Perfect Codes for Uniform Chains Poset Metrics

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***Abstract*—The class of poset metrics is very large and contains some interesting families of metrics. A family of metrics, based on posets which are formed from disjoint chains which have the same size, is examined. A necessary and sufficient condition, for the existence of perfect single-error-correcting codes for such poset metrics, is proved.**

***Index Terms*—Disjoint uniform chains, perfect codes, poset codes.**

I. Introduction

The classic coding theory consider codes in the Hamming scheme [1]. Let $\mathbb{F}_q$ denotes a finite field with q elements. For two words $X = (x_1, x_2, \ldots, x_n),\ Y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{F}_q^n$, the Hamming distance between X and Y, $d_H(X, Y)$, is the number of positions in which X and Y differ. An (n, M, d) code $\mathcal{C}$ over $\mathbb{F}_q$ is a subset of size M of $\mathbb{F}_q^n$, such that for each two codewords $X, Y \in \mathcal{C}$, we have $d_H(X, Y) \geq d$. This definition was generalized by Niederreiter [2], [3], [4], [5] as defined in the sequel.

The poset metric was defined by Bruladi, Graves, and Lawrence [5]. Let P be an arbitrary finite poset of cardinality n whose partial order relation is denoted by $\leq$. If $A \subseteq P$ then $\langle A \rangle$ denotes the smallest ideal in P which contains A, i.e.,

$$\langle A \rangle \stackrel{\text{def}}{=} \{x \ : \ (\exists y)(y \in A,\ x \leq y)\}$$

For a word $x = (x_1, x_2, \ldots, x_n) \in \mathbb{F}_q^n$, let supp$(x)$ denotes the support of x, i.e. supp$(x) = \{i \ : \ x_i \neq 0\}$. We define the P-*weight* of x, $w_P(x)$, to be the cardinality of $\langle \text{supp}(x) \rangle$, i.e. $w_P(x) = |\langle \text{supp}(x) \rangle|$. For two vectors $X, Y \in \mathbb{F}_q^n$ the P-distance, $d_P(X, Y)$, is defined by $d_P(X, Y) = w_P(X - Y)$. An (n, M, d_P) P-code $\mathcal{C}$ over $\mathbb{F}_q$ is a subset of size M of $\mathbb{F}_q^n$, such that for each two codewords $X, Y \in \mathcal{C}$, we have $d_P(X, Y) \geq d_P$, i.e., the minimum P-distance of the code is d_P. If P is an antichain (isolated points) then these definitions coincide with those of the Hamming metric. The definition of poset codes given in [5] is a generalization of a different definition given by Niederreiter [2], [3], [4] for a generalization to the Hamming metric. His generalization relates to posets which consist of disjoint chains. Rosenbloom and Tsfasman [6] considered the posets in which all the disjoint chains have the same length. In this paper we will also consider these posets. This family of posets will be called in the sequel the *uniform chains poset*.

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For a word $X \in \mathbb{F}_q^n$, the sphere with radius R centered at X is defined as the set of words in $\mathbb{F}_q^n$ whose P-distance from X is at most R, i.e. $\{Y \ : \ Y \in \mathbb{F}_q^n,\ d_P(X, Y) \leq R\}$. A code $\mathcal{C}$ is an R-perfect P-code if the spheres with radius R centered at the codewords of $\mathcal{C}$ form a partition of $\mathbb{F}_q^n$ (the spheres are disjoint and their union is $\mathbb{F}_q^n$). Perfect codes is one of the most fascinating topics in coding theory. They were considered for various metrics and especially for the Hamming scheme. Perfect poset codes were considered first in [5], where it is proved that there are no such codes if the poset consists of two disjoint chains with equal sizes. In the Hamming scheme the only family of nontrivial perfect codes are 1-perfect codes over $\mathbb{F}_q$. They exist for each length $m = \frac{q^k - 1}{q - 1}$, $k \geq 1$, and their size is q^{m-k}. If the poset consists of one chain of length n then an $(n, q^k, n - k + 1)$ MDS code [1] is an $(n - k)$-perfect code [5], [7]. Perfect codes were also considered for other various posets by [8], [9], [10], [11].

