SOJOURN TIMES AND THE FRAGILITY INDEX

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Abstract. We investigate the sojourn time above a high threshold of a continuous stochastic process $\boldsymbol{Y} = (Y_t)_{t\in[0,1]}$ on $[0,1]$. It turns out that the limit, as the threshold increases, of the expected sojourn time given that it is positive, exists if the copula process corresponding to $\boldsymbol{Y}$ is in the functional domain of attraction of of an extreme value process. This limit coincides with the limit of the fragility index corresponding to $(Y_{i/n})_{1\le i\le n}$ as n and the threshold increase.

If the process is in a certain neighborhood of a generalized Pareto process, then we can replace the constant threshold by a general threshold function and we can compute the asymptotic sojourn time distribution. An extreme value process is a prominent example. Given that there is an exceedance at t_0 above the threshold, we can also compute the asymptotic distribution of the time cluster length, which the process spends above the threshold function.

1. Introduction

Let $\boldsymbol{Y} = (Y_t)_{t\in[0,1]}$ be a stochastic process with continuous sample paths, i.e., $\boldsymbol{Y} \in C[0,1]$, and identical continuous marginal distribution functions (df) F, say. We investigate in this paper the sojourn time of $\boldsymbol{Y}$ above a threshold s

$$S(s) := \int_0^1 1(Y_t > s)\, dt,$$

under the condition that there is an exceedance, i.e., $S(s) > 0$. Sojourn times of stochastic processes have been extensively studied in the literature, with emphasis on Gaussian processes and Markov random fields, we refer to Berman [3] and the literature given therein. We will investigate the sojourn time under the condition that the copula process $\boldsymbol{C} := (F(Y_t))_{t\in[0,1]}$ corresponding to $\boldsymbol{Y}$ is in the functional domain of attraction of an extreme value process $\boldsymbol{\eta}$, say.

Denote by $N_s := \sum_{i=1}^n 1_{(s,\infty)}(Y_{i/n})$ the number of exceedances among $(Y_{i/n})_{1\le i\le n}$ above the threshold s. The *fragility index* (FI) corresponding to $(Y_{i/n})_{1\le i\le n}$ is defined as the asymptotic expectation of the number of exceedances given that there

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is at least one exceedance:

$$FI := \lim_{s\nearrow} E(N_s \mid N_s > 0)$$

The FI was introduced in Geluk et al. [7] to measure the stability of a stochastic system. The system is called stable if $FI = 1$, otherwise it is called fragile. The collapse of a bank, symbolized by an exceedance, would be a typical example, illustrating the FI as a measure of joint stability among a portfolio of banks.

It turns out that the limit, as the threshold increases, of the expected sojourn time given that it is positive, exists if the copula process corresponding to $\boldsymbol{Y}$ is in the functional domain of attraction of an extreme value process. This limit coincides with the limit of the FI corresponding to $(Y_{i/n})_{1\le i\le n}$ as n and the threshold increase.

For such processes, which are in a certain neighborhood of a generalized Pareto process, we can replace the constant threshold by a threshold function and we can compute the asymptotic sojourn time distribution above a high threshold function. An extreme value process is a prominent example. Given that there is an exceedance $Y_{t_0} > s$ at t_0 above the threshold s, we can also compute the asymptotic distribution of the cluster length, that the process spends above the threshold function.

This paper is organized as follows. In Section 2.1 we recall some mathematical framework from functional extreme value theory and provide basic definitions and tools. In particular we consider a functional domain of attraction approach for stochastic processes, which is more general than the usual one based on weak convergence. In Section 2.3 we apply the framework from Section 2.1 to copula processes and derive characterizations of the domain of attraction condition for copula processes. In Section 3 we use the results from Section 2.3 to compute the limit $\lim_{s\nearrow} E(S(s) \mid S(s) > 0)$ as the threshold s increases of the mean sojourn time, conditional on the assumption that it is positive. We show that this limit coincides with the FI. In Section 4 we replace the constant threshold by a threshold function and we compute the limit distribution of the sojourn time for those processes, which are in a certain neighborhood of a generalized Pareto process. Given that there is an exceedance at t_0, we compute in Section 5 the asymptotic distribution of the cluster length that the process spends above a high threshold function.

To improve the readability of this paper we use bold face such as $\boldsymbol{\xi}$, $\boldsymbol{Y}$ for stochastic processes and default font f, a_n etc. for non stochastic functions. Operations on functions such as $\boldsymbol{\xi} < a$ or $(\boldsymbol{\xi} - b_n)/a_n$ are meant componentwise. The usual abbreviations *df, fidis, iid* and *rv* for the terms *distribution function, finite dimensional distributions, independent and identically distributed* and *random variable*, respectively, are used.

2. Definitions and Preliminaries

2.1. **Extreme Value Processes and the Functional D-Norm.** An *extreme value process* (EVP) $\boldsymbol{\xi} = (\xi_t)_{t\in[0,1]}$ in $C[0,1] := \{f : [0,1] \to \mathbb{R} : \ f \text{ continous}\}$, equipped with the sup-norm $\|f\|_\infty = \sup_{t\in[0,1]} |f(t)|$, is a stochastic process with the characteristic property that its distribution is max-stable, i.e. $\boldsymbol{\xi}$ has the same distribution as $\max_{1\le i\le n}(\boldsymbol{\xi}_i - b_n)/a_n$ for independent copies $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \dots$ of $\boldsymbol{\xi}$ and some $a_n, b_n \in C[0,1]$, $a_n > 0$, $n \in \mathbb{N}$ (cf. de Haan and Ferreira [9]).

We call a process $\boldsymbol{\eta} \in C^-[0,1] := \{f \in C[0,1] : \ f < 0\}$ a *standard* EVP, if it is an EVP with standard negative exponential (one-dimensional) margins, $P(\eta_t \le x) = \exp(x)$, $x \le 0$, $t \in [0,1]$.

We denote in what follows by $\bar{C}^-[0,1] := \{f \in C[0,1] : \ f \le 0\}$ the set of all continuous function on $[0,1]$ that do not attain positive values.

The following characterization is essentially due to Giné et al. [8]; we refer also to Aulbach et al. [2].

Proposition 2.1. *A process $\boldsymbol{\eta} \in C^-[0,1]$ is a standard EVP if, and only if there exists a number $m \ge 1$ and a stochastic process $\boldsymbol{Z} \in \bar{C}^+[0,1] := \{f \in C[0,1] : \ f \ge 0\}$ with the properties*

$$\max_{t\in[0,1]} Z_t = m, \qquad E(Z_t) = 1, \quad t \in [0,1], \tag{1}$$

such that for compact subsets $K_1, \dots, K_d$ of $[0,1]$ and $x_1, \dots, x_d \le 0$, $d \in \mathbb{N}$,

$$P(\eta_t \le x_j,\, t \in K_j,\, 1 \le j \le d) = \exp\left(-E\left(\max_{1\le j\le d}\left(|x_j| \max_{t\in K_j} Z_t\right)\right)\right). \tag{2}$$

Conversely, every stochastic process $\boldsymbol{Z} \in \bar{C}^+$ satisfying (1) *gives rise to a standard EVP. The connection is via* (2)*. We call $\boldsymbol{Z}$ generator of $\boldsymbol{\eta}$.*

According to de Haan and Ferreira [9, Corollary 9.4.5] the condition $\max_{t\in[0,1]} Z_t = m$ in (1) can be replaced by the condition $E\left(\max_{t\in[0,1]} Z_t\right) < \infty$. The number $m = E\left(\max_{t\in[0,1]} Z_t\right)$ is uniquely determined, see Remark 3.3. Therefore, we call m the *generator constant* of $\boldsymbol{\eta}$.

