

# FORMAL PLETHORIES

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TL;DR. Unstable operations in a generalized cohomology theory  $E$  give rise to a functor from the category of algebras over  $E$  to itself which is a colimit of representable functors and a comonoid with respect to composition of such functors. In this paper I set up a framework to study the algebra of such functors, which I call formal plethories. I show that the “logarithmic” functors of primitives and indecomposables give linear approximations of formal plethories by bimonoids in the 2-monoidal category of bimodules over a ring.

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## 1. INTRODUCTION

Let  $k$  be a commutative ring and denote by  ${}_k\text{Alg}_k$  the category of representable endofunctors  $\text{Alg}_k \rightarrow \text{Alg}_k$  of  $k$ -algebras. From an algebro-geometric point of view, these can be considered as affine schemes over  $k$  with a structure of a  $k$ -algebra on them. Composition of such representable endofunctors constitutes a non-symmetric monoidal structure  $\circ$  on  ${}_k\text{Alg}_k$ ; a *plethory* is an object  $F \in {}_k\text{Alg}_k$  which is a comonoid with respect to  $\circ$ , i.e. a representable functor  $F: \text{Alg}_k \rightarrow \text{Alg}_k$  with natural transformations  $F \rightarrow \text{id}$  and  $F \rightarrow F \circ F$  such that coassociativity and counitality conditions are satisfied. The algebra of plethories was first studied by Tall and Wraith [TW70] and then extended by Borger and Wieland [BW05]. The aim of this paper is to extend the theory of plethories to the setting of graded *formal* schemes and to study linearizations of them. The motivation for doing this comes from topology.

Let  $K$  be a homotopy commutative ring spectrum representing a cohomology theory  $K^*$ . For any space  $X$ ,  $K^*(X)$  is naturally a algebra over the ring of coefficient  $K_*$  of  $K$ ; furthermore, there is an action

$$K^n(\underline{K}_m) \times K^m(X) \rightarrow K^n(X)$$

by unstable operations. Here  $\underline{K}_m$  denotes the  $m$ th space in the  $\Omega$ -spectrum associated to  $K$ . The bigraded  $K_*$ -algebra  $K^*(\underline{K}_*)$  almost qualifies as the representing object of a plethory, but not quite. In order for  $K^*(\underline{K}_*)$  to have the required structure maps (the ring structure on the spectrum of this ring must come from a coaddition and a comultiplication, for instance), one would have to assume that  $K^*(\underline{K}_n)$  is a finitely generated free  $K_*$ -module in order to have a Künneth isomorphism, but this is almost never the case. A solution to this is to pass to the category of pro- $K_*$ -algebras. If  $X$  is a CW-complex, we *define*  $K^*(X) \in \text{Pro-Alg}_{K_*}$  to be the system  $\{K^*(F)\}_{F \subseteq X}$  indexed by all finite sub-CW-complexes  $F$  of  $X$ . We then assume:

$$(1.1) \quad K^*(\underline{K}_n) \text{ is pro-finitely generated free for all } n.$$

Note that we do not require that  $K^*(F)$  be free for all finite sub-CW-complexes  $F \subseteq X$ , or even for any such  $F$ . We merely require that  $K^*(F)$  is pro-isomorphic to a system that consists of free  $K_*$ -modules.

This passing to pro-objects gives the theory of plethories a whole new flavor.

**Definition.** Let  $k$  be a graded commutative ring. A *formal scheme over  $k$*  is a functor  $F: \text{Alg}_k \rightarrow \text{Set}$  which is a filtered colimit of representable functors. A *formal  $k$ -algebra scheme over  $k$*  is a functor  $F: \text{Alg}_k \rightarrow \text{Alg}_k$  whose underlying functor to  $\text{Set}$  is a formal scheme.

**Theorem 1.2.** *The category  ${}_k\widehat{\text{Alg}}_k$  of formal algebra schemes is complete and cocomplete and has a monoidal structure  $\circ$  given by composition of functors. Similarly, the category  $\text{Sch}_k$  of formal schemes has an action of  ${}_k\widehat{\text{Alg}}_k$ .*

In contrast, the category  ${}_k\text{Alg}_k$  is in fact not complete – it does not have an initial object, for example.

**Definition.** A *formal plethory* is a comonoid in  ${}_k\widehat{\text{Alg}}_k$  with respect to  $\circ$ . A *comodule* over a formal plethory  $P$  is a formal scheme  $X$  with a coaction  $X \rightarrow P \circ X$ .

**Example 1.3.** Let  $K$  be a ring spectrum satisfying (1.1). Then  $K^*(\underline{K}_n)$  represents a formal plethory, and  $K^*(X)$  represents a comodule over this plethory for any space  $X$ .

Thus formal plethories provide an algebraic framework for studying unstable cohomology operations. Algebraic descriptions of unstable cohomology operations are not new: in [BJW95], unstable algebras and unstable modules were studied in great depth from a monadic point of view. Plethories were first introduced in [TW70], and a framework similar to our formal plethories has been independently developed by Stacey and Whitehouse [SW09] using filtrations instead of pro-objects.

Note that there is an unstable Adams-Novikov spectral sequence

$$\mathrm{Ext}_{K^*(\underline{K}_*)}(K^*X, K_*) \implies \pi_*(X_{\hat{K}})$$

converging conditionally to the homotopy groups of the unstable  $K$ -completion of  $X$ . The Ext group of the left is the nonlinear comonad-derived functor of homomorphisms of comodules over the plethory represented by  $K^*(\underline{K}_*)$ . Since this is generally very difficult to compute, it is desirable to find good linear approximations to the formal plethory represented by  $K^*(\underline{K}_n)$  and its comodules, such that the Ext computation takes place in an abelian category. For this, it is useful to introduce the concept of a 2-monoidal category.

**Definition** ([AM10]). A *2-monoidal category* is a category  $\mathcal{C}$  with two monoidal structures  $(\otimes, I)$  and  $(\circ, J)$  with natural transformations

$$\zeta: (A \circ B) \otimes (C \circ D) \rightarrow (A \otimes C) \circ (B \otimes D)$$

and

$$\Delta_I: I \rightarrow I \circ I, \quad \mu_J: J \otimes J \rightarrow J, \quad \iota_J = \epsilon_I: I \rightarrow J,$$

satisfying various compatibility conditions explicated in Section 8.

A *bilax monoidal functor*  $F: \mathcal{C} \rightarrow \mathcal{D}$  between 2-monoidal categories is functor which is lax monoidal with respect to  $\otimes$ , oplax monoidal with respect to  $\circ$ , and whose lax and oplax monoidal structures satisfy certain compatibility conditions (cf. Section 8).

Loosely speaking, a 2-monoidal category is the most general setting in which one can define a bimonoid (an object with a multiplication and a comultiplication that are compatible with each other). A bilax monoidal functor is the most general notion of functor which sends bimonoids to bimonoids.

**Example 1.4.** If  $(\mathcal{C}, \otimes, I)$  is a cocomplete monoidal category then  $(\mathcal{C}, \otimes, I, \sqcup, \emptyset)$  becomes a 2-monoidal category (the other monoidal structure being the categorical coproduct). In particular, the category of formal algebra schemes is 2-monoidal with the composition product and the coproduct.

**Example 1.5.** The category  ${}_k\mathrm{Mod}_k$  of bimodules over a (graded) commutative ring  $k$  is 2-monoidal with respect to the two-sided tensor product  ${}_k\otimes_k$  and the left-right tensor product  $\otimes_k$ .

**Example 1.6.** The category  ${}_k\widehat{\mathrm{Mod}}_k$  of *formal bimodules*, i.e. ind-representable additive functors  $\mathrm{Mod}_k \rightarrow \mathrm{Mod}_k$ , is 2-monoidal.

The main result of this paper about linearization of formal plethories concerns the functors of primitives  $P: \text{Coalg}_k^+ \rightarrow \text{Mod}_k$  and of indecomposables  $Q: \text{Alg}_k^+ \rightarrow \text{Mod}_k$ .

**Theorem 1.7.** *The functors of primitives and indecomposables extend to bilax monoidal functors*

$$\mathcal{P}, \mathcal{Q}: {}_k\widehat{\text{Alg}}_k \rightarrow {}_k\widehat{\text{Mod}}_k.$$

This means that a formal plethory  $F$  gives rise to two bimonoids  $\mathcal{P}(F)$ ,  $\mathcal{Q}(F)$ . The usefulness for cohomology operations comes from the existence of an edge map

$$\text{Ext}_{QK_*\underline{K}_*}(QK^*X, K_*) \rightarrow \text{Ext}_{K_*\underline{K}_*}(K^*X, K_*)$$

which is an isomorphism in interesting cases. We will study these topological applications and computations made possible by this theory in a forthcoming paper.

Finally, if one desires to leave the worlds of pro-categories, one can do so after dualization because of the following theorem.

**Theorem 1.8.** *Assume that  $k \otimes k$  is flat over  $k$ . Then the full subcategory  ${}_k\widehat{\text{Mod}}_k'$  of  ${}_k\widehat{\text{Mod}}_k$  consisting of pro-finitely generated free  $k$ -modules is equivalent, as a 2-monoidal category, to the full subcategory of right  $k$ -flat modules in  ${}_k\text{Mod}_k$  (cf. Example 1.5).*

**Corollary 1.9.** *Let  $K$  be a multiplicative homology theory such that  $K_*(\underline{K}_n)$  and  $PK_*(\underline{K}_n)$  are flat  $K_*$ -modules for all  $n$ . Assume that  $K_* \otimes K_*$  is flat over  $K_*$ . Then  $PK_*\underline{K}_*$  is a bimonoid in  ${}_k\text{Mod}_k$ , and for any pointed space  $X$ ,  $PK_*(X)$  is a comodule over it.*

*Proof.* By the flatness condition and the Lazard-Govorov theorem [Laz69, Gov65],  $\{K_*(F)\}_{F \subseteq \underline{K}_n}$ , where  $F$  runs through all finite sub-CW-complexes of  $\underline{K}_n$ , is ind-finitely generated free. By the universal coefficient theorem [Boa95, Thm 4.14], this implies that condition (1.1) is also satisfied. Thus  $K^*(\underline{K}_*)$  represents a formal plethory by Example 1.3, and applying the functor  $\mathcal{Q}$  gives a bimonoid  $QK^*(\underline{K}_*)$  by Thm. 1.7. Under the additional flatness assumption on  $PK_*(\underline{K}_*)$ ,  $QK^*(\underline{K}_*)$  is in fact pro-finitely generated free, and Thm. 1.8 yields that  $PK_*(\underline{K}_*)$  is a bimonoid in  ${}_k\text{Mod}_k$ . With a similar reasoning,  $PK_*(X)$  is a comodule over this bimonoid.  $\square$

**1.1. Outline of the paper.** In Section 2, we set up notation and terminology to deal with the kind of monoidal categories that come up in this context, and with pro-objects and ind-representable functors. In Sections 3–7, we study formal bimodules and formal plethories and the structure of their categories and prove the first part of Thm. 1.2. In Section 8, we recall the definition of 2-monoidal categories and functors between them, define the objects of the title of this paper, along with their linearizations, formal rings and bimonoid, and prove the second part of Thm.1.2. The long Section 9 is devoted to the study of the linearizing functors of primitives and indecomposables and culminates in a proof of Thm. 1.7. The final Section 10 deals with dualization and the proof of Thm. 1.8. There are two appendices: in Appendix A, we review some background on ind- and pro-categories and how the indexing categories can be simplified. This is needed for Appendix B, which contains an exposition of how enrichments of categories lift to pro- and ind-categories.

## 2. FUNCTORS WHICH ARE FILTERED COLIMITS OF REPRESENTABLE FUNCTORS

In this section we will set up some category-theoretic terminology to talk about enrichments, monoidal structures, and ind-representable functors.

## 2.1. Two-algebra.

**Definition.** A 2-ring is a bicomplete closed symmetric monoidal category. If  $(\mathcal{V}, \otimes, I)$  is a 2-ring, define a *left 2-module* over  $\mathcal{V}$  to be a category  $\mathcal{C}$  which is enriched and tensored over  $\mathcal{V}$  and which has all colimits. A *right 2-module* over  $\mathcal{V}$  is a category  $\mathcal{C}$  which is enriched and cotensored over  $\mathcal{V}$  with all limits. A *2-bimodule* is a category  $\mathcal{C}$  which is both a left and a right 2-module with respect to the same enrichment.

**Notation.** We will typically denote the internal hom object of a 2-ring by  $\mathcal{V}(X, Y)$  and the enrichment of a 2-module by  $\mathcal{C}(X, Y)$ . We will use the symbol  $\otimes$  for the symmetric monoidal structure of  $\mathcal{V}$ ,  $\overline{\otimes}$  for the left module structure of  $\mathcal{C}$  over  $\mathcal{V}$ , and  $\text{hom}(-, -)$  for the right module structure of  $\mathcal{C}$  over  $\mathcal{V}$ .

**Remark 2.1.** If  $\mathcal{V}$  is a 2-ring and  $\mathcal{C}$  is a left 2-module over  $\mathcal{V}$  then the opposite category  $\mathcal{C}^{\text{op}}$  is a right 2-module over  $\mathcal{V}$  and vice versa.

**Example 2.2.** Any bicomplete category is a 2-bimodule over the category of sets: the left and right 2-module structures are given by

$$S \overline{\otimes} X = \coprod_{s \in S} X \quad \text{and} \quad \text{hom}(S, X) = \prod_{s \in S} X.$$

**Example 2.3.** The category  $\text{Set}^{\mathbf{Z}}$  of  $\mathbf{Z}$ -graded sets is a 2-ring with enrichment given by

$$\text{Set}^{\mathbf{Z}}(X, Y)(n) = \prod_{i \in \mathbf{Z}} \text{Set}(X(i), Y(i+n))$$

and tensor product by

$$(X \times Y)(n) = \prod_{i+j=n} X(i) \times Y(j).$$

The unit object is the singleton in degree 0.

A left 2-module (right 2-module, bimodule) over  $\text{Set}^{\mathbf{Z}}$  is a precisely a cocomplete (resp. complete, bicomplete) category  $\mathcal{C}$  with a  $\mathbf{Z}$ -action on objects by a shift functor  $\Sigma^n: \mathcal{C} \rightarrow \mathcal{C}$  ( $n \in \mathbf{Z}$ ). The  $\mathbf{Z}$ -grading on its morphism sets is determined by the shift functor:  $\mathcal{C}(X, Y)(n) = \mathcal{C}_0(X, \Sigma^n Y)$ , where the right hand side denotes the unenriched homomorphism sets.

**Definition.** A *morphism of 2-bimodules*  $F: \mathcal{C} \rightarrow \mathcal{D}$  over a fixed 2-ring  $\mathcal{V}$  is an enriched functor  $\mathcal{C} \rightarrow \mathcal{D}$ . This implies the existence of canonical morphisms  $\alpha: L \overline{\otimes} F(X) \rightarrow F(L \overline{\otimes} X)$  and  $\beta: F(\text{hom}(L, X)) \rightarrow \text{hom}(L, F(X))$  given by the adjoints of

$$L \rightarrow \mathcal{C}(X, L \overline{\otimes} X) \xrightarrow{F} \mathcal{D}(F(X), F(L \overline{\otimes} X))$$

and

$$L \rightarrow \mathcal{C}(\text{hom}(L, X), X) \xrightarrow{F} \mathcal{D}(F(\text{hom}(L, X)), F(X)),$$

respectively. We call  $F$  *left strict* if  $\alpha$  is a natural isomorphism, and *right strict* if  $\beta$  is a natural isomorphism.

**Example 2.4.** Let  $\mathcal{C}, \mathcal{D}$  be two bicomplete categories with a  $\mathbf{Z}$ -action by a shift functor  $\Sigma^n$ . By Example 2.3, this is equivalent with being a 2-bimodule over  $\text{Set}^{\mathbf{Z}}$ . Then a functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is a morphism of 2-bimodules if  $F$  commutes with the shift functor, i.e. if the maps  $\alpha$  and  $\beta$  give mutually inverse maps between  $\Sigma^n F(X)$  and  $F(\Sigma^n X)$ .

**Definition.** A 2-algebra  $\mathcal{C}$  over a 2-ring  $(\mathcal{V}, \otimes, I)$  is a 2-bimodule  $\mathcal{C}$  with a monoidal structure  $(\boxtimes, J)$  such that the functors  $- \boxtimes X$  and  $X \boxtimes -: \mathcal{C} \rightarrow \mathcal{C}$  are enriched functors for all  $X \in \mathcal{C}$ .

Although  $\mathcal{C} \times \mathcal{C}$  is a  $\mathcal{V}$ -category by the diagonal enrichment, we do not require the functor  $\boxtimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  to be thus enriched.

To make the structure maps more explicit, the enrichment gives

$$(2.5) \quad \text{a natural map } \alpha: L \overline{\otimes} (X \boxtimes Y) \rightarrow (L \overline{\otimes} X) \boxtimes Y$$

$$(2.6) \quad \text{a natural map } \beta: \text{hom}(L, X) \boxtimes Y \rightarrow \text{hom}(L, X \boxtimes Y)$$

Note that a 2-algebra, even if it is symmetric, is not required to be closed monoidal ( $\boxtimes$  need not have a right adjoint) and thus is not necessarily a 2-ring.

Note also that the definition of a 2-algebra is symmetric: if  $\mathcal{C}$  is a 2-algebra over  $\mathcal{V}$  then so is  $\mathcal{C}^{\text{op}}$ .

**Example 2.7.** A 2-algebra over the category of (ungraded) sets is simply a monoidal category. A 2-algebra over  $\mathbf{Z}$ -graded sets is a  $\mathbf{Z}$ -graded category with a monoidal structure  $\boxtimes$  which is equivariant under grading shifts in either variable.

**Example 2.8.** Let  $k$  be a (not necessarily commutative) ring,  $\mathcal{V}$  the category of abelian groups, and  $\mathcal{C} = {}_k\text{Mod}_k$  the category of  $k$ -bimodules. Then  ${}_k\text{Mod}_k$  is a 2-bimodule over  $\mathcal{V}$ , and the tensor product of  $k$ -bimodules  $\otimes_k$ , using the right module structure on the left and vice versa, makes  ${}_k\text{Mod}_k$  into a 2-algebra.

**Lemma 2.9.** Let  $(\mathcal{C}, \boxtimes, J)$  be a 2-algebra over  $(\mathcal{V}, \otimes, I)$ . Let  $X, Y \in \mathcal{C}$  and  $K, L \in \mathcal{V}$ . There are natural maps

$$\mu: I \overline{\otimes} J \rightarrow J, \quad \zeta: (K \otimes L) \overline{\otimes} (M \boxtimes N) \rightarrow (K \overline{\otimes} M) \boxtimes (L \overline{\otimes} N)$$

and

$$\mu': J \rightarrow \text{hom}(I, J), \quad \zeta': \text{hom}(K, M) \boxtimes \text{hom}(L, N) \rightarrow \text{hom}(K \otimes L, M \boxtimes N)$$

which make  $\overline{\otimes}$  and  $\text{hom}$  monoidal functors  $\mathcal{V} \times \mathcal{C} \rightarrow \mathcal{C}$ .

*Proof.* The map  $\mu$  is adjoint to the map  $I \rightarrow \mathcal{D}(J, J)$  classifying the unit map of  $\mathcal{D}$ . The map  $\zeta$  is the composite

$$(2.10) \quad (K \otimes L) \overline{\otimes} (M \boxtimes N) \cong K \overline{\otimes} (L \overline{\otimes} (M \boxtimes N)) \\ \xrightarrow{K \overline{\otimes} \alpha} K \overline{\otimes} (M \boxtimes (L \overline{\otimes} N)) \xrightarrow{\alpha} (K \overline{\otimes} M) \boxtimes (L \overline{\otimes} N).$$

The assertion about  $\zeta'$  follows from passing to the 2-algebra  $\mathcal{C}^{\text{op}}$ .  $\square$

**Definition.** A lax morphism of 2-algebras  $F: \mathcal{C} \rightarrow \mathcal{D}$  over a fixed 2-ring  $\mathcal{V}$  is a morphism of 2-modules with a natural transformation  $\phi: F(X) \boxtimes F(Y) \rightarrow F(X \boxtimes Y)$  and a morphism  $\phi_0: J_{\mathcal{D}} \rightarrow F(J_{\mathcal{C}})$  which make  $F$  into a lax monoidal functor, and which

is compatible with the enrichment in the sense that the following diagrams commute:

$$\begin{array}{ccccc}
 I & \xrightarrow{\text{id}_{J_{\mathcal{D}}}} & \mathcal{C}(J_{\mathcal{C}}, J_{\mathcal{C}}) & \xrightarrow{F} & \mathcal{D}(F(J_{\mathcal{C}}), F(J_{\mathcal{C}})) \\
 \downarrow \text{id}_{J_{\mathcal{C}}} & & & & \swarrow \phi_0^* \\
 \mathcal{D}(J_{\mathcal{D}}, J_{\mathcal{D}}) & \xrightarrow{(\phi_0)^*} & \mathcal{D}(J_{\mathcal{D}}, F(J_{\mathcal{C}})) & & 
 \end{array}$$
  

$$\begin{array}{ccccc}
 \mathcal{C}(X, X') & \xrightarrow{F} & \mathcal{D}(FX, FX') & \xrightarrow{- \boxtimes FY} & \mathcal{D}(FX \boxtimes FY, FX' \boxtimes FY) \\
 \downarrow - \boxtimes Y & & & & \downarrow \phi_* \\
 \mathcal{C}(X \boxtimes Y, X' \boxtimes Y) & \xrightarrow{F} & \mathcal{D}(F(X \boxtimes Y), F(X' \boxtimes Y)) & \xrightarrow{\phi^*} & \mathcal{D}(FX \boxtimes FY, F(X' \boxtimes Y)).
 \end{array}$$

Similarly, the previous diagram with application of  $Y \boxtimes -$  and  $FY \boxtimes -$  instead of  $- \boxtimes Y$  and  $- \boxtimes FY$  is required to commute.

An *oplax morphism of 2-algebras* is a morphism  $F: \mathcal{C} \rightarrow \mathcal{D}$  such that  $F^{\text{op}}: \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}^{\text{op}}$  is a lax morphism of 2-algebras. More explicitly, it is a morphism of 2-modules with a natural transformation  $\psi: F(X \boxtimes Y) \rightarrow F(X) \boxtimes (Y)$  and a morphism  $\psi_0: F(J_{\mathcal{C}}) \rightarrow J_{\mathcal{D}}$  making  $F$  into an oplax monoidal functor, and which is compatible with the enrichment in a similar sense as above.

A *strict morphism of 2-algebras* is a morphism  $F$  as above with is both lax and oplax with  $\phi = \psi^{-1}$  and  $\phi_0 = \psi_0^{-1}$ .

It will be useful later to express the conditions for being a lax/oplax morphism of 2-algebras in terms of the maps  $\mu, \zeta$  of Lemma 2.9 and  $\alpha, \beta$  from the definition of a morphism of 2-modules.