An $[n, k]$ code $\mathcal{C}$ over $\mathbb{F}_q$ is a linear subspace with dimension k of $\mathbb{F}_q^n$. A *coset* of $\mathcal{C}$ is a translate of $\mathcal{C}$, i.e. given a word $X \in \mathbb{F}_q^n$, $X + \mathcal{C} \stackrel{\text{def}}{=} \{X + Y \ : \ Y \in \mathcal{C}\}$ is a coset of $\mathcal{C}$. $\mathcal{C}$ has q^{n-k} disjoint cosets, each one of size q^k. The union of these cosets in $\mathbb{F}_q^n$, i.e. these q^{n-k} cosets form a partition of $\mathbb{F}_q^n$. A *coset leader* is a word with minimum P-weight in the coset. An $[n, k, d_P]$ P-code $\mathcal{C}$ over $\mathbb{F}_q$ is an $[n, k]$ code having a minimum P-distance d_P. An (n, M, d_P) P-code $\mathcal{C}$ over $\mathbb{F}_q$ is a set of M word all have length n, such that the P-distance between any two distinct codewords of $\mathcal{C}$ is at least d_P.

The rest of this paper is organized as follows. In Section II we prove the result of the paper, that a q-ary perfect single-error-correcting uniform poset code with m chains of length $\ell \geq 1$ exists if and only if $m = \frac{q^k - 1}{q - 1}$. The section consists of three parts. First, we prove the necessary condition for the existence of a perfect code. This condition apply to both linear and nonlinear codes. Next, we present two type of codes which are essential in our construction: perfect single-error-correcting codes in the Hamming scheme and MDS code with minimum Hamming distance two. In the third part of the section we prove that the necessary condition is also sufficient by presenting a construction of perfect codes for the related parameters. The constructed codes are linear. Finally, in Section III we conclude with problems for further research.

II. 1-Perfect Uniform Chains Poset Codes

In this section we will prove a necessary and sufficient condition for the existence of 1-perfect codes in the uniform chains poset. Throughout this section a perfect code is always an 1-perfect code. Assume that we have m chains, each one of length ℓ, i.e., the length of the code is $n = m\ell$. This poset will be denoted throughout this section by P. The poset related to the Hamming metric will be denoted in this section by H. A code in the Hamming metric will be denoted by $\mathcal{C}$, while a code in the uniform chains poset will be denoted by $\mathbb{C}$.

A. Necessary conditions

In this subsection we derive a necessary condition for the existence of a perfect P-code.

Lemma 1: The size of a sphere, in the uniform chains poset, with radius one does not depend on the center of the sphere, and this size is equal $1 + (q-1)m$.

Proof: The fact that the size of the sphere does not depend on its center is easily verified. Hence, w.l.o.g. (without loss of generality) we can compute the size of a sphere centered at the allzero word. There is exactly one word with P-weight zero. Words with P-weight one are words with exactly one nonzero coordinate in one of the positions which corresponds to the bottom of a chain. Each nonzero alphabet letter can be used in these positions. There are m chains and $q-1$ nonzero alphabet letters for a total of $(q-1)m$ words of weight one. Thus, the size of a sphere with radius one is $1 + (q-1)m$. ∎

Since the size of sphere should divide the size of the space $\mathbb{F}_q^n$, $|\mathbb{F}_q^n| = q^n$, it follows that $1+(q-1)m = p^t$, where $q = p^r$ and p is a prime. It implies that $(p^r-1)m = p^t - 1$ and hence r must divides t. Therefore, $p^t = p^{rk} = q^k$, i.e. $1 + (q-1)m = q^k$. Thus, we have

Theorem 1: If $\mathbb{C}$ is a P-code with ℓ chains of length m then $m = \frac{q^k-1}{q-1}$ and $|\mathbb{C}| = q^{m\ell-k}$.

Finally, we characterize perfect P-codes. The first lemma is a simple observation.

Lemma 2: If $\mathbb{C}$ is a P-code with minimum P-distance three then the spheres with radius one centered at the codewords of $\mathbb{C}$ are disjoint.

Theorem 2: Let $\mathbb{C}$ be a P-code of length $n = m\ell = \frac{q^k-1}{q-1}\ell$ with q^{m-k} codewords. If the minimum P-distance of $\mathbb{C}$ is three then $\mathbb{C}$ is a perfect P-code.

Proof: By Lemma 2, in a P-code $\mathbb{C}$ with minimum P-distance three the spheres with radius one centered at the codewords of $\mathbb{C}$ are disjoint. By Theorem 1, a perfect P-code $\mathbb{C}$ of length $n = m\ell = \frac{q^k-1}{q-1}\ell$ has q^{m-k} codeword. Since by Lemma 1, the size of an 1-sphere does not depend on its center, it follows that a P-code $\mathbb{C}$ of length $n = \frac{q^k-1}{q-1}\ell$ with q^{m-k} codewords and minimum P-distance three is perfect. ∎

B. Codes from the Hamming scheme

Two types of codes in the Hamming scheme are the ingredient for our construction of perfect P-codes. The finite field $\mathbb{F}_{q^k}$ has an important role in their construction. For these constructions let α be a primitive element in $\mathbb{F}_{q^k}$.

