, then using the obtained results it is not difficult to derive the three-term recurrence relations for both the deformation structure function and energy eigenspectrum:

$$\phi(n+1) = \frac{2(n+1)}{n}\phi(n) - \frac{n+1}{n-1}\phi(n-1),$$
$$E_{n+1} = \frac{4n^2+4n-4}{2n^2-1}E_n - \frac{2n^2+4n+1}{2n^2-1}E_{n-1}.$$

The latter equality has typical form of the so-called quasi-Fibonacci [15] relation for the eigenenergies. Note that general case of deformed oscillators with polynomial structure functions $\phi(N)$ (these are quasi-Fibonacci as well) was studied in [28].

2.2. Treatment of quasiboson number operator

The quasiboson number operator N_α can be introduced in different ways. Its definition is dictated by the requirements $G_0 \cong 0$, $G_1 \cong 0$ and also by self-consistency of the realization. Possible definition could be given by the relation $N_\alpha \stackrel{def}{=} \phi^{-1}(A_\alpha^\dagger A_\alpha)$, or by

$N_\alpha \overset{def}{=} \phi^{-1}(A_\alpha A_\alpha^\dagger) - 1$. We'll not choose some of the two forms of definition, but consider the general possibility:

$$N_\alpha \overset{def}{=} \chi(A_\alpha^\dagger A_\alpha, \varepsilon_\alpha), \quad \text{where} \quad \varepsilon_\alpha \equiv 1 - \Delta_{\alpha\alpha} = [A_\alpha, A_\alpha^\dagger]. \tag{27}$$

As we have mentioned above, it remains to satisfy the relations (19), which in turn serve as defining for the function χ. Note that the second of them stems by conjugation from the first one,

$$[N_\alpha, A_\alpha^\dagger] \cong A_\alpha^\dagger. \tag{28}$$

Since we assume *independence* of different modes, see (11), we consider the case $\gamma_1 = \gamma_2 = \ldots = \alpha$ in the definition (7).

It is useful to denote by L_n the operators

$$L_0 = N, \quad L_{n+1} = [L_n, A_\alpha^\dagger] = [\ldots[N_\alpha, A_\alpha^\dagger]\ldots A_\alpha^\dagger]. \tag{29}$$

With account of that, the condition (28) is written as

$$L_1|O\rangle = A_\alpha^\dagger|O\rangle, \quad L_n|O\rangle = 0, \quad n > 1. \tag{30}$$

Now consider three useful statements statements.

Proposition 1. The following relations are true:

$$\begin{array}{lll} [\Delta_{\alpha\alpha}, A_\alpha^\dagger] = f A_\alpha^\dagger, & [\Delta_{\alpha\alpha}, A_\alpha] = -\overline{f} A_\alpha, & f = 2\,\mathrm{Tr}(\Phi_\alpha^\dagger \Phi_\alpha \Phi_\alpha^\dagger \Phi_\alpha), \\ [\varepsilon_\alpha, A_\alpha^\dagger] = -f A_\alpha^\dagger, & [\Delta_{\alpha\alpha}, N_\alpha] \cong 0, & \Delta_{\alpha\alpha} = \Delta_{\alpha\alpha}^\dagger. \end{array}$$

This statement is proven straightforwardly.

Proposition 2. For each $n \geq 0$ we have the equalities:

$$\left[(A_\alpha^\dagger A_\alpha)^n, A_\alpha^\dagger\right] = A_\alpha^\dagger \left[(A_\alpha^\dagger A_\alpha + \varepsilon_\alpha)^n - (A_\alpha^\dagger A_\alpha)^n\right], \tag{31}$$

$$\left[\varepsilon_\alpha^n, A_\alpha^\dagger\right] = A_\alpha^\dagger[(-f + \varepsilon_\alpha)^n - \varepsilon_\alpha^n]. \tag{32}$$

Using the propositions 1,2 and the exact commuting of $A_\alpha^\dagger A_\alpha$ and ε_α we come to the following

Proposition 3. For N_α defined as $N_\alpha = \chi(A_\alpha^\dagger A_\alpha, \varepsilon)$, and $n \geq 0$, there is the following equality for the n-fold commutator (29):

$$L_n = (A_\alpha^\dagger)^n \chi(A_\alpha^\dagger A_\alpha + n\varepsilon_\alpha - \sigma_n f, \varepsilon_\alpha - nf) - \sum_{k=0}^{n-1} C_n^k (A_\alpha^\dagger)^{n-k} L_k,$$

$$\sigma_n = \frac{n(n-1)}{2}.$$

The proof of the propositions 2 and 3 is given in appendices.

Then the conditions (30) turn into equalities

$$\begin{cases} A_\alpha^\dagger \chi(A_\alpha^\dagger A_\alpha + \varepsilon_\alpha, \varepsilon_\alpha - f)|O\rangle = A_\alpha^\dagger|O\rangle, \\ (A_\alpha^\dagger)^n \chi(A_\alpha^\dagger A + n\varepsilon_\alpha - \sigma_n f, \varepsilon_\alpha - nf)|O\rangle = \\ \qquad = C_n^1 (A_\alpha^\dagger)^{n-1} L_1|O\rangle = n(A_\alpha^\dagger)^n|O\rangle, \quad n > 1. \end{cases} \tag{33}$$

To satisfy these, it is necessary that

$$\chi(n - \sigma_n f, 1 - nf) = n, \quad n \geq 1. \tag{34}$$

So, the condition (34) guarantees the validity of commutation relations (19), and therefore the consistency of the whole representation of quasibosons by deformed bosons. As one can see, both definitions $N_\alpha \overset{def}{=} \phi^{-1}(A^\dagger_\alpha A_\alpha)$ and $N_\alpha \overset{def}{=} \phi^{-1}(A_\alpha A^\dagger_\alpha) - 1$ satisfy (34). Also there are other definitions like $N_\alpha \overset{def}{=} (1-p)\phi^{-1}(A^\dagger_\alpha A_\alpha) + p(\phi^{-1}(A_\alpha A^\dagger_\alpha) - 1)$, $0 < p < 1$, which satisfy (34) and lead, as it can be checked, to self-consistent representation of quasibosons.