The preceding characterization implies in particular that the fidis of $\boldsymbol{\eta}$ are multivariate negative EVD with standard negative exponential margins: We have for $0 \le t_1 < t_2 \dots < t_d \le 1$

$$-\log(G_{t_1,\dots,t_d})(\boldsymbol{x}) = E\left(\max_{1\le i\le d}(|x_i|\, Z_{t_i})\right) =: \|\boldsymbol{x}\|_{D_{t_1,\dots,t_d}}\,,\ \boldsymbol{x} \le \boldsymbol{0} \in \mathbb{R}^d, \tag{3}$$

where $\|\cdot\|_{D_{t_1,\dots,t_d}}$ is a *D-norm* on $\mathbb{R}^d$ (cf. Falk et al. [5]).

Let $E[0,1]$ be the set of all bounded real-valued functions on $[0,1]$ which are discontinuous at a finite set of points. Moreover, denote by $\bar{E}^-[0,1]$ the set of those functions in $E[0,1]$ which do not attain positive values.

For a generator process $\boldsymbol{Z} \in \bar{C}^+[0,1]$ as in Proposition 2.1 and all $f \in E[0,1]$ set

$$\|f\|_D := E\left(\sup_{t\in[0,1]} (|f(t)|\, Z_t)\right).$$

Obviously, $\|\cdot\|_D$ defines a norm on $E[0,1]$, called a *D-norm with generator* $\boldsymbol{Z}$; see Aulbach et al. [2] for further details.

The following result is established in Aulbach et al. [2].

Lemma 2.2. *Let $\boldsymbol{\eta}$ be a standard EVP with generator $\boldsymbol{Z}$. Then we have for each $f \in \bar{E}^-[0,1]$*

$$P(\boldsymbol{\eta} \le f) = \exp\left(-\|f\|_D\right) = \exp\left(-E\left(\sup_{t\in[0,1]} (|f(t)|\, Z_t)\right)\right). \tag{4}$$

Conversely, if there is some $\boldsymbol{Z}$ with properties (1) *and some $\boldsymbol{\eta} \in C^-[0,1]$ which satisfies* (4)*, then $\boldsymbol{\eta}$ is standard max-stable with generator $\boldsymbol{Z}$.*

The representation $P(\boldsymbol{\eta} \le f) = \exp\left(-\|f\|_D\right)$, $f \in \bar{C}^-[0,1]$, of a standard EVP is in complete accordance with the df of a multivariate EVD with standard negative exponential margins via a D-norm on $\mathbb{R}^d$ as developed in Falk et al. [5, Section 4.4].

Note that for $d \in \mathbb{N}$ the function

$$f(t) = \sum_{i=1}^d x_i 1_{\{t_i\}}(t), \qquad t_i \in [0,1],\ x_i < 0,\ i = 1, \dots, d$$

is an element of $\bar{E}^-[0,1]$ with the property

$$P(\boldsymbol{\eta} \le f) = \exp\left(-\|\boldsymbol{x}\|_{D_{t_1,\dots,t_d}}\right).$$

So representation (4) incorporates all fidis of $\boldsymbol{\eta}$.

Just like in the uni- or multivariate case, we might consider

$$H(f) := P(\boldsymbol{Y} \le f), \qquad f \in \bar{E}^-[0,1],$$

as the df of a stochastic process $\boldsymbol{Y}$ in $\bar{C}^-[0,1]$.

2.2. **Functional Domain of Attraction.** According to Aulbach et al. [2] we say that a stochastic process $\boldsymbol{Y} \in C[0,1]$ is *in the functional domain of attraction of a standard EVP* $\boldsymbol{\eta}$, denoted by $\boldsymbol{Y} \in \mathcal{D}(\boldsymbol{\eta})$, if there are functions $a_n \in C^+[0,1]$, $b_n \in C[0,1]$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} P\left(\frac{\boldsymbol{Y} - b_n}{a_n} \le f\right)^n = P(\boldsymbol{\eta} \le f) = \exp\left(-\|f\|_D\right) \tag{FuDA}$$

for any $f \in \bar{E}^-[0,1]$. This is equivalent to

$$\text{(FuDA')} \qquad \lim_{n\to\infty} P\left(\max_{1\le i\le n} \frac{\boldsymbol{Y}_i - b_n}{a_n} \le f\right) = P(\boldsymbol{\eta} \le f)$$

for any $f \in \bar{E}^-[0,1]$, where $\boldsymbol{Y}_1, \boldsymbol{Y}_2, \dots$ are independent copies of $\boldsymbol{Y}$.

There should be no risk of confusion with the notation of domain of attraction in the sense of weak convergence of stochastic processes as investigated in de Haan and Lin [10]. But to distinguish between these two approaches we will consistently speak of *functional* domain of attraction in this paper, when the above definition is meant. Actually, this definition of domain of attraction is less restrictive as the next lemma shows; it is established in Aulbach et al. [2].

Lemma 2.3. *Suppose that $\boldsymbol{Y}$ is a continuous process in $\bar{C}^-[0,1]$. If the sequence of continuous processes $\boldsymbol{X}_n := \max_{1\le i\le n}((\boldsymbol{Y}_i - b_n)/a_n))$ converges weakly in $\bar{C}^-[0,1]$, equipped with the sup-norm $\|\cdot\|_\infty$, to the standard EVP $\boldsymbol{\eta}$, then $\boldsymbol{Y} \in \mathcal{D}(\boldsymbol{\eta})$ in the sense of condition* (FuDA).

2.3. **Domain of Attraction for Copula Processes.** The sojourn time distribution of a stochastic process with identical continuous univariate margins does not depend on this marginal df but on the corresponding copula process. We, therefore, recall in this section results for copula processes established in Aulbach et al. [2].

Let $\boldsymbol{Y} = (Y_t)_{t\in[0,1]} \in C[0,1]$ be a stochastic process with identical continuous marginal df F. Set

$$\boldsymbol{U} = (U_t)_{t\in[0,1]} := (F(Y_t))_{t\in[0,1]},$$

which is the *copula process* corresponding to $\boldsymbol{Y}$.

We conclude from de Haan and Lin [10] that the process $\boldsymbol{Y}$ is in the domain of attraction of an EVP if, and only if each Y_t is in the domain of attraction of a univariate extreme value distribution together with the condition that the copula process converges in distribution to a standard EVP $\boldsymbol{\eta}$, that is

$$\left(\max_{1\le i\le n} n(U_t^{(i)} - 1)\right)_{t\in[0,1]} \to_D \boldsymbol{\eta}$$

in $C[0,1]$, where $\boldsymbol{U}^{(i)}$, $i \in \mathbb{N}$, are independent copies of $\boldsymbol{U}$. Note that the univariate margins determine the norming constants, so the norming functions are necessarily the constant functions $a_n = 1/n$, $b_n = 1$, $n \in \mathbb{N}$. Lemma 2.3 implies that $\boldsymbol{U}$ is in the functional domain of attraction of $\boldsymbol{\eta}$.