The first type of code is a perfect $[m = \frac{q^k-1}{q-1}, m-k, 3]$ H-code over $\mathbb{F}_q$, $k \geq 1$. Such a code $\mathcal{C}_1$ has q^k disjoint cosets. Each coset leader is a word of length m and H-weight less than two. If X is such a word then the coset $\mathcal{C}_1^X$ is defined by $\mathcal{C}_1^X \overset{\text{def}}{=} \{X + Y \ : \ Y \in \mathcal{C}_1\}$. Each vector X of H-weight less than two can be identified by a different element of $\gamma \in \mathbb{F}_{q^k}$.

Lemma 3: If $m = \frac{q^k-1}{q-1}$ and α is a primitive element in the field $\mathbb{F}_{q^k}$ then the matrix

$$\mathcal{H} = \begin{bmatrix} \alpha^0 & \alpha^1 & \alpha^2 & \cdots & \alpha^{m-1} \end{bmatrix} ,$$

where α^i is represented by a vector of length k over $\mathbb{F}_q$, is a parity-check matrix of a perfect $[m, m-k, 3]$ H-code over $\mathbb{F}_q$.

Proof: $\mathcal{H}$ is a parity-check matrix of a perfect H-code with length $\frac{q^k-1}{q-1}$ if and only if in the $\frac{q^k-1}{q-1}$ columns of $\mathcal{H}$ there are no two distinct columns γ_1, γ_2, and $\beta \in \mathbb{F}_q$ such that $\gamma_2 = \beta\gamma_1$. Clearly, $\alpha^i = \beta\alpha^j$ if and only if $\alpha^{i+1} = \beta\alpha^{j+1}$. Therefore, since $\alpha^i \neq \alpha^j$, $0 \leq i < j \leq q^k - 2$, it follows that $\alpha^m = \hat{\beta}$ for a primitive element $\hat{\beta} \in \mathbb{F}_q$. Thus, $\mathcal{H}$ is a parity-check matrix of perfect $[m, m-k, 3]$ H-code. ∎

By lemma 3 if follows that each coset $\mathcal{C}_1^\gamma$ is identified with an element $\gamma \in \mathbb{F}_{q^k}$. Hence, we will denote $\mathcal{C}_1^\gamma$ instead of $\mathcal{C}_1^X$ and write $\gamma + Y$ instead of $X + Y$.

Lemma 4: If X_1 is a word in $\mathcal{C}_1^{\gamma_1}$ and X_2 is a word in $\mathcal{C}_1^{\gamma_2}$ then $X_1 + X_2$ is a word in $\mathcal{C}_1^{\gamma_3}$, where $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{F}_{q^k}$ and $\gamma_3 = \gamma_1 + \gamma_2$.

Proof: We distinguish between four cases.

Case 1: If $\gamma_1 = -\gamma_2$ then $X_1 = Y_1 + \gamma_1$, $X_2 = Y_2 + \gamma_2$, $Y_1, Y_2 \in \mathcal{C}_1$. Hence, $X_1 + X_2 = Y_1 + Y_2 \in \mathcal{C}_1$.

Case 2: If $\gamma_1 = \lambda\gamma_2$, $\lambda \in \mathbb{F}_q \setminus \{0, -1\}$, then $X_1 = Y_1 + \gamma_1$, $X_2 = Y_2 + \gamma_2$, $Y_1, Y_2 \in \mathcal{C}_1$. Hence, $X_1 + X_2 = Y_1 + Y_2 + (\lambda+1)\gamma_2 \in \mathcal{C}_1^{\gamma_3}$, $\gamma_3 = (\lambda+1)\gamma_2$, since the coset leaders γ_1 and γ_2 are both of weight one and share the same nonzero coordinate.

Case 3: If $\gamma_1 \neq \gamma_2$ and $\gamma_1 = 0$ then $X_1 = Y_1$, $X_2 = Y_2 + \gamma_2$, $Y_1, Y_2 \in \mathcal{C}_1$. Hence, $X_1 + X_2 = Y_1 + Y_2 + \gamma_2 \in \mathcal{C}_1^{\gamma_2}$.