**Lemma 2.11.** *Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  be a lax morphism of 2-algebras. Then the following diagrams commute:*

$$\begin{array}{ccccc}
 I \overline{\otimes} J_{\mathcal{D}} & \xrightarrow{\text{id} \overline{\otimes} \phi_0} & I \overline{\otimes} F(J_{\mathcal{C}}) & \xrightarrow{\alpha} & F(I \overline{\otimes} J_{\mathcal{C}}) \\
 \downarrow \mu & & & & \swarrow F(\mu) \\
 J_{\mathcal{D}} & \xrightarrow{\phi_0} & F(J_{\mathcal{C}}) & & 
 \end{array}$$
  

$$\begin{array}{ccccc}
 (K \otimes L) \overline{\otimes} (F(X) \boxtimes F(Y)) & \xrightarrow{\text{id} \overline{\otimes} \phi} & (K \otimes L) \overline{\otimes} F(X \boxtimes Y) & \xrightarrow{\alpha} & F((K \otimes L) \overline{\otimes} (X \boxtimes Y)) \\
 \downarrow \zeta_{\mathcal{D}} & & & & \downarrow F(\zeta_{\mathcal{C}}) \\
 (K \overline{\otimes} F(X)) \boxtimes (L \overline{\otimes} F(Y)) & \xrightarrow{\alpha \boxtimes \alpha} & F(K \overline{\otimes} X) \boxtimes F(L \overline{\otimes} Y) & \xrightarrow{\phi} & F((K \overline{\otimes} X) \boxtimes (L \overline{\otimes} Y))
 \end{array}$$

We leave the formulation of the analogous other three assertions (for oplax morphisms, and for  $\zeta'$  and  $\beta$ , and hom for lax and oplax morphisms) to the reader, along with the proofs, which are standard exercises in adjunctions.  $\square$

**2.2. Ind-representable functors.** For any category  $\mathcal{C}$ , denote by  $\text{Ind-}\mathcal{C}$  its ind-category, whose objects are diagrams  $I \rightarrow \mathcal{C}$  with  $I$  a small filtering category, and by  $\text{Pro-}\mathcal{C}$  its pro-category, whose objects are diagrams  $J \rightarrow \mathcal{C}$  with  $J$  a small cofiltering category. See App. A for the definition of morphisms, along with possible simplifications on the type of indexing categories we need to allow.

We collect some easy limit and colimit preservation properties and adjoints in the following lemma.

**Lemma 2.12.** *Let  $\mathcal{V}$  be a 2-ring with a set of small generators.*

- (1) *Filtered colimits commute with finite limits in  $\mathcal{V}$ .*
- (2) *Let  $\text{Small}: \mathcal{V} \rightarrow \text{Ind-}\mathcal{V}$  be the functor that sends an object  $V \in \mathcal{V}$  to the ind-object consisting of all small subobjects of  $\mathcal{V}$ . Then there are adjunctions*

$$\text{Small} \vdash (\text{colim}: \text{Ind-}\mathcal{V} \rightarrow \mathcal{V}) \vdash (\mathcal{V} \hookrightarrow \text{Ind-}\mathcal{V})$$

*In particular, the inclusion functor  $\mathcal{V} \rightarrow \text{Ind-}\mathcal{V}$  commutes with limits, the colimit functor  $\text{Ind-}\mathcal{V} \rightarrow \mathcal{V}$  commutes with all limits and colimits, and  $\text{Small}$  commutes with all colimits.*

- (3) *If  $\mathcal{C}$  is a  $\mathcal{V}$ -2-bimodule then the inclusion functor  $\mathcal{C} \rightarrow \text{Pro-}\mathcal{C}$  commutes with finite limits.*

*Proof.* (1) holds because  $\mathcal{V}$  has a set of small generators. Indeed, if  $J$  is a filtered category,  $F$  is a finite category, and  $X: J \times F \rightarrow \mathcal{V}$  is a functor then

$$\mathcal{V}(S, \text{colim}_J \lim_F X) = \text{colim}_J \lim_F \mathcal{V}(S, X) = \lim_F \text{colim}_J \mathcal{V}(S, X) = \mathcal{V}(S, \lim_F \text{colim}_J X)$$

for each small generator  $S$ .

The first adjunction of (2) also uses this fact: let  $X: I \rightarrow \mathcal{V}$  be an element of  $\text{Ind-}\mathcal{V}$  and  $Y \in \mathcal{V}$ . Then

$$\begin{aligned} \mathcal{C}(Y, \text{colim}_i X(i)) &= \mathcal{C}(\text{colim Small}(Y), \text{colim } X(i)) = \lim_{K < Y} \mathcal{C}(K, \text{colim } X(i)) \\ &= \lim_{K \text{ small } K < Y} \text{colim}_i \mathcal{C}(K, X(i)) = \text{Ind-}\mathcal{C}(\text{Small}(Y), X), \end{aligned}$$

where  $K$  runs through all small subobjects of  $Y$ . Finally, (3) is a standard fact for pro-categories [AM69].  $\square$

The following result is proved in Appendix B:

**Theorem 2.13.** *If  $\mathcal{V}$  is a 2-ring then so is  $\text{Ind-}\mathcal{V}$ . If  $\mathcal{C}$  is a 2-bimodule over  $\mathcal{V}$  then  $\text{Pro-}\mathcal{C}$  is a 2-bimodule over  $\text{Ind-}\mathcal{V}$ . If  $\mathcal{C}$  is a 2-algebra over  $\mathcal{V}$  then  $\text{Pro-}\mathcal{C}$  is also a 2-algebra over  $\text{Ind-}\mathcal{V}$ .*  $\square$

**Corollary 2.14.** *Let  $\mathcal{V}$  be a 2-ring with a set of small generators and  $\mathcal{C}$  be a 2-bimodule (2-algebra) over  $\mathcal{V}$ . Then also  $\text{Pro-}\mathcal{C}$  is a 2-bimodule (2-algebra) over  $\mathcal{V}$ .*

*Proof.* The enrichment of  $\text{Pro-}\mathcal{C}$  over  $\text{Ind-}\mathcal{V}$  becomes an enrichment over  $\mathcal{V}$  by passing to the colimit. The left 2-module structure is given by

$$L \bar{\otimes} X = \text{Small}(L) \bar{\otimes} X,$$

where the left hand symbol  $\bar{\otimes}$  is being defined and the right hand symbol is the left 2-module structure from Thm. 2.13. Lemma 2.12(2) shows that this is indeed a left 2-module structure. The right 2-module structure must therefore be defined as

$$\text{hom}(L, X) = \text{hom}(\text{Small}(L), X). \quad \square$$

In our situation (of a 2-ring  $\mathcal{V}$  with a set of small generators), we thus have an enrichment over  $\mathcal{V}$  and one over  $\text{Ind-}\mathcal{V}$ , but we will use the  $\mathcal{V}$ -enrichment much more often. The notation  $\text{Pro-}\mathcal{C}(X, Y)$  will always refer to the  $\mathcal{V}$ -enrichment.



Recall that a  $\mathcal{V}$ -functor  $F: \mathcal{C} \rightarrow \mathcal{V}$  is called *representable* if there is a (necessarily unique) object  $A \in \mathcal{C}$  such that  $F(X) = \text{map}(A, X) \in \mathcal{V}$  for all  $X$ .

**Definition.** Let  $\mathcal{C}$  be a 2-bimodule over  $\mathcal{V}$ . A  $\mathcal{V}$ -functor  $F: \mathcal{C} \rightarrow \mathcal{V}$  is called *ind-represented* by  $A \in \text{Pro-}\mathcal{C}$  if  $F = \mathcal{C}(A, \iota(-))$  for some  $A \in \text{Pro-}\mathcal{C}$ , where  $\iota: \mathcal{C} \hookrightarrow \text{Pro-}\mathcal{C}$  denotes the inclusion as constant pro-objects.

An ind-representable functor is the same as an ordinary representable  $\mathcal{V}$ -functor  $F': \text{Pro-}\mathcal{C} \rightarrow \mathcal{V}$ . Indeed, a representable functor  $F'$  gives rise to an ind-representable functor  $F = F' \circ \iota$ . On the other hand, since any representable functor commutes with all limits and any object  $X: I \rightarrow \mathcal{C}$  in  $\text{Pro-}\mathcal{C}$  is the  $I$ -limit in  $\text{Pro-}\mathcal{C}$  of the diagram  $X: I \rightarrow \mathcal{C} \hookrightarrow \text{Pro-}\mathcal{C}$ ,  $F'$  is uniquely determined by its images on constant pro-objects. In conjunction with the enriched Yoneda Lemma [Kel05], this also shows that the ind-representing object  $A$  of an ind-representable functor  $F$  is uniquely determined by  $F$ .

**Lemma 2.15.** *The category of ind-representable functors  $\mathcal{C} \rightarrow \mathcal{V}$  has all limits and finite colimits and the inclusion into the category of all functors preserves and reflects them.*

*Proof.* Since the category of ind-representable functors  $\mathcal{C} \rightarrow \mathcal{V}$  is equivalent to  $(\text{Pro-}\mathcal{C})^{\text{op}}$  by assigning to an ind-representable functor its ind-representing object, the statement about limits is the universal property of the colimit. The existence of finite (in fact all) colimits in  $(\text{Pro-}\mathcal{C})^{\text{op}}$  follows from the completeness of  $\mathcal{C}$  and [Isa02], and the preservation and reflection is a consequence of the fact that finite limits in  $\text{Pro-}\mathcal{C}$  can be computed levelwise.  $\square$

Infinite colimits in the category of ind-representable functors generally differ from the colimit in the category of all functors.

### 3. FORMAL BIMODULES

Fix a commutative base ring  $K$ . Let  $k$  be a graded commutative  $K$ -algebra. We denote by  $\text{Mod}_k$  the category of graded right  $k$ -modules. This is a 2-bimodule over  $\text{Mod}_K$ , the category of graded  $K$ -modules.

For  $K$ -algebras  $k$  and  $l$ , a  $k$ - $l$ -bimodule is an object in  $\text{Mod}_k$  with a left  $l$ -action on the underlying object in  $\text{Mod}_K$  which commutes with the  $k$ -action. Alternatively, it is a  $k \otimes_K l$ -module. This can also be thought of as a representable  $K$ -linear functor  $M: \text{Mod}_k \rightarrow \text{Mod}_l$ . Namely, given a bimodule  $M$ , the functor given by  $\text{Mod}_k(M, -): \text{Mod}_k \rightarrow \text{Ab}$  obtains a right  $l$ -module structure from the left  $l$ -structure on  $M$ . Conversely, if  $M$  is a  $k$ -module representing a functor into  $l$ -modules, consider the map

$$l \xrightarrow{\text{id} \otimes \eta} l \otimes \text{Mod}_k(M, M) \rightarrow \text{Mod}_k(M, M),$$

where  $\eta: K \rightarrow \text{Mod}_k(M, M)$  maps 1 to the identity map. The adjoint of this map gives a left module structure on  $M$ .

This leads to the following generalizing definition:

**Definition.** Let  $k, l$  be  $K$ -algebras. A *formal  $k$ - $l$ -bimodule* is an ind-representable functor  $F: \text{Mod}_k \rightarrow \text{Mod}_l$ . Denote the category of formal bimodules by  ${}_k\widehat{\text{Mod}}_l$ .

We will now study the structure given by formal bimodules explicitly.

**Lemma 3.1.** *For any formal  $k$ - $l$ -bimodule  $F$ , there is a unique bigraded pro- $k$ -module  $M = \{M(i)\}_{i \in I} = \{M_p^q(i)\}_{i \in I, p, q \in \mathbb{Z}}$  such that*

$$F(X) = \text{Pro-Mod}_k(M, X)$$

*with the structure of a left  $l$ -module, i.e. a  $K$ -algebra map*

$$\mu: l \rightarrow \text{Pro-Mod}_k(M, M).$$

If  $F$  is a formal bimodule, we will denote the associated pro- $k$ -module  $M$  by  $\mathcal{O}_F$ , and conversely we will write  $F = \text{Spf}(M)$ . Here we are borrowing notation from the nonlinear situation of formal schemes discussed in the next section.

Note that there are proper inclusions

$$\{\text{pro-}k\text{-}l\text{-bimodules}\} \subsetneq \{l\text{-module objects in } \text{Pro-Mod}_k\} \subsetneq {}_k\widehat{\text{Mod}}_l$$

An object in all three categories is given by a diagram  $\{M(i)\}_{i \in I}$  of  $k$ -modules, but:

- In bimodules, compatible maps  $l \otimes_K M(i) \rightarrow M(i)$  are required;
- in  $l$ -module objects in  $\text{Pro-Mod}_k$ , certain maps  $l \otimes M(i) \rightarrow M(j)$  are required;
- in  ${}_k\widehat{\text{Mod}}_l$ , maps  $\text{Small}(l) \otimes_K M(i) \rightarrow M(j)$  are required.

We want to spell out the last point in the following description, which is a direct result of Theorem 2.13. Let  $\overline{\otimes}_K$  denote the left 2-module structure of  $\text{Pro-Mod}_k$  over  $\text{Mod}_K$  given by Corollary 2.14, and let  $\text{hom}_K$  denote the right 2-module structure. Note that for  $N: I \rightarrow \text{Mod}_k$ ,  $\text{hom}_K(M, N)$  is *not* the same as the objectwise application  $i \mapsto \text{hom}_K(M, N(i))$ , rather it is  $\text{hom}_K(\text{Small}(M), N)$ .

**Corollary 3.2.** *A formal  $k$ - $l$ -bimodule  $F$  is given by a pro- $k$ -module  $M = \mathcal{O}_F$  together with a map*

$$l \overline{\otimes}_K M \rightarrow M$$

*such that  $M \cong K \overline{\otimes}_K M \rightarrow l \overline{\otimes}_K M \rightarrow M$  is the identity and the two maps*

$$l \otimes_K l \overline{\otimes}_K M \rightrightarrows l \overline{\otimes}_K M \rightarrow M$$

*are equal.*

**Example 3.3.** Let  $M \in \text{Mod}_l$  and  $N \in \text{Pro-Mod}_k$ . Then the cotensor  $\text{hom}(M, N) \in \text{Pro-Mod}_k$  (cf. Cor. 2.14) represents an object in  ${}_k\widehat{\text{Mod}}_l$ . The left  $l$ -action is given, using Corollary 3.2, as the adjoint of the map

$$M \otimes_K l \overline{\otimes}_K \text{hom}_K(M, N) \xrightarrow{M \otimes_K l \rightarrow M} M \overline{\otimes}_K \text{hom}_K(M, N) \rightarrow N.$$

#### 4. FORMAL ALGEBRA AND MODULE SCHEMES

In the previous section, the definition of an algebra/module was lifted from  $\text{Mod}_K$ , but we can make this more general and also lift the additive structure from sets.

Let  $k, l$  be graded commutative rings. Let  $\text{Alg}_k$  be the category of graded commutative unital  $k$ -algebras. We denote by  $\text{Sch}_k$  the category of affine formal schemes over  $k$ , i.e.  $\text{Sch}_k = (\text{Pro-Alg}_k)^{\text{op}}$ . In this section we will study certain functors from  $\text{Alg}_k$  to  $\text{Mod}_l$  and to  $\text{Alg}_l$ , i.e.  $l$ -module and  $l$ -algebra structures on formal schemes.

**Definition.** A formal  $l$ -module scheme over  $k$  is an ind-representable functor

$$F: \text{Alg}_k \rightarrow \text{Mod}_l.$$

Denote by  ${}_k\widehat{\text{Hopf}}_l$  the category of formal module schemes.

This is an  $l$ -enriched version of commutative formal group schemes. A formal  $\mathbf{Z}$ -module scheme is precisely a commutative formal group scheme over  $k$ . This definition of a formal module scheme generalizes the notion of *formal  $A$ -modules* [Haz78, Chapter 21], which is the special case where  $k$  is an  $l$ -algebra.

**Definition.** A formal  $l$ -algebra scheme over  $k$  is an ind-representable functor

$$F: \text{Alg}_k \rightarrow \text{Alg}_l.$$

Denote by  ${}_k\widehat{\text{Alg}}_l$  the category of formal algebra schemes.

This is a pro-version of what is called a  $k$ - $l$ -biring in [TW70, BW05], but that terminology suggests a similarity with bialgebras, which is something completely different, so we will stick to our terminology.

We will now study the structure given by formal module schemes and formal algebra schemes explicitly. Note that there is a forgetful functor  $U: {}_k\widehat{\text{Alg}}_l \rightarrow {}_k\widehat{\text{Hopf}}_l$ , which means that the object representing a formal algebra scheme is equal to the object representing the underlying formal module scheme, but has more structure.

**Lemma 4.1.** *For any formal  $l$ -module scheme  $F$  over  $k$ , there is a unique bigraded pro- $k$ -algebra  $A = \{A(i)\}_{i \in \mathbf{I}} = \{A_p^q(i)\}_{i \in \mathbf{I}, p, q \in \mathbf{Z}}$  such that*

$$F(R) = \text{Pro-Alg}_k(A, R) \quad \text{for any } R \in \text{Alg}_k.$$

This pro-algebra  $A$  comes with the structure of a co- $l$ -module, i.e. with pro- $k$ -algebra maps

$$\begin{aligned} \psi_+ : A_p^q &\rightarrow A_p^{q'} \otimes_k A_p^{q''} \quad (\text{coaddition}) \\ \epsilon_0 : A_p^q &\rightarrow k_p \quad (\text{cozero}) \end{aligned}$$

as well as an additive and multiplicative map

$$\lambda : l \rightarrow \text{Pro-Alg}_k(A_*^*, A_*^*) \quad (l\text{-module structure}).$$

These maps are such that  $\epsilon_0$  is the counit for  $\psi_+$ ,  $\lambda_0 = \eta \circ \epsilon_0$ ,  $\lambda_{-1}$  is the antipode for  $\psi_+$ , and such that  $\psi_+$  is associative and (graded) commutative. Furthermore,  $\lambda$  takes values in the sub-graded set of pro-algebra maps that commute with  $\psi_+$  and  $\epsilon_0$ .

A formal  $l$ -algebra scheme  $F$  consists of the same data and in addition an associative, (graded) commutative pro- $k$ -algebra map

$$\psi_\times : A_p^q \rightarrow A_{p'}^{q'} \otimes_k A_{p''}^{q''} \quad (\text{comultiplication})$$

as well as an additive and multiplicative map extending  $\epsilon_0$ :

$$\epsilon : l \rightarrow \text{Pro-Alg}_k(A_*^*, k) \quad (\text{unit}).$$

such that  $\lambda_a$  is comultiplication with  $\epsilon_a$ ,  $\epsilon_1$  the counit for  $\psi_\times$ , and such that  $\psi_\times$  distributes over  $\psi_+$ .

As before for modules, we write  $A = \mathcal{O}_F$  and  $F = \mathrm{Spf}(A)$  if  $F$  is ind-represented by  $A$ , thinking of  $A$  as the ring of functions on the formal scheme  $F$  and of  $F$  as the formal spectrum of the pro-algebra  $A$ .

Since the constant functor  $F(R) = l$  is not ind-representable, we cannot phrase the  $l$ -module and unit data as a map on representing objects. However, just as in Corollary 3.2, it follows from Cor.2.14 that we can describe  $\lambda$  and  $\epsilon$  in adjoint form as maps  $l \overline{\otimes} A \rightarrow A$  and  $l \overline{\otimes} A \rightarrow k$ , respectively, satisfying additional properties.

Note that  $k \xrightarrow{\eta} A \xrightarrow{\epsilon_\lambda} k$  is a  $k$ -algebra map, hence the identity, for all  $\lambda \in l$ . Denote by  $A_+$  the kernel of  $\epsilon_0$ . Since the short exact sequence of pro- $k$ -modules

$$0 \rightarrow A_+ \rightarrow A \xrightarrow{\epsilon_0} k \rightarrow 0$$

splits, we can and will identify  $A_+$  with the cokernel of  $k \rightarrow A$ .

The terminal example of an  $l$ -algebra or module scheme  $\mathcal{O}$  over  $k$  is the trivial  $l$ -algebra (or module)  $\mathcal{O}(A) = 0$ . Here  $\mathcal{O}_{\mathcal{O}} = k$ ,  $\psi_+$  and  $\psi_\times$  are the identity, and  $\epsilon_\lambda = \mathrm{id}_k$  for all  $\lambda \in l$ .

Although the initial  $l$ -algebra is clearly  $l$  itself, it is not obvious what the initial  $l$ -algebra scheme might be, if it exists. The following construction gives the somewhat surprising answer.

**Definition.** Let  $M \in \mathrm{Mod}_l$  be an  $l$ -module. Then

$$\mathrm{hom}(M, k) \in \mathrm{Pro}\text{-}\mathrm{Alg}_k$$

represents a formal  $l$ -module scheme over  $k$ . (In this formula, the  $\mathrm{hom}$  is the right module structure of  $\mathrm{Pro}\text{-}\mathrm{Alg}_k$  over  $\mathrm{Set}$ .) Its structure is given as follows: if  $+: M(i) \times M(i) \rightarrow M(j)$  is a component of the addition map on  $M$ , it gives rise to a coaddition

$$\psi_+: \mathrm{hom}(M(j), k) \rightarrow \mathrm{hom}(M(i) \times M(i), k) \rightarrow \mathrm{hom}(M(i), k) \otimes_k \mathrm{hom}(M(i), k).$$

If  $0 \in M(i)$ , the cozero is given by

$$\epsilon_0: \mathrm{hom}(M(i), k) \rightarrow \mathrm{hom}(\{0\}, k) = k.$$

The  $l$ -module structure  $l \overline{\otimes} \mathrm{hom}(M, k) \rightarrow \mathrm{hom}(M, k)$  is the adjoint of

$$(l \times M) \overline{\otimes} \mathrm{hom}(M, k) \xrightarrow{\mu \otimes \mathrm{id}} M \overline{\otimes} \mathrm{hom}(M, k) \xrightarrow{\mathrm{eval}} k.$$

Note that this last evaluation map is adjoint to a map of  $l$ -modules

$$e_k: M \rightarrow \mathrm{Pro}\text{-}\mathrm{Alg}_k(\mathrm{hom}(M, k), k)$$

which can be extended to a map

$$(4.2) \quad e_A: M \rightarrow \mathrm{Spf}(\mathrm{hom}(M, k))(A) = \mathrm{Pro}\text{-}\mathrm{Alg}_k(\mathrm{hom}(M, k), A)$$

by the unique  $k$ -algebra map  $k \rightarrow A$ .

**Lemma 4.3.** *Let  $M$  be an  $l$ -module. Then the formal module scheme  $\mathrm{Spf}(\mathrm{hom}(M, k))$  is the best possible representable approximation to the constant functor  $\underline{M}$  with value  $M$ . More precisely, let  $\mathfrak{F}$  denote the category of all functors  $\mathrm{Alg}_k \rightarrow \mathrm{Mod}_l$ . Then for any  $F \in {}_k\widehat{\mathrm{Hopf}}_l$ , the map  $e$  induces an isomorphism*

$${}_k\widehat{\mathrm{Hopf}}_l(\mathrm{Spf}(\mathrm{hom}(M, k)), F) \xrightarrow{e^*} \mathfrak{F}(\underline{M}, F).$$

*Proof.* Let  $f: M \rightarrow F(k) = \text{Pro-Alg}_k(\mathcal{O}_F, k)$  be the evaluation at the pro- $k$ -algebra  $k$  of a natural transformation in  $\mathfrak{F}(\underline{M}, F)$ . This map is adjoint to a map of pro-algebras  $M \otimes \mathcal{O}_F \rightarrow k$ , which in turn is adjoint to a map of pro-algebras  $\mathcal{O}_F \rightarrow \text{hom}(M, k)$ . Since the original map was a map of  $l$ -modules, this map represents a map of  $l$ -module schemes.  $\square$

**Example 4.4** (The initial formal algebra scheme). The functor  $\text{Spf}(\text{hom}(l, k))$  is a formal  $l$ -algebra scheme over  $k$ . We have already seen that it is a formal module scheme, and the comultiplication occurs in the same way. The unit is given by the evaluation map  $l \otimes \text{hom}(l, k) \rightarrow k$ .