2.3. General solution for matrices Φ_α

In this subsection we describe how to find admissible $d_a \times d_b$ matrices Φ_α. These should satisfy the system

$$\begin{cases} \mathrm{Tr}(\Phi_\alpha \Phi^\dagger_\beta) = \delta_{\alpha\beta}, \\ \Phi_\alpha \Phi^\dagger_\alpha \Phi_\alpha = \dfrac{f}{2}\Phi_\alpha, \\ \Phi_\beta \Phi^\dagger_\alpha \Phi_\gamma + \Phi_\gamma \Phi^\dagger_\alpha \Phi_\beta = 0. \end{cases} \tag{35}$$

<u>*Consider first the case $f \neq 0$*</u>. If the matrix Φ_α is nondegenerate (that means $d_a = d_b$ and $\det \Phi_\alpha \neq 0$) for some α, the second equation yields $\Phi_\alpha \Phi^\dagger_\alpha = \frac{f}{2}\mathbb{1}$. From the third equation at $\gamma = \alpha$ we obtain: $\Phi_\beta = 0, \quad \forall \beta \neq \alpha$. Then it follows that only one value of α is possible for which $\det \Phi_\alpha \neq 0$. In that case Φ_α is an arbitrary unitary matrix. All the rest $\Phi_\beta = 0$, $\beta \neq \alpha$. That gives partial non-degenerate solution of the system. Note that other solutions will be degenerate for all α.

Let us go over to the analysis of degenerate solutions. At $\gamma = \alpha$ the last equation in (35) reduces to $\Phi_\beta \Phi^\dagger_\alpha \Phi_\alpha + \Phi_\alpha \Phi^\dagger_\alpha \Phi_\beta = 0$; multiplying it by $\Phi^\dagger_\alpha$ and utilizing the second equation we (note that f is real) infer

$$K \Phi_\beta \Phi^\dagger_\alpha \equiv \left(\Phi_\alpha \Phi^\dagger_\alpha + \frac{f}{2}\mathbb{1} \right) \Phi_\beta \Phi^\dagger_\alpha = 0, \qquad K \equiv \Phi_\alpha \Phi^\dagger_\alpha + \frac{f}{2}\mathbb{1}. \tag{36}$$

From the second equation of the system (35) we also obtain:

$$\forall x \in \mathrm{Im}\,\Phi_\alpha: \quad \Phi_\alpha \Phi^\dagger_\alpha x = \frac{f}{2}x \quad \Rightarrow \quad \dim \mathrm{Im}\,\Phi_\alpha \Phi^\dagger_\alpha \geq \dim \mathrm{Im}\,\Phi_\alpha.$$

With the account of the last statement and the fact that $\mathrm{Im}\,\Phi_\alpha \Phi^\dagger_\alpha \subseteq \mathrm{Im}\,\Phi_\alpha$ we find

$$\mathrm{Im}\,\Phi_\alpha \Phi^\dagger_\alpha = \mathrm{Im}\,\Phi_\alpha. \tag{37}$$

Applying the Fredholm theorem first to Φ_α and then to $\Phi_\alpha \Phi^\dagger_\alpha$ and using (37) we arrive at the decompositions

$$\begin{aligned} \forall \alpha: \quad \mathbb{C}^{d_a} &= \mathrm{Im}\,\Phi_\alpha \oplus \mathrm{Ker}\,\Phi^\dagger_\alpha = \mathrm{Im}\,\Phi_\alpha \Phi^\dagger_\alpha \oplus \mathrm{Ker}\,\Phi_\alpha \Phi^\dagger_\alpha, \\ \mathbb{C}^{d_a} &= \mathrm{Im}\,\Phi_\alpha \oplus \mathrm{Ker}\ \Phi_\alpha \Phi^\dagger_\alpha. \end{aligned}$$

On each of subspaces $\mathrm{Im}\,\Phi_\alpha$ and $\mathrm{Ker}\ \Phi_\alpha \Phi^\dagger_\alpha$, which are eigenspaces for K, the operator K is nondegenerate:

$$\forall x \in \mathrm{Im}\,\Phi_\alpha: \quad Kx = fx, \quad \text{and} \quad \forall y \in \mathrm{Ker}\,\Phi_\alpha \Phi^\dagger_\alpha: \quad Ky = \frac{f}{2}y. \tag{38}$$

Consequently the operator K is nondegenerate on the whole $\mathbb{C}^{d_a}$. Using (36) we find

$$\forall\alpha\neq\beta:\quad \Phi_\beta\Phi_\alpha^\dagger=0 \quad \text{or} \quad \Phi_\alpha\Phi_\beta^\dagger=0. \tag{39}$$

As result we come to the system which is equivalent to the initial one (35) and to the respective (for each equations) implications ($\alpha\neq\beta$)

$$\begin{cases} \mathrm{Tr}(\Phi_\alpha\Phi_\alpha^\dagger)=1, & \Rightarrow\ \dim\mathrm{Im}\,\Phi_\alpha\Phi_\alpha^\dagger=\mathrm{rank}\,\Phi_\alpha=2/f\equiv m,\ \text{ for all }\alpha,\\ \Phi_\alpha\Phi_\alpha^\dagger\cdot\Phi_\alpha=(f/2)\cdot\Phi_\alpha, & \Rightarrow\ \mathrm{Im}\,\Phi_\alpha - \text{eigen subspace of } \Phi_\alpha\Phi_\alpha^\dagger,\\ \Phi_\alpha\Phi_\alpha^\dagger\cdot\Phi_\beta=0, & \Rightarrow\ \forall\beta\neq\alpha\ \mathrm{Im}\,\Phi_\beta\subset\mathrm{Ker}\,\Phi_\alpha\Phi_\alpha^\dagger=\mathrm{Ker}\,\Phi_\alpha^\dagger,\\ \Phi_\alpha\Phi_\beta^\dagger=0. & \Rightarrow\ \mathrm{Im}\,\Phi_\beta^\dagger\subset\mathrm{Ker}\,\Phi_\alpha. \end{cases}$$

So, the deformation parameter f has a discrete range of values determined by m:

$$f=\frac{2}{m}. \tag{40}$$

The set of the solutions depends on the relation between $\sum_\alpha m$ and $\min(d_a,d_b)$. If $\sum_\alpha m>\min(d_a,d_b)$, the set of solutions is empty. If $\sum_\alpha m\le\min(d_a,d_b)$, then, according to the relations

$$\mathbb{C}^{d_a}=\mathrm{Im}\,\Phi_\alpha\oplus\mathrm{Ker}\,\Phi_\alpha^\dagger,\quad \mathrm{Im}\,\Phi_\beta\subset\mathrm{Ker}\,\Phi_\alpha^\dagger,\quad \forall\beta\neq\alpha, \tag{41}$$

the space $\mathbb{C}^{d_a}$ ($\mathbb{C}^{d_b}$) decomposes into the direct sum of linearly independent subspaces:

$$\mathbb{C}^{d_a}=\Big(\bigoplus_\alpha\mathrm{Im}\,\Phi_\alpha\Big)\oplus R,\quad \dim R=n-\sum_\alpha m,\quad \Phi_\alpha^\dagger R=0;$$
$$\mathbb{C}^{d_b}=\Big(\bigoplus_\alpha\mathrm{Im}\,\Phi_\alpha^\dagger\Big)\oplus\tilde{R},\quad \dim\tilde{R}=n-\sum_\alpha m,\quad \Phi_\alpha\tilde{R}=0.$$