Case 4: If $\gamma_1 \neq \gamma_2$ and $\gamma_1, \gamma_2 \in \mathbb{F}_{q^k} \setminus \{0\}$ then $X_1 = Y_1 + \gamma_1$, $X_2 = Y_2 + \gamma_2$, $Y_1, Y_2 \in \mathcal{C}_1$. Hence, $X_1 + X_2 = Y_1 + Y_2 + \gamma_1 + \gamma_2$. Let $\gamma_3 = \gamma_1 + \gamma_2$, i.e. $\gamma_3 - \gamma_1 - \gamma_2 = 0$. Let $-\gamma_1 = \lambda_1\alpha^{j_1}$, $-\gamma_2 = \lambda_2\alpha^{j_2}$, $\gamma_3 = \lambda_3\alpha^{j_3}$, where $0 \leq j_1, j_2, j_3 \leq m-1$ and $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{F}_q \setminus \{0\}$. Hence, the word with H-weight three with λ_i at position j_i, $1 \leq i \leq 3$ is a codeword in $\mathcal{C}_1$. Therefore $X_1 + X_2$ are contained in the same coset as $X_1 + X_2 + \gamma_3 - \gamma_1 - \gamma_2 = Y_1 + Y_2 + \gamma_3$. Thus $X_1 + X_2 \in \mathcal{C}_1^{\gamma_3}$. ∎

The second code $\mathcal{C}_2$ is an $[n, n-1, 2]$ code over $\mathbb{F}_{q^k}$. This code also known to be an MDS code is easily constructed. For the construction, of perfect P-codes, which follows we need another property to be satisfied by this code. If $(x, y, 0 \ldots, 0)$ is a codeword of $\mathcal{C}_2$ such that $x, y \in \mathbb{F}_{q^k} \setminus \{0\}$, then there is no $\beta \in \mathbb{F}_q$ such that $y = \beta x$. A generator matrix for such a code is given for example by the following $(n-1) \times n$ matrix.

$$G = \begin{bmatrix} 1 & \alpha & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & \alpha \\ 0 & 0 & 1 & \cdots & 0 & \alpha \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \alpha \end{bmatrix}.$$

Lemma 5: The code formed by the generator matrix G is an $[n, n-1, 2]$ code over $\mathbb{F}_{q^k}$ in which for each codeword of the form $(x, y, 0 \ldots, 0)$, $x, y \in \mathbb{F}_{q^k} \setminus \{0\}$ there is no $\beta \in \mathbb{F}_q$ such that $y = \beta x$.

Proof: The length of the code, its dimension and minimum H-distance are easily verified from the structure of the generator matrix G. All the codewords of the form $(x, y, 0 \ldots, 0)$, $x, y \in \mathbb{F}_{q^k} \setminus \{0\}$ are contained in the set $\{(\alpha^i, \alpha^{i+1}, 0, \ldots, 0) \ : \ 0 \leq i \leq q^k - 2\}$. It is readily verified that there is no $\beta \in \mathbb{F}_q$ such that $\alpha^{i+1} = \beta\alpha^i$. ■

C. A Product Construction for Perfect Codes

There are several product constructions for non-binary perfect codes in the Hamming scheme [12], [13], [14], [15], [16], [17]. The construction that we present for perfect P-codes is a generalization for the constructions in [16], [17]. For our construction we will use the two codes $\mathcal{C}_1$ and $\mathcal{C}_2$ in the Hamming scheme. We construct the following code $\mathbb{C}$:

$$\mathbb{C} \stackrel{\text{def}}{=} \{(x_{1,1}, \ldots, x_{1,\ell}, x_{2,1}, \ldots, x_{2,\ell}, \ldots, x_{m,1}, \ldots, x_{m,\ell}) \ : \ (x_{1,j}, x_{2,j}, \ldots, x_{m,j}) \in \mathcal{C}_1^{i_j}, \ (i_1, i_2, \ldots, i_\ell) \in \mathcal{C}_2\}$$

Lemma 6: The code $\mathbb{C}$ is a linear code.

Proof: We only have to prove that for any two codewords $X, Y \in \mathbb{C}$ also $X + Y$ is a codeword in $\mathbb{C}$. Let $X = (x_{1,1}, \ldots, x_{1,\ell}, x_{2,1}, \ldots, x_{2,\ell}, \ldots, x_{m,1}, \ldots, x_{m,\ell})$, where $(x_{1,j}, x_{2,j}, \ldots, x_{m,j}) \in \mathcal{C}_1^{i_j}$, and $(i_1, i_2, \ldots, i_\ell) \in \mathcal{C}_2$. Let $Y = (y_{1,1}, \ldots, y_{1,\ell}, y_{2,1}, \ldots, y_{2,\ell}, \ldots, y_{m,1}, \ldots, y_{m,\ell})$, where $(y_{1,j}, y_{2,j}, \ldots, y_{m,j}) \in \mathcal{C}_1^{t_j}$, and $(t_1, t_2, \ldots, t_\ell) \in \mathcal{C}_2$. Let $Z = X + Y = (z_{1,1}, \ldots, z_{1,\ell}, z_{2,1}, \ldots, z_{2,\ell}, \ldots, z_{m,1}, \ldots, z_{m,\ell})$.