**Lemma 4.5.** *The formal algebra scheme  $\text{Spf}(\text{hom}(l, k))$  is the initial object in  ${}_k\widehat{\text{Alg}}_l$ .*

*Proof.* The adjoint of the unit map  $l \otimes \mathcal{O}_F \rightarrow k$  for a formal algebra scheme  $F$  according to Lemma 2.14 gives the unique map  $\mathcal{O}_F \rightarrow \text{hom}(l, k)$ .  $\square$

The functor  $\text{Spf}(\text{hom}(l, k))$  can be described explicitly. It assigns to a  $k$ -algebra  $A$  the set of all  $l$ -tuples of complete idempotent orthogonal elements of  $A$ , i.e. tuples  $(a_i)_{i \in l}$  with  $a_i = 0$  for almost all  $i \in l$ ,  $\sum_i a_i = 1$ ,  $a_i a_j = 0$  if  $i \neq j$  and  $a_i^2 = a_i$ . The addition and multiplication are defined by

$$((a_i)_{i \in l} + (b_j)_{j \in l})_m = \sum_{i+j=m} a_i b_j, \quad ((a_i)_{i \in l} (b_j)_{j \in l})_m = \sum_{ij=m} a_i b_j.$$

The reader can check that this indeed defines a complete set of idempotent orthogonal elements if  $(a_i)$  and  $(b_j)$  are so.

If the algebra  $A$  has no zero divisors, i.e. no nontrivial orthogonal elements, then an  $l$ -tuple of elements as above has to be of the form  $\delta_i$ , where  $(\delta_i)_j = \delta_{ij}$  ( $i, j \in l$ ). Thus in this case,  $\text{Spf}(\text{hom}(l, k))(A) = l$ , independently of  $A$ . Thus for these  $A$ , the map  $e_A$  of (4.2) is an isomorphism.

**Example 4.6** (The identity functor). The identity functor  $\text{id}: \text{Alg}_k \rightarrow \text{Alg}_k$  is represented by the bigraded  $k$ -algebra

$$(\mathcal{O}_{\text{id}})_p = k[e_p]$$

Here  $e_p$  has bidegree  $(p, p)$ , and  $k[e_p]$  denotes the free graded commutative algebra on  $e_p$ , i.e. polynomial if  $p$  is even and exterior if  $p$  is odd. The coaddition is given by  $\psi_+(e_p) = e_p \otimes 1 + 1 \otimes e_p$ , and the comultiplication  $(\mathcal{O}_{\text{id}})_p^q \rightarrow (\mathcal{O}_{\text{id}})_{p'}^{q'} \otimes (\mathcal{O}_{\text{id}})_{p''}^{q''}$  is given by  $\psi_\times(e_p) = e_{p'} \otimes e_{p''}$ . The unit map is the canonical isomorphism  $k \rightleftarrows \text{Alg}_k(k[e_0], k)$ .

**Example 4.7** (The completion-at-zero functor). The functor  $F(X) = \text{Nil}(X)$  is a non-unital  $k$ -algebra scheme over  $k$  represented by the pro- $k$ -algebra scheme

$$(\mathcal{O}_{\text{Nil}})_p = k[[e_p]] =_{\text{def}} k[e_p]/(e_p^n)_n$$

with the structure induced by the canonical map  $\mathcal{O}_{\text{id}} \rightarrow \mathcal{O}_{\text{Nil}}$ . Obviously this formal  $k$ -module scheme cannot have a unital multiplication since the unit element in a ring is never nilpotent. In an algebro-geometric picture, a nontrivial (formal) ring scheme always needs at least two geometric points, 0 and 1, whereas the kind of formal groups that appear in topology as cohomology rings of connected spaces only have one geometric point, given by the augmentation ideal.

**Example 4.8** (The formal completion functor). The lack of a unit of the previous example can be remedied in much the same way as for ordinary algebras, namely, by taking a direct product with a copy of the base ring. We define the functor

$$F = \mathrm{Spf}(\mathrm{hom}(k, k)) \times \mathrm{Nil}: \mathrm{Alg}_k \rightarrow \mathrm{Alg}_k$$

Obviously, this functor is represented by  $\mathrm{hom}(k, k) \otimes_k k[[e_p]]$ . The addition is defined componentwise, whereas the multiplication is defined as follows. Since  $\mathrm{Nil}$  is a formal  $k$ -module scheme, it has a ring map

$$\mu: \mathrm{hom}(k, k) \xrightarrow{e_0} k \rightarrow \mathrm{Pro}\text{-}\mathrm{Alg}_k(\mathrm{Nil}, \mathrm{Nil}).$$

Using this module structure of  $\mathrm{Nil}$  over  $\mathrm{Ind}(\mathrm{hom}(k, k), -)$ , we define the multiplication by

$$(\lambda_1, x_1)(\lambda_2, x_2) = (\lambda_1 \lambda_2, \mu(\lambda_1, x_2) + \mu(\lambda_2, x_1) + x_1 x_2).$$

The unit for the functor  $F$  is given by

$$(\epsilon, \epsilon_0): k \rightarrow \mathrm{Pro}\text{-}\mathrm{Alg}_k(\mathrm{hom}(k, k), k) \times \mathrm{Pro}\text{-}\mathrm{Alg}_k(k[[e_p]], k).$$

**Example 4.9** (The divided power algebra). Another example of a non-unital  $\mathbf{Z}$ -algebra scheme over  $k$  is given by the divided power algebra. Let  $H = \bigoplus_{i=0}^{\infty} k\langle x_i \rangle$  denote the divided polynomial algebra, i.e. the Hopf algebra with

$$\begin{aligned} x_i x_j &= \binom{i+j}{i} x_{i+j} \\ \psi_+(x_n) &= \sum_{i+j=n} x_i \otimes x_j \end{aligned}$$

Let  $H(n)$  denote the quotient algebra  $H/(x_{n+1}, x_{n+2}, \dots)$ . Then  $\Gamma = \{H(n)\}_{n \geq 0}$  represents a formal  $\mathbf{Z}$ -module scheme over  $k$ . For a generalized construction along these lines, see Section 6. This can be given the structure of a *non-unital* algebra scheme by defining  $\psi_{\times}(x_n) = n!(x_n \otimes x_n)$ .

**Example 4.10** (The  $\Lambda$ -algebra). Let  $\Lambda = k[[c_1, c_2, \dots]]$  be the power series pro-algebra, where we think of  $c_n$  as the  $n$ th symmetric polynomial in  $x_1, x_2, \dots$  (with  $c_0 = 1$ ). We then define a formal ring scheme structure on  $\mathrm{Spf}(\Lambda)$  by

$$\prod_{i,j=1}^{\infty} (1 + t(x_i \otimes 1))(1 + t(1 \otimes x_j)) = \sum_{n=0}^{\infty} \psi_+(c_n) t^n \in (\Lambda \otimes_k \Lambda)[[t]]$$

and

$$\prod_{i,j=1}^{\infty} (1 + t(x_i \otimes x_j)) = \sum_{n=0}^{\infty} \psi_{\times}(c_n) t^n \in (\Lambda \otimes_k \Lambda)[[t]]$$

For  $a \in \mathbf{Z}$ , the unit is given by

$$(1 + t)^a = \sum_{n=0}^{\infty} \epsilon_a(c_n) t^n, \quad \text{or,} \quad \epsilon_a(n) = \binom{a}{n}.$$

The polynomial version of this construction represents the functor which associates to a ring its ring of big Witt vectors [Haz78, Chapter 17.2]. From a topological point of view, this formal ring scheme is isomorphic with  $\mathrm{Spf}(K^0(BU))$ , with the addition and multiplication induced by the maps  $BU \times BU \rightarrow BU$  classifying direct sums resp. tensor products of vector bundles.

For the sake of concreteness, the coaddition in  $\Lambda$  is easily described:

$$\psi_+(c_n) = \sum_{i+j=n} c_i \otimes c_j,$$

whereas the comultiplication does not have a handy closed formula:

$$\begin{aligned} \psi_\times(c_1) &= c_1 \otimes c_1 \\ \psi_\times(c_2) &= c_1^2 \otimes c_2 + c_2 \otimes c_1^2 - 2c_2 \otimes c_2 \\ \psi_\times(c_3) &= c_1^3 \otimes c_3 + c_3 \otimes c_1^3 - 3c_3 \otimes c_1 c_2 - 3c_1 c_2 \otimes c_3 + c_1 c_2 \otimes c_1 c_2 \quad \text{etc.} \end{aligned}$$

**Example 4.11** (Operations of cohomology theories). Let  $R, S$  be homotopy commutative ring spectra. Denote by  $\underline{S}_n$  the  $n$ th space in the  $\Omega$ -spectrum associated to  $S$ . Assume that  $\underline{S}_n$  is a filtered colimit of CW-complexes  $S_{n,i}$  such that  $R^*(S_{n,i})$  is a finitely generated free  $R_*$ -module for all  $i$  and  $n$ . Let  $F_n = \text{Spf}((R^*(S_{n,i}))_i)$ . This is a formal scheme over  $R_*$ , and it acquires the structure of a formal  $S_*$ -algebra scheme by means of the maps

$$\underline{S}_n \times \underline{S}_n \rightarrow \underline{S}_n \quad (\text{loop structure})$$

and

$$\underline{S}_i \times \underline{S}_j \rightarrow \underline{S}_{i+j} \quad (\text{multiplication}).$$

The condition on the  $S_n$  is necessary to ensure that these maps translate to coalgebra structures by a Künneth isomorphism. The unit map  $\epsilon: S_* \rightarrow \text{Pro-Alg}_{R_*}(R^* \underline{S}_*, R_*)$  is induced by application of  $R^*$  to an element of  $S_* = \pi_* S = [\mathbf{S}^0, \underline{S}_n]$ .

**Remark 4.12** (The forgetful functor). An object  $F \in {}_k \widehat{\text{Alg}}_l$  is represented by a pro- $k$ -algebra  $\mathcal{O}_F$  with additional structure, which in particular equips  $\mathcal{O}_F$  with a comultiplication

$$(4.13) \quad l \xrightarrow{\epsilon} \text{Pro-Alg}_k(\mathcal{O}_F, k) \xrightarrow{\text{unit}} \text{Pro-Alg}_k(\mathcal{O}_F, \mathcal{O}_F) \xrightarrow{\text{forget}} \text{Pro-Mod}_k(\mathcal{O}_F, \mathcal{O}_F)$$

At first glance one might think that this equips us with a forgetful functor  ${}_k \widehat{\text{Alg}}_l \rightarrow {}_k \widehat{\text{Mod}}_l$ . This is not true since (4.13) is not a map of  $K$ -modules (or even abelian groups) in general. There are interesting functors from algebra schemes to bimodules (defined in Section 9), but the forgetful functor is not one of them.

However, a functor  $\text{Alg}_k \rightarrow \text{Alg}_l$  can of course be composed with the forgetful functor to  $\text{Mod}_l$ , giving a functor from algebra schemes to module schemes.

## 5. THE STRUCTURE OF THE CATEGORY OF FORMAL BIMODULES

In this section we will study the algebraic structure of  ${}_k \widehat{\text{Mod}}_l$ , the category of formal bimodules, in more detail. The main points are that this category is a 2-bimodule over  $\text{Mod}_l$  (Lemma 5.1) and the existence of an approximation to the objectwise tensor product of modules over  $l$ , making it into a 2-algebra (Thm. 5.5).

Recall from Cor. 2.14 that  $\text{Pro-Mod}_k$  is a 2-bimodule over  $\text{Mod}_k$ .

**Lemma 5.1.** *The category  ${}_k \widehat{\text{Mod}}_l$  is a 2-bimodule over  $\text{Mod}_l$ .*

We will denote the left 2-module structure by  $\overline{\otimes}_l$  and the right 2-module structure by  $\text{hom}_l$ .

*Proof.* Let  $F, G: \text{Mod}_k \rightarrow \text{Mod}_l$  be objects of  ${}_k\widehat{\text{Mod}}_l$ . Then the enrichment is given by the  $l$ -module of  $l$ -linear natural transformation  $G \rightarrow F$ , i.e. by the equalizer

$${}_k\widehat{\text{Mod}}_l(G, F) \rightarrow \text{Pro-Mod}_k(\mathcal{O}_F, \mathcal{O}_G) \rightrightarrows \text{Pro-Mod}_k(l \overline{\otimes}_K \mathcal{O}_F, \mathcal{O}_G),$$

where the two maps are given by the map  $l \overline{\otimes}_K \mathcal{O}_F \rightarrow \mathcal{O}_F$  of Cor. 3.2 and by

$$\text{Pro-Mod}_k(\mathcal{O}_F, \mathcal{O}_G) \rightarrow \text{Pro-Mod}_k(l \overline{\otimes}_K \mathcal{O}_F, l \overline{\otimes}_K \mathcal{O}_G) \rightarrow \text{Pro-Mod}_k(l \overline{\otimes}_K \mathcal{O}_F, \mathcal{O}_G).$$

By the commutativity of  $l$ , this is again an  $l$ -module.

The right 2-module structure  $\text{hom}_l(M, F)$  is given objectwise:  $\text{hom}_l(M, F)(R) = \text{Mod}_l(M, F(R))$ . To see that this is representable, we can write  $\text{hom}_l(M, F) = \text{Spf}(M \odot_l \mathcal{O}_F)$ , where  $M \odot_l \mathcal{O}_F$  is the coequalizer

$$M \otimes_K l \overline{\otimes}_K \mathcal{O}_F \rightrightarrows M \overline{\otimes}_K \mathcal{O}_F \rightarrow M \odot_l \mathcal{O}_F.$$

Here the first map is the adjoint  $\mu^\#: l \overline{\otimes}_K \mathcal{O}_F \rightarrow \mathcal{O}_F$  of the left module structure map  $\mu: l \rightarrow \text{Pro-Mod}_k(\mathcal{O}_F, \mathcal{O}_F)$  (Cor. 3.2) and the second map is induced by the right action  $M \otimes_K l \rightarrow M$ .

The left 2-module structure  $M \overline{\otimes}_l F$  is given by  $M \overline{\otimes}_l F = \text{Spf}(\text{hom}_l(M, \mathcal{O}_F))$ , which is represented by the equalizer

$$\text{hom}_l(M, \mathcal{O}_F) \rightarrow \text{hom}(M, \mathcal{O}_F) \rightrightarrows \text{hom}(M \otimes_K l, \mathcal{O}_F). \quad \square$$

The left 2-module structure approximates the functor  $N \mapsto M \otimes_l F(N)$  in the following sense. Clearly  $N \mapsto M \otimes_l F(N)$  is not representable in general because it is not right exact if  $M$  is not flat. Denote by  $\mathfrak{F}$  the category of all additive functors  $\text{Mod}_k \rightarrow \text{Mod}_l$ , representable or not.

**Lemma 5.2.** *For  $F, G \in {}_k\widehat{\text{Mod}}_l$  and  $M \in \text{Mod}_l$ , there is a natural adjunction isomorphism*

$$\mathfrak{F}(M \otimes_l F(-), G) \cong {}_k\widehat{\text{Mod}}_l(M \overline{\otimes}_l F, G)$$

*Proof.*

$$\begin{aligned} \mathfrak{F}(M \otimes_l F(-), G) &\cong \mathfrak{F}(F, \text{Mod}_l(M, G(-))) \\ &= {}_k\widehat{\text{Mod}}_l(F, \text{hom}_l(M, G)) \cong {}_k\widehat{\text{Mod}}_l(M \overline{\otimes}_l F, G). \quad \square \end{aligned}$$

For formal bimodules, we will consider a tensor product which is a variant of the Sweedler product [Swe74]:

**Definition.** Let  $F, G \in {}_k\widehat{\text{Mod}}_l$ . Define their *tensor product*  $F \otimes_l G$  as

$$F \otimes_l G = \text{Spf}(\mathcal{O}_F \overset{l}{\otimes}_k \mathcal{O}_G),$$

where  $\overset{l}{\otimes}_k$ , the *Sweedler product*, is the following equalizer in  $\text{Pro-Mod}_k$ :

$$(5.3) \quad \mathcal{O}_F \overset{l}{\otimes}_k \mathcal{O}_G \rightarrow \mathcal{O}_F \otimes_k \mathcal{O}_G \rightrightarrows \text{hom}_K(l, \mathcal{O}_F \otimes_k \mathcal{O}_G).$$

The maps are given by

$$\mathcal{O}_F \otimes_k \mathcal{O}_G \rightarrow \text{hom}(l, \mathcal{O}_F) \otimes_k \mathcal{O}_G \xrightarrow{\beta} \text{hom}(l, \mathcal{O}_F \otimes_k \mathcal{O}_G)$$

(and similarly for  $\mathcal{O}_G$ ), where the first map is the adjoint of the structure map  $l \rightarrow \text{Pro-Mod}_k(\mathcal{O}_F, \mathcal{O}_F)$ , and the second map is the natural transformation (2.6).

The  $l$ -action on the pro- $k$ -module is given by either of the composites of (5.3), which actually factors through  $\mathcal{O}_F \overset{l}{\otimes}_k \mathcal{O}_G \rightarrow \text{hom}(l, \mathcal{O}_F \overset{l}{\otimes}_k \mathcal{O}_G)$  because the two possible composites into  $\text{hom}(l, \text{hom}(l, \mathcal{O}_F \otimes_k \mathcal{O}_G))$  coincide.



One should think about the Sweedler product as the submodule of  $\mathcal{O}_F \otimes_k \mathcal{O}_G$  where the  $l$ -actions on the left and the right factor agree.

As a corollary of Lemma 5.2 we get the following result:

**Corollary 5.4.** *Let  $F, G \in {}_k\widehat{\text{Mod}}_l$ . Let  $P \in {}_k\widehat{\text{Mod}}_{l \otimes l}$  be the functor  $P = \text{Spf}(\mathcal{O}_F \otimes_k \mathcal{O}_G)$ . Then  $F \otimes_k G$  “tries to be” the functor  $\tilde{P}$  given by  $\tilde{P}(M) = \text{coeq}(l \otimes P(M) \rightrightarrows P(M))$ , where the two maps are given by the two  $l$ -actions on  $P(M)$ . That is, for  $H \in {}_k\widehat{\text{Mod}}_l$ , there is an adjunction isomorphism*

$$\mathfrak{F}(\tilde{P}, H) \cong {}_k\widehat{\text{Mod}}_l(F \otimes_l G, H).$$

The tensor product equips  ${}_k\widehat{\text{Mod}}_l$  with a symmetric monoidal structure (with unit  $\text{hom}_K(l, k)$ ). This symmetric structure is compatible with the enrichment:

**Theorem 5.5.** *The symmetric monoidal category  $({}_k\widehat{\text{Mod}}_l, \otimes_l, \text{hom}_K(l, k))$  is a symmetric 2-algebra over  $(\text{Mod}_l, \otimes_l, l)$ .*

*Proof.* We have already seen that  ${}_k\widehat{\text{Mod}}_l$  is a 2-module over  $\text{Mod}_l$  and that  $\otimes_l$  is a symmetric monoidal structure. It remains to show that the functor  $F \mapsto F \otimes_l G$  is enriched over  $\text{Mod}_l$ , i.e. that the  $\mathbf{Z}$ -linear map

$${}_k\widehat{\text{Mod}}_l(F, F') \rightarrow {}_k\widehat{\text{Mod}}_l(F \otimes_l G, F' \otimes_l G)$$

is in fact  $l$ -linear. But this is true by the construction of the  $l$ -action on  $F \otimes_l G$  and  $F' \otimes_l G$ , where we are free to choose the map induced by the  $l$ -action on  $F$  and  $F'$ .  $\square$

## 6. THE STRUCTURE OF THE CATEGORY OF FORMAL MODULE SCHEMES

In this section we will study enrichments and free/cofree adjunctions for formal module schemes. Only in this section,  $l$  is a *not necessarily commutative* ring.

**Lemma 6.1.** *The forgetful functor  $V: {}_k\widehat{\text{Hopf}}_{\mathbf{Z}} \rightarrow \text{Sch}_k$  has a left adjoint  $\text{Fr}$ .*

*Proof.* Let  $F \in \text{Sch}_k$ . We will construct the left adjoint  $\text{Fr}$  as  $\text{Fr} = \text{Spf}(\Gamma(\mathcal{O}_F))$  for a certain functor  $\Gamma$ . This should be the cofree cocommutative cogroup object in pro-algebras. Since as a right adjoint,  $\Gamma$  has to commute with limits, it suffices to construct  $\Gamma(A)$  for a constant pro-algebra  $A \in \text{Alg}_k$ . In the setting of algebras instead of pro-algebras, an explicit description of the cofree bialgebra on an algebra is quite hard (see [Swe69, Chapter VI], [PT80] for fields, [Tak82] for characteristic  $p$ , [Fox93] for the general noncocommutative case) unless one restricts to *irreducible* coalgebras [NR79], and the antipode is yet another problem [Ago09]. The situation for pro-algebras is actually much simpler and resembles the irreducible case.

Note that the inclusion  $\text{Alg}_k \rightarrow \text{Pro-Mod}_k$  does *not* send the cofree construction to the cofree construction!

For a  $k$ -module  $M$ , denote by  $\Gamma_{\mathbf{N}}(M)$  the coalgebra in pro- $k$ -modules

$$\Gamma_{\mathbf{N}}(M) = \prod_{n=0}^{\infty} \Gamma_{\mathbf{N}}^n(M) = \prod_{n=0}^{\infty} (M^{\otimes_k n})^{\Sigma_n},$$

where the product is taken in the category  $\text{Pro-Mod}_k$  and the symmetric group  $\Sigma_n$  acts by signed permutation of the tensor factors. Since

$$\Gamma_{\mathbf{N}}(M) \otimes_k \Gamma_{\mathbf{N}}(M) \cong \prod_{m,n=0}^{\infty} (M^{\otimes_k m})^{\Sigma_m} \otimes_k (M^{\otimes_k n})^{\Sigma_n} = \prod_{m,n=0}^{\infty} (M^{\otimes_k m+n})^{\Sigma_m \times \Sigma_n},$$

the canonical restriction maps

$$\Gamma_{\mathbf{N}}^{m+n}(M) = (M^{\otimes_k m+n})^{\Sigma_{m+n}} \rightarrow (M^{\otimes_k m+n})^{\Sigma_m \times \Sigma_n} = \Gamma_{\mathbf{N}}^m(M) \otimes_k \Gamma_{\mathbf{N}}^n(M)$$

induce a cocommutative comultiplication  $\Gamma_{\mathbf{N}}(M) \rightarrow \Gamma_{\mathbf{N}}(M) \otimes_k \Gamma_{\mathbf{N}}(M)$ . Furthermore, the projection  $\varepsilon_0: \Gamma_{\mathbf{N}}(M) \rightarrow \Gamma_{\mathbf{N}}^0(M) = k$  is a cozero. There is also natural transformation  $\pi: \Gamma_{\mathbf{N}}(M) \rightarrow M$  given by projection to the  $\Gamma_{\mathbf{N}}^1$ -factor. Note that due to the failure of the tensor product to commute with infinite products,  $\lim \Gamma_{\mathbf{N}}(M)$  is not a coalgebra.