Let $\{e_{1\alpha},\ldots,e_{m\alpha}\}$ be the orthonormal basis in the space $\mathrm{Im}\,\Phi_\alpha$, and $U_1(d_a)$ be the corresponding transition matrix to these bases from the initial one of $\mathbb{C}^{d_a}$. Likewise, let $\{f_{1\alpha},\ldots,f_{m\alpha}\}$ be the orthonormal basis in the space $\mathrm{Im}\,\Phi_\alpha^\dagger$, and $U_2(d_b)$ the corresponding transition matrix from the initial basis in $\mathbb{C}^{d_b}$. In the new bases, the transition matrix Φ_α is block-diagonal:

$$U_1^\dagger(d_a)\Phi_\alpha U_2(d_b)=\begin{pmatrix}0&0&0\\0&\tilde{\Phi}_\alpha&0\\0&0&0\end{pmatrix}. \tag{42}$$

The $m\times m$ matrix $\tilde{\Phi}_\alpha$ satisfies the equation $\tilde{\Phi}_\alpha\tilde{\Phi}_\alpha^\dagger=\frac{f}{2}\mathbb{1}_m$. Its general solution can be given through unitary matrix: $\tilde{\Phi}_\alpha=\sqrt{f/2}\ U_\alpha(m)$. Thus the general solution of the initial system (35) is given in the form

$$\Phi_\alpha=U_1(d_a)\,\mathrm{diag}\Big\{0,\sqrt{\frac{f}{2}}U_\alpha(m),0\Big\}U_2^\dagger(d_b). \tag{43}$$

In this formula, for every matrix Φ_α, the block $\sqrt{\frac{f}{2}}U_\alpha(m)$ is at its α-th place, and does not intersect with the corresponding block of any other matrix Φ_β for $\beta\neq\alpha$. We conclude: thus we have got all possible quasibosonic composite operators, expressed by (2) and (43), which can be realized by the algebra of deformed oscillators.

The case $f=0$ in (35). It can be shown that Φ_α should be zero for such f. That follows by applying singular value decomposition formula for each of the matrices in eq. $\Phi_\alpha\Phi_\alpha^\dagger\Phi_\alpha = 0$. The fact that $\Phi_\alpha = 0$ means, see eq. (2) and the normalization just after it, that pure boson being a special $f=0$ case of deformed boson with the DSF (26) is unsuitable for realization of the two-fermion composite quasiboson.

3. Quasibosons with q-deformed constituent fermions

Now let us go over to the q-generalization of the model considered above. Namely, we adopt nontrivial q-deformation for the constituents, the other assumptions being left as above. So, we use the set of q-fermions ($q \neq 1$), see [30], independent in fermionic sense:

$$a_\mu a_{\mu'}^\dagger + q^{\delta_{\mu\mu'}} a_{\mu'}^\dagger a_\mu = \delta_{\mu\mu'}, \qquad b_\nu b_{\nu'}^\dagger + q^{\delta_{\nu\nu'}} b_{\nu'}^\dagger b_\nu = \delta_{\nu\nu'}. \tag{44}$$

$$a_\mu a_{\mu'} + a_{\mu'} a_\mu = 0, \quad \mu \neq \mu', \quad b_\nu b_{\nu'} + b_{\nu'} b_\nu = 0, \quad \nu \neq \nu'. \tag{45}$$

The $q=1$ case (i.e., usual fermions with well-known nilpotency of their creation/annihilation operators) was completely analyzed in the preceding section. Here, the new feature appears: we have

$$(a_\mu^\dagger)^k \neq 0, \quad (b_\nu^\dagger)^k \neq 0 \tag{46}$$

if $(-q)$ is not a root of unity up to k-th degree. So we restrict ourselves with the case that $(-q)$ is not root of unity of any degree (including $q \neq 1$). Hence (46) holds for any k. The composite quasibosons' creation and annihilation operators are defined as

$$A_\alpha^\dagger = \sum_{\mu\nu} \Phi_\alpha^{\mu\nu} a_\mu^\dagger b_\nu^\dagger, \quad A_\alpha = \sum_{\mu\nu} \overline{\Phi}_\alpha^{\mu\nu} b_\nu a_\mu, \tag{47}$$

that is, similarly to eq. (2). The requirements of self-consistency of the realization (by deformed bosons) remain intact, see (10) and (11):

$$A_\alpha^\dagger A_\alpha \cong \phi(N_\alpha), \quad A_\alpha A_\alpha^\dagger \cong \phi(N_\alpha + 1), \tag{48}$$

$$[A_\alpha^\dagger, A_\beta^\dagger] \cong 0 \;\Leftrightarrow\; [A_\alpha, A_\beta] \cong 0, \quad [A_\alpha, A_\beta^\dagger] \cong 0, \;\alpha \neq \beta, \tag{49}$$

$$[N_\alpha, A_\alpha^\dagger] \cong A_\alpha^\dagger, \quad [N_\alpha, A_\alpha] \cong -A_\alpha. \tag{50}$$

In the present case the requirement of independence $[A_\alpha^\dagger, A_\beta^\dagger] \cong 0$, as one can easily check, leads to the following condition on matrices Φ_α:

$$\Phi_\alpha^{\mu\nu}\Phi_\beta^{\mu\nu'} = \Phi_\alpha^{\mu\nu'}\Phi_\beta^{\mu\nu}, \quad \Phi_\alpha^{\mu\nu}\Phi_\beta^{\mu'\nu} = \Phi_\alpha^{\mu'\nu}\Phi_\beta^{\mu\nu}. \tag{51}$$

The second relation in (48) implies there should be

$$A_\alpha (A_\alpha^\dagger)^n |O\rangle = \phi(N_\alpha + 1)(A_\alpha^\dagger)^{n-1}|O\rangle, \quad n = 1, 2, 3.... \tag{52}$$

Using (50) we obtain:

$$\phi(N_\alpha + 1)(A_\alpha^\dagger)^{n-1}|O\rangle = (A_\alpha^\dagger)^{n-1}\phi(N_\alpha + n)|O\rangle.$$