Since $\mathcal{C}_2$ is a linear code it follows that $(s_1, s_2, \ldots, s_\ell) = (i_1 + t_1, i_2 + t_2, \ldots, i_\ell + t_\ell)$ is also a codeword in $\mathcal{C}_2$. By Lemma 4 we have that $(z_{1,j}, z_{2,j}, \ldots, z_{m,j}) = (x_{1,j}, x_{2,j}, \ldots, x_{m,j}) + (y_{1,j}, y_{2,j}, \ldots, y_{m,j}) \in \mathcal{C}_1^{s_j}$, $1 \leq j \leq \ell$. Thus, Z is constructed in $\mathbb{C}$. ■

Theorem 3: The code $\mathbb{C}$ is a q-ary perfect P-code of length $m\ell$.

Proof: Obviously, the length of $\mathbb{C}$ is $m\ell$. The number of codewords in $\mathcal{C}_2$ is $(q^k)^{\ell-1}$. Each coset of $\mathcal{C}_1$ has q^{m-k} codewords substituted for each one of the ℓ alphabet letters in a codeword of $\mathcal{C}_2$. Thus, the total number of codewords in $\mathbb{C}$ is $(q^k)^{\ell-1}(q^{m-k})^\ell = q^{m\ell-k}$.

By Theorem 2, to complete the proof we have to show that there are no nonzero codewords in $\mathbb{C}$ with P-weight less than 3. A codeword with H-weight three has P-weight at least three. Since $\mathcal{C}_2$ is a code with minimum Hamming distance two, it follows that each nonzero codeword of $\mathbb{C}$ has H-weight at least two. Assume that $X = (x_{1,1}, \ldots, x_{1,\ell}, x_{2,1}, \ldots, x_{2,\ell}, \ldots, x_{m,1}, \ldots, x_{m,\ell})$, where $(x_{1,j}, x_{2,j}, \ldots, x_{m,j}) \in \mathcal{C}_1^{i_j}$, $1 \leq j \leq \ell$, and $(i_1, i_2, \ldots, i_\ell) \in \mathcal{C}_2$. X has H-weight two and P-weight two only in one of the following two cases:

Case 1: $x_{s,1}$ and $x_{t,1}$ are the only nonzero entries in X for some $1 \leq s < t \leq m$.

In this case $(0, \ldots, 0, x_{s,1}, 0, \ldots, 0, x_{t,1}, 0, \ldots, 0) \in \mathcal{C}_1^{i_1}$ for a codeword $(i_1, i_2, \ldots, i_\ell) \in \mathcal{C}_2$ which has H-weight at least two. W.l.o.g. assume that $i_2 \neq 0$ and hence $(x_{1,2}, x_{2,2}, \ldots, x_{m,2}) \in \mathcal{C}_1^{i_2}$ is a nonzero word. Therefore, the H-weight of X is at least three, a contradiction.

Case 2: $x_{s,1}$ and $x_{s,2}$ are the only nonzero entries in X for some $1 \leq \alpha \leq m$.

In this case $(0, \ldots, 0, x_{s,1}, 0, \ldots, 0) \in \mathcal{C}_1^{i_1}$, $(0, \ldots, 0, x_{s,2}, 0, \ldots, 0) \in \mathcal{C}_1^{i_2}$, where $(i_1, i_2, \ldots, i_\ell) \in \mathcal{C}_2$. Hence, the words of weight one in the cosets $\mathcal{C}_1^{i_1}$ and $\mathcal{C}_1^{i_2}$ have the nonzero entry in the same coordinate. It implies that $i_2 = \gamma i_1$ for some $\gamma \in \mathbb{F}_q$, a contradiction to Lemma 5.

Thus, the code $\mathbb{C}$ is a q-ary perfect P-code of length $m\ell$. ■

Corollary 1: A q-ary perfect P-code with m chains of length $\ell \geq 1$ exists if and only if $m = \frac{q^k-1}{q-1}$.

III. Conclusions

We settled the existence problem of 1-perfect codes with the uniform chains poset metric. Over an alphabet with q letters, q a power of a prime, these codes with m chains of length $\ell \geq 1$ exist if and only if $m = \frac{q^k-1}{q-1}$. Other perfect codes for these posets are known when there is only one chain or when each chain is of length one (the Hamming metric). The main open problem for future research is to consider R-perfect codes, $R > 1$, for the uniform chains poset metrics in which there is more than one chain and the length of the chains is greater than one.

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