To see that this is the cofree cocommutative coalgebra, let  $C$  be a pro- $k$ -module with a cocommutative comultiplication  $\psi: C \rightarrow C \otimes C$ . Given a coalgebra map  $C \rightarrow \Gamma_{\mathbf{N}}(M)$ , composing with  $\pi$  gives a  $k$ -module map  $C \rightarrow M$ . Conversely, given a  $k$ -module map  $f: C \rightarrow M$ , define a map  $\tilde{f} = (\tilde{f})_n: C \rightarrow \Gamma_{\mathbf{N}}(M) = \prod_n \Gamma_{\mathbf{N}}^n(M)$  by  $(\tilde{f})_n = (f \otimes \cdots \otimes f) \circ \psi^n$ , where  $\psi^n: C \rightarrow (C^{\otimes_k n})^{\Sigma_n}$  denotes the  $(n-1)$ -fold comultiplication of  $C$ . It is clear that this gives an adjunction isomorphism.

If  $A$  is an algebra in  $\text{Alg}_k$  then there is a map

$$\mu': \Gamma_{\mathbf{N}}(A) \otimes_k \Gamma_{\mathbf{N}}(A) \rightarrow A; \quad \mu'(x \otimes y) = \pi(x)\pi(y)$$

which, by the universal property of  $\Gamma_{\mathbf{N}}(A)$  as the cofree coalgebra, lifts to a unique coalgebra map

$$\mu: \Gamma_{\mathbf{N}}(A) \otimes_k \Gamma_{\mathbf{N}}(A) \rightarrow \Gamma_{\mathbf{N}}(A).$$

This makes  $\Gamma_{\mathbf{N}}(A)$  a bialgebra (or, more precisely, a formal commutative monoid scheme) and  $\pi$  a  $k$ -algebra map. It is easy to see that it is the cofree object on the algebra  $A$ .

This cofree bialgebra does in general not have an antipode. Note that the category of Hopf algebras is a full subcategory of the category of bialgebras: an antipode, if it exists, is unique, and any bialgebra map between Hopf algebras automatically respects the antipode (much like a monoid map between groups is a group map).

It is straightforward that the union of any increasing chain of Hopf subalgebras  $H_i$  of a bialgebra  $A$  is again a Hopf subalgebra of  $A$ . Also, given two Hopf subalgebras  $H_1, H_2$  of a bialgebra  $A$ , let  $H$  be the image of  $H_1 \otimes_k H_2$  under the map  $H_1 \otimes_k H_2 \rightarrow A \otimes_k A \xrightarrow{\mu} A$ . Then  $H$  is a Hopf subalgebra of  $A$  which contains  $H_1$  and  $H_2$  as Hopf subalgebras. Therefore, the union of all Hopf subalgebras of  $\Gamma_{\mathbf{N}}(A)$  is itself a Hopf subalgebra  $\Gamma(A) \subseteq \Gamma_{\mathbf{N}}(A)$ .

Any map of bialgebras  $f: H \rightarrow \Gamma_{\mathbf{N}}(A)$  with  $H$  a Hopf algebra has its image in  $\Gamma(A)$  since otherwise the image of  $f$  in  $\Gamma_{\mathbf{N}}(A)$  would be a Hopf subalgebra of  $A$  not contained in  $\Gamma(A)$ . Therefore,  $\text{Fr} = \text{Spf}(\Gamma(\mathcal{O}_{(-)}))$  is the left adjoint of  $V$ .  $\square$

Denote by  $\mathfrak{F}$  the category of all functors  $\text{Alg}_k \rightarrow \text{Mod}_{\mathbf{Z}}$ , representable or not. Let  $\text{Fr}: \text{Set} \rightarrow \text{Mod}_{\mathbf{Z}}$  also denote the free  $\mathbf{Z}$ -module functor. The following lemma says that  $\text{Fr}(F)$  is the best representable approximation to the functor  $R \mapsto \text{Fr}(F(R))$  for a formal scheme  $F \in \text{Sch}_k$ .

**Corollary 6.2.** *For  $F \in \text{Sch}_k$ ,  $G \in {}_k\widehat{\text{Hopf}}_{\mathbf{Z}}$ , there is a natural adjunction isomorphism*

$$\mathfrak{F}(\text{Fr} \circ F, G) \cong {}_k\widehat{\text{Hopf}}_{\mathbf{Z}}(\text{Fr}(F), G).$$

*Proof.*  $\mathfrak{F}(\text{Fr} \circ F, G) \cong \text{Sch}_k(F, G) \underset{\text{Lemma 6.1}}{\cong} {}_k\widehat{\text{Hopf}}_{\mathbf{Z}}(\text{Fr}(F), G).$   $\square$

**Corollary 6.3.**  $\text{Fr}(\text{Spf}(k)) = \text{Spf}(\text{hom}(\mathbf{Z}, k))$ .

*Proof.* Both functors are characterized as a best approximations to the constant functor  $\mathbf{Z}$ , hence equal. More precisely, note that  $\text{Spf}(k)$  is the constant functor with value the singleton, thus for any  $G \in {}_k\widehat{\text{Hopf}}_{\mathbf{Z}}$  we have

$${}_k\widehat{\text{Hopf}}_{\mathbf{Z}}(\text{Fr}(\text{Spf}(k)), G) \underset{\text{Cor. 6.2}}{\cong} \mathfrak{F}(\mathbf{Z}, G) \underset{\text{Lemma 4.3}}{\cong} {}_k\widehat{\text{Hopf}}_{\mathbf{Z}}(\text{Spf}(\text{hom}(\mathbf{Z}, k)), G).$$

□

**Lemma 6.4.** *The forgetful functor  $V: {}_k\widehat{\text{Hopf}}_l \rightarrow \text{Sch}_k$  creates  $\text{Sch}_k$ -split coequalizers.*

*This means: Let  $f, g: F \rightarrow G \in {}_k\widehat{\text{Hopf}}_l$  be two maps of formal module schemes with coequalizer  $\epsilon: V(G) \rightarrow Q$  in  $\text{Sch}_k$ . Given right inverses  $s: V(G) \rightarrow V(F)$  of  $f, g$  in  $\text{Sch}_k$  as well as  $t: Q \rightarrow V(G)$  of  $\epsilon$ , then  $Q$  carries the structure of a formal  $l$ -module scheme and is, with this structure, the coequalizer in  ${}_k\widehat{\text{Hopf}}_l$ .*

*Proof.* Since  $\mathcal{O}_Q \rightarrow \mathcal{O}_G \rightrightarrows \mathcal{O}_F$  is a split equalizer in  $\text{Pro} - \text{Alg}_k$ , it is in particular an absolute coequalizer, and tensoring with any other pro- $k$ -algebra will preserve this property. Thus the comultiplication on  $l$  takes values in

$$\ker(\mathcal{O}_{G \times G} \rightarrow \mathcal{O}_{F \times F}) = \mathcal{O}_{Q \times G} \times_{\mathcal{O}_{G \times G}} \mathcal{O}_{G \times Q} = \mathcal{O}_Q \otimes_k \mathcal{O}_Q,$$

where the last identity holds again because  $F \rightarrow Q$  splits. □

As a result of Duskin's formulation of the Beck monadicity theorem [Dus69], see also [Bor94, Theorem 4.4.4], Lemmas 6.1 and 6.4 imply

**Corollary 6.5.** *The forgetful functor  $V$  is monadic, i. e. the category  ${}_k\widehat{\text{Hopf}}_{\mathbf{Z}}$  is equivalent to the category of algebras for the monad  $\text{Fr}$ .* □

**Lemma 6.6.** *The forgetful functor  $V$  commutes with filtered colimits.*

*Proof.* Let  $\{A_i\}_{i \in I}$  in  $\text{Pro} - \text{Alg}_k$  be a cofiltered system of pro-algebras representing formal  $l$ -module schemes. In particular, for each  $i \in I$  there is a cofiltered category  $J_i$  and a functor  $A'_i: J_i \rightarrow \text{Alg}_k$  representing  $A_i \in \text{Pro} - \text{Alg}_k$ . By [AM69, Prop. A.4.4], its limit in  $\text{Pro} - \text{Alg}_k$  can be presented as the functor  $A': K \rightarrow \text{Alg}_k$ , where  $K$  is the category with objects  $(i, j)$  with  $i \in I$  and  $j \in J_i$ , and where a morphism from  $(i, j)$  to  $(i', j')$  consist of a morphism  $\phi: i \rightarrow i'$  and a map  $A'_{i'}(j') \rightarrow A_i(j)$  representing  $A_\phi$ . Then  $A': K \rightarrow \text{Alg}_k$  is given by  $A'(i, j) = A'_i(j)$  and it represents the limit of  $A_i$ . The coaddition on every  $A_i$  is represented by a map  $\mu_i: J_i \rightarrow J_i$  and a  $k$ -algebra map  $A'(i, \mu_i(j)) \rightarrow A'(i, j) \otimes_k A'(i, j)$ . This equips  $A'$  (as a pro- $k$ -algebra indexed by  $K$ ) with a coaddition. The cozero and linearity maps descend to  $A'$  in a similar way. □

**Theorem 6.7.** *The category  ${}_k\widehat{\text{Hopf}}_{\mathbf{Z}}$  is complete and cocomplete.*

*Proof.* A category of algebras over a monad is always complete if the base category is (in our case  $\text{Sch}_k$ ). For cocompleteness, by a standard theorem for monads [Bor94, Prop. 4.3.4], it suffices to construct coequalizers. Thus let  $F \rightrightarrows G$  be two morphisms in  ${}_k\widehat{\text{Hopf}}_{\mathbf{Z}}$  with coequalizer  $Q$  in  $\text{Sch}_k$ . Consider the category of formal module schemes  $C'$  with an epimorphism  $Q \rightarrow C'$  of underlying formal schemes such that  $G \rightarrow Q \rightarrow C'$  is a map of formal module schemes. Since small limits exist in  ${}_k\widehat{\text{Hopf}}_{\mathbf{Z}}$ , the limit of these  $C'$  will be the equalizer once we show that  $\text{Pro} - \text{Alg}_k$  is well-powered, i.e. the category of  $C'$  is in fact small. It suffices to show that the

category  $\text{Pro} - \text{Mod}_k$  is well-powered since every sub-pro-algebra is in particular a sub-pro-module. By [AM69, Prop. 4.6], a monomorphism in  $\text{Pro} - \text{Mod}_k$  can be represented as a levelwise monomorphism. Thus if  $A: I \rightarrow \text{Mod}_k$  represents a pro- $k$ -module with  $\#\text{Sub}(A(i)) = \alpha_i$  for some cardinals  $\alpha_i$  then  $\#\text{Sub}(A) \leq \prod_{i \in I} \alpha_i$ ; in particular, it is a set.  $\square$

**Corollary 6.8.** *Let  $F: I \rightarrow {}_k\widehat{\text{Hopf}}_{\mathbf{Z}}$  be a diagram of formal module schemes with colimit  $\text{colim} F$ . Denote by  $\text{colim}^{\mathfrak{F}} F$  the colimit of  $F$  as a diagram of functors in  $\mathfrak{F}$ . Then for any formal module scheme  $G \in {}_k\widehat{\text{Hopf}}_{\mathbf{Z}}$ , we have a natural isomorphism*

$$\mathfrak{F}(\text{colim}^{\mathfrak{F}} F, G) \cong {}_k\widehat{\text{Hopf}}_{\mathbf{Z}}(\text{colim} F, G).$$

*Proof.*  $\mathfrak{F}(\text{colim}^{\mathfrak{F}} F, G) \cong \lim \mathfrak{F}(F, G) = \lim {}_k\widehat{\text{Hopf}}_{\mathbf{Z}}(F, G) = {}_k\widehat{\text{Hopf}}_{\mathbf{Z}}(\text{colim} F, G)$ .  $\square$

**Corollary 6.9.** *The category  ${}_k\widehat{\text{Hopf}}_{\mathbf{Z}}$  is a 2-bimodule over  $\text{Set}$ , and the forgetful functor  $V: {}_k\widehat{\text{Hopf}}_{\mathbf{Z}} \rightarrow \text{Sch}_k$  is a functor of 2-bimodules.*

*Proof.* By Example 2.3, a bicomplete category with a shift functor  $\Sigma^n$  is a 2-bimodule over sets, and a functor between two such categories is a functor of 2-modules if it commutes with shifts. The category  ${}_k\widehat{\text{Hopf}}_{\mathbf{Z}}$  has a shift functor given by  $(\Sigma^n F)(R) = \Sigma^n(F(R))$ , using the shift on  $\text{Mod}_l$ , and the forgetful functor obviously commutes with this shift.  $\square$

The morphisms in  ${}_k\widehat{\text{Hopf}}_{\mathbf{Z}}$ , i.e. the set of natural transformations of  $\text{Mod}_{\mathbf{Z}}$ -valued functors, obviously form a  $\mathbf{Z}$ -module, and if  $F \in {}_k\widehat{\text{Hopf}}_{\mathbf{Z}}$  and  $G \in {}_k\widehat{\text{Hopf}}_l$  then  ${}_k\widehat{\text{Hopf}}_{\mathbf{Z}}(F, G)$  is naturally an  $l$ -module.

For  $M \in \text{Mod}_l$  and  $F \in {}_k\widehat{\text{Hopf}}_l$ , let  $\text{hom}_l(M, F)$  denote the functor  $\text{Alg}_k \rightarrow \text{Mod}_{\mathbf{Z}}$  which sends  $R$  to  $\text{Hom}_l(M, F(R))$ . Thus  $\text{hom}_l(M, F)$  can be expressed as the simultaneous equalizer of

$$\text{hom}(M, F) \rightrightarrows \text{hom}(M \times M, F)$$

and

$$\text{hom}(M, F) \rightrightarrows \text{hom}(M \times l, F),$$

where the maps in the first diagram are induced by the addition on  $M$  and by

$$\text{hom}(M, F) \xrightarrow{\Delta} \text{hom}(M, F)^2 \rightarrow \text{hom}(M^2, F^2) \xrightarrow{\text{hom}(M^2, +)} \text{hom}(M^2, F),$$

and the maps in the second diagram are the map induced by scalar multiplication on  $M$  and by

$$\text{hom}(M, F) \rightarrow \text{hom}(M \times l, F \times l) \xrightarrow{\text{hom}(M \times l, \cdot)} \text{hom}(M \times l, F).$$

As a limit of representable functors, this functor is representable as well. Denote the representing object of  $\text{hom}_l(M, F)$  by  $M \odot_l F$ .

We have a adjunctions

$$(6.10) \quad {}_k\widehat{\text{Hopf}}_{\mathbf{Z}}(G, \text{hom}_l(M, F)) \cong \mathfrak{F}_l(G(-) \otimes_{\mathbf{Z}} M, F) \cong \text{Mod}_l(M, {}_k\widehat{\text{Hopf}}_{\mathbf{Z}}(G, F)).$$

The functor  $G(-) \otimes_{\mathbf{Z}} M$  in the middle is not representable unless  $M$  is flat. In any case, there is an optimal approximation by a representable functor.

**Lemma 6.11.** *There is a functor  $\overline{\otimes}_{\mathbf{Z}}: {}_k\widehat{\text{Hopf}}_{\mathbf{Z}} \times \text{Mod}_l \rightarrow {}_k\widehat{\text{Hopf}}_l$  with a natural transformation of  $l$ -modules*

$$(G \overline{\otimes}_{\mathbf{Z}} M)(R) \rightarrow G(R) \otimes_{\mathbf{Z}} M$$

which produces an adjunction isomorphism for  $F \in {}_k\widehat{\text{Hopf}}_l$ :

$${}_k\widehat{\text{Hopf}}_l(G \overline{\otimes}_{\mathbf{Z}} M, F) \cong \mathfrak{F}_l(G(-) \otimes_{\mathbf{Z}} M, F).$$

*Proof.* Let  $N$  be a  $\mathbf{Z}$ -module. If  $N \otimes S$  denotes the left module structure of  $\text{Mod}_{\mathbf{Z}}$  over sets, we can express the tensor product  $N \otimes_{\mathbf{Z}} M$  as a coequalizer

$$N \otimes (M \times M) \rightrightarrows N \otimes M,$$

where one map is the addition on  $M$  and the other map is given by

$$N \otimes (M^2) \xrightarrow{\Delta \otimes (M^2)} (N^2) \otimes (M^2) \rightarrow (N \otimes M)^2 \xrightarrow{+} N \otimes M$$

Modelled by this, we define in the category  ${}_k\widehat{\text{Hopf}}_{\mathbf{Z}}$ :

$$G \overline{\otimes}_{\mathbf{Z}} M = \text{coeq}(F \overline{\otimes} (M \times M) \rightrightarrows F \overline{\otimes} M)$$

The claim then follows from Corollaries 6.8 and 6.9.  $\square$

**Corollary 6.12.** *The forgetful functor  $\text{Res}_{\mathbf{Z}}^l: {}_k\widehat{\text{Hopf}}_l \rightarrow {}_k\widehat{\text{Hopf}}_{\mathbf{Z}}$  has both a left and a right adjoint.*

*Proof.* For  $G \in {}_k\widehat{\text{Hopf}}_{\mathbf{Z}}$ , the left adjoint is given by  $G \overline{\otimes}_{\mathbf{Z}} l$ . This follows from (6.10) and Lemma 6.11 along with the observation that  $\text{Res}_{\mathbf{Z}}^l = \text{hom}_l(l, -)$ .

The right adjoint is given by  $\text{hom}_{\mathbf{Z}}(l, G)$ , which becomes a formal  $l$ -module scheme by the left action of  $l$  on itself by multiplication.  $\square$

**Corollary 6.13.** *The forgetful functor  $V: {}_k\widehat{\text{Hopf}}_l \rightarrow \text{Sch}_k$  has a left adjoint  $\text{Fr}_l$ . For  $F \in \text{Sch}_k$ ,  $G \in {}_k\widehat{\text{Hopf}}_l$ , there is a natural adjunction isomorphism*

$$\mathfrak{F}(\text{Fr}_l \circ F, G) \cong {}_k\widehat{\text{Hopf}}_l(\text{Fr}_l(F), G).$$

The category  ${}_k\widehat{\text{Hopf}}_l$  is complete, cocomplete, and equivalent to the category of algebras for the monad  $\text{Fr}_l$ .

*Proof.* The forgetful functor factors as  ${}_k\widehat{\text{Hopf}}_l \xrightarrow{U} {}_k\widehat{\text{Hopf}}_{\mathbf{Z}} \rightarrow \text{Sch}_k$  and both have left adjoints by Cor. 6.12 and Lemma 6.1. The adjunction follows from Cor. 6.2 and Lemma 6.11. Since by Cor. 6.12,  $U$  has left and right adjoints, it commutes with all limits and colimits and thus creates  ${}_k\widehat{\text{Hopf}}_{\mathbf{Z}}$ -split coequalizers. By the monadicity theorem,  $V$  is therefore monadic, and using Lemma 6.6,  $V$  commutes with filtered colimits. Therefore [Bor94, Prop. 4.3.6],  ${}_k\widehat{\text{Hopf}}_l$  is complete and cocomplete.  $\square$

By the functoriality in the  $l$ -module variable, the right 2-module structure of  ${}_k\widehat{\text{Hopf}}_l$  extends to bimodules:

$$\text{hom}_l: {}_l\text{Mod}_{l'} \times {}_k\widehat{\text{Hopf}}_l \rightarrow {}_k\widehat{\text{Hopf}}_{l'}.$$

Similarly, there is a tensor-type functor

$$\overline{\otimes}_{l'}: {}_{l'}\text{Mod}_l \times {}_k\widehat{\text{Hopf}}_{l'} \rightarrow {}_k\widehat{\text{Hopf}}_l.$$

given by the coequalizer in  ${}_k\widehat{\text{Hopf}}_{l'}$

$$F \overline{\otimes}_{\mathbf{Z}} l' \overline{\otimes}_{\mathbf{Z}} M \rightrightarrows F \overline{\otimes}_{\mathbf{Z}} M \rightarrow F \overline{\otimes}_{l'} M$$

These are of particular interest if  $l, l'$  are commutative,  $\alpha: l \rightarrow l'$  is a ring map, and  $l'$  is considered an  $l' - l$ -bimodule by means of right and left multiplication.

**Corollary 6.14.** *Let  $l \rightarrow l'$  be a map of commutative rings, and let  $\text{Res}_l^{l'}$  denote the restriction functor from  ${}_k\widehat{\text{Hopf}}_{l'}$  to  ${}_k\widehat{\text{Hopf}}_l$ . Then  $-\otimes_l l'$  is left adjoint and  $\text{hom}_l(l', -)$  is right adjoint to  $\text{Res}_l^{l'}$ .  $\square$*

For  $l = l'$ , we obtain thus:

**Corollary 6.15.** *For a graded commutative ring  $l$ , the category  ${}_k\widehat{\text{Hopf}}_l$  is a 2-bimodule over  $\text{Mod}_l$ .  $\square$*

**6.1. Tensor products of formal module schemes.** In this subsection, all rings are again assumed to be graded commutative. Our next objective is to construct the tensor product  $F \otimes_l G$  of two formal  $l$ -module schemes in  ${}_k\widehat{\text{Hopf}}_l$ . There are two equivalent characterizations of what this tensor product is supposed to accomplish.

**Definition.** A functor  $\otimes_l: {}_k\widehat{\text{Hopf}}_l^2 \rightarrow {}_k\widehat{\text{Hopf}}_l$  together with an  $l$ -bilinear natural transformation  $F \times G \rightarrow F \otimes_l G$  is called a *tensor product of formal module schemes* if the following two equivalent conditions hold for  $F, G, H \in {}_k\widehat{\text{Hopf}}_l$ :

- (1) There is an adjunction  ${}_k\widehat{\text{Hopf}}_l(F \otimes_l G, H) \cong \mathfrak{F}(F(-) \otimes_l G(-), H)$ , where the  $F(-) \otimes_l G(-)$  is the objectwise tensor product of  $l$ -modules.
- (2) Any  $l$ -bilinear natural transformation of formal module schemes  $F \times G \rightarrow H$  factors uniquely through a morphism of formal modules schemes  $F \otimes_l G \rightarrow H$ .

This concept is a dual and  $l$ -module enhanced version of the tensor product of bicommutative Hopf algebras [Goe99]. It is immediate from this definition that if a tensor product exists then it will be unique.

**Theorem 6.16.** *The tensor product of formal  $l$ -module schemes over  $k$  exists.*

*Proof.* As in several proofs before, it suffices to write the tensor product of two  $l$ -modules as a combination of limits, colimits, and free  $l$ -module functors. To get an idea for the construction, first consider tensor products of ordinary  $l$ -modules. For sets  $X, Y$ , define

$$\text{Fr}_l(X, Y) = \text{colim}(\text{Fr}_l(X) \times Y \leftarrow X \times Y \rightarrow X \times \text{Fr}_l(Y)).$$

There is a canonical map  $\eta: \text{Fr}_l(X, Y) \rightarrow \text{Fr}_l(X \times Y)$ . On the left hand summand, this is given the map  $\text{Fr}_l(X) \times Y \rightarrow \text{Fr}_l(X \times Y)$  which is adjoint to

$$Y \rightarrow \text{map}(X, X \times Y) \xrightarrow{\text{Fr}_l} \text{map}(\text{Fr}_l(X), \text{Fr}_l(X \times Y)),$$

and similarly on the right hand side. Furthermore, if  $M, N$  are  $l$ -modules, their module structure can be described as monad actions  $\text{Fr}_l(M) \rightarrow M, \text{Fr}_l(N) \rightarrow N$ , and those give rise to a map  $\mu: \text{Fr}_l(M, N) \rightarrow M \times N$ .

Then the tensor product  $M \otimes_l N$  of two  $l$ -modules can be constructed as the coequalizer of

$$\text{Fr}_l(\text{Fr}_l(M, N)) \rightrightarrows \text{Fr}_l(M \times N)$$

where one map uses  $\eta$  followed by the multiplication on  $l$  and the other is  $l(\mu)$ .