As a result we come to

$$A_\alpha (A_\alpha^\dagger)^n |O\rangle = \phi(n)(A_\alpha^\dagger)^{n-1}|O\rangle, \quad n = 1, 2, 3.... \tag{53}$$

It can be checked by induction that

$$A_\alpha (A^\dagger_\alpha)^n = (-1)^{\left[\frac{n-1}{2}\right]} \overline{\Phi^{\mu\nu}_\alpha} \prod_{j=1}^{n} \Phi^{\mu_1\nu_1}_\alpha \cdot$$
$$\cdot \Big[\sum_{i=1}^{n} (-1)^{i-1} \delta_{\mu\mu_i} q^{\sum_{s=1}^{i-1}\delta_{\mu\mu_s}} \prod_{\substack{r=1\\ r\neq i}}^{n} a^\dagger_{\mu_r} + (-1)^n q^{\sum_{s=1}^{n}\delta_{\mu\mu_s}} \prod_{r=1}^{n} a^\dagger_{\mu_r} \cdot a_\mu \Big] \cdot$$
$$\cdot \Big[\sum_{k=1}^{n} (-1)^{k-1} \delta_{\nu\nu_k} q^{\sum_{s=1}^{k-1}\delta_{\nu\nu_s}} \prod_{\substack{r=1\\ r\neq k}}^{n} b^\dagger_{\nu_r} + (-1)^n q^{\sum_{s=1}^{n}\delta_{\nu\nu_s}} \prod_{r=1}^{n} b^\dagger_{\nu_r} \cdot b_\nu \Big].$$

Then, using the equation (53) we come to

$$\phi(n) \prod_{l=1}^{n-1} \Phi^{\mu_l\nu_l}_\alpha a^\dagger_{\mu_l} b^\dagger_{\nu_l} |O\rangle = (-1)^{\left[\frac{n-1}{2}\right]} \overline{\Phi^{\mu\nu}_\alpha} \prod_{l=1}^{n} \Phi^{\mu_l\nu_l}_\alpha \cdot$$
$$\cdot \Big[\sum_{i=1}^{n} (-1)^{i-1} \delta_{\mu\mu_i} q^{\sum_{s=1}^{i-1}\delta_{\mu\mu_s}} \prod_{\substack{r=1\\ r\neq i}}^{n} a^\dagger_{\mu_r} + (-1)^n q^{\sum_{s=1}^{n}\delta_{\mu\mu_s}} \prod_{r=1}^{n} a^\dagger_{\mu_r} \cdot a_\mu \Big] \cdot$$
$$\cdot \Big[\sum_{k=1}^{n} (-1)^{k-1} \delta_{\nu\nu_k} q^{\sum_{s=1}^{k-1}\delta_{\nu\nu_s}} \prod_{\substack{r=1\\ r\neq k}}^{n} b^\dagger_{\nu_r} + (-1)^n q^{\sum_{s=1}^{n}\delta_{\nu\nu_s}} \prod_{r=1}^{n} b^\dagger_{\nu_r} \cdot b_\nu \Big] |O\rangle. \qquad (54)$$

Note that if eq. (54) holds on the vacuum, the following equality holds on any state:

$$(-1)^{\left[\frac{n-1}{2}\right]} \overline{\Phi^{\mu\nu}_\alpha} \prod_{l=1}^{n} \Phi^{\mu_l\nu_l}_\alpha \cdot \Big[\sum_{i=1}^{n} (-1)^{i-1} \delta_{\mu\mu_i} q^{\sum_{s=1}^{i-1}\delta_{\mu\mu_s}} \prod_{\substack{r=1\\ r\neq i}}^{n} a^\dagger_{\mu_r} \Big] \cdot$$
$$\cdot \Big[\sum_{k=1}^{n} (-1)^{k-1} \delta_{\nu\nu_k} q^{\sum_{s=1}^{k-1}\delta_{\nu\nu_s}} \prod_{\substack{r=1\\ r\neq k}}^{n} b^\dagger_{\nu_r} \Big] = \phi(n) \prod_{l=1}^{n-1} \Phi^{\mu_l\nu_l}_\alpha a^\dagger_{\mu_l} b^\dagger_{\nu_l}. \qquad (55)$$

As a recursive step, let us consider the following relation valid for $n+1$:

$$A_\alpha (A^\dagger_\alpha)^{n+1} = (-1)^{\left[\frac{n}{2}\right]} \overline{\Phi^{\mu\nu}_\alpha} \prod_{l=1}^{n} \Phi^{\mu_l\nu_l}_\alpha \cdot \Big[\sum_{i=1}^{n+1} (-1)^{i-1} \delta_{\mu\mu_i} q^{\sum_{s=1}^{i-1}\delta_{\mu\mu_s}} \prod_{\substack{r=1\\ r\neq i}}^{n+1} a^\dagger_{\mu_r} \Big] \cdot$$
$$\cdot \Big[\sum_{k=1}^{n+1} (-1)^{k-1} \delta_{\nu\nu_k} q^{\sum_{s=1}^{k-1}\delta_{\nu\nu_s}} \prod_{\substack{r=1\\ r\neq k}} b^\dagger_{\nu_r} \Big] \Phi^{\mu_{n+1}\nu_{n+1}}_\alpha + (-1)^{\left[\frac{n-1}{2}\right]} \overline{\Phi^{\mu\nu}_\alpha} \prod_{l=1}^{n} \Phi^{\mu_l\nu_l}_\alpha \cdot$$
$$\Big[\Big(\sum_{i=1}^{n+1} (-1)^{i-1} \delta_{\mu\mu_i} q^{\sum_{s=1}^{i-1}\delta_{\mu\mu_s}} \prod_{\substack{r=1\\ r\neq i}} a^\dagger_{\mu_r} \Big) \cdot (-1)^n q^{\sum_{s=1}^{n+1}\delta_{\nu\nu_s}} \prod_{r=1}^{n} b^\dagger_{\nu_r} \cdot b_\nu +$$
$$+ (-1)^n q^{\sum_{s=1}^{n+1}\delta_{\mu\mu_s}} \prod_{r=1}^{n} a^\dagger_{\mu_r} \cdot a_\mu \Big(\sum_{k=1}^{n+1} (-1)^{k-1} \delta_{\nu\nu_k} q^{\sum_{s=1}^{k-1}\delta_{\nu\nu_s}} \prod_{\substack{r=1\\ r\neq k}}^{n} b^\dagger_{\nu_r} \Big) +$$