Suppose that the rv $(Y_{i/d})_{i=1}^d$ is in the ordinary domain of attraction of a multivariate EVD (see, for instance, Falk et al. [5, Section 5.2]). Then we know from Aulbach et al. [1] that the copula C_d corresponding to the rv $(Y_{i/d})_{i=1}^d$ satisfies the equation

$$C_d(\boldsymbol{y}) = 1 - \|\mathbf{1}-\boldsymbol{y}\|_{D_d} + o\left(\|\mathbf{1}-\boldsymbol{y}\|_\infty\right), \tag{5}$$

as $\|\mathbf{1}-\boldsymbol{y}\|_\infty \to \mathbf{0}$, uniformly in $\boldsymbol{y}\in[0,1]^d$, where the D-norm is given by

$$\|\boldsymbol{x}\|_{D_d} = E\left(\max_{1\le i\le d}\left(|x_i|\, Z_{i/d}\right)\right), \qquad \boldsymbol{x}\in\mathbb{R}^d.$$

The following analogous result for the functional domain of attraction was established in Aulbach et al. [2].

Proposition 2.4. *Suppose that $\boldsymbol{U}\in\bar{C}^+[0,1]$ is a copula process. The following equivalences hold:*

$$\begin{aligned}&\boldsymbol{U}\in\mathcal{D}(\boldsymbol{\eta}) \text{ in the sense of condition (FuDA)}\\ &\iff P\left(\boldsymbol{U}-1\le\frac{f}{n}\right) = 1-\left\|\frac{f}{n}\right\|_D + o\left(\frac{1}{n}\right), \quad f\in\bar{E}^-[0,1], \text{ as } n\to\infty,\\ &\iff P(\boldsymbol{U}-1\le |c|\, f) = 1 + c\,\|f\|_D + o(c), \quad f\in\bar{E}^-[0,1], \text{ as } c\uparrow 0, \qquad (6)\end{aligned}$$

Note that condition (6) holds if

$$P(\boldsymbol{U}-1\le g) = 1-\|g\|_D + o(\|g\|_\infty) \tag{6'}$$

as $\|g\|_\infty\to 0$, uniformly for all $g\in\bar{E}^-[0,1]$ with $\|g\|_\infty\le 1$. It is an open problem whether (6') and (6) are, actually, equivalent conditions.

3. Sojourn Times and the Fragility Index

Let $\boldsymbol{Y} = (Y_t)_{t\in[0,1]}\in C[0,1]$ be a stochastic process with identical continuous marginal df F. We investigate in this section the mean of the sojourn time of $\boldsymbol{Y}$ above a threshold s

$$S(s) = \int_0^1 1(Y_t > s)\, dt,$$

under the condition that there is an exceedance, i.e., $S(s)>0$. In particular we establish its asymptotic equality with the limit of the FI corresponding to $(Y_{i/n})_{1\le i\le n}$.

Before we present the main results of this section we need some auxiliary results. Put for $n\in\mathbb{N}$

$$S_n(s) := \frac{1}{n}\sum_{i=1}^n 1(Y_{i/n} > s),$$

which is a Riemann sum of the integral $S(s)$. We have

$$S_n(s) \to_{n\to\infty} S(s)$$

and, thus,

$$P(S_n(s)\le x) \to_{n\to\infty} P(S(s)\le x)$$

for each $x \geq 0$ such that $P(S(s) = x) = 0$. As a consequence we obtain

$$\begin{aligned} P(S_n(s) \leq x \mid S_n(s) > 0) &= \frac{P(0 < S_n(s) \leq x)}{P(S_n(s) > 0)} \\ &\to_{n\to\infty} \frac{P(0 < S(s) \leq x)}{P(S(s) > 0)} \\ &= P(S(s) \leq x \mid S(s) > 0) \end{aligned}$$

for each such $x > 0$. This conclusion requires the following argument.

Lemma 3.1. *We have*

$$P(S_n(s) = 0) \to_{n\to\infty} P(S(s) = 0),$$

Proof. We have

$$P(S_n(s) = 0) \leq P(S_n(s) \leq \varepsilon) \to_{n\to\infty} P(S(s) \leq \varepsilon) = P(S(s) = 0) + \delta,$$

where $\varepsilon, \delta > 0$ can be made arbitrarily small. This implies $\limsup_{n\to\infty} P(S_n(s) = 0) \leq P(S(s) = 0)$. We have, on the other hand,

$$P(S(s) = 0) = P\left(\bigcap_{n\in\mathbb{N}} \{S_n(s) = 0\}\right) \leq \liminf_{n\to\infty} P(S_n(s) = 0),$$

which implies the assertion. □

We have

$$\begin{aligned} S_n(s) &= \frac{1}{n}\sum_{i=1}^{n} 1(F(Y_{i/n}) > F(s)) \\ &= \frac{1}{n}\sum_{i=1}^{n} 1(U_{i/n} > c) \end{aligned}$$

almost surely, where $c := F(s)$.

Note that

$$\begin{aligned} FI_n(s) &:= E(nS_n(s) \mid S_n(s) > 0) \\ &= E\left(\sum_{i=1}^{n} 1(U_{i/n} > c) \mid S_n(s) > 0\right) \\ &= \sum_{i=1}^{n} P\left(U_{i/n} > c \mid S_n(s) > 0\right) \\ &= \sum_{i=1}^{n} \frac{P(U_{i/n} > c)}{P\left(S_n(s) > 0\right)} \\ &= n\frac{1 - c}{1 - P\left(S_n(s) = 0\right)} \end{aligned}$$

is the FI of level s corresponding to $Y_{i/n}$, $1 \le i \le n$. The FI was introduced in Geluk et al. [7] to measure the stability of a stochastic system. The system is called *stable* if $FI = 1$, indicating tail independence of the $Y_{i/n}$, $1 \le i \le n$, otherwise it is called *fragile*. For an extensive investigation and extension of the FI we refer to Falk and Tichy [6]. The following theorem is the first main result of this section.