We mimick this construction for formal module schemes as follows. Let

$$(6.17) \quad \mathrm{Fr}_l(F, G) = \mathrm{colim}(\mathrm{Fr}_l(F) \times G \leftarrow F \times G \rightarrow F \times \mathrm{Fr}_l(G))$$

be the pushout in the category  ${}_k\widehat{\mathrm{Hopf}}_l$ , which exists by Cor. 6.13. By the adjunction of the same corollary, there is a map  $\tilde{\eta}: \mathrm{Fr}_l(F, G) \rightarrow \mathrm{Fr}_l(F \times G)$  compatible with  $\eta$  as well as a map  $\tilde{\mu}: \mathrm{Fr}_l(F, G) \rightarrow F \times G$ . Define  $F \otimes_l G$  as the coequalizer in  ${}_k\widehat{\mathrm{Hopf}}_l$  of

$$(6.18) \quad \mathrm{Fr}_l(\mathrm{Fr}_l(F, G)) \rightrightarrows \mathrm{Fr}_l(F \times G).$$

Property (1) follows immediately from the adjunction in Cor. 6.13.  $\square$

We record how  $\otimes_l$  behaves on free formal module schemes:

**Lemma 6.19.** *Let  $X, Y \in \mathrm{Sch}_k$ . Then*

$$\mathrm{Fr}_l(X) \otimes_l \mathrm{Fr}_l(Y) \cong \mathrm{Fr}_l(X \times Y).$$

*Proof.* After applying  $\mathrm{Fr}_l$  once more, the natural maps

$$X \times Y \rightarrow \mathrm{Fr}_l(X) \times \mathrm{Fr}_l(Y) \rightrightarrows \mathrm{Fr}_l(\mathrm{Fr}_l(X), \mathrm{Fr}_l(Y))$$

split the coequalizer defining  $\mathrm{Fr}_l(X) \otimes_l \mathrm{Fr}_l(Y)$  and therefore give an isomorphism with  $\mathrm{Fr}_l(X \times Y)$ .  $\square$

We will denote the representing object of  $F \otimes_l G$  by  $\mathcal{O}_F \overset{l}{\boxtimes}_k \mathcal{O}_G =_{\mathrm{def}} \mathcal{O}_{F \otimes_l G}$ .

Using Example 2.7, we thus summarize:

**Lemma 6.20.** *The symmetric monoidal category  $({}_k\widehat{\mathrm{Hopf}}_l, \otimes_l, \mathrm{hom}(l, k))$  is a 2-algebra over  $\mathrm{Set}$ .  $\square$*

**Theorem 6.21.** *The symmetric monoidal category  $({}_k\widehat{\mathrm{Hopf}}_l, \otimes_l, \mathrm{hom}(l, k))$  is a 2-algebra over  $(\mathrm{Mod}_l, \otimes_l, l)$ .*

*Proof.* We need to see that the map in  $\mathrm{Set}$

$${}_k\widehat{\mathrm{Hopf}}_l(F, F') \xrightarrow{-\otimes_l G} {}_k\widehat{\mathrm{Hopf}}_l(F \otimes_l G, F' \otimes_l G)$$

is in fact a map in  $\mathrm{Mod}_l$ . To see this, note that it factors as a map

$$\begin{aligned} {}_k\widehat{\mathrm{Hopf}}_l(F, F') &= \mathfrak{F}_l(F, F') \\ &\xrightarrow{\otimes_l G(-)} \mathfrak{F}_l(F(-) \otimes_l G(-), F'(-) \otimes_l G(-)) \\ &\rightarrow \mathfrak{F}_l(F(-) \otimes_l G(-), F' \otimes_l G) \\ &\xrightarrow[\cong]{\text{Characterization (1) of } \otimes_l} {}_k\widehat{\mathrm{Hopf}}_l(F \otimes_l G, F' \otimes_l G). \end{aligned}$$

All maps in this diagram are  $l$ -module maps.  $\square$

It is useful to have an alternative description of  $\otimes_l$  in terms of  $\otimes_{\mathbf{Z}}$ :

**Lemma 6.22.** *There is a coequalizer diagram in  ${}_k\widehat{\mathrm{Hopf}}_l$*

$$l \overline{\otimes}_{\mathbf{Z}}(F \otimes_{\mathbf{Z}} G) \rightrightarrows F \otimes_{\mathbf{Z}} G \rightarrow F \otimes_l G.$$

*The two maps are given by the (2.5) and the  $l$ -action on  $F$  and  $G$ , respectively.  $\square$*

## 7. THE STRUCTURE OF THE CATEGORY OF FORMAL ALGEBRA SCHEMES

The category of formal algebra schemes behaves differently from the category of module schemes or bimodules in that it does not have an enrichment over  $\text{Mod}_l$  or  $\text{Alg}_l$ . It does not fit into our framework of 2-algebras or 2-modules. We are already lacking a 2-ring structure on the category  $\text{Alg}_l$ .

As an immediate consequence of the tensor product construction (Subsection 6.1) we obtain:

**Corollary 7.1.** *For  $F \in {}_k\widehat{\text{Alg}}_l$ , the multiplication  $F(-) \times F(-) \rightarrow F(-)$  can be uniquely extended to a map of formal  $l$ -module schemes*

$$\mu_\times: F \otimes_l F \rightarrow F.$$

*This map is in fact a map of formal  $l$ -algebra schemes (by the commutativity of the multiplication of  $F$ ).  $\square$*

The 2-module structure on  ${}_k\widehat{\text{Hopf}}_l$  over  $\text{Mod}_l$  does not descend to a 2-module structure on  ${}_k\widehat{\text{Alg}}_l$ . The only remnant of it is a cotensor-like functor which pairs an algebra and an algebra scheme and gives a formal scheme. For this, let  $F \in {}_k\widehat{\text{Alg}}_l$  and let  $R \in \text{Alg}_l$ . Recall from Section 6 the construction of a right module structure  $\text{hom}_l(M, F)$  for an  $l$ -module  $M$  and  $F \in {}_k\widehat{\text{Hopf}}_l$ . Define  $\text{hom}(R, F)$  to be the functor which sends a  $k$ -algebra  $T$  to the set of algebra maps  $R \rightarrow F(T)$ . We can thus write  $\text{hom}(R, F)$  as the simultaneous equalizer of

$$\text{hom}_l(R, F) \rightrightarrows \text{hom}_l(R \times R, F)$$

and

$$\text{hom}_l(R, F) \rightrightarrows F,$$

where the first couple of maps are given by the multiplication on  $R$  and

$$\text{hom}_l(R, F) \xrightarrow{\Delta} \text{hom}_l(R, F)^2 \rightarrow \text{hom}_l(R \times R, F \times F) \xrightarrow{\text{hom}_l(R \times R, \cdot)} \text{hom}_l(R \times R, F),$$

respectively, and the second couple of maps are given by evaluation at  $1 \in R$  and the constant map with value  $\eta(1) \in F$ . As a limit of representable functors,  $\text{hom}(R, F)$  is representable. We write  $R \odot_l \mathcal{O}_F$  for the pro- $k$ -algebra representing  $\text{hom}(R, F)$ .

## 8. FORMAL RINGS AND FORMAL PLETHORIES

We have seen in the previous sections that the categories  ${}_k\widehat{\text{Mod}}_l$ ,  ${}_k\widehat{\text{Hopf}}_l$ , and  ${}_k\widehat{\text{Alg}}_l$  are symmetric monoidal categories with respect to the tensor product of formal bimodules resp. formal module schemes. On  ${}_k\widehat{\text{Alg}}_l$ , this tensor product is actually the categorical coproduct; on  ${}_k\widehat{\text{Hopf}}_l$ , it is not. Furthermore, we have various left and right 2-module structures over  $\text{Mod}_l$  and  $\text{Alg}_l$ . The aim of this section is to construct a second monoidal structure  $\circ$  on the categories  ${}_k\widehat{\text{Mod}}_k$  and  ${}_k\widehat{\text{Alg}}_k$  which corresponds to composition of ind-representable functors. These monoidal structures have an interesting compatibility with the tensor product monoidal structure.

The situation of a category with two such monoidal structures has been studied, although not with our examples in mind [AM10, Val08, JS93], and they are known as *2-monoidal categories*. As always with higher categorical concepts, there



is much leeway in the definitions as to what level of strictness one wants to require; Aguiar and Mahajan's definition of a 2-monoidal category [AM10, Section 6] is the laxest in the literature and fits our application, although in our case more strictness assumptions could be made.

**Definition** (Aguiar-Mahajan). A 2-monoidal category is a category  $\mathcal{C}$  with two monoidal structures  $(\otimes, I)$  and  $(\circ, J)$  with natural transformations

$$\zeta: (A \circ B) \otimes (C \circ D) \rightarrow (A \otimes C) \circ (B \otimes D)$$

and

$$\Delta_I: I \rightarrow I \circ I, \quad \mu_J: J \otimes J \rightarrow J, \quad \iota_J = \epsilon_I: I \rightarrow J,$$

such that:

- (1) the functor  $\circ$  is a lax monoidal functor with respect to  $\otimes$ , the structure maps being given by  $\zeta$  and  $\mu_J$ ;
- (2) the functor  $\otimes$  is an oplax monoidal functor with respect to  $\circ$ , the structure maps being given by  $\zeta$  and  $\Delta_I$ ;
- (3)  $(J, \mu_J, \iota_J)$  is a  $\otimes$ -monoid;
- (4)  $(I, \Delta_I, \epsilon_I)$  is a  $\circ$ -comonoid.

A 2-monoidal category is the most general categorical setup where a bialgebra can be defined, although in this context it is more common to call it a bimonoid.

**Definition** (Aguiar-Mahajan). A *bimonoid* in a 2-monoidal category  $\mathcal{C}$  as above is an object  $H$  with a structure of a monoid in  $(\mathcal{C}, \otimes, I)$  and a structure of a comonoid in  $(\mathcal{C}, \circ, J)$  satisfying the compatibility condition that the monoid structure maps are comonoid maps and the comonoid structure maps are monoid maps.

To make sense of the compatibility condition, notice that if  $H$  is an  $\otimes$ -monoid with multiplication  $\mu$  and unit  $\iota$ , then so is  $H \circ H$  by virtue of the maps

$$(H \circ H) \otimes (H \circ H) \xrightarrow{\zeta} (H \otimes H) \circ (H \otimes H) \xrightarrow{\mu \circ \mu} H \circ H$$

and

$$I \xrightarrow{\Delta_I} I \circ I \xrightarrow{\iota \circ \iota} H \circ H;$$

similarly if  $H$  is a  $\circ$ -comonoid with comultiplication  $\Delta$  and counit  $\epsilon$  then so is  $H \otimes H$  by virtue of the maps

$$H \otimes H \xrightarrow{\Delta \otimes \Delta} (H \circ H) \otimes (H \circ H) \xrightarrow{\zeta} (H \otimes H) \circ (H \otimes H)$$

and

$$H \otimes H \xrightarrow{\epsilon \otimes \epsilon} J \otimes J \xrightarrow{\mu_J} J.$$

Finally, we define the correct notion of a functor between 2-monoidal categories in order to map bimonoids to bimonoids:

**Definition** (Aguiar-Mahajan). A *bilax monoidal functor*  $F: \mathcal{C} \rightarrow \mathcal{D}$  between 2-monoidal categories is a functor which is lax monoidal with respect to  $\otimes$  and oplax monoidal with respect to  $\circ$  and whose lax structure  $\phi$  and oplax structure  $\psi$  are compatible

in the sense that the following diagrams commute:

$$(8.1) \quad \begin{array}{ccccc} I & \xrightarrow{\iota_I} & J & & I & \xrightarrow{\phi_0} & F(I) & \xrightarrow{F(\Delta_I)} & F(I \circ I) & & F(J) \otimes F(J) & \xrightarrow{\psi_0 \otimes \psi_0} & J \otimes J \\ \downarrow \phi_0 & & \psi_0 \uparrow & & \downarrow \Delta_I & & & & \downarrow \psi & & \downarrow \phi & & \downarrow \mu_J \\ F(I) & \xrightarrow{F(\iota_I)} & F(J) & & I \circ I & \xrightarrow{\phi_0 \circ \phi_0} & F(I) \circ F(I) & & & & F(J \otimes J) & \xrightarrow{F(\mu_J)} & F(J) & \xrightarrow{\psi_0} & J \end{array}$$

$$(8.2) \quad \begin{array}{ccc} F(A \circ B) \otimes F(C \circ D) & \xrightarrow{\psi \otimes \psi} & (FA \circ FB) \otimes (FC \circ FD) & \xrightarrow{\zeta} & (FA \otimes FC) \circ (FB \otimes FD) \\ \downarrow \phi & & & & \downarrow \phi \circ \phi \\ F((A \circ B) \otimes (C \circ D)) & \xrightarrow{F(\zeta)} & F((A \otimes C) \circ (B \otimes D)) & \xrightarrow{\psi} & F(A \otimes C) \circ F(B \otimes D) \end{array}$$

**Proposition 8.3** ([AM10, Cor. 6.53]). *Bilax monoidal functors preserve bimonoids and morphisms between them.*  $\square$

**Proposition 8.4.** *Let  $F: \mathcal{C} \rightarrow \mathcal{D}$ ,  $G: \mathcal{D} \rightarrow \mathcal{C}$  be functors between 2-monoidal categories. Assume*

- (1)  $F$  is left adjoint to  $G$ ;
- (2)  $F$  is strictly monoidal with respect to  $(\otimes, I)$ ;
- (3)  $G$  is strictly monoidal with respect to  $(\circ, J)$ .

*Then  $F$  is bilax if and only if  $G$  is.*

*Proof.* Let  $\eta: \text{id} \rightarrow G \circ F$  denote the unit and  $\epsilon: F \circ G \rightarrow \text{id}$  the counit of the adjunction.

Firstly,  $F$  will be oplax monoidal with respect to  $\circ$  by means of the maps  $F(J) \rightarrow J$  adjoint to  $J \xrightarrow{\cong} G(J)$  and the adjoint of

$$A \circ B \xrightarrow{\eta \circ \eta} G(F(A)) \circ G(F(B)) \xrightarrow{\psi_G^{-1}} G(F(A) \circ F(B)),$$

and similarly  $G$  will be lax monoidal with respect to  $\otimes$ .

Under the given conditions, the following composites are identities:

$$\begin{array}{c} I \xrightarrow[\cong]{(\phi_F)_0} F(I) \xrightarrow{F((\phi_G)_0)} F(G(I)) \xrightarrow{\epsilon} I \\ J \xrightarrow{\eta} G(F(J)) \xrightarrow{G((\psi_F)_0)} G(J) \xrightarrow[\cong]{(\psi_G)_0} J. \end{array}$$

If  $G$  is bilax then the first diagram in (8.1) commutes for  $F$  because it can be factored as:

$$\begin{array}{ccccc} I & \xlongequal{\quad} & I & \xrightarrow{\iota_I} & J \\ \downarrow (\phi_F)_0 & & \epsilon \uparrow & & \downarrow \epsilon \\ F(I) & \xrightarrow{F((\phi_G)_0)} & F(G(I)) & & \\ \downarrow F(\iota_I) & & F(G(\iota_I)) \downarrow & & \\ F(J) & \xrightarrow{F((\psi_G)_0^{-1})} & F(G(J)) & & \end{array}$$

The lower rectangle commutes because  $G$  was assumed to be bilax monoidal. The other diagrams follow by similar exercises in adjunctions.  $\square$

In the context of this paper, the new monoidal structures  $\circ$  are defined as follows.

**Definition.** Let  $k$  be a commutative ring. Consider the following setups:

- (1)  $l$  is a  $K$ -algebra,  $M \in \text{Pro-Mod}_l$ , and  $F \in {}_k\widehat{\text{Mod}}_l$ .
- (2)  $l$  is a commutative ring,  $M \in \text{Pro-Mod}_l$ , and  $F \in {}_k\widehat{\text{Hopf}}_l$ .
- (3)  $l$  is a commutative ring,  $M \in \text{Pro-Alg}_l$ , and  $F \in {}_k\widehat{\text{Alg}}_l$ .

In each of these cases, define a functor  $\circ_l$  by

$$\text{Spf}(M) \circ_l F = \text{colim} \text{hom}(M(i), F),$$

where the colimit is taken in  $(\text{Pro-Mod}_k)^{\text{op}}$  and  $\text{Sch}_k$ , respectively, and  $\text{hom}$  denotes  $\text{hom}_l$  in the first two cases and the algebra homomorphism object  $\text{hom}$  in the last case.

We denote the corresponding operation on representing objects by

$$M \hat{\circ}_l \mathcal{O}_F = \mathcal{O}_{\text{Spf}(M) \circ_l F} = \lim M(i) \circ_l \mathcal{O}_F.$$

As a consequence of the various 2-module structures exhibited in Sections 5, 6, and 7, we obtain:

**Corollary 8.5.** (1) For  $l$  a  $K$ -algebra,  $M \in \text{Pro-Mod}_l$ ,  $F \in {}_k\widehat{\text{Mod}}_l$ , and  $N \in \text{Mod}_k$ , there is a natural isomorphism

$$\text{Pro-Mod}_k(\mathcal{O}_{\text{Spf}(M) \circ_l F}, N) \rightarrow \text{Pro-Mod}_l(M, F(N))$$

(2) For  $l$  a ring,  $M \in \text{Pro-Mod}_l$ ,  $F \in {}_k\widehat{\text{Hopf}}_l$ , and  $N \in \text{Alg}_k$ , there is a natural isomorphism

$$\text{Pro-Alg}_k(\mathcal{O}_{\text{Spf}(M) \circ_l F}, N) \rightarrow \text{Pro-Mod}_l(M, F(N))$$

(3) For  $l$  a ring,  $M \in \text{Pro-Alg}_l$ ,  $F \in {}_k\widehat{\text{Alg}}_l$ , and  $N \in \text{Alg}_k$ , there is a natural isomorphism

$$\text{Pro-Alg}_k(\mathcal{O}_{\text{Spf}(M) \circ_l F}, N) \rightarrow \text{Pro-Alg}_l(M, F(N))$$

*Proof.* For the proof of (1), note that  $\text{Spf}(N)$  is a small object in the category  ${}_k\widehat{\text{Mod}}_{\mathbf{Z}} = (\text{Pro-Mod}_k)^{\text{op}}$  and thus

$$\begin{aligned} {}_k\widehat{\text{Mod}}_{\mathbf{Z}}(\text{Spf}(N), \text{Spf}(M) \circ_l F) &= {}_k\widehat{\text{Mod}}_{\mathbf{Z}}\left(\text{Spf}(N), \text{colim}_i \text{hom}_l(M(i), F)\right) \\ &= \text{colim}_i {}_k\widehat{\text{Mod}}_{\mathbf{Z}}\left(\text{Spf}(N), \text{hom}(M(i), F)\right) \\ &= \text{colim}_i \text{Mod}_l(M(i), F(N)) = \text{Pro-Mod}_l(M, F(N)). \end{aligned}$$

For (2) and (3), the proofs are formally the same.  $\square$

If  $k, l, m$  are  $K$ -algebras then by Cor. 8.5, additional structure on  $M$  gives rise to additional structure in the target: the product  $\circ_l$  extends to products

$$\begin{aligned} \circ_l: {}_l\widehat{\text{Mod}}_m \times {}_k\widehat{\text{Mod}}_l &\rightarrow {}_k\widehat{\text{Mod}}_m \\ \circ_l: {}_l\widehat{\text{Mod}}_m \times {}_k\widehat{\text{Hopf}}_l &\rightarrow {}_k\widehat{\text{Hopf}}_m \\ \circ_l: {}_l\widehat{\text{Hopf}}_m \times {}_k\widehat{\text{Alg}}_l &\rightarrow {}_k\widehat{\text{Hopf}}_m \\ \circ_l: {}_l\widehat{\text{Alg}}_m \times {}_k\widehat{\text{Alg}}_l &\rightarrow {}_k\widehat{\text{Alg}}_m. \end{aligned}$$

All of these products represent compositions  $G \circ F$  of functors. By associativity of the composition of functors, all of these products are associative in the appropriate sense, i.e. the first and last operations are associative, the second is a tensoring over the first, and the third is a tensoring over the fourth.

When  $k = l = m$ , the first and fourth versions of the composition product define monoidal structures on  ${}_k\widehat{\text{Mod}}_k$  and  ${}_k\widehat{\text{Alg}}_k$ , respectively, but they are neither symmetric nor closed. The identity functors in the various categories are units for  $\circ_l$ . We summarize this in the following lemma.

**Lemma 8.6.** *The category  $({}_k\widehat{\text{Mod}}_k, \circ_k, \text{id})$  is a 2-algebra over  $\text{Mod}_k$ . The category  $({}_k\widehat{\text{Alg}}_k, \circ_k, \text{id})$  is a 2-algebra over  $\text{Set}$ . Finally, the category  ${}_k\widehat{\text{Hopf}}_k$  is tensored over both  ${}_k\widehat{\text{Mod}}_k$  and  ${}_k\widehat{\text{Alg}}_k$ .  $\square$*

We can finally define the object of the title of this paper.

**Definition.** A *formal  $k$ -coalgebra* is a comonoid in  $({}_k\widehat{\text{Mod}}_k, \circ_k, \text{id})$ . A *formal plethory* is a comonoid in  $({}_k\widehat{\text{Alg}}_k, \circ_k, \text{id})$ .

The rest of this section is devoted to proving the following theorem:

**Theorem 8.7.** *The categories  $({}_k\widehat{\text{Mod}}_k, \otimes_k, \circ_k)$  and  $({}_k\widehat{\text{Alg}}_k, \otimes_k, \circ_k)$  are 2-monoidal categories.*

In the next section, we will produce bilax monoidal functors between these two 2-monoidal categories.

*Proof.* The case of  ${}_k\widehat{\text{Alg}}_k$  follows from abstract nonsense because in that category,  $\otimes_k$  is the categorical coproduct, and any monoidal category is automatically 2-monoidal with respect to the categorical coproduct [AM10, Example 6.19], cf. Example 1.4. The case of  ${}_k\widehat{\text{Mod}}_k$  (Example 1.6, however, requires some work. To define  $\zeta$ , consider more generally the functor  $\circ_l: {}_l\widehat{\text{Mod}}_m \times {}_k\widehat{\text{Mod}}_l \rightarrow {}_k\widehat{\text{Mod}}_m$  and first assume that  $m = \mathbf{Z}$ . Let  $F_1, G_1 \in {}_l\widehat{\text{Mod}}_{\mathbf{Z}}$  and  $F_2, G_2 \in {}_k\widehat{\text{Mod}}_l$ . Let  $\mathcal{O}_{F_1}: I \rightarrow \text{Mod}_l$ ,  $\mathcal{O}_{G_1}: J \rightarrow \text{Mod}_l$  be representations of the corresponding pro- $l$ -modules. Then we have a map

$$\begin{aligned} (F_1 \circ_{\mathbf{Z}} F_2) \otimes_m (G_1 \circ_l G_2) &= \text{colim}_{i \in I, j \in J} \text{hom}(\mathcal{O}_{F_1}(i), F_2) \otimes_m \text{hom}(\mathcal{O}_{G_1}(j), G_2) \\ &\xrightarrow{\text{Lemmas 5.5, 2.9}} \text{colim}_{i, j} \text{hom}(\mathcal{O}_{F_1}(i) \otimes_{\mathbf{Z}} \mathcal{O}_{G_1}(j), F_2 \otimes_l G_2) \\ &= (F_1 \otimes_{\mathbf{Z}} G_1) \circ_l (F_2 \otimes_l G_2). \end{aligned}$$

The general case follows from writing  $F \otimes_l G$  as the coequalizer of  $l \otimes_{\mathbf{Z}} F \otimes_{\mathbf{Z}} G \rightrightarrows F \otimes_{\mathbf{Z}} G$  (Lemma 6.22).