$$+q^{\sum_{s=1}^{n+1}\delta_{\mu\mu_s}+\delta_{\nu\nu_s}}\prod_{r=1}^{n}a^\dagger_{\mu_r}\cdot a_\mu\prod_{r=1}^{n}b^\dagger_{\nu_r}\cdot b_\nu\Big]\Phi_\alpha^{\mu_{n+1}\nu_{n+1}}a^\dagger_{\mu_{n+1}}b^\dagger_{\nu_{n+1}}=$$

$$=\phi(n)\prod_{l=1}^{n}\Phi_\alpha^{\mu_l\nu_l}a^\dagger_{\mu_l}b^\dagger_{\nu_l}+(-1)^{\left[\frac{n}{2}\right]}\overline{\Phi_\alpha^{\mu\nu}}\prod_{l=1}^{n+1}\Phi_\alpha^{\mu_l\nu_l}\cdot$$

$$\Big[(-1)^n\Big(\sum_{i=1}^{n}(-1)^{i-1}\delta_{\mu\mu_i}q^{\sum_{s=1}^{i-1}\delta_{\mu\mu_s}}\prod_{\substack{r=1\\r\neq i}}a^\dagger_{\mu_r}\Big)\Big(\delta_{\nu\nu_{n+1}}q^{\sum_{s=1}^{n}\delta_{\nu\nu_s}}\prod_{r=1}^{n}b^\dagger_{\nu_r}-q^{\sum_{s=1}^{n+1}\delta_{\nu\nu_s}}\prod_{r=1}^{n+1}b^\dagger_{\nu_r}\cdot b_\nu\Big)+$$

$$+(-1)^n\Big(\delta_{\mu\mu_{n+1}}q^{\sum_{s=1}^{n}\delta_{\mu\mu_s}}\prod_{r=1}^{n}a^\dagger_{\mu_r}-q^{\sum_{s=1}^{n+1}\delta_{\mu\mu_s}}\prod_{r=1}^{n+1}a^\dagger_{\mu_r}\cdot a_\mu\Big)\Big(\sum_{k=1}^{n}(-1)^{k-1}\delta_{\nu\nu_k}q^{\sum_{s=1}^{k-1}\delta_{\nu\nu_s}}\prod_{\substack{r=1\\r\neq k}}^{n+1}b^\dagger_{\nu_r}\Big)+$$

$$+\Big(\delta_{\mu\mu_{n+1}}q^{\sum_{s=1}^{n}\delta_{\mu\mu_s}}\prod_{r=1}^{n}a^\dagger_{\mu_r}-q^{\sum_{s=1}^{n+1}\delta_{\mu\mu_s}}\prod_{r=1}^{n+1}a^\dagger_{\mu_r}\cdot a_\mu\Big)\Big(\delta_{\nu\nu_{n+1}}q^{\sum_{s=1}^{n}\delta_{\nu\nu_s}}\prod_{r=1}^{n}b^\dagger_{\nu_r}-q^{\sum_{s=1}^{n+1}\delta_{\nu\nu_s}}\prod_{r=1}^{n+1}b^\dagger_{\nu_r}\cdot b_\nu\Big)\Big]$$

where at the last stage we have used eq. (55). Substituting the last expression for $A_\alpha(A^\dagger_\alpha)^{n+1}$ into (53) written for $n+1$ we deduce the following relation that involves linear combination:

$$B^{\mu_1\ldots\mu_n,\nu_1\ldots\nu_n}(\Phi_\alpha,q)\cdot e_{\mu_1\ldots\mu_n,\nu_1\ldots\nu_n}=0, \tag{56}$$

where the coefficients are

$$B^{\mu_1\ldots\mu_n,\nu_1\ldots\nu_n}(\Phi_\alpha,q)=\sum_{i=1}^{n}(-1)^{n+i-1}q^{\sum_{s=1}^{i-1}(\delta_{\mu\mu_s}+\delta_{\nu\nu_s})}\overline{\Phi_\alpha^{\mu\nu}}\prod_{r=1}^{n-1}\Phi_\alpha^{\mu_r\nu_r}\Phi_\alpha^{\mu\nu_n}\Phi_\alpha^{\mu_n\nu}\cdot$$

$$\Big(q^{\sum_{s=i}^{n}\delta_{\nu\nu_s}}(-1)^{\sum_{r=i}^{n-1}\delta_{\nu_r\nu_{r+1}}}+q^{\sum_{s=i}^{n}\delta_{\mu\mu_s}}(-1)^{\sum_{r=1}^{n-1}\delta_{\mu_r\mu_{r+1}}}\Big)+$$

$$+q^{\sum_{s=1}^{n}(\delta_{\mu\mu_s}+\delta_{\nu\nu_s})}\overline{\Phi_\alpha^{\mu\nu}}\Phi_\alpha^{\mu\nu}\prod_{l=1}^{n}\Phi_\alpha^{\mu_l\nu_l}-[\phi(n+1)-\phi(n)]\prod_{l=1}^{n}\Phi_\alpha^{\mu_l\nu_l}$$

and the basis elements are

$$e_{\mu_1\ldots\mu_n,\nu_1\ldots\nu_n}=a^\dagger_{\mu_1}b^\dagger_{\nu_1}\ldots a^\dagger_{\mu_n}b^\dagger_{\nu_n}|O\rangle.$$

These basis elements are independent for differing sets of indices $\mu_1\ldots\mu_n$ and $\nu_1\ldots\nu_n$ regardless of any permutations within each set. So let us extract in (56) the terms with $\mu_1=\ldots=\mu_n$ and $\nu_1=\ldots=\nu_n$; using their linear independence from the others, we infer $B^{\mu_1\ldots\mu_1,\nu_1\ldots\nu_1}(\Phi_\alpha,q)=0$, that can be rewritten in the next form:

$$\sum_{i=1}^{n}(-1)^{n+i-1}\Big(2+(\delta_{\mu\mu_1}+\delta_{\nu\nu_1})(q^n-q^i-2)+2\delta_{\mu\mu_1}\delta_{\nu\nu_1}(q^n-1)(q^{i-1}-1)\Big)\Phi_\alpha^{\mu_1\nu}\overline{\Phi_\alpha^{\mu\nu}}\Phi_\alpha^{\mu\nu_1}(\Phi_\alpha^{\mu_1\nu_1})^{n-1}+$$

$$+\Big(1+(\delta_{\mu\mu_1}+\delta_{\nu\nu_1})(q^n-1)+\delta_{\mu\mu_1}\delta_{\nu\nu_1}(q^n-1)^2\Big)\overline{\Phi_\alpha^{\mu\nu}}\Phi_\alpha^{\mu\nu}(\Phi_\alpha^{\mu_1\nu_1})^n=$$