Theorem 3.2. *Let $\boldsymbol{Y}$ be a stochastic process in $C[0,1]$ with identical continuous marginal df F. Suppose that the copula process $\boldsymbol{U} = (F(Y_t))_{t\in[0,1]}$ corresponding to $\boldsymbol{Y}$ is in the functional domain of attraction of an EVP $\boldsymbol{\eta}$ with generator constant $m \ge 1$ as in Proposition 2.1. Then we have*

$$\lim_{n\to\infty}\lim_{s\nearrow}\frac{FI_n(s)}{n} = \lim_{s\nearrow}\lim_{n\to\infty}\frac{FI_n(s)}{n} = \lim_{s\nearrow} E(S(s) \mid S(s) > 0) = \frac{1}{m}.$$

Proof. Expansion (5) implies for $n \in \mathbb{N}$

$$\begin{aligned}
&P(S_n(s) > 0)\\
&= 1 - P\left(\sum_{i=1}^n 1(U_{i/n} > c) = 0\right)\\
&= 1 - P(U_{i/n} \le c,\, 1 \le i \le n)\\
&= 1 - C_n(c,\dots,c)\\
&= (1-c)\left\|(1,\dots,1)\right\|_{D_n} + o\left((1-c)\left\|(1,\dots,1)\right\|_{D_n}\right)\\
&= (1-c)E\left(\max_{1\le i\le n} Z_{i/n}\right) + o\left((1-c)E\left(\max_{1\le i\le n} Z_{i/n}\right)\right)
\end{aligned}$$

as $c \uparrow 1$ and, thus,

$$\begin{aligned}
\frac{FI_n(s)}{n} &= \frac{1-c}{P(S_n(s) > 0)}\\
&= \frac{1}{E\left(\max_{1\le i\le n} Z_{i/n}\right) + o\left(E\left(\max_{1\le i\le n} Z_{i/n}\right)\right)}
\end{aligned}$$

as $c \uparrow 1$. We, thus, obtain

$$\lim_{n\to\infty}\lim_{s\nearrow}\frac{FI_n(s)}{n} = \lim_{n\to\infty}\frac{1}{E\left(\max_{1\le i\le n} Z_{i/n}\right)} = \frac{1}{E\left(\max_{0\le t\le 1} Z_t\right)} = \frac{1}{m}.$$

We have, on the other hand,

$$\lim_{n\to\infty}\frac{FI_n(s)}{n} = \lim_{n\to\infty}\frac{1-c}{1-P\left(S_n(s)=0\right)} = \frac{1-c}{1-P\left(S(s)=0\right)}.$$

Since $\boldsymbol{U} \in \mathcal{D}(\boldsymbol{\eta})$, we obtain from the equivalent condition (6)

$$\begin{aligned}
\lim_{s\nearrow}\lim_{n\to\infty}\frac{FI_n(s)}{n} &= \lim_{s\nearrow}\frac{1-c}{1-P\left(S(s)=0\right)}\\
&= \lim_{s\nearrow}\frac{1-c}{1-P\left(\boldsymbol{Y} \le s\right)}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{s\nearrow} \frac{1-c}{1-P\left(\boldsymbol{U}\le c\right)}\\
&= \lim_{s\nearrow} \frac{1-c}{1-\left(1-(1-c)\left\|1\right\|_D + o(1-c)\right)}\\
&= \frac{1}{\left\|1\right\|_D}\\
&= \frac{1}{E\left(\max_{0\le t\le 1} Z_t\right)}\\
&= \frac{1}{m},
\end{aligned}
$$

where 1 is the constant function on $[0,1]$. Moreover, by the dominated convergence theorem

$$
\begin{aligned}
\frac{FI_n(s)}{n} &= E(S_n(s) \mid S_n(s) > 0)\\
&= \frac{E(S_n(s))}{P(S_n(s)>0)}\\
&\to_{n\to\infty} \frac{E(S(s))}{P(S(s)>0)}\\
&= E(S(s) \mid S(s)>0).
\end{aligned}
$$

□

Remark 3.3. While the generator $\boldsymbol{Z}$ of a standard EVP $\boldsymbol{\eta}$ is in general not uniquely determined, the generator constant $m = E\left(\sup_{t\in[0,1]} Z_t\right) = \left\|1\right\|_D$ is.

Remark 3.4. Under the conditions of Theorem 3.2 we have

$$
P(S(s)>0) = (1-c)m + o(1-c) \quad \text{as } c\nearrow 1 \qquad \text{and} \qquad E(S(s)) = 1-F(s).
$$

To apply the preceding result to generalized Pareto processes defined below, we add an extension of Theorem 3.2. It is shown by repeating the preceding arguments.

We call a copula process $\boldsymbol{U} = (U_t)_{t\in[0,1]}$ (upper) *tail continuous*, if the process $\boldsymbol{U}_{c_0} := (\max(c_0, U_t))_{t\in[0,1]}$ is a.s. continuous for some $c_0 < 1$. Note that in this case $\boldsymbol{U}_c$ is a.s. continuous for each $c \ge c_0$.

A stochastic process $\boldsymbol{Y} = (Y_t)_{t\in[0,1]}$ is said to have ultimately identical and continuous marginal df F_t, $t\in[0,1]$, if $F_t(x) = F_s(x)$, $0\le s,t\le 1$, $x\ge x_0$ with $F_1(x_0) < 1$, and $F_1(x)$ is continuous for $x \ge x_0$.

Theorem 3.5. *Let $\boldsymbol{Y} = (Y_t)_{t\in[0,1]}$ be a stochastic process with ultimately identical and continuous marginal df. Suppose that the copula process pertaining to $\boldsymbol{Y}$ is tail continuous and that it is in the functional domain of attraction of an EVP $\boldsymbol{\eta}$, whose finite dimensional marginal distributions are given by*

$$
G_{t_1,\dots,t_d}(\boldsymbol{x}) = \exp\left(-E\left(\max_{1\le i\le d} |x_i|\, Z_{t_i}\right)\right),
$$

$0 \le t_1 < \cdots < t_d \le 1$, $\boldsymbol{x} \le \boldsymbol{0} \in \mathbb{R}^d$, $d \in \mathbb{N}$. *We require that the stochastic process* $\boldsymbol{Z} = (Z_t)_{t\in[0,1]}$ *is a.s. continuous and that its components satisfy* $0 \le Z_t \le m$ *a.s.,* $E(Z_t) = 1$, $t \in [0,1]$, *for some* $m \ge 1$. *Then we have*

$$\begin{aligned}\lim_{n\to\infty}\lim_{s\nearrow}\frac{FI_n(s)}{n} &= \lim_{s\nearrow}\lim_{n\to\infty}\frac{FI_n(s)}{n}\\ &= \lim_{s\nearrow} E(S(s) \mid S(s) > 0)\\ &= \frac{1}{E\left(\max_{0\le t\le 1} Z_t\right)}.\end{aligned}$$

Example 3.6. Consider the d-dimensional EVD $G(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_p)$, $\boldsymbol{x} \le \boldsymbol{0} \in \mathbb{R}^d$, $d \ge 2$, where the D-norm is the usual p-norm $\|\boldsymbol{x}\|_D = \left(\sum_{i=1}^d |x_i|^p\right)^{1/p} = \|\boldsymbol{x}\|_p$, $\boldsymbol{x} \in \mathbb{R}^d$, with $1 \le p \le \infty$. The case $p = \infty$ yields the maximum-norm $\|\boldsymbol{x}\|_\infty$. Let the rv $(Z_1, \dots, Z_d)$ be a generator of $\|\cdot\|_p$, i.e., $0 \le Z_i \le c$ a.s., $E(Z_i) = 1$, $1 \le i \le d$ with some $c \ge 1$, and $\|\boldsymbol{x}\|_p = E\left(\max_{1\le i\le d}(|x_i|\, Z_i)\right)$, $\boldsymbol{x} \in \mathbb{R}^d$. The rv $(Z_1, \dots, Z_d)$ can be extended by linear interpolation to a generator $\boldsymbol{Z} = (Z_t)_{t\in[0,1]}$ of a standard EVP $\boldsymbol{\eta}$: Put for $i = 1, \dots, d-1$

$$Z_{(1-\vartheta)\frac{i-1}{d-1}+\vartheta\frac{i}{d-1}} := (1-\vartheta)Z_{i-1} + \vartheta Z_i, \qquad 0 \le \vartheta \le 1,$$

which yields a continuous generator $\boldsymbol{Z} = (Z_t)_{t\in[0,1]}$. In this case we have

$$\frac{1}{E\left(\max_{0\le t\le 1} Z_t\right)} = \frac{1}{E\left(\max_{1\le i\le d} Z_i\right)} = \frac{1}{\|(1,\dots,1)\|_p} = \frac{1}{d^{1/p}},$$

i.e., the generator constant is $d^{1/p}$. This example implies, in particular, that a standard EVP $\boldsymbol{\eta}$, whose finite dimensional marginal distributions $G_{t_1,\dots,t_d}$ are given by $G_{t_1,\dots,t_d} = \exp(-\|\boldsymbol{x}\|_p)$, $\boldsymbol{x} \le 0 \in \mathbb{R}^d$, $d \ge 1$, does not exist.