The remaining structure maps are easier:

- $\iota_J = \epsilon_I: I \rightarrow J$ : this map  $\text{Spf}(\text{hom}_K(k, k)) \rightarrow \text{Spf}(k)$  is adjoint to the multiplication  $k \otimes_K k \rightarrow k$ . Note that this map has a left inverse  $\pi: J \rightarrow I$  given by the evaluation-at-1 map  $\text{hom}(k, k) \rightarrow k$ .

- $\Delta_I: I \rightarrow I \circ I$ : We define  $\Delta_I$  on representing objects  $\mathcal{O}_I \hat{\circ}_k \mathcal{O}_I \rightarrow \mathcal{O}_I = \text{hom}_K(k, k)$  as the adjoint of

$$\begin{aligned} k \circledast_k (\text{hom}_K(k, k) \hat{\circ}_k \text{hom}_K(k, k)) &\xrightarrow[\text{Lemma 8.6}]{\text{enrichment}} (k \circledast_k \text{hom}_K(k, k)) \hat{\circ}_k \text{hom}_K(k, k) \\ &\xrightarrow{\text{eval} \hat{\circ}_k \text{id}} k \hat{\circ}_k \text{hom}_K(k, k) = k \circledast_k \text{hom}_K(k, k) \\ &\xrightarrow{\text{eval}} k. \end{aligned}$$

- $\mu_J: J \otimes_k J \rightarrow J$ : Since  $J = \text{Spf}(k)$  and  $k \circledast_k k \cong k$ ,  $\Delta_I$  is defined as this isomorphism.

That  $\zeta$  is compatible with the associativity isomorphisms of  $\circ$  and  $\otimes_k$  follows easily from Lemma 2.9 and the definition. We will verify the various unitality conditions required in a 2-monoidal category.

- (1)  **$J$  is a monoid with respect to  $\otimes$** : Since  $\text{Spf}(k) \otimes_k \text{Spf}(k) \xrightarrow[\cong]{\mu_J} \text{Spf}(k)$ , associativity is obvious. For unitality, observe that

$$\text{Spf}(k) \cong \text{Spf}(\text{hom}_K(k, k)) \otimes_k \text{Spf}(k) \xrightarrow{\iota_J \otimes \text{id}} \text{Spf}(k) \otimes_k \text{Spf}(k) \cong \text{Spf}(k)$$

is the identity map as well.

- (2)  **$I$  is a comonoid with respect to  $\circ$** : The associativity is immediate from the definition of  $\Delta_I$  and the counitality follows from the fact that

$$k \cong k \circledast_k k \xrightarrow{\text{id} \circledast_k \mathcal{O}_I} k \circledast_k \text{hom}_K(k, k) \xrightarrow{\text{eval}} k$$

is the identity map, and similarly for the left counit. □

It makes little sense to talk about bimonoids in  $\widehat{{}_k\text{Alg}}_k$  because every plethory is a bimonoid in a unique way:

**Proposition 8.8.** *The forgetful functor from bimonoids in  $\widehat{{}_k\text{Alg}}_k$  to formal plethories is an equivalence of categories.*

*Proof.* This follows again from the fact that  $\otimes_k$  is the categorical coproduct in  $\widehat{{}_k\text{Alg}}_k$  and that a monoid structure with respect to the categorical coproduct always exists uniquely (cf. [AM10, Example 6.42]). □

## 9. PRIMITIVES AND INDECOMPOSABLES

For any Hopf algebra  $A$  over  $k$ , there are two particularly important  $k$ -modules: the module of primitive elements  $PA = \{a \in A \mid \psi(a) = a \otimes 1 + 1 \otimes a\}$  and the module of indecomposable elements  $QA = A_+ / (A_+)^2$ , where  $A_+$  denotes the augmentation ideal with respect to the counit  $\epsilon: A \rightarrow k$ . In this section, I will consider the analogous notions for pro-algebras and formal module and algebra schemes, and prove compatibility with the various products studied so far, making these functors into morphisms of 2-algebras and into bilax monoidal functors of 2-monoidal categories.

Throughout this section, we will restrict to the case  $K = \mathbf{Z}$  for formal bimodules.

**9.1. Indecomposables.** Let  $\text{Alg}_k^+$  be the category of augmented graded commutative unital  $k$ -algebras, i.e. algebras  $A$  with an algebra morphism  $A \xrightarrow{\epsilon} k$  such that the composite with the algebra unit  $k \xrightarrow{\eta} A \xrightarrow{\epsilon} k$  is the identity. Passing to pro-categories, the inclusion

$$\text{Pro}-(\text{Alg}_k^+) \rightarrow (\text{Pro}-\text{Alg}_k)^+$$

is an equivalence. We denote its opposite category, the category of pointed formal schemes, by  $\text{Sch}_k^+$ .

Now let  $M \in \text{Pro}-\text{Mod}_k$  and  $k \oplus M \in \text{Pro}-\text{Alg}_k^+$  the *square-zero extension*, i.e. the algebra where  $\eta: k \hookrightarrow k \oplus M$  is a unit map and the multiplication is given by decreeing that  $M^2 = 0$ . The augmentation is given by the projection  $\epsilon: k \oplus M \rightarrow k$ .

**Definition.** The left adjoint of  $k \oplus -: \text{Pro}-\text{Mod}_k \rightarrow \text{Pro}-\text{Alg}_k^+$  is denoted by  $A \mapsto Q(A)$ , the module of indecomposables of the augmented pro-algebra  $A$ .

If  $A_+ = \ker(\epsilon: A \rightarrow k)$  is the augmentation ideal, then  $QA \cong A_+/(A_+)^2$ , in particular, the left adjoint exists and is defined levelwise. If  $F \in {}_k\widehat{\text{Hopf}}_l$  is a formal module scheme then it induces a new functor

$$F_+ = \text{Pro}-\text{Alg}_k^+(\mathcal{O}_F, -): \text{Alg}_k^+ \rightarrow \text{Mod}_l.$$

The left hand side of

$$F_+(k \oplus M) = \text{Pro}-\text{Alg}_k^+(\mathcal{O}_F, k \oplus M) \cong \text{Pro}-\text{Mod}_k(Q(\mathcal{O}_F), M).$$

is thus in  $\text{Mod}_l$ , thus  $Q(\mathcal{O}_F)$  actually represents an object of  ${}_k\widehat{\text{Mod}}_l$  and  $Q$  yields a functor

$$\mathcal{Q} = \text{Spf}(Q): {}_k\widehat{\text{Hopf}}_l \rightarrow {}_k\widehat{\text{Mod}}_l.$$

This functor  $\mathcal{Q}$  is also a right adjoint, as we will see now. Recall from Cor. 6.13 that the functor forgetful functor  $V: {}_k\widehat{\text{Hopf}}_l \rightarrow \text{Sch}_k$  has a left adjoint  $\text{Fr}_l$ . There is also a pointed version, i.e. a left adjoint  $\text{Fr}_l^+ = \text{Spf}(\Gamma_l^+)$  of  $V^+: \text{Sch}_k^+ \rightarrow {}_k\widehat{\text{Hopf}}_l$ , given by  $\text{Fr}_l^+(F) = \text{Spf}(\Gamma_l(\mathcal{O}_F^+))$ . In particular,  $\Gamma_l^+(k \oplus M) = \Gamma_l(M)$  as  $k$ -coalgebras.

For  $B = \text{Spf}(M) \in {}_k\widehat{\text{Mod}}_l$  we obtain a ring map

$$l \xrightarrow{\mu} \text{Pro}-\text{Mod}_k(M, M) \xrightarrow{k \oplus -} \text{Pro}-\text{Alg}_k^+(k \oplus M, k \oplus M) \xrightarrow{\Gamma^+} \text{Pro}-\text{Alg}_k(\Gamma M, \Gamma M)$$

In this way,  $\text{Fr}(B) = \text{Fr}_{\mathbf{Z}}(B)$  obtains an  $l$ -module scheme structure. We obtain an adjunction

$${}_k\widehat{\text{Hopf}}_l(\text{Fr}(B), F) \cong {}_k\widehat{\text{Mod}}_l(B, \mathcal{Q}(F)).$$

**Lemma 9.1.** For  $B \in {}_k\widehat{\text{Mod}}_{\mathbf{Z}}$ , we have

$$\text{Fr}_l(B) \cong \text{Fr}(l \overline{\otimes}_{\mathbf{Z}} B) \quad (\text{cf. Lemma 5.1})$$

*Proof.* Let  $F \in {}_k\widehat{\text{Hopf}}_l$ . Then we have

$$\begin{aligned} {}_k\widehat{\text{Hopf}}_l(\text{Fr}_l(B), F) &\cong {}_k\widehat{\text{Mod}}_{\mathbf{Z}}(B, \mathcal{Q}(F)) \\ &\cong {}_k\widehat{\text{Mod}}_l(l \overline{\otimes}_{\mathbf{Z}} B, \mathcal{Q}(F)) \cong {}_k\widehat{\text{Hopf}}_l(\text{Fr}(l \overline{\otimes}_{\mathbf{Z}} B), F). \quad \square \end{aligned}$$

**Lemma 9.2.** The functor  $\mathcal{Q}: {}_k\widehat{\text{Hopf}}_l \rightarrow {}_k\widehat{\text{Mod}}_l$  is a right strict morphism of 2-modules over  $\text{Mod}_l$ . The functor  $\text{Fr}: {}_k\widehat{\text{Mod}}_l \rightarrow {}_k\widehat{\text{Hopf}}_l$  is a left strict morphism of 2-modules over  $\text{Mod}_l$ .

More explicitly, for  $M \in \text{Mod}_l$ ,  $F \in {}_k\widehat{\text{Hopf}}_l$ , and  $B \in {}_k\widehat{\text{Mod}}_l$ , the canonical maps

$$\beta: \mathcal{Q}(\text{hom}_l(M, F)) \rightarrow \text{hom}_l(M, \mathcal{Q}(F))$$

and

$$\alpha: M \overline{\otimes}_l \text{Fr}(B) \rightarrow \text{Fr}(M \overline{\otimes}_l B)$$

are isomorphisms.

*Proof.* Let  $N \in \text{Pro-Mod}_k$  be a test object. Then

$$\begin{aligned} \mathcal{Q}(\text{hom}_l(M, F))(N) &= \text{Pro-Mod}_k(Q(M \circ_l \mathcal{O}_F), N) \\ &\cong {}_k\widehat{\text{Hopf}}_{\mathbf{Z}}^{\text{op}}(M \circ_l \mathcal{O}_F, \Gamma(N)) \\ &\cong \text{Mod}_l(M, {}_k\widehat{\text{Hopf}}_{\mathbf{Z}}^{\text{op}}(\mathcal{O}_F, \Gamma(N))) \\ &\cong \text{Mod}_l(M, \text{Pro-Mod}_k(Q\mathcal{O}_F, N)) \\ &\cong \text{Pro-Mod}_k(M \circ_l Q\mathcal{O}_F, N) = \text{hom}_l(M, \mathcal{Q}(F))(N). \end{aligned}$$

The statement about  $\text{Fr}$  follows by adjointness.  $\square$

The functor  $\mathcal{Q}$  is not strict with respect to the tensor product  $\otimes_l$  of formal module schemes (Thm. 6.21). This can already be seen by observing that for the unit of  $\otimes_l$ ,  $\text{Spf}(\text{hom}(l, k))$ ,

$$Q(\text{hom}(l, k)) = \left( \prod_{\lambda \in l - \{0\}} k \right) / \left( \prod_{\lambda \in l - \{0\}} k \right)^2 = 0,$$

which is different from the unit  $\text{hom}_{\mathbf{Z}}(l, k)$  of  ${}_k\widehat{\text{Mod}}_l$ .

**Lemma 9.3.** *The functor  $\text{Fr}: {}_k\widehat{\text{Mod}}_l \rightarrow {}_k\widehat{\text{Hopf}}_l$  is a strict morphism of 2-algebras, i.e. there are natural isomorphisms*

$$\text{hom}(l, k) \rightarrow \Gamma(\text{hom}_{\mathbf{Z}}(l, k))$$

and

$$\psi: \Gamma(M) {}^l\boxtimes_k \Gamma(N) \rightarrow \Gamma(M {}^l\otimes_k N)$$

making  $\Gamma = \mathcal{O}_{\text{Fr}}$  into a strict monoidal functor.

*Proof.* For  $l = \mathbf{Z}$ , this follows from Cor. 6.3 and Lemma 6.19. For general  $l$ , the first map is an isomorphism since

$$\Gamma(\text{hom}_{\mathbf{Z}}(l, k)) \underset{\text{Lemma 9.2}}{\cong} \text{hom}_{\mathbf{Z}}(l, \Gamma(k)) \cong \text{hom}_{\mathbf{Z}}(l, \text{hom}(\mathbf{Z}, k)) \cong \text{hom}(l, k).$$

For the second isomorphism recall from Lemma 6.22 that the top row in the following diagram is an equalizer:

$$\begin{array}{ccccc} \Gamma M {}^l\boxtimes_k \Gamma N & \longrightarrow & \Gamma M {}^{\mathbf{Z}}\boxtimes_k \Gamma N & \rightrightarrows & \text{hom}(l, \Gamma M {}^{\mathbf{Z}}\boxtimes_k \Gamma N) \\ & & \downarrow & & \downarrow \\ \Gamma(M {}^l\otimes_k N) & \longrightarrow & \Gamma(M {}^{\mathbf{Z}}\otimes_k N) & \rightrightarrows & \text{hom}(l, \Gamma(M {}^{\mathbf{Z}}\otimes_k N)) \end{array}$$

The bottom row is also an equalizer because  $\Gamma$  commutes with limits and is right strict by Lemma 9.2. The desired isomorphism is thus induced on the left.  $\square$

**Corollary 9.4.** *The functor  $\mathcal{Q}: {}_k\widehat{\text{Hopf}}_l \rightarrow {}_k\widehat{\text{Mod}}_l$  is a lax morphism of 2-algebras.*

*Proof.* On representing objects, we need to produce natural transformations

$$\psi_0: Q(\text{hom}(l, k)) \rightarrow \text{hom}_{\mathbf{Z}}(l, k)$$

and

$$\psi: Q(A \overset{l}{\boxtimes}_k B) \rightarrow QA \overset{l}{\otimes}_k QB.$$

making  $Q$  into a lax monoidal functor, and satisfying the compatibility relation for the enrichments. Since  $Q(\text{hom}(l, k)) = 0$ ,  $\psi_0$  is the zero map.

The map  $\psi$  comes from the fact that  $Q$  is left adjoint to  $\Gamma$ . Explicitly, it is given as the adjoint of

$$A \overset{l}{\boxtimes}_k B \xrightarrow{\eta_A \overset{l}{\boxtimes}_k \eta_B} (\Gamma QA) \overset{l}{\boxtimes}_k (\Gamma QB) \xrightarrow[\cong]{\text{Lemma 9.3}} \Gamma(QA \overset{l}{\otimes}_k QB). \quad \square$$

**9.2. Primitives.** The situation for primitives is almost, but not quite dual to that of indecomposables. Whereas for indecomposables, we considered a pair of adjoint functors  $(\text{Fr}, \mathcal{Q})$  where  $\mathcal{Q}$  was defined levelwise, there will be a pair  $(\mathcal{P}, \text{Cof})$  of functors where  $\text{Cof}$  is defined levelwise.

A functor  $B: \text{Mod}_k \rightarrow \text{Mod}_l$  in  $\widehat{k\text{Mod}}_l$  can be propagated to a functor

$$\text{Cof}(B): \text{Alg}_k \rightarrow \text{Mod}_l \quad \text{in } \widehat{k\text{Hopf}}_l$$

by forgetting the  $k$ -algebra structure. To see that this is representable, note that the forgetful functor  $\text{Pro-Alg}_k \rightarrow \text{Pro-Mod}_k$  has a left adjoint given levelwise by  $\text{Sym}: \text{Pro-Mod}_k \rightarrow \text{Pro-Alg}_k$ , the free commutative algebra on a pro- $k$ -module. Explicitly, for  $M \in \text{Mod}_k$ ,  $\text{Sym}(M)$  is the the  $k$ -algebra  $\bigoplus_{i=0}^{\infty} \text{Sym}^i(M)$ , where  $\text{Sym}^i(M) = M^{\otimes_k i} / (m_1 \otimes m_2 - m_2 \otimes m_1)$ . The adjunction

$$(9.5) \quad \text{Pro-Alg}_k(\text{Sym}(M), R) \cong \text{Pro-Mod}_k(M, R)$$

shows that it indeed represents  $\text{Cof}(B)$  if  $M = \mathcal{O}_B$ .

This algebra has a cozero  $\text{Sym}(M) \rightarrow \text{Sym}^0(M) = k$  and a coaddition given by decreeing that all elements of  $M = \text{Sym}^1(M)$  are primitive. Furthermore, if  $M = \mathcal{O}_B$  for a formal bimodule  $B \in \widehat{k\text{Mod}}_l$ , then it has an  $l$ -action defined by

$$l \rightarrow \text{Pro-Mod}_k(M, M) \rightarrow \text{Pro-Alg}_k(\text{Sym}(M), \text{Sym}(M)),$$

which makes the formal  $l$ -module scheme structure over  $k$  explicit. Note that  $\text{Sym}(M)$  is in fact a pro-Hopf algebra. Summarizing, we have constructed an enriched functor

$$\text{Cof}: \widehat{k\text{Mod}}_l \rightarrow \widehat{k\text{Hopf}}_l.$$

**Definition.** The left adjoint of  $\text{Cof}$  is denoted by  $\mathcal{P} = \mathcal{O}_P$ , the formal bimodule of primitives of a formal module scheme.

Of course, one could define the primitives  $P(C)$  for a pointed coalgebra  $C$ , but we will make no use of that. Dually to the situation for indecomposables, we can think of  $P(A)$  explicitly as those elements in  $A$  such that  $\psi_+(a) = a \otimes 1 + 1 \otimes a$ , but it would require unnecessary elaboration to say what this means in our pro-setting.

For  $F \in \widehat{k\text{Hopf}}_{\mathbf{Z}}$ , there is an alternative useful construction of  $\mathcal{P}(F)$ . Note that there are two maps

$$\text{Fr}(F) \rightrightarrows F$$



given by the module scheme structure (Cor. 6.5) and by the map represented by the inclusion  $\mathcal{O}_F \hookrightarrow \Gamma_{\mathbf{Z}}(\mathcal{O}_F)$ . Then  $\mathcal{P}(F)$  is the coequalizer of these two maps in the category  ${}_k\widehat{\text{Mod}}_{\mathbf{Z}}$ .

**Lemma 9.6.** *The functor  $\mathcal{P}$  is left inverse to  $\text{Fr}$  and the functor  $\mathcal{Q}$  is left inverse to  $\text{Cof}$ . More precisely, for  $B \in {}_k\widehat{\text{Mod}}_I$ , the natural map*

$$\text{Fr}(B) \rightarrow \text{Cof}(B)$$

*has adjoints  $\mathcal{P}(\text{Fr}(B)) \rightarrow B$  and  $B \rightarrow \mathcal{Q}(\text{Cof}(B))$  which are isomorphisms. Moreover, if  $X \in \text{Sch}_k$  then  $\mathcal{P}(\text{Fr}(X)) = U(X)$ , where  $U$  denotes the forgetful functor  $\text{Sch}_k \rightarrow (\text{Pro} - \text{Mod}_k)^{\text{op}}$ .*

*Proof.* One adjoint is an isomorphism if and only if the other is, so it suffices to check this for  $\mathcal{P}(\text{Fr}(B))$ , which follows from the second claim on a general formal scheme  $X$ . In that case, note that the coequalizer defining  $\mathcal{P}(\text{Fr}(X))$ ,

$$\text{Fr}(\text{Fr}(X)) \rightrightarrows \text{Fr}(X) \rightarrow \mathcal{P}(\text{Fr}(X))$$

splits by the canonical inclusion  $X \rightarrow \text{Fr}(X)$ , thus the composite  $X \rightarrow \mathcal{P}(\text{Fr}(X))$  gives the claimed isomorphism.  $\square$

**Lemma 9.7.** *The functor  $\text{Cof}: {}_k\widehat{\text{Mod}}_I \rightarrow {}_k\widehat{\text{Hopf}}_I$  is right strict morphism of 2-modules over  $\text{Mod}_I$ , and the functor  $\mathcal{P}: {}_k\widehat{\text{Hopf}}_I \rightarrow {}_k\widehat{\text{Mod}}_I$  is left strict, i.e. for  $M \in \text{Mod}_I$ ,  $F \in {}_k\widehat{\text{Hopf}}_I$ , and  $B \in {}_k\widehat{\text{Mod}}_I$ , the natural maps*

$$\beta: \text{hom}_I(M, \text{Cof}(B)) \rightarrow \text{Cof}(\text{hom}_I(M, B))$$

*and*

$$\alpha: M \otimes_I \mathcal{P}(F) \rightarrow \mathcal{P}(M \otimes_I F)$$

*are isomorphisms.*

*Proof.* Let  $T \in {}_k\widehat{\text{Hopf}}_{\mathbf{Z}}$  be a test object. Then

$$\begin{aligned} {}_k\widehat{\text{Hopf}}_{\mathbf{Z}}(T, \text{Cof}(\text{hom}_I(M, B))) &\cong \text{Pro} - \text{Mod}_k(M \otimes_I \mathcal{O}_B, P(\mathcal{O}_T)) \\ &= \text{Mod}_I(M, {}_k\widehat{\text{Mod}}_{\mathbf{Z}}(\mathcal{P}(T), B)) \\ &= \text{Mod}_I(M, {}_k\widehat{\text{Hopf}}_{\mathbf{Z}}(T, \text{Cof}(B))) \\ &= {}_k\widehat{\text{Hopf}}_{\mathbf{Z}}(T, \text{hom}_I(M, \text{Cof}(B))). \end{aligned}$$

The statement about  $\mathcal{P}$  follows from adjointness.  $\square$

**Proposition 9.8.** *The functor  $\mathcal{P}: {}_k\widehat{\text{Hopf}}_I \rightarrow {}_k\widehat{\text{Mod}}_I$  is an oplax morphism of 2-algebras over  $\text{Mod}_I$ . The oplax structure  $\psi: \mathcal{P}(F \otimes_I G) \rightarrow \mathcal{P}(F) \otimes_I \mathcal{P}(G)$  is an isomorphism if  $F, G, \mathcal{P}(F)$ , and  $\mathcal{P}(G)$  are represented by pro-flat  $k$ -modules.*

*Proof.* Recall that the unit of  $\otimes_I$  in  ${}_k\widehat{\text{Hopf}}_I$  is  $\text{Spf}(\text{hom}(l, k))$ , whereas the unit of  $\otimes_I$  in  ${}_k\widehat{\text{Mod}}_I$  is  $\text{hom}_{\mathbf{Z}}(l, k)$ . The augmentation ideal of  $\text{hom}(l, k)$  consists those functions  $f: l \rightarrow k$  such that  $f(0) = 0$ ; the primitives consist of those functions that are a fortiori additive. Thus,

$$P(\text{hom}(l, k)) = \text{hom}_{\mathbf{Z}}(l, k).$$

The tensor product in  ${}_k\widehat{\text{Hopf}}_I$  was defined in (6.18) as

$$F \otimes_I G = \text{coeq}\left(\text{Fr}_I(F \times G) \rightrightarrows \text{Fr}_I(\text{Fr}_I(F, G))\right)$$

where  $\text{Fr}_l(F, G) = \text{Fr}_l(F) \times G \cup F \times \text{Fr}_l(G)$  (cf. (6.17)). Let us first assume that  $l = \mathbf{Z}$  and denote the forgetful functor  ${}_k\widehat{\text{Hopf}}_l \rightarrow (\text{Pro-Mod}_k)^{\text{op}} = {}_k\widehat{\text{Mod}}_{\mathbf{Z}}$  by  $U$ . Then we have the adjunctions

$$\begin{aligned} & {}_k\widehat{\text{Mod}}_{\mathbf{Z}}(\mathcal{P}(F \otimes_{\mathbf{Z}} G), B) \\ & \cong {}_k\widehat{\text{Hopf}}_{\mathbf{Z}}(F \otimes_{\mathbf{Z}} G, \text{Cof}(B)) \\ & = \text{eq}\left({}_k\widehat{\text{Hopf}}_{\mathbf{Z}}(\text{Fr}_{\mathbf{Z}}(F \times G), \text{Cof}(B)) \rightrightarrows {}_k\widehat{\text{Hopf}}_{\mathbf{Z}}(\text{Fr}_{\mathbf{Z}}(\text{Fr}_{\mathbf{Z}}(F, G)), \text{Cof}(B))\right) \\ & \stackrel{\text{Lemma 9.6}}{=} \text{eq}\left({}_k\widehat{\text{Mod}}_{\mathbf{Z}}(U(F \times G), B) \rightrightarrows {}_k\widehat{\text{Mod}}_{\mathbf{Z}}(U(\text{Fr}_{\mathbf{Z}}(F, G)), B)\right) \\ & = {}_k\widehat{\text{Mod}}_{\mathbf{Z}}(\text{coeq}(U(\text{Fr}_{\mathbf{Z}}(F, G)) \rightrightarrows U(F \times G)), B). \end{aligned}$$

Since the coequalizer of  $U(\text{Fr}_{\mathbf{Z}}(F)) \rightrightarrows U(F)$  is  $\mathcal{P}(F)$ , there is a canonical fork

$$U(\text{Fr}_{\mathbf{Z}}(F, G)) \rightrightarrows U(F \times G) \rightarrow \mathcal{P}(F) \otimes_{\mathbf{Z}} \mathcal{P}(G),$$

or in terms of representing objects, a fork of pro- $k$ -modules

$$P\mathcal{O}_F \otimes_k P\mathcal{O}_G \rightarrow \mathcal{O}_F \otimes_k \mathcal{O}_G \rightrightarrows (\Gamma\mathcal{O}_F \otimes_k \mathcal{O}_G) \times_{\mathcal{O}_F \otimes_k \mathcal{O}_G} (\mathcal{O}_F \otimes_k \Gamma\mathcal{O}_G)$$

which is an equalizer if  $\mathcal{O}_F, \mathcal{O}_G, P\mathcal{O}_F, P\mathcal{O}_G$  are all pro-flat  $k$ -modules.