$$=[\phi(n+1)-\phi(n)](\Phi_\alpha^{\mu_1\nu_1})^n.$$

Performing the summation over i, μ, ν in the left-hand side we find

$$((-1)^n-1)\left(\Phi_\alpha\Phi_\alpha^\dagger\Phi_\alpha\right)^{\mu_1\nu_1}(\Phi_\alpha^{\mu_1\nu_1})^{n-1}+\Big(\frac{(-1)^n+1}{2}q^n-(-1)^n\Big)\left[(\Phi_\alpha^\dagger\Phi_\alpha)^{\nu_1\nu_1}+(\Phi_\alpha\Phi_\alpha^\dagger)^{\mu_1\mu_1}\right]\cdot$$

$$\cdot(\Phi_\alpha^{\mu_1\nu_1})^n + \frac{q-1}{q+1}(q^n-1)\,(q^n-(-1)^n)\,|\Phi_\alpha^{\mu_1\nu_1}|^2(\Phi_\alpha^{\mu_1\nu_1})^n = \\ = [\phi(n+1)-\phi(n)-1](\Phi_\alpha^{\mu_1\nu_1})^n. \qquad (57)$$

For all the indices (μ_1,ν_1), for which $\Phi_\alpha^{\mu_1\nu_1}\neq 0$, the last equation can be divided by $(\Phi_\alpha^{\mu_1\nu_1})^n$. Summing eq. (57) over n from $n=1$ to $n=s$ and then replacing in the resulting equality $s+1\to n$ we obtain:

$$\Big(\frac{1-(-1)^n}{2}-n\Big)\frac{(\Phi_\alpha\Phi_\alpha^\dagger\Phi_\alpha)^{\mu_1\nu_1}}{\Phi_\alpha^{\mu_1\nu_1}}+\left(\frac{q-1}{q+1}\right)\left([n]_{q^2}-[n]_q-[n]_{-q}+\frac{1}{2}(1-(-1)^n)\right)\cdot|\Phi_\alpha^{\mu_1\nu_1}|^2+ \\ +\frac{1}{2}\Big([n]_q+[n]_{-q}-(-1)^n-1\Big)\left[(\Phi_\alpha^\dagger\Phi_\alpha)^{\nu_1\nu_1}+(\Phi_\alpha\Phi_\alpha^\dagger)^{\mu_1\mu_1}\right]=\phi(n)-n,\ n\geq 2.$$

Here the notation $[n]_q\equiv(q^n-q^{-n})/(q-q^{-1})$ is introduced. Discriminant of this system is not zero for the admissible values of q. Hence $(\Phi_\alpha\Phi_\alpha^\dagger\Phi_\alpha)^{\mu_1\nu_1}/\Phi_\alpha^{\mu_1\nu_1}$, $|\Phi_\alpha^{\mu_1\nu_1}|^2$ and $[(\Phi_\alpha^\dagger\Phi_\alpha)^{\nu_1\nu_1}+(\Phi_\alpha\Phi_\alpha^\dagger)^{\mu_1\mu_1}]$ do not depend on (μ_1,ν_1) if $\Phi_\alpha^{\mu_1\nu_1}\neq 0$:

$$(\Phi_\alpha\Phi_\alpha^\dagger\Phi_\alpha)^{\mu_1\nu_1}/\Phi_\alpha^{\mu_1\nu_1}=p_1, \\ |\Phi_\alpha^{\mu_1\nu_1}|^2=p_2, \\ (\Phi_\alpha^\dagger\Phi_\alpha)^{\nu_1\nu_1}+(\Phi_\alpha\Phi_\alpha^\dagger)^{\mu_1\mu_1}=p_3$$

where p_1, p_2 and p_3 are some numerical parameters. Thus we obtain

$$\phi(n)=n-\Big(n-\frac{1-(-1)^n}{2}\Big)p_1+\left(\frac{q-1}{q+1}\right)\left([n]_{q^2}-[n]_q-[n]_{-q}+\frac{1-(-1)^n}{2}\right)p_2+ \\ +\frac{1}{2}\Big([n]_q+[n]_{-q}-(-1)^n-1\Big)p_3. \qquad (58)$$

Let us now consider the terms in equation (56) with basis elements $e_{\mu_1\ldots\mu_n,\nu_1\ldots\nu_n}$, where $\mu_1=\ldots=\mu_n$, and among indices $(\nu_1,\ldots,\nu_n)$ there are $n-1$ identical, the remaining one being different. Let the $n-1$ identical indices be equal to ν_1, and the remaining one is ν_2 and sits at the k-th position (i.e. in place of ν_k). Due to linear independence of the mentioned terms from the others we get the equation

$$\sum_{k=1}^n B^{\mu_1\ldots\mu_1,\nu_1..\nu_k..\nu_1}e_{\mu_1\ldots\mu_1,\nu_1..\nu_k..\nu_1}|_{\nu_k\to\nu_2}=0\ \text{ i.e. }\ \sum_{k=1}^n(-1)^k B^{\mu_1\ldots\mu_1,\nu_1..\nu_k..\nu_1}|_{\nu_k\to\nu_2}=0. \qquad (59)$$

Introducing auxiliary notations

$$\begin{cases} X=\Phi_\alpha^{\mu_1\nu_1}\Phi_\alpha^{\mu_1\nu_2}, \\ Y=\Phi_\alpha^{\mu_1\nu_1}\Phi_\alpha^{\mu_1\nu_2}(\Phi_\alpha\Phi_\alpha^\dagger)^{\mu_1\mu_1}, \\ Z=(\Phi_\alpha^{\mu_1\nu_1})^2(\Phi_\alpha^\dagger\Phi_\alpha)^{\nu_1\nu_2}, \end{cases} \qquad (60)$$

after performing all the summations in (59) we obtain:

$$\begin{aligned}&[Xp_2](-1)^n q^{2n}+\\&[(-q^3+2q^2-3q+4)p_2X]q^{2n}+\\&[((q^2-2-q)p_2+2p_3)X+(-2-q)Y]nq^n+\\&[((-3q^3+17q^2+q^4-26-5q)p_2+(-4q+2q^2+2)p_3)X+(6+5q-q^3-2q^2)Y+\\&\qquad+(4q-10q^2+6)Z]q^n+\\&[((-q^3+q+2q^2-2)p_2+(2-2q)p_3)X+(q^2+3q-2)Y+(-2-2q)Z](-q)^n+\\&[((q^2+3-4q)p_2+(-4q^2+2q+2)p_3)X+(4q^2-q-5)Y+(3q^2+1)Z](-1)^n+\\&[(2p_1+(-3q+5)p_2-2p_3)X+Y+(3q-3)Z]n+\\&[((8-8q^2)p_1+(23-3q-19q^2+7q^3)p_2+(8q-4q^3+2q^2-6)p_3)X+\\&\qquad+(4q^3-5-12q+5q^2)Y+(-3q^3-11+3q+11q^2)Z]=0.\end{aligned}$$