Example 3.7 (Generalized Pareto Process (GPP)). Let $\boldsymbol{Z} = (Z_t)_{t\in[0,1]} \in \bar{C}^+[0,1]$ with $0 \le Z_t \le m$ a.s., $E(Z_t) = 1$, $t \in [0,1]$, for some $m \ge 1$, and let U be a rv that is uniformly on $(0,1)$ distributed and which is independent of $\boldsymbol{Z}$. Then the process

$$\boldsymbol{Y} := \frac{1}{U}\boldsymbol{Z} \in \bar{C}^+[0,1]$$

is an example of a *generalized Pareto process (GPP)* (cf. Buishand et al. [4]), as its univariate margins are (in its upper tails) standard Pareto distributions:

$$\begin{aligned}F_t(x) &= P(Z_t \le xU)\\ &= \int_0^m P\left(\frac{z}{x} < U\right)(P * Z_t)(dz)\\ &= 1 - \frac{1}{x}E(Z_t)\\ &= 1 - \frac{1}{x}, \qquad x \ge m,\ t \in [0,1].\end{aligned}$$

We have, moreover, by Fubini's theorem

$$P\left(-\frac{1}{\boldsymbol{Y}} \le f\right) = 1 - \|f\|_D$$

for all $f \in \bar{E}^-[0,1]$ with $\|f\|_\infty \le 1/m$, i.e., the GPP $\boldsymbol{V} := -1/\boldsymbol{Y} = -U/\boldsymbol{Z}$ has the property that its df is in its upper tail equal to

$$W(f) := P(\boldsymbol{V} \le f) \;=\; 1 + \log(G(f)), \quad f \in \bar{E}^-[0,1],\ \|f\|_\infty \le 1/m,$$

where $G(f) = P(\boldsymbol{\eta} \le f)$ is the df of the EVP $\boldsymbol{\eta}$ with D-norm $\|\cdot\|_D$ and generator $\boldsymbol{Z}$ (cf. Aulbach et al. [2, Section 4]).

The preceding representation of the upper tail of the df of a GPP $\boldsymbol{V}$ in terms of $1 + \log(G)$ is in complete accordance with the unit- and multivariate case (see, for example, Falk et al. [5, Chapter 5]).

We call in general a stochastic process $\boldsymbol{V} \in \bar{C}^-[0,1]$ a *standard* GPP, if there is an $\varepsilon_0 > 0$ such that $P(\boldsymbol{V} \le f) = P(-U/\boldsymbol{Z} \le f)$ for all $f \in \bar{E}^-[0,1]$ with $\|f\|_\infty \le \varepsilon_0$.

Note that the copula process pertaining to the GPP $\boldsymbol{Z}/U$ is in its upper tail given by the shifted standard GPP $1 + \boldsymbol{V}$, which satisfies the conditions of Theorem 3.5. We, therefore, obtain for the GPP process $\boldsymbol{Z}/U$

$$\lim_{n\to\infty} \lim_{s\nearrow} \frac{FI_n(s)}{n} = \lim_{s\nearrow} E(S(s) \mid S(s) > 0) \;=\; \frac{1}{E\left(\max_{0\le t\le 1} Z_t\right)}.$$

4. Sojourn Time Distribution

In this section we compute the asymptotic sojourn time distribution of such processes, which are in a certain neighborhood of a standard GPP. A standard EVP is a prominent example. In this setup we can replace the constant threshold s by a threshold function.

The sojourn time distribution of a standard GPP is easily computed as the following lemma shows. This distribution is independent of the threshold level s, which reveals another exceedance stability of a GPP. Note that we replace the constant threshold line s in what follows by a *threshold function* $sf(t)$, where $f \in \bar{E}^-[0,1]$ is fixed and s is the variable *threshold level*.

Lemma 4.1. *Let $\boldsymbol{V} \in \bar{C}^-[0,1]$ be a standard GPP, i.e. there is an $\varepsilon_0 > 0$ such that $P(\boldsymbol{V} \le g) = P(-U/\boldsymbol{Z} \le g)$ for all $g \in \bar{E}^-[0,1]$ with $\|g\|_\infty \le \varepsilon_0$, where U is uniformly on $(0,1)$ distributed and independent of the generator $\boldsymbol{Z} = (Z_t)_{t\in[0,1]}$, which is continuous and satisfies $0 \le Z_t \le m$, $E(Z_t) = 1$, $t \in [0,1]$, for some $m \ge 1$. Choose $f \in \bar{E}^-[0,1]$. Then there is an $s_0 > 0$ such that the sojourn time* *df H_f of $\boldsymbol{V}$ above sf is given by*

$$\begin{aligned}
&P\left(\int_0^1 1\left(V_t > sf(t)\right)\, dt > y \mid \int_0^1 1\left(V_t > sf(t)\right)\, dt > 0\right)\\
&= \frac{\int_0^{m\|f\|_\infty} P\left(\int_0^1 1\left(|f(t)|\, Z_t > u\right)\, dt > y\right)\, du}{\int_0^{m\|f\|_\infty} P\left(\int_0^1 1\left(|f(t)|\, Z_t > u\right)\, dt > 0\right)\, du}\\
&=: 1 - H_f(y), \qquad 0 \le y \le 1,\ s_0 \le s < 0,
\end{aligned}$$

provided the denominator is greater than zero. Note that $H_f(0) = 0$, $H_f(1) = 1$.

Example 4.2. Any continuous df F on $[0,1]$ can occur as a sojourn time df. Take $Z_t = 1$, $0 \le t \le 1$, which provides the case of complete dependence of the margins of the corresponding standard EVP $\boldsymbol{\eta}$. Choose a continuous df $F : [0,1] \to [0,1]$ and put $f(t) = F(t) - 1$, $0 \le t \le 1$. Then the sojourn time df equals F, $H_f(y) = F(y)$, $y \in [0,1]$.

If we take, on the other hand, $f(t) = -1$, $t \in [0,1]$, then H_f has all its mass at 1, i.e., $H_f(y) = 0$, $y < 0$, and $H_f(1) = 1$. These examples show in particular that the sojourn time df H_f can be continuous as well as discrete.