For general  $l$ , we can write  $F \otimes_l G$  as a coequalizer (Lemma 6.22)

$$l \otimes_{\mathbf{Z}} F \otimes_{\mathbf{Z}} G \rightrightarrows F \otimes_{\mathbf{Z}} G \rightarrow F \otimes_l G,$$

which by the previous steps and Lemma 9.7, and since  $\mathcal{P}$ , as a left adjoint, commutes with coequalizers, gives an equalizer

$$P(\mathcal{O}_F \overset{l}{\otimes}_k \mathcal{O}_G) \rightarrow P\mathcal{O}_F \otimes_k P\mathcal{O}_G \rightrightarrows \text{hom}_{\mathbf{Z}}(l, P\mathcal{O}_F \otimes_k P\mathcal{O}_G),$$

which is, by definition,  $P\mathcal{O}_F \overset{l}{\otimes}_k P\mathcal{O}_G$ .  $\square$

The following corollary follows immediately from the adjunctions.

**Corollary 9.9.** *The functor  $\text{Cof}$  is a lax morphism of 2-algebras, i.e. for  $B, C \in {}_k\widehat{\text{Mod}}_l$ , there are natural transformations*

$$\begin{aligned} \phi_0: \text{Spf}(\text{hom}(l, k)) &\rightarrow \text{Cof}(\text{Spf}(\text{hom}_{\mathbf{Z}}(l, k))) \quad \text{and} \\ \phi: \text{Cof}(B) \otimes_l \text{Cof}(C) &\rightarrow \text{Cof}(B \otimes_l C) \end{aligned}$$

turning  $\text{Cof}$  into a lax monoidal functor.  $\square$

**9.3. Primitives and indecomposables of formal algebra schemes.** We will now consider what the additional structure of a formal algebra scheme maps to under the functors  $\mathcal{P}$  and  $\mathcal{Q}$ .

Let  $F \in {}_k\widehat{\text{Alg}}_l$  be a formal  $l$ -algebra scheme over  $k$  and let  $R \in \text{Alg}_l^+$  be an  $l$ -algebra augmented by  $\epsilon_R: R \rightarrow l$ . Then  $\text{hom}(R, F)$  is a pointed formal scheme over  $k$  by the map  $\text{Spf}(k) \rightarrow \text{hom}(R, F)$  adjoint to  $R \xrightarrow{\epsilon_R} l \xrightarrow{\epsilon} F(k)$ . This produces thus a functor

$$\text{hom}(-, F): \text{Alg}_l^+ \rightarrow (\text{Sch}_k^+)^{\text{op}}$$

which has a right adjoint

$$F_+ = l \times_{F(k)} F(-): \text{Alg}_k^+ \rightarrow \text{Alg}_l^+,$$

an augmented version of  $F$  with  $F_+(k) = l$ .

**Proposition 9.10.** *Given  $F \in {}_k\widehat{\text{Alg}}_l$  and  $B \in {}_k\widehat{\text{Mod}}_l$ , there is a natural  $\text{Alg}_l^+$ -isomorphism*

$$F_+(k \oplus \mathcal{O}_B) \cong l \oplus \mathcal{Q}(F)(\mathcal{O}_B).$$

*Proof.* Equivalently, we need to see that

$$(9.11) \quad \text{Pro} - \text{Alg}_k^+(\mathcal{O}_F, k \oplus \mathcal{O}_B) \cong \text{Pro} - \text{Alg}_k^+(\mathcal{O}_F, k) \oplus \text{Pro} - \text{Mod}_k(Q\mathcal{O}_F, \mathcal{O}_B).$$

An augmented pro-algebra map  $\mathcal{O}_F \rightarrow k \oplus \mathcal{O}_B$  consists of a pair  $(f, g)$  with  $f: \mathcal{O}_F \rightarrow k$  and  $g: \mathcal{O}_F \rightarrow B$  such that  $f$  is an algebra map,  $g$  is a  $k$ -module map, and  $g(ab) = f(a)g(b) + g(a)f(b)$ . We produce the isomorphism (9.11) by sending  $(f, g)$  to  $(f, a \mapsto g(a - f(a)))$ . To see that this is an isomorphism of  $l$ -algebras, with the square-zero structure on the right hand side, we note that the comultiplication  $\psi_\times$  restricts to a map

$$\psi_\times: (\mathcal{O}_F)_+ \rightarrow (\mathcal{O}_F)_+ \otimes_k (\mathcal{O}_F)_+$$

because, dually, multiplication with 0 on either side gives 0. Thus the product of any two maps in  $\text{Pro} - \text{Mod}_k(Q\mathcal{O}_F, B)$  is zero.  $\square$

**Corollary 9.12.** *For  $R \in \text{Alg}_l^+$  and  $F \in {}_k\widehat{\text{Alg}}_l$ ,*

$$\mathcal{Q}(\text{hom}(R, F)) \cong \text{hom}_l(Q(R), \mathcal{Q}(F)).$$

*Proof.* This follows from the adjunction of the previous proposition: for  $M \in \text{Mod}_k$ ,

$$\begin{aligned} \text{Pro} - \text{Mod}_k(Q(R \circ_l \mathcal{O}_F), M) &\cong \text{Pro} - \text{Alg}_k^+(R \circ_l \mathcal{O}_F, k \oplus M) \\ &\cong \text{Alg}_l^+(R, \text{Pro} - \text{Alg}_k^+(\mathcal{O}_F, k \oplus M)) \\ &\cong \text{Alg}_l^+(R, l \oplus \text{Pro} - \text{Mod}_k(Q\mathcal{O}_F, M)) \\ &\cong \text{Mod}_l(QR, \text{Pro} - \text{Mod}_k(Q\mathcal{O}_F, M)) \\ &\cong \text{Pro} - \text{Mod}_k(QR \circ_l Q\mathcal{O}_F, M). \end{aligned} \quad \square$$

To understand how  $\mathcal{P}$  behaves with respect to the pairing

$$\text{hom}: \text{Alg}_l \times {}_k\widehat{\text{Alg}}_l \rightarrow \text{Sch}_k$$

of Section 7, we need the following lemma, whose proof is a short series of standard adjunctions and will be left to the reader.

**Lemma 9.13.** *Let  $M \in \text{Mod}_l$ ,  $F \in {}_k\widehat{\text{Alg}}_l$ , and denote by  $U(F) \in {}_k\widehat{\text{Hopf}}_l$  the formal module scheme obtained by forgetting the multiplicative structure. Then there is a natural isomorphism*

$$\text{hom}(\text{Sym}(M), F) \cong \text{hom}_l(M, U(F)). \quad \square$$

To even make sense of a compatibility statement of  $\mathcal{P}$  with  $\text{hom}(R, F)$ , the latter has to have a Hopf algebra structure. By naturality in the  $R$ -variable, this happens if  $R$  is a Hopf algebra. Denote the category of  $l$ -Hopf algebras (or  $\mathbf{Z}$ -module schemes over  $l$ ) by  ${}_Z\text{Hopf}_l$ .

**Proposition 9.14.** *For  $R \in {}_Z\text{Hopf}_l$  and  $F \in {}_k\widehat{\text{Alg}}_l$ , there is a natural map*

$$\beta: \mathcal{P}(\text{hom}(R, F)) \rightarrow \text{hom}_l(\mathcal{P}(R), \mathcal{P}(F)).$$

*Proof.* The natural map  $\beta$  is given as the adjoint of the map

$$\mathrm{hom}(R, F) \rightarrow \mathrm{Cof}(\mathrm{hom}_l(P(R), \mathcal{P}(F)))$$

which is the composite

$$\begin{aligned} \mathrm{hom}(R, F) &\xrightarrow{\epsilon^*} \mathrm{hom}(\mathrm{Sym}(P(R)), F) \xrightarrow[\cong]{\text{Lemma 9.13}} \mathrm{hom}_l(P(R), U(F)) \\ &\xrightarrow{\eta} \mathrm{hom}_l(P(R), \mathrm{Cof}(\mathcal{P}(F))) \xrightarrow[\cong]{\text{Lemma 9.7}} \mathrm{Cof}(\mathrm{hom}_l(P(R), \mathcal{P}(F))). \end{aligned}$$

where  $\epsilon: \mathrm{Sym}(P(R)) \rightarrow R$  and  $\eta: F \rightarrow \mathrm{Cof}(\mathcal{P}(F))$  are counit and unit of the respective adjunctions.  $\square$

The transformation  $\beta$  will in general not be an isomorphism. Indeed, if we choose  $R = l[z]$  as a polynomial ring then  $\mathrm{hom}(R, F) = \mathrm{hom}_l(l, U(F)) = U(F)$  by Lemma 9.13 and thus  $\mathcal{P}(\mathrm{hom}(R, F)) \cong \mathcal{P}(F)$ . On the other hand,  $P(R) = P(l[z])$  is in general greater than  $l$  (for instance if  $l$  has positive characteristic), and thus  $\mathrm{hom}_l(P(R), \mathcal{P}(F)) \neq \mathcal{P}(F)$ .

**Lemma 9.15.** *The following diagram commutes:*

$$\begin{array}{ccc} \mathcal{P}(\mathrm{hom}_l(PR, U(F))) & \xleftarrow{\text{Lemma 9.13}} & \mathcal{P}(\mathrm{hom}(\mathrm{Sym} PR, F)) \\ \beta \uparrow \text{Mod}_l\text{-enrichment} & & \uparrow \epsilon^* \\ \mathrm{hom}_l(PR, \mathcal{P}(F)) & \xleftarrow{\beta} & \mathcal{P}(\mathrm{hom}(R, F)) \end{array}$$

*Proof.* By adjointness of  $\mathcal{P}$  and  $\mathrm{Cof}$ , it suffices to show that the exterior of the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{Cof} \mathcal{P}(\mathrm{hom}_l(PR, U(F))) & \xleftarrow{\eta} & \mathrm{hom}_l(PR, U(F)) & \xleftarrow[\cong]{\text{Lemma 9.13}} & \mathrm{hom}(\mathrm{Sym} PR, F) \\ \uparrow \mathrm{Cof}(\beta) & & \parallel & & \uparrow \epsilon^* \\ \mathrm{Cof} \mathrm{hom}(PR, \mathcal{P}(F)) & & & & \mathrm{hom}(R, F) \\ \uparrow \cong \text{Lemma 9.7} & & & & \downarrow \eta \circ \mathrm{id} \\ \mathrm{hom}_l(PR, \mathrm{Cof} \mathcal{P}(F)) & \xleftarrow{\epsilon^*} & \mathrm{hom}_l(PR, U(F)) & \xleftarrow[\cong]{\text{Lemma 9.13}} & \mathrm{hom}(\mathrm{Sym} PR, F) \end{array}$$

In fact, the smaller pentagon on the left already commutes because  $\mathrm{Cof}$  and  $\mathcal{P}$  are enriched functors.  $\square$

From Corollary 7.1 we find that the multiplication  $\mu_\times$  on  $F$  gives rise to morphisms of bimodule schemes

$$\mathcal{Q}(F) \otimes_l \mathcal{Q}(F) \xrightarrow{\text{Cor. 9.4}} \mathcal{Q}(F \otimes_l F) \xrightarrow{\mathcal{Q}(\mu_\times)} \mathcal{Q}(F)$$

and, under the conditions in Prop. 9.8 for  $\mathcal{P}$  being strict,

$$\mathcal{P}(F) \otimes_l \mathcal{P}(F) \xrightarrow{\text{Prop. 9.8}} \mathcal{P}(F \otimes_l F) \xrightarrow{\mathcal{P}(\mu_\times)} \mathcal{P}(F)$$

Thus both  $\mathcal{Q}(F)$  and  $\mathcal{P}(F)$  are algebras in  $({}_k \widehat{\mathrm{Mod}}_l, \otimes_l, \mathrm{Spf}(\mathrm{hom}_{\mathbf{Z}}(l, k)))$ .

**Proposition 9.16.** *Assume  $F, G \in {}_k\widehat{\text{Alg}}_l$  are such that  $F, G, \mathcal{P}(F), \mathcal{P}(G), \mathcal{Q}(F)$ , and  $\mathcal{Q}(G)$  are pro-flat  $k$ -modules. Then the lax structure of  $\mathcal{Q}$  of Cor. 9.4 factors as a map*

$$\mathcal{Q}(F) \otimes_l \mathcal{Q}(G) \rightarrow \mathcal{Q}(F) \otimes_l \mathcal{P}(G) \times_{\mathcal{Q}(F) \otimes_l \mathcal{Q}(G)} \mathcal{P}(F) \otimes_l \mathcal{Q}(G) \rightarrow \mathcal{Q}(F \otimes_l G).$$

For  $F = G$ , this makes  $\mathcal{Q}(F)$  into a two-sided module over the algebra  $\mathcal{P}(F)$ .

*Proof.* By commutativity, it suffices to produce the factorization

$$\mathcal{Q}(F) \otimes_l \mathcal{Q}(G) \rightarrow \mathcal{Q}(F) \otimes_l \mathcal{P}(G) \rightarrow \mathcal{Q}(F \otimes_l G)$$

First assume  $l = \mathbf{Z}$ . The first step is to produce a factorization in  $\text{Pro-Mod}_k$

$$Q(\mathcal{O}_F \mathbb{Z}_{\boxtimes k} \mathcal{O}_G) \rightarrow Q\mathcal{O}_F \otimes_k (\mathcal{O}_G)_+ \rightarrow Q\mathcal{O}_F \otimes_k Q\mathcal{O}_G,$$

where  $(\mathcal{O}_G)_+$  denotes the augmentation ideal of  $\mathcal{O}_G$ .

The adjunction between  $V^+ : {}_k\widehat{\text{Hopf}}_{\mathbf{Z}}^{\text{op}} \rightarrow \text{Pro-Alg}_k^+$  and  $\Gamma_+$  gives a unit map

$$\mathcal{O}_G \xrightarrow{\eta'} \Gamma^+(V^+(\mathcal{O}_G)) = \Gamma((\mathcal{O}_G)_+)$$

which together with the unit adjunction  $\mathcal{O}_F \xrightarrow{\eta} \Gamma Q\mathcal{O}_F$  gives a map

$$(9.17) \quad \mathcal{O}_F \mathbb{Z}_{\boxtimes k} \mathcal{O}_G \rightarrow \Gamma^+(k \oplus Q\mathcal{O}_F \mathbb{Z}_{\boxtimes k} \Gamma^+ V^+ \mathcal{O}_G) \\ \xrightarrow[\cong]{\text{Lemma 6.19}} \Gamma^+((k \oplus Q\mathcal{O}_F) \otimes_k^+ V^+ \mathcal{O}_G).$$

Here  $\otimes_k^+$  denotes the tensor product of augmented pro- $k$ -algebras, i.e. the operation classifying the smash product of two pointed formal schemes. Since the augmented algebra  $k \oplus Q\mathcal{O}_F$  is a square-zero extension, so is

$$(k \oplus Q\mathcal{O}_F) \otimes_k^+ V^+ \mathcal{O}_G \cong k \oplus (Q\mathcal{O}_F \otimes_k (\mathcal{O}_G)_+).$$

The composite of (9.17) with the isomorphism

$$\Gamma^+((k \oplus Q\mathcal{O}_F) \otimes_k^+ V^+ \mathcal{O}_G) \cong \Gamma^+(k \oplus (Q\mathcal{O}_F \otimes_k (\mathcal{O}_G)_+)) \cong \Gamma(Q\mathcal{O}_F \otimes_k (\mathcal{O}_G)_+)$$

has an adjoint  $\tilde{\psi} : Q(\mathcal{O}_F \mathbb{Z}_{\boxtimes k} \mathcal{O}_G) \rightarrow Q\mathcal{O}_F \otimes_k (\mathcal{O}_G)_+$ , which is the desired factorization.

To show that this map factors further through  $Q\mathcal{O}_F \otimes_k P\mathcal{O}_G$ , it suffices by the flatness assumption on  $Q\mathcal{O}_F$  to show that

$$Q(\mathcal{O}_F \mathbb{Z}_{\boxtimes k} \mathcal{O}_G) \rightarrow Q\mathcal{O}_F \otimes_k (\mathcal{O}_G)_+ \xrightarrow{\text{id} \otimes \psi_+} Q\mathcal{O}_F \otimes_k (\mathcal{O}_G)_+ \otimes_k (\mathcal{O}_G)_+$$

is null. This map fits into a diagram

$$\begin{array}{ccccc} Q\mathcal{O}_F \otimes_k (\mathcal{O}_G)_+ \otimes_k (\mathcal{O}_G)_+ & \xleftarrow{\tilde{\psi}} & Q(\mathcal{O}_F \mathbb{Z}_{\boxtimes k} (\mathcal{O}_G \otimes \mathcal{O}_G)) & \xleftarrow{\cong} & Q((\mathcal{O}_F \mathbb{Z}_{\boxtimes k} \mathcal{O}_G)^{\otimes 2}) \\ \uparrow QA \otimes \psi_+ & & \uparrow Q(\text{id} \mathbb{Z}_{\boxtimes k} \psi_+) & \nearrow Q(\psi_+) & \uparrow \cong \\ Q\mathcal{O}_F \otimes_k (\mathcal{O}_G)_+ & \xleftarrow{\tilde{\psi}} & Q(\mathcal{O}_F \mathbb{Z}_{\boxtimes k} \mathcal{O}_G) & \xrightarrow{\Delta} & Q(\mathcal{O}_F \mathbb{Z}_{\boxtimes k} \mathcal{O}_G)^2. \end{array}$$

The composites  $Q(\mathcal{O}_F \mathbb{Z}_{\boxtimes k} \mathcal{O}_G) \rightarrow Q(\mathcal{O}_F \mathbb{Z}_{\boxtimes k} (\mathcal{O}_G \otimes \mathcal{O}_G))$  from the lower right hand corner to the upper middle entry are induced by the inclusions  $\eta \times \text{id}, \text{id} \times \eta : \mathcal{O}_G \rightarrow \mathcal{O}_G \otimes \mathcal{O}_G$  and thus both compose to the zero map when followed with  $\tilde{\psi}$ .

It remains to produce a formal bimodule map  $\mathcal{Q}(F) \otimes_l \mathcal{P}(G) \rightarrow \mathcal{Q}(F \otimes_l G)$  for arbitrary rings  $l$ . For this, consider the diagram

$$\begin{array}{ccccc}
 \mathcal{Q}(F \otimes_l G) & \longleftarrow & \mathcal{Q}(F \otimes_{\mathbb{Z}} G) & \longleftarrow & \mathcal{Q}(l \otimes_{\mathbb{Z}} F \otimes_{\mathbb{Z}} G) \\
 \uparrow \text{dotted} & & \uparrow \tilde{\psi} & & \uparrow \text{Lemma 9.2} \\
 & & & & l \otimes_{\mathbb{Z}} \mathcal{Q}(F \otimes_{\mathbb{Z}} G) \\
 & & & & \uparrow \tilde{\psi} \\
 \mathcal{Q}(F) \otimes_l \mathcal{Q}(G) & \longleftarrow & \mathcal{Q}(F) \otimes_{\mathbb{Z}} \mathcal{Q}(G) & \longleftarrow & l \otimes \mathcal{Q}(F) \otimes_{\mathbb{Z}} \mathcal{Q}(G).
 \end{array}$$

The upper row is not necessarily a coequalizer, but a fork, coming from the characterization of the tensor product from Lemma 6.22. The lower row is the coequalizer defining the tensor product of formal bimodules. Therefore the dotted map exists.  $\square$

**9.4. Primitives and indecomposables of formal plethories.** Now let  $F \in {}_k\widehat{\text{Alg}}_k$  be a formal plethory with the flatness assumptions of Prop. 9.16. By Lemma 9.4 and Prop. 9.8,  $\mathcal{P}(F)$  and  $\mathcal{Q}(F)$  are both algebras with respect to the  $\otimes_k$ -product. By Prop. 9.16,  $\mathcal{Q}(F)$  is also  $\mathcal{P}(F)$ -bimodule algebra with respect to  $\otimes_k$ .