Extracting the coefficients of this system at the independent functions $(-1)^n q^{2n}$, q^{2n}, nq^n, q^n, $(-q)^n, (-1)^n, n, 1$, we come to the following system:

$$\begin{cases}Xp_2=0,\\ [-q^3+2q^2-3q+4]p_2X=0,\\ [(q^2-2-q)p_2+2p_3]X+[-2-q]Y=0,\\ [(-3q^3+17q^2+q^4-26-5q)p_2+(-4q+2q^2+2)p_3]X+[6+5q-q^3-2q^2]Y+\\ \qquad+[4q-10q^2+6]Z=0,\\ [(-q^3+q+2q^2-2)p_2+(2-2q)p_3]X+[q^2+3q-2]Y+[-2-2q]Z=0,\\ [(q^2+3-4q)p_2+(-4q^2+2q+2)p_3]X+[4q^2-q-5]Y+[3q^2+1]Z=0,\\ [2p_1+(-3q+5)p_2-2p_3]X+Y+[3q-3]Z=0,\\ [(8-8q^2)p_1+(23-3q-19q^2+7q^3)p_2+(8q-4q^3+2q^2-6)p_3]X+\\ \qquad+[4q^3-5-12q+5q^2]Y+[-3q^3-11+3q+11q^2]Z=0.\end{cases}$$

The solution of this system is $(q\neq 1)$

$$\begin{cases}X=\Phi_\alpha^{\mu_1\nu_1}\Phi_\alpha^{\mu_1\nu_2}=0,\\ Y=\Phi_\alpha^{\mu_1\nu_1}\Phi_\alpha^{\mu_1\nu_2}(\Phi_\alpha\Phi_\alpha^\dagger)^{\mu_1\mu_1}=0,\\ Z=(\Phi_\alpha^{\mu_1\nu_1})^2(\Phi_\alpha^\dagger\Phi_\alpha)^{\nu_1\nu_2}=0.\end{cases}$$

This set of conditions is equivalent to such one:

$$\Phi_\alpha^{\mu_1\nu_1}\Phi_\alpha^{\mu_1\nu_2}=0, \tag{61}$$

which means that the matrix Φ_α can not contain two nonzero elements in any one raw.

In a similar way we can obtain the condition

$$\Phi_\alpha^{\mu_1\nu_1}\Phi_\alpha^{\mu_2\nu_1}=0, \tag{62}$$

implying that the matrix Φ_α can not contain two nonzero elements in any one column.

Next perform the same analysis for the terms, when among indices $(\mu_1,\ldots,\mu_n)$ there is one (denoted by μ_2) different from the rest $(n-1)$ identical ones (which we denote by μ_1), and likewise for indices ν: among indices $(\nu_1, \ldots, \nu_n)$ there is one (denoted by ν_2) different from the rest identical ones (denoted by ν_1). As result we derive

$$\Phi_\alpha^{\mu_1\nu_1}\Phi_\alpha^{\mu_2\nu_2} = 0.$$

That is, the matrix Φ_α cannot have two nonzero elements in differing raws and columns. And, using the previous conditions (61),(62), we obtain that the matrix Φ_α cannot contain two nonzero elements. As a consequence, we obtain the following values for the parameters p_1, p_2, p_3:

$$p_1 = p_2 = 1, \quad p_3 = 2.$$

Then the following expression for the deformation structure function results from (58):

$$\phi(n) = \frac{1 + 2q^n(-1)^{n-1} + q^{2n}}{1 + 2q + q^2} = \left(\frac{1 - (-q)^n}{1 + q}\right)^2 = \left([n]_{-q}\right)^2. \tag{63}$$

Independence conditions contained in (51) enable us to determine the solution for Φ_α: the only non-zero elements in matrices Φ_α and Φ_β are situated at intersection of different raws and different columns: $\Phi_\alpha^{\mu\nu} = \delta_{\mu\mu_0(\alpha)}\delta_{\nu\nu_0(\alpha)}$.

4. Conclusions and outlook

As shown in our preceding paper and in section 2 above, the problem of realization of "fermion+fermion" quasibosons by means of deformed oscillators has nontrivial solutions. In this case of pure fermions as constituents, the structure function ϕ of the relevant deformation is of the form (26), i.e. quadratic in the number operator N, and with one deformation parameter $f = 2/m$. This is the only DSF for which the realization (isomorphism) is possible. In addition, necessary and sufficient conditions on the matrices Φ_α involved in the construction (2) of quasibosons, for such representation to be self-consistent, are obtained and the expression (43) gives their general solution.

Then, novel generalization was carried out, as described in section 3. This is the case of the quasibosons made of two constituents which are not of Fermi type, but the q-deformed fermions. For this case, again, we have derived the relations for the defining matrices Φ_α and solved them. Detailed analysis led us to the resulting structure function (63) of deformed oscillator which provides exact realization of the quasibosons made from two q-fermions. The essential distinction of the situation treated here from the case considered in Section 2 (and in short in [26]) is such that, while the pure fermions are nilpotent, for the q-deformed fermions we required absence of any order of nilpotency, see eq. (46). Since the second order nilpotency of usual fermions (as the non-deformed limit of q-fermions) reappears at $q \to 1$ abruptly, there is no direct connection (direct limit) which would lead to DSF (26) from the alternative DSF (63).

General strategy of the developed approach is to explore deformed bosons as tools to realize quasibosons, which should give considerable simplification (in algebraic sense) in subsequent applications achieved when the algebra, representing initial system of composite particles, reduces to the algebra corresponding to some deformed oscillator.

The application of the obtained results to specific physical situations or systems with composite particles is clearly of importance. Such results can be useful for modeling physical particles or quasi-particles (mesons, higgson, light even nuclei, excitons, etc.).

As the next steps we are planning to study more complicated situations, say, the quasibosons composed from two (deformed) bosons, or from four (deformed) fermions. Yet another path of extension is to treat quasi-independent quasibosons. Also, in our nearest plans there is the analysis of composite (quasi-)fermions.