Proof. The assertion is an immediate consequence of standard rules of integration together with conditioning on $U = u$:

$$\begin{aligned}
&P\left(\int_0^1 1\left(V_t > sf(t)\right)\, dt > y\right)\\
&= P\left(\int_0^1 1\left(U < s\,|f(t)|\, Z_t\right)\, dt > y\right)\, du\\
&= \int_0^1 P\left(\int_0^1 1\left(u < s\,|f(t)|\, Z_t\right)\, dt > y\right)\, du,
\end{aligned}$$

where substituting u by su yields

$$\begin{aligned}
&= s\int_0^{1/s} P\left(\int_0^1 1\left(|f(t)|\, Z_t > u\right)\, dt > y\right)\\
&= s\int_0^{m\|f\|_\infty} P\left(\int_0^1 1\left(|f(t)|\, Z_t > u\right)\, dt > y\right)\, du
\end{aligned}$$

if $s \le 1/(m\,\|f\|_\infty)$. This implies the assertion. □

Next we will extend the preceding lemma to processes $\boldsymbol{\xi} \in \bar{C}^-[0,1]$ which are in certain neighborhoods of a standard GPP $\boldsymbol{V}$. Precisely, we require that for a given function $f \in \bar{E}_1^-[0,1] := \left\{f \in \bar{E}^-[0,1] :\ \|f\|_\infty = 1\right\}$

$$P(\boldsymbol{\xi} > sf) = P(\boldsymbol{V} > sf) + o(s) \tag{7}$$

and

$$P(\boldsymbol{\xi} \le sf) = P(\boldsymbol{V} \le sf) + o(s) \tag{8}$$

as $s \downarrow 0$.

An example of a process satisfying conditions (7) and (8) is a standard EVP $\boldsymbol{\eta}$, which follows by Lemma 4.5 below together with equation (4). The next lemma follows from elementary computations.

Lemma 4.3. *For each standard GPP* $\boldsymbol{V}$ *there exists* $s_0 > 0$ *such that for* $0 \le s \le s_0$ *and for each* $f \in \bar{E}^-[0,1]$ *with* $\|f\|_\infty \le 1$

(i)
$$P(\boldsymbol{V} \le sf) = 1 - sE\left(\max_{t\in[0,1]} (|f(t)|\, Z_t)\right) = 1 - s\,\|f\|_D\,,$$

(ii)
$$P(\boldsymbol{V} > sf) = sE\left(\min_{t\in[0,1]} (|f(t)|\, Z_t)\right).$$

The next result extends the Lemma 4.1 to processes, which satisfy condition (7) and (8).

Proposition 4.4. *Suppose that* $\boldsymbol{\xi} \in \bar{C}^-[0,1]$ *has identical univariate margins and that it satisfies condition* (7) *as well as* (8)*. Choose* $f \in \bar{E}_1^-[0,1]$*. Then the asymptotic sojourn time distribution of* $\boldsymbol{\xi}$*, conditional on the assumption that it is positive, is given by*

$$P\left(\int_0^1 1\left(\xi_t > sf(t)\right)\, dt > y \mid \int_0^1 1\left(\xi_t > sf(t)\right)\, dt > 0\right) \to_{s\downarrow 0} 1 - H_f(y),$$

where the sojourn time df H_f *is given in Lemma 4.1.*

Proof. We establish this result via establishing convergence of characteristic functions. Put $I_s := \int_0^1 1(\xi_t > sf(t))\,dt$, $s > 0$. Without loss of generality we can assume that the function f is one-sided continuous at its points of discontinuity. The characteristic function of the rv I_s, conditional on the event that it is positive, is

$$E\left(\exp\left(itI_s\right) \mid I_s > 0\right) = \frac{\int_{\{I_s>0\}} \exp(itI_s)\, dP}{P(I_s > 0)}.$$

Note that $0 \le I_s \le 1$. By the dominated convergence theorem we have

$$\begin{aligned}\int_{\{I_s>0\}} \exp(itI_s)\, dP &= \int_{\{I_s>0\}} \sum_{k=0}^\infty \frac{(itI_s)^k}{k!}\, dP\\ &= \sum_{k=0}^\infty \frac{(it)^k}{k!} \int_{\{I_s>0\}} I_s^k\, dP\end{aligned}$$

$$= P(I_s > 0) + \sum_{k=1}^{\infty} \frac{(it)^k}{k!} \int_\Omega I_s^k \, dP$$

$$= P(I_s > 0) + \sum_{k=1}^{\infty} \frac{(it)^k}{k!} E\left(I_s^k\right). \tag{9}$$

From condition (8) we obtain

$$\begin{aligned} P(I_s > 0) &= 1 - P(I_s = 0) \\ &= 1 - P(\boldsymbol{\xi} \le sf) \\ &= 1 - P(\boldsymbol{V} \le sf) + o(s) \end{aligned}$$

$$= s\left(E\left(\max_{t\in[0,1]} |f(t)Z_t|\right) + o(1)\right) \tag{10}$$

as $s \downarrow 0$.

From Fubini's theorem and condition (7) we obtain for $k \in \mathbb{N}$

$$\begin{aligned} E(I_s^k) &= E\left(\left(\int_0^1 1(\xi_t > sf(t))\, dt\right)^k\right) \\ &= E\left(\int_0^1 \dots \int_0^1 \prod_{i=1}^k 1\left(\xi_{t_i} > sf(t_i)\right)\, dt_1 \dots dt_k\right) \\ &= \int_0^1 \dots \int_0^1 E\left(\prod_{i=1}^k 1\left(\xi_{t_i} > sf(t_i)\right)\right)\, dt_1 \dots dt_k \\ &= \int_0^1 \dots \int_0^1 P\left(\xi_{t_i} > sf(t_i),\, 1 \le i \le k\right)\, dt_1 \dots dt_k. \end{aligned}$$

We have by condition (7)

$$P\left(\xi_{t_i} > sf(t_i),\, 1 \le i \le k\right) \le P(\xi_{t_1} > s) = P(\xi_0 > s) = P(V_0 > s) + o(s)$$

uniformly for $t_1, \dots, t_k \in [0,1]$ and, thus, $P\left(\xi_{t_i} > sf(t_i),\, 1 \le i \le k\right)/s$ is uniformly bounded. Condition (7) together with the dominated convergence theorem now implies

$$\begin{aligned} \frac{E(I_s^k)}{s} &= \int_0^1 \dots \int_0^1 \frac{P\left(\xi_{t_i} > sf(t_i),\, 1 \le i \le k\right)}{s}\, dt_1 \dots dt_k \\ &= \int_0^1 \dots \int_0^1 \frac{P\left(V_{t_i} > sf(t_i),\, 1 \le i \le k\right) + o(s)}{s}\, dt_1 \dots dt_k \end{aligned}$$

$$\to_{s\downarrow 0} \int_0^1 \dots \int_0^1 E\left(\min_{1\le i\le k} |f(t_i)Z_{t_i}|\right)\, dt_1 \dots dt_k. \tag{11}$$