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Appendix A. Proof of Proposition 2

As our treatment below concerns only one mode α, we will omit the index α. Let us first prove the equality (31). For $n = 0$ this is trivial. Put $n = 1$:

$$[A^\dagger A, A^\dagger] = A^\dagger [A, A^\dagger] = A^\dagger (1 - \Delta_{\alpha\alpha}) = A^\dagger \varepsilon = A^\dagger \left[(A^\dagger A + \varepsilon)^1 - (A^\dagger A)^1 \right].$$

Then we proceed by induction. Assuming that the equality holds for n, let us prove that it is valid for $n + 1$:

$$\begin{aligned}\left[(A^\dagger A)^{n+1}, A^\dagger\right] &= \left[A^\dagger A (A^\dagger A)^n, A^\dagger\right] = [A^\dagger A, A^\dagger](A^\dagger A)^n + A^\dagger A[(A^\dagger A)^n, A^\dagger] = \\ &= A^\dagger \varepsilon (A^\dagger A)^n + A^\dagger A A^\dagger \left[(A^\dagger A + \varepsilon)^n - (A^\dagger A)^n\right] = A^\dagger \varepsilon (A^\dagger A)^n + \\ &+ A^\dagger (A^\dagger A + \varepsilon)^{n+1} - A^\dagger (A^\dagger A)^{n+1} - A^\dagger \varepsilon (A^\dagger A)^n = A^\dagger \left[(A^\dagger A + \varepsilon)^{n+1} - (A^\dagger A)^{n+1}\right].\end{aligned}$$

Consider the second equation. When $n = 0$ it is also trivial. For $n = 1$ we have

$$[\varepsilon, A^\dagger] = -f A^\dagger = A^\dagger [(-f + \varepsilon) - \varepsilon].$$

The step of induction is:

$$\begin{aligned}\left[\varepsilon^{n+1}, A^\dagger\right] &= \left[\varepsilon \varepsilon^n, A^\dagger\right] = -f A^\dagger \varepsilon^n + \varepsilon A^\dagger [(-f + \varepsilon)^n - \varepsilon^n] = \\ &= -f A^\dagger \varepsilon^n + (-f A^\dagger + A^\dagger \varepsilon)[(-f + \varepsilon)^n - \varepsilon^n] = A^\dagger [(-f + \varepsilon)^{n+1} - \varepsilon^{n+1}].\end{aligned}$$

Thus, the proposition is proven.

Appendix B. Proof of Proposition 3

When $n = 0$ the equality reduces to the definition of N. Let us prove it for $n = 1$. Present χ as formal series:

$$\chi(x, y) = \sum_{n,m=1}^{\infty} b_{nm} x^n y^m, \quad [x, y] = 0.$$

Then

$$L_1 = [\chi(A^\dagger A,\varepsilon),A^\dagger] = \sum_{n,m=1}^{\infty} b_{nm}[(A^\dagger A)^n,A^\dagger]\,\varepsilon^m + \sum_{n,m=1}^{\infty} b_{nm}(A^\dagger A)^n\,[\varepsilon^m,A^\dagger] =$$

$$= \sum_{n,m=1}^{\infty} b_{nm}\left[(A^\dagger A+\varepsilon)^n-(A^\dagger A)^n\right]\varepsilon^m + \sum_{n,m=1}^{\infty} b_{nm}(A^\dagger A)^n A^\dagger[(-f+\varepsilon)^m-\varepsilon^m] =$$

$$= A^\dagger\left[\chi(A^\dagger A+\varepsilon,\varepsilon-f)-\chi(A^\dagger A,\varepsilon)\right] = A^\dagger\chi(A^\dagger A+\varepsilon,\varepsilon-f)-A^\dagger N.$$

Next, proceed by induction. The induction step is:

$$L_{n+1} = [L_n,A^\dagger] = (A^\dagger)^n[\chi(A^\dagger A+n\varepsilon-\alpha_n f,\varepsilon-nf),A^\dagger] - \sum_{k=0}^{n-1} C_n^k (A^\dagger)^{n-k}[L_k,A^\dagger] =$$

$$= (A^\dagger)^n[\chi(A^\dagger A+n\varepsilon-\alpha_n f,\varepsilon-nf),A^\dagger] - \sum_{k=0}^{n-1} C_n^k (A^\dagger)^{n-k} L_{k+1}. \tag{B.1}$$

Let us transform the commutator in the last expression:

$$[\chi(A^\dagger A+n\varepsilon-\alpha_n f,\varepsilon-nf),A^\dagger] = \sum_{n,m=1}^{\infty} b_{nm}[(A^\dagger A+n\varepsilon-\alpha_n f)^n,A^\dagger](\varepsilon-nf)^m +$$

$$+ \sum_{n,m=1}^{\infty} b_{nm}(A^\dagger A+n\varepsilon-\alpha_n f)^n[(\varepsilon-nf)^m,A^\dagger] =$$

$$= A^\dagger\chi(A^\dagger A+(n+1)\varepsilon-\alpha_{n+1}f,\varepsilon-(n+1)f) - A^\dagger\chi(A^\dagger A+n\varepsilon-\alpha_n f,\varepsilon-nf);$$

$$\alpha_{n+1} = \alpha_n + n.$$

Consequently,

$$(A^\dagger)^n[\chi(A^\dagger A+n\varepsilon-\alpha_n f,\varepsilon-nf),A^\dagger] = (A^\dagger)^{n+1}\chi(A^\dagger A+(n+1)\varepsilon-\alpha_{n+1}f,\varepsilon-(n+1)f) -$$
$$- A^\dagger\cdot(A^\dagger)^n\chi(A^\dagger A+n\varepsilon-\alpha_n f,\varepsilon-nf). \tag{B.2}$$

Using the induction assumption, the last term takes the form:

$$A^\dagger\cdot(A^\dagger)^n\chi(A^\dagger A+n\varepsilon-\alpha_n f,\varepsilon-nf) = A^\dagger L_n + A^\dagger\sum_{k=0}^{n-1} C_n^k (A^\dagger)^{n-k} L_k.$$

Substituting this expression into (B.2), and then the resulting expression into (B.1), we obtain

$$L_{n+1} = (A^\dagger)^{n+1}\chi(A^\dagger A+(n+1)\varepsilon-\alpha_{n+1}f,\varepsilon-(n+1)f) - A^\dagger L_n - A^\dagger\sum_{k=0}^{n-1} C_n^k (A^\dagger)^{n-k} L_k - \sum_{k=0}^{n-1} C_n^k (A^\dagger)^{n-k} L_{k+1}$$

$$= (A^\dagger)^{n+1}\chi(A^\dagger A+(n+1)\varepsilon-\alpha_{n+1}f,\varepsilon-(n+1)f) - \sum_{k=0}^{n} C_{n+1}^k (A^\dagger)^{n+1-k} L_k.$$

Thus, the proposition is proven.

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