From equations (9)-(11) we obtain

$$\int_{\{I_s>0\}} \exp(itI_s)\, dP$$

$$= s(1+o(1))\left(E\left(\max_{t\in[0,1]}|f(t)Z_t|\right)\right.$$
$$\left.+\sum_{k=1}^{n}\frac{(it)^k}{k!}\left(\int_0^1\dots\int_0^1 E\left(\min_{1\le i\le k}|f(t_i)Z_{t_i}|\right)dt_1\dots dt_k\right)\right)$$
$$+\sum_{k=n+1}^{\infty}\frac{(it)^k}{k!}E(I_s^k),$$

where $n\in\mathbb{N}$ is chosen such that for a given $\varepsilon>0$ we have $\sum_{k=m+1}^{\infty}1/k!\le\varepsilon$. As $I_s\in[0,1]$, we obtain $E(I_s^k)\le E(I_s)=s\left(E\left(\min_{t\in[0,1]}|f(t)Z_t|\right)+o(1)\right)$ by condition (8) and, thus,

$$\int_{\{I_s>0\}}\exp(itI_s)\,dP$$
$$= s(1+o(1))\left(E\left(\max_{t\in[0,1]}|f(t)Z_t|\right)\right.$$
$$\left.+\sum_{k=1}^{n}\frac{(it)^k}{k!}\left(\int_0^1\dots\int_0^1 E\left(\min_{1\le i\le k}|f(t_i)Z_{t_i}|\right)dt_1\dots dt_k\right)+O(\varepsilon)\right)$$

as $s\downarrow 0$. Since $\varepsilon>0$ was arbitrary we obtain

$$\lim_{s\downarrow 0}\frac{\int_{\{I_s>0\}}\exp(itI_s)\,dP}{P(I_s>0)}$$
$$=1+\frac{\sum_{k=1}^{\infty}\frac{(it)^k}{k!}\left(\int_0^1\dots\int_0^1 E\left(\min_{1\le i\le k}|f(t_i)Z_{t_i}|\right)dt_1\dots dt_k\right)}{E\left(\max_{t\in[0,1]}|f(t)Z_t|\right)}$$
$$=:\varphi(t),\qquad t\in\mathbb{R}.$$

An inspection of the preceding arguments shows that φ is the characteristic function of the sojourn time df H_f, which completes the proof. □

We conclude this section by showing that a standard EVP $\boldsymbol{\eta}$ satisfies condition (7) and (8) and, thus, Proposition 4.4 applies.

Lemma 4.5. *Let $\boldsymbol{\eta}$ be a standard EVP with generator $\boldsymbol{Z}$. Then we obtain for $f\in\bar{E}^-[0,1]$*

(i) $P(\boldsymbol{\eta}>f)\ge 1-\exp\left(-E\left(\inf_{0\le t\le 1}(|f(t)|\,Z_t)\right)\right)$,

(ii) $\lim_{s\downarrow 0}\frac{P(\boldsymbol{\eta}>sf)}{s}=E\left(\inf_{0\le t\le 1}(|f(t)|\,Z_t)\right)$.

Proof. Due to the continuity of $\boldsymbol{\eta}$ and $\boldsymbol{Z}$ it is sufficient to consider $f\in\bar{E}^-[0,1]$ with $\sup_{0\le t\le 1}f(t)<0$. Let $\{t_1,t_2,\dots\}$ be a denumerable dense subset of $[0,1]$, which contains those finitely many points t_i at which the function $f(t)$ is discontinuous.

The continuity from above of an arbitrary probability measure implies

$$P\left(\bigcap_{j\in\mathbb{N}}\left\{\eta_{t_j}>f(t_j)\right\}\right)=\lim_{m\to\infty}P\left(\eta_{t_j}>f(t_j),\,1\le j\le m\right).$$

The first assertion of the lemma follows if we show that for arbitrary $m\in\mathbb{N}$

$$P\left(\eta_{t_j}>f(t_j),\,1\le j\le m\right)\ge 1-\exp\left(-E\left(\min_{1\le j\le m}\left(\left|f(t_j)\right|Z_{t_j}\right)\right)\right). \tag{12}$$

From Aulbach et al. [2] we deduce that

$$\xi_t:=-\frac{1}{\eta_t},\qquad 0\le t\le 1,$$

defines a continuous max-stable process $\boldsymbol{\xi}=(\xi_t)_{0\le t\le 1}$ on $[0,1]$ with standard Fréchet margins. By Proposition 3.2 in Giné et al. [8] we know that

$$\boldsymbol{\xi}=_D\max_i \boldsymbol{Y}_i$$

in $\bar{C}^+[0,1]$, where $\boldsymbol{Y}_1,\boldsymbol{Y}_2,\dots$ are the points (functions in $\bar{C}^+[0,1]$) of a Poisson process N with intensity measure ν given by $d\nu=d\sigma\times dr/r^2$ on $\bar{C}^+[0,1]\times(0,\infty)=: C[0,1]^+=\{h\in C[0,1]:\,h\ge 0,\,h\ne 0\}$. By $\bar{C}_1^+[0,1]$ we denote the space of those functions h in $\bar{C}^+[0,1]$ with $\|h\|_\infty=\sup_{0\le t\le 1}|h(t)|=1$. The (finite) measure σ is given by $\sigma(\cdot)=mP(\tilde{\boldsymbol{Z}}\in\cdot)$, where $\tilde{\boldsymbol{Z}}:=\boldsymbol{Z}/m$ and m is the generator constant pertaining to $\boldsymbol{Z}$. Note that m coincides with the total mass of σ.

We, therefore, obtain

$$\begin{aligned}
&P(\eta_{t_j}>f(t_j),\,1\le j\le m)\\
&=P\left(-\frac{1}{\eta_{t_j}}>\frac{1}{|f(t_j)|},\,1\le j\le m\right)\\
&=P\left(\xi_{t_j}>\frac{1}{|f(t_j)|},\,1\le j\le m\right)\\
&=P\left(\bigcap_{j=1}^m\bigcup_i\left\{Y_i(t_j)>\frac{1}{|f(t_j)|}\right\}\right)\\
&=P\left(\bigcap_{j=1}^m\left\{N\left(\left\{g\in C[0,1]^+:\,g(t_j)>\frac{1}{|f(t_j)|}\right\}\right)>0\right\}\right)\\
&=1-P\left(\bigcup_{j=1}^m\left\{N\left(\left\{g\in C[0,1]^+:\,g(t_j)>\frac{1}{|f(t_j)|}\right\}\right)=0\right\}\right)\\
&=1-P\left(\omega:\,\exists j\,\forall i:\,Y_i(t_j)\le\frac{1}{|f(t_j)|}\right)\\
&\ge 1-P\left(\omega:\,\forall i\,\exists j:\,Y_i(t_j)\le\frac{1}{|f(t_j)|}\right)
\end{aligned}$$

$$= 1 - P\left(\left\{N\left(\left\{g \in C[0,1]^+ : \max_{1\le j\le m}\left(\frac{1}{|f(t_j)|\, g(t_j)}\right) < 1\right\}\right) = 0\right\}\right)$$
$$= 1 - \exp\left(-\nu\left(\left\{g \in C[0,1]^+ : \max_{1\le j\le m}\left(\frac{1}{|f(t_j)|\, g(t_j)}\right) < 1\right\}\right)\right)$$
$$= 1 - \exp\left(-\nu\left(\left\{(h,r) \in \bar{C}_1^+[0,1]\times(0,\infty) : r > \max_{1\le j\le m}\left(\frac{1}{|f(t_j)|\, h(t_j)}\right)\right\}\right)\right)$$
$$= 1 - \exp\left(-\int_{\bar{C}_1^+[0,1]}\int_{\max_{1\le j\le m}(1/(|f(t_j)|h(t_j)))}^{\infty} \frac{1}{r^2}\, dr\, \sigma(dh)\right)$$
$$= 1 - \exp\left(-\int_{\bar{C}_1^+[0,1]} \min_{1\le j\le m}\left(|f(t_j)|\, h(t_j)\right)\sigma(dh)\right)$$
$$= 1 - \exp\left(-E\left(\min_{1\le j\le m}\left(|f(t_j)|\, Z_{t_j}\right)\right)\right),$$

which is inequality (12). Next we establish the inequality

$$\limsup_{s\downarrow 0}\frac{P(\boldsymbol{\eta} > sf)}{s} \le E\left(\min_{1\le j\le m}\left(|f(t_j)|\, Z_{t_j}\right)\right). \tag{13}$$

The inclusion-exclusion theorem implies

$$\begin{aligned}
&P(\boldsymbol{\eta} > sf)\\
&\le P\left(\bigcap_{j=1}^{m}\left\{\eta_{t_j} > sf(t_j)\right\}\right)\\
&= 1 - P\left(\bigcup_{j=1}^{m}\left\{\eta_{t_j} \le sf(t_j)\right\}\right)\\
&= 1 - \sum_{\emptyset\ne T\subset\{1,\dots,m\}}(-1)^{|T|-1} P\left(\bigcap_{j\in T}\left\{\eta_{t_j} \le sf(t_j)\right\}\right)\\
&= 1 - \sum_{\emptyset\ne T\subset\{1,\dots,m\}}(-1)^{|T|-1}\exp\left(-sE\left(\max_{j\in T}\left(|f(t_j)|\, Z_{t_j}\right)\right)\right)\\
&=: 1 - H(s)\\
&= H(0) - H(s),
\end{aligned}$$

where the function H is differentiable and, thus,

$$\begin{aligned}
&\limsup_{s\downarrow 0}\frac{P(\boldsymbol{\eta} > sf)}{s}\\
&\le -\lim_{s\downarrow 0}\frac{H(s)-H(0)}{s}\\
&= -H'(0)
\end{aligned}$$

$$= \sum_{\emptyset \neq T \subset \{1,\dots,m\}} (-1)^{|T|-1} E\left(\max_{j \in T} \left(|f(t_j)|\, Z_{t_j}\right)\right)$$
$$= E\left(\min_{1 \le j \le m} \left(|f(t_j)|\, Z_{t_j}\right)\right),$$

since $\sum_{\emptyset \neq T \subset \{1,\dots,m\}} (-1)^{|T|-1} \max_{j \in T} a_j = \min_{1 \le j \le m} a_j$ for arbitrary numbers $\{a_1, \dots, a_m\} \in \mathbb{R}$, which can be seen by induction. This implies equation (13). Part (ii) is a straightforward consequence of (i) and (13). □

5. Cluster Length

The considerations in the previous section enable us also to compute the limit distribution of the cluster length above the threshold sf of a process $\boldsymbol{\xi} \in \bar{C}^-[0,1]$, which is in the neighborhood of a standard GPP. Precisely, we require the following variant of condition (7). Choose $0 \le a \le b \le 1$, and denote by $\bar{C}^-[a,b]$ the set of continuous functions $f : [a,b] \to (-\infty, 0]$. We suppose that for $f \in \bar{C}^-[a,b]$

$$P(\xi_t > sf(t),\, t \in [a,b]) = P(V_t > sf(t),\, t \in [a,b]) + o(s) \tag{14}$$

as $s \downarrow 0$, where $\boldsymbol{V} = (V_t)_{t \in [0,1]}$ is a standard GPP. Note that

$$P(V_t > sf(t),\, t \in [a,b]) = sE\left(\min_{a \le t \le b} (|f(t)|\, Z_t)\right) + o(s), \qquad s \in (0, s_0),$$

and that we allow the case $a = b$. We do not require $\boldsymbol{\xi}$ to have identical marginal distributions. A standard EVP $\boldsymbol{\eta}$ satisfies condition (14); an inspection of the proof of Lemma 4.5 shows that it remains true with the interval $[0,1]$ replaced by $[a,b]$.

The *cluster length above sf* of the process $\boldsymbol{\xi}$ with inspection point $t_0 \in [0,1)$ is defined by

$$\tau_{t_0}(s) := \sup\{L \in (0, 1 - t_0] : \xi_t > sf(t), t \in [t_0, t_0 + L)\}$$

under the condition that $\xi_{t_0} > sf(t_0)$.

Proposition 5.1. *Suppose that $\boldsymbol{\xi} \in \bar{C}^-[0,1]$ satisfies condition* (14)*. Then we have for $u \in [0, 1 - t_0)$ and $f \in \bar{C}^-[a,b]$ with $f(t_0) < 0$*

$$\lim_{s \downarrow 0} P\left(\tau_{t_0}(s) > u \mid \xi_{t_0} > sf(t_0)\right) = \frac{E\left(\min_{t_0 \le t \le u}(|f(t)|\, Z_t)\right)}{|f(t_0)|}.$$

Proof. We have for $u \in [0, 1 - t_0)$

$$\begin{aligned} P\left(\tau_{t_0}(s) > u \mid \xi_{t_0} > sf(t_0)\right) &= \frac{P(\xi_t > sf(t),\, t \in [t_0, t_0 + u])}{P\left(\xi_{t_0} > sf(t_0)\right)} \\ &= \frac{P(V_t > sf(t),\, t \in [t_0, t_0 + u]) + o(s)}{P\left(V_{t_0} > sf(t_0)\right) + o(s)} \\ &= \frac{E\left(\min_{t_0 \le t \le t_0 + u}(|f(t)|\, Z_t)\right)}{|f(t_0)|} + o(1) \end{aligned}$$

as $s \downarrow 0$. □

The *asymptotic* cluster length τ_{t_0}, as $s \downarrow 0$, with inspection point $t_0 \in [0,1)$ has, consequently, the continuous df

$$P(\tau_{t_0} \le u) = 1 - \frac{E\left(\min_{t_0 \le t \le t_0+u}(|f(t)|\, Z_t)\right)}{|f(t_0)|}$$

for $0 \le u < 1 - t_0$, and possibly positive mass at $1 - t_0$:

$$P(\tau_{t_0} = 1 - t_0) = \frac{E\left(\min_{t_0 \le t \le 1}(|f(t)|\, Z_t)\right)}{|f(t_0)|}.$$

Its expected value is, therefore, given by

$$\begin{aligned} E(\tau_{t_0}) &= \int_0^{1-t_0} P(\tau_{t_0} > u)\, du \\ &= \frac{1}{|f(t_0)|} \int_0^{1-t_0} E\left(\min_{t_0 \le t \le t_0+u}(|f(t)|\, Z_t)\right) du \\ &= \frac{1}{|f(t_0)|} E\left(\int_{t_0}^1 \min_{t_0 \le t \le u}(|f(t)|\, Z_t)\, du\right). \end{aligned}$$

References

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