**Lemma 9.18.** *The functor  $\text{Cof}: {}_k\widehat{\text{Mod}}_k \rightarrow {}_k\widehat{\text{Alg}}_k$  is strict monoidal with respect to the  $\circ_k$ -product.*

*Proof.* The unit isomorphism  $\text{Spf}(k[z]) \rightarrow \text{Cof}(\text{Spf}(k))$  is represented by the canonical isomorphism  $\text{Sym}(k) \cong k[z]$ . If  $B, C \in {}_k\widehat{\text{Mod}}_k$ , we construct an isomorphism

$$\begin{aligned}
 \text{Cof}(B) \circ_k \text{Cof}(C) &= \text{colim}_i \text{hom}(\text{Sym}(\mathcal{O}_B(i)), \text{Cof}(C)) \\
 &\cong \text{colim}_i \text{hom}_k(\mathcal{O}_B(i), U(\text{Cof}(C))) \\
 &\stackrel{\text{Lemma 9.13}}{\cong} \text{colim}_i \text{Cof}(\text{hom}_k(\mathcal{O}_B(i), C)) \\
 &\stackrel{\text{Lemma 9.7}}{\cong} \text{Cof}(\text{colim}_i \text{hom}_k(\mathcal{O}_B(i), C)) \cong \text{Cof}(B \circ_k C).
 \end{aligned}$$

Although  $\text{Cof}$  is a right adjoint, it does commute with filtered colimits (used in the last line) because its representing functor  $\text{Sym}$  is induced up from  $\text{Mod}_k \rightarrow \text{Alg}_k$  to  $\text{Pro-Mod}_k \rightarrow \text{Pro-Alg}_k$  by objectwise application.  $\square$

**Theorem 9.19.** *The functor  $\text{Cof}$  is a bilax monoidal functor.*

*Proof.* It was shown in Lemma 9.18 and Corollary 9.9 that  $\text{Cof}$  is strict monoidal with respect to  $\circ_k$  and lax monoidal with respect to  $\otimes_k$ , and it remains to check the compatibility conditions (8.1) and (8.2).

For the unitality diagrams in (8.1), first recall that the comparison map between the units  $\iota_J: I \rightarrow J$  is given as

$$k \rightarrow \text{hom}_{\mathbb{Z}}(k, k); \quad 1 \mapsto \text{id} \quad \text{in } {}_k\widehat{\text{Mod}}_k$$

and

$$k[x] = \text{Sym}(k) \rightarrow \text{hom}(k, k); \quad x \mapsto \text{id} \quad \text{in } {}_k\widehat{\text{Alg}}_k;$$

note that since the source,  $I$ , is a constant pro-object, we can disregard the pro-structure in the target and simply map to its inverse limit.

The commutativity of the first diagram in (8.1) follows from the factorization

$$k \xrightarrow{t_{\text{hom}_{\mathbf{Z}}(k,k)}} \text{hom}_{\mathbf{Z}}(k, k) \hookrightarrow U(\text{hom}(k, k)),$$

where  $U$  is the forgetful functor from  $\text{Pro} - \text{Alg}_k$  to  $\text{Pro} - \text{Mod}_k$  and the last map is the inclusion of linear maps into all maps.

For the second diagram in (8.1), we proceed in two steps. First we use the tensoring  $\hat{\circ}_l$  between  $\widehat{l\text{Mod}}_m$  and  $\widehat{k\text{Hopf}}_l$  from Lemma 8.6 to prove that the following diagram commutes:

$$\begin{array}{ccc}
 & & \text{Sym}(\text{hom}_{\mathbf{Z}}(k, k)) \\
 & \swarrow \phi_0 & \uparrow F(\Delta_k) \\
 \text{hom}(k, k) & & \text{Sym}(\text{hom}_{\mathbf{Z}}(k, k) \hat{\circ}_l \text{hom}_{\mathbf{Z}}(k, k)) \\
 \uparrow \mu & & \uparrow \beta \\
 \text{hom}_{\mathbf{Z}}(k, k) \hat{\circ}_l U \text{hom}(k, k) & \xleftarrow{\text{id} \circ \phi_0} & \text{hom}_{\mathbf{Z}}(k, k) \hat{\circ}_l U \text{Sym}(\text{hom}_{\mathbf{Z}}(k, k)) \\
 \uparrow \cong & & \uparrow \cong \\
 \text{Sym} \text{hom}_{\mathbf{Z}}(k, k) \hat{\circ}_l \text{hom}(k, k) & \xleftarrow{\text{id} \circ \phi_0} & \text{Sym} \text{hom}_{\mathbf{Z}}(k, k) \hat{\circ}_l \text{Sym} \text{hom}_{\mathbf{Z}}(k, k)
 \end{array}$$

The commutativity of the top rectangle is Lemma 2.11, while the commutativity of the lower square is the naturality of the isomorphism of Lemma 9.13. The result then follows from the commutativity of the diagram

$$\begin{array}{ccc}
 \text{hom}(k, k) & \xleftarrow{\mu} & \text{hom}_{\mathbf{Z}}(k, k) \hat{\circ}_l U \text{hom}(k, k) \\
 \Delta_k \uparrow & & \cong \uparrow \\
 \text{hom}(k, k) \hat{\circ}_k \text{hom}(k, k) & \xleftarrow{\phi_0 \hat{\circ}_k \text{id}} & \text{Sym} \text{hom}_{\mathbf{Z}}(k, k) \hat{\circ}_l \text{hom}(k, k)
 \end{array}$$

The commutativity of the third unitality diagram in (8.1) is formal because  $\otimes_k$  is the categorical coproduct in  $\widehat{k\text{Alg}}_k$ .

It remains to show the commutativity of (8.2). We proceed similarly as above and first note that the following diagram commutes for  $A, B, C, D$  representing objects of  $\widehat{k\text{Mod}}_k$  because of Lemma 2.11:

$$\begin{array}{ccc}
 \text{Sym}(A \hat{\circ}_k B) \overset{k}{\boxtimes}_k \text{Sym}(C \hat{\circ}_k D) & \xleftarrow{\phi} & \text{Sym}((A \hat{\circ}_k B) \overset{k}{\otimes}_k (C \hat{\circ}_k D)) \\
 \cong \uparrow & & \text{Sym}(\zeta) \uparrow \\
 (A \hat{\circ}_k \text{Sym}(B)) \overset{k}{\boxtimes}_k (C \hat{\circ}_k \text{Sym}(D)) & & \text{Sym}((A \overset{k}{\otimes}_k C) \hat{\circ}_k (B \overset{k}{\otimes}_k D)) \\
 \zeta \uparrow & & \cong \uparrow \\
 (A \overset{k}{\otimes}_k C) \hat{\circ}_k (\text{Sym}(B) \overset{k}{\boxtimes}_k \text{Sym}(D)) & \xleftarrow{\text{id} \hat{\circ}_k \phi} & (A \overset{k}{\otimes}_k C) \hat{\circ}_k \text{Sym}(B \overset{k}{\otimes}_k D)
 \end{array}$$

The result then follows from the commutativity of the following diagram:

$$\begin{array}{ccc}
(\mathrm{Sym} A \hat{\circ}_k \mathrm{Sym} B) \overset{k}{\boxtimes}_k (\mathrm{Sym} C \hat{\circ}_k \mathrm{Sym} D) & \xrightarrow{\cong} & (A \hat{\circ}_k \mathrm{Sym} B) \overset{k}{\boxtimes}_k (C \hat{\circ}_k \mathrm{Sym} D) \\
\uparrow \zeta & & \uparrow \zeta \\
(\mathrm{Sym} A \overset{k}{\boxtimes}_k \mathrm{Sym} C) \hat{\circ}_k (\mathrm{Sym} B \overset{k}{\boxtimes}_k \mathrm{Sym} D) & & \\
\uparrow \phi \hat{\circ}_k \mathrm{id} & & \\
(\mathrm{Sym}(A \overset{k}{\otimes}_k C)) \hat{\circ}_k (\mathrm{Sym} B \overset{k}{\boxtimes}_k \mathrm{Sym} D) & \xrightarrow{\cong} & (A \overset{k}{\otimes}_k C) \hat{\circ}_k (\mathrm{Sym} B \overset{k}{\boxtimes}_k \mathrm{Sym} D) \\
\uparrow \mathrm{id} \hat{\circ}_k \phi & & \uparrow \mathrm{id} \hat{\circ}_k \phi \\
\mathrm{Sym}(A \overset{k}{\otimes}_k C) \hat{\circ}_k \mathrm{Sym}(B \overset{k}{\otimes}_k D) & \xrightarrow{\cong} & (A \overset{k}{\otimes}_k C) \hat{\circ}_k \mathrm{Sym}(B \overset{k}{\otimes}_k D)
\end{array}$$

□

**Lemma 9.20.** *The functor  $\mathcal{Q}: {}_k\widehat{\mathrm{Alg}}_k \rightarrow {}_k\widehat{\mathrm{Mod}}_k$  is a strict monoidal functor with respect to the  $\circ_k$ -product.*

*Proof.* The unit isomorphism  $\mathrm{Spf}(k) \rightarrow \mathcal{Q}(\mathrm{Spf}(k[z]))$  is the canonical isomorphism  $\mathcal{Q}(k[z]) \cong k$ . If  $F, G \in {}_k\widehat{\mathrm{Alg}}_k$ , without loss of generality  $F \in \mathrm{Pro}\text{-}\mathrm{Alg}_k^+$ , we construct an isomorphism

$$\begin{aligned}
\mathcal{Q}(F) \circ_k \mathcal{Q}(G) &= \mathrm{colim}_i \mathrm{hom}(\mathcal{Q}F(i), \mathcal{Q}(G)) \\
&\cong \underset{\mathrm{Cor. 9.12}}{\mathrm{colim}_i} \mathcal{Q}(\mathrm{hom}_k(\mathcal{O}_F(i), G)) \\
&\cong \mathcal{Q}(\mathrm{colim}_i \mathrm{hom}_k(\mathcal{O}_F(i), G)) \cong \mathcal{Q}(F \circ_k G).
\end{aligned}$$

As in the proof of Lemma 9.18,  $\mathcal{Q}$  commutes with filtered colimits because its representing functor  $\mathcal{Q}$  is induced up from  $\mathrm{Alg}_k^+ \rightarrow \mathrm{Mod}_k$  to  $\mathrm{Pro}\text{-}\mathrm{Alg}_k^+ \rightarrow \mathrm{Pro}\text{-}\mathrm{Mod}_k$  by objectwise application. □

**Theorem 9.21.** *The functor  $\mathcal{Q}$  is a bilax monoidal functor.*

*Proof.* It was shown in Lemma 9.20 and Cor. 9.4 that  $\mathcal{Q}$  is strictly monoidal with respect to  $\circ_k$  and lax monoidal with respect to  $\otimes_k$ . As in Thm. 9.19, it remains to check the compatibility conditions (8.1) and (8.2). The proofs proceed in the same way, utilizing Lemma 2.11 throughout. □

**Corollary 9.22.** *The functors  $\mathcal{P}: {}_k\widehat{\mathrm{Alg}}_k \rightarrow {}_k\widehat{\mathrm{Mod}}_k$  and  $\mathrm{Fr}: {}_k\widehat{\mathrm{Mod}}_k \rightarrow {}_k\widehat{\mathrm{Alg}}_k$  are bilax monoidal functors.*

*Proof.* This follows immediately from Thm. 9.19 and Thm. 9.21 in conjunction with Prop. 8.4. □

It follows that  $\mathcal{P}(F)$  and  $\mathcal{Q}(F)$  are formal  $k$ -coalgebras if  $F$  is a formal plethory. In fact, by Lemma 8.3, more is true:

**Theorem 9.23.** *Let  $F \in {}_k\widehat{\mathrm{Alg}}_k$  be a formal plethory. Then  $\mathcal{P}(F)$  and  $\mathcal{Q}(F) \in {}_k\widehat{\mathrm{Mod}}_k$  are bimonoids.* □

As a corollary of Prop. 9.16, we also get a stronger statement for  $\mathcal{Q}(F)$ :

**Corollary 9.24.** *The bimonoid  $\mathcal{Q}(F)$  is also a two-sided module over  $\mathcal{P}(F)$  with respect to the  $\otimes_k$ -product.*



## 10. DUALIZATION

In this section, we will consider the dualization of bimodules. We will show that the subcategory of  ${}_k\widehat{\text{Mod}}_l$  consisting of objects represented by pro-finitely generated free  $k$ -modules is equivalent to the category  ${}_k\text{Mod}_l$  of ordinary  $k$ - $l$ -bimodules.

**10.1. Bimodules.** Let  $F \in {}_k\widehat{\text{Mod}}_l$  be a formal bimodule. Define  $F^\vee$  to be the object in  ${}_l\text{Mod}_k$

$$(F^\vee)_p^q = F(k[q])_p = \text{Pro-Mod}_k((\mathcal{O}_F)_p^q, k).$$

From this description, it is obvious that it is a  $k$ -module (because  $\text{Mod}_k(\mathcal{O}_F, k)$  is) and that it is an  $l$ -module (because  $F: \text{Mod}_k \rightarrow \text{Mod}_l$ ).

Conversely, given an object  $B \in {}_l\text{Mod}_k$ , we can define a dual

$$({}^\vee B)_p^q = \text{Spf}(\text{hom}_k(B_p^q, k)) \in {}_k\widehat{\text{Mod}}_l \quad (\text{cf. Example 3.3}).$$

To be explicit, this is the pro- $k$ -module  $\{\text{Hom}_k(F, k)\}$  where  $F$  runs through all finitely generated  $k$ -submodules of  $B$ .

These functors are not equivalences of categories, but suitable restrictions are. Denote by  ${}_k\widehat{\text{Mod}}_l'$  the full subcategory of formal bimodules which are represented by pro-(finitely generated free)  $k$ -modules, and  ${}_k\text{Mod}_l'$  the full subcategory of  $k$ - $l$ -bimodules which are flat as  $k$ -modules.

**Lemma 10.1.** *The restrictions of the functors  $(-)^{\vee}: {}_k\widehat{\text{Mod}}_l' \rightarrow {}_k\text{Mod}_l'$  and  $(-)^{\vee}: {}_k\text{Mod}_l' \rightarrow {}_k\widehat{\text{Mod}}_l'$  are inverse equivalences of categories.*

*Proof.* First assume  $l = \mathbf{Z}$ . By the Govorov-Lazard theorem [Laz69, Gov65], the category of flat  $k$ -modules  ${}_k\widehat{\text{Mod}}_{\mathbf{Z}}'$  is equivalent to the category of ind-finitely generated free  $k$ -modules. Since the category of finitely generated free  $k$ -modules is self-dual by the functor  $\text{Mod}_k(-, k)$ , the opposite category of  ${}_k\widehat{\text{Mod}}_{\mathbf{Z}}$  is thus isomorphic to pro-finitely generated free  $k$ -modules by the functors  ${}^\vee(-)$  and  $(-)^{\vee}$ . The case for general  $l$  follows because the forgetful functors  ${}_k\widehat{\text{Mod}}_l' \rightarrow {}_k\widehat{\text{Mod}}_{\mathbf{Z}}'$  and  ${}_k\text{Mod}_l' \rightarrow {}_k\text{Mod}_{\mathbf{Z}}'$  are faithful.  $\square$

**Lemma 10.2.** *The category  ${}_k\text{Mod}_l$  is a 2-bimodule over  $\text{Mod}_l$ . The functor  ${}^\vee(-)$  is a left strict and the functor  $(-)^{\vee}$  is a right strict morphism of 2-bimodules.*

*Proof.* The homomorphism set  ${}_k\text{Mod}_l(B, C)$  is naturally an  $l$ -module because  $l$  is commutative. The left action of  $\text{Mod}_l$  is given by the usual tensor product  $M \otimes_l B$ , which is again a left  $l$ -module by the commutativity of  $l$ . The right action is given by  $\text{hom}_l(M, B)$ .

It is straightforward to check that  $(-)^{\vee}$  and  ${}^\vee(-)$  are enriched functors. To see that  $(-)^{\vee}$  is right strict, we need to verify that

$$\text{hom}(M, F)^{\vee} \cong \text{hom}(M, F^{\vee}),$$

which is true since both sides are given by  $\text{Hom}_l(M, F(k))$ . To see that  ${}^\vee(-)$  is left strict, we check the identity

$$M \overline{\otimes}_l {}^\vee B \cong {}^\vee(M \otimes_l B)$$

by noting that the left hand side is represented by  $\text{hom}_l(M, \text{hom}_k(B, k))$ , whereas the right hand side is represented by  $\text{hom}_k(M \otimes_l B, k)$ . These are equal by the standard hom-tensor adjunction.  $\square$

Note that the  $\text{Mod}_l$ -enrichment of  ${}_k\text{Mod}_l'$  and  ${}_k\widehat{\text{Mod}}_l'$  does not descend to a 2-bimodule structure on these categories. For example, if  $B \in {}_k\text{Mod}_l'$  then  $M \otimes_l B$  is in general not flat unless  $M$  is a flat  $l$ -module. On the other hand, for  $B, C \in {}_k\text{Mod}_l'$ ,  ${}_k\text{Mod}_l(B, C)$  need not be a flat  $l$ -module.

Let  $F \in {}_k\widehat{\text{Mod}}_l$  and  $B \in {}_m\text{Mod}_l$ . In Section 5 we defined an object  $B \overline{\otimes}_l F \in {}_k\widehat{\text{Mod}}_m$  as part of the left module structure of  ${}_k\widehat{\text{Mod}}_l$  over  $\text{Mod}_l$ .

**Lemma 10.3.** *Let  $F \in {}_k\widehat{\text{Mod}}_l$  and  $B \in {}_m\text{Mod}_l$ . Then there is a natural map*

$$B \overline{\otimes}_l F \rightarrow ({}^\vee B) \circ F$$

*which is an isomorphism if  $B \in {}_m\text{Mod}_l'$ . Similarly, there is a natural map*

$$B \circ F^\vee \rightarrow (B \overline{\otimes}_l F)^\vee$$

*which is an isomorphism if  $B \in {}_m\text{Mod}_l'$ .*

*Proof.* Let  $\text{ev}: B \circ_m \text{hom}(B, l) \rightarrow l$  be the adjoint of the identity of  $\text{hom}(B, l)$ . By Corollary B.3(1), there is a natural map in  ${}_k\widehat{\text{Mod}}_l^{\text{op}}$ :

$$B \circ_m (\text{hom}(B, l) \hat{\otimes}_l \mathcal{O}_F) \rightarrow (B \circ_m \text{hom}(B, l)) \hat{\otimes}_l \mathcal{O}_F \xrightarrow{\text{ev} \hat{\otimes} \text{id}_{\mathcal{O}_F}} \mathcal{O}_F.$$

This map adjoints to a map in  ${}_k\widehat{\text{Mod}}_m$

$$\text{hom}(B, l) \hat{\otimes}_l \mathcal{O}_F \rightarrow \text{hom}_l(B, \mathcal{O}_F),$$

which induces the desired map.

Note that as functors of  $B$ , both sides of the morphism map colimits of  $l$ -modules to limits in  $\text{Pro-Mod}_k$ . Thus to show that the map is an isomorphism for  $l$ -flat  $B$ , it suffices to let  $B = l^n$  be a single finitely generated free  $l$ -module. In that case the map is the identity on  $\mathcal{O}_F^n$ .

The second map is given by the canonical map  $B \otimes_l F(k) \rightarrow (B \overline{\otimes}_l F)(k)$  given by Lemma 5.2. Since  $N \mapsto B \otimes F(N)$  is representable if  $B$  is  $l$ -flat, this map is an isomorphism for  $B \in {}_l\text{Mod}_m'$ .  $\square$

**Lemma 10.4** ([AM10, Example 6.18]). *For  $B, C \in {}_k\text{Mod}_k$ , define  $B \circ C = B \otimes_k C$ , where the tensor product uses the right module structure on  $B$  and the left module structure on  $C$ . Also define  $B {}_k\otimes_k C$  to be the tensor product over  $k \otimes k$ . Then the category  ${}_k\text{Mod}_k$  is a 2-monoidal category with respect to  $\circ$  and  ${}_k\otimes_k$ .*

**Theorem 10.5.** *The functors  $(-)^{\vee}$  and  ${}^\vee(-)$  are bilax monoidal functors.*

*Proof.* The unit of  $\circ$  in  ${}_k\widehat{\text{Mod}}_k$  is  $\text{Spf}(k)$ , the unit of  $\circ$  in  ${}_k\text{Mod}_k$  is  $k$ , and we have that  $\text{Spf}(k)^\vee = k$  and  ${}^\vee k = \text{Spf}(k)$ . Thus we find that  ${}^\vee(-)$  and  $(-)^{\vee}$  strictly preserve the units.

We now show that  ${}^\vee(-)$  is lax monoidal with respect to the  $\circ$ -products. There is a natural transformation

$$\begin{aligned} {}^\vee(B \circ C) &= \text{Spf}(\text{hom}(B \circ C, k)) \\ &= \text{Spf}(\text{hom}(B, \text{hom}(C, k))) = B \overline{\otimes}_k {}^\vee C \xrightarrow{\text{Lemma 10.3}} {}^\vee B \circ {}^\vee C, \end{aligned}$$

and this map is an isomorphism if  $B \in {}_k\text{Mod}_k'$ .

We now show that  $(-)^{\vee}$  is lax monoidal with respect to the  $\circ$ -products. There is a natural transformation

$$\begin{aligned} F^{\vee} \circ G^{\vee} &= F(k) \circ G(k) \\ &\rightarrow \text{Pro-Mod}_k(\mathcal{O}_F \otimes_k \mathcal{O}_G, k) \\ &\xrightarrow{\text{Lemma B.3(2)}} \text{Pro-Mod}_k(\mathcal{O}_F, \underline{\text{Pro}}_k(\mathcal{O}_G, k)) = (F \circ G)^{\vee}. \end{aligned}$$

The arrow is an isomorphism if  $F \in {}_k\widehat{\text{Hopf}}_k'$ .

We now study the compatibility with the other monoidal structure, i.e.  $\otimes_k$  on  ${}_k\widehat{\text{Mod}}_k$  and  ${}_k\otimes_k$  on  ${}_k\text{Mod}_k$ . Note that the unit in  ${}_k\widehat{\text{Mod}}_k$  is  $\text{Spf}(\text{hom}(k, k))$ , whereas the unit in  ${}_k\text{Mod}_k$  is  $k \otimes_K k$ . We have

$$\mathcal{O}_{\vee(k \otimes k)} = \text{hom}_k(k \otimes k, k) = \text{hom}(k, k).$$

For  $(-)^{\vee}$ , we have a comparison map

$$k \otimes_K k \rightarrow \text{Spf}(\text{hom}(k, k))^{\vee} = \text{hom}_k(\text{hom}(k, k), k)$$

which is adjoint to

$$k \otimes_K k \otimes_k \overline{\text{hom}}(k, k) = k \otimes \overline{\text{hom}}(k, k) \xrightarrow{\text{evaluate}} k,$$

but this map is not necessarily an isomorphism. It is if  $k \otimes k$  is flat over  $k$  (in this case,  $\text{hom}(k, k) = \text{hom}_k(k \otimes k, k) \in {}_k\widehat{\text{Mod}}_k'$ ).

To show that  ${}^{\vee}(-)$  is oplax monoidal with respect to  $\otimes_k$  and  ${}_k\otimes_k$ , consider the diagram

$$\begin{array}{ccccc} \mathcal{O}_{\vee(B \otimes_k C)} & \longrightarrow & \text{hom}_k(B \otimes_k C, k) & \xrightarrow{\cong} & \text{hom}_k(k \otimes B \otimes_k C, k) \\ \uparrow \text{dashed} & & \uparrow & & \parallel \\ \mathcal{O}_{\vee B} \otimes_k \mathcal{O}_{\vee C} & \rightarrow & \text{hom}_k(B, k) \otimes_k \text{hom}_k(C, k) & \xrightarrow{\cong} & \text{hom}_K(k, \text{Mod}_k(B \otimes_k C, k)) \\ & & & & \uparrow \\ & & & & \text{hom}_K(k, \text{hom}_k(B, k) \otimes_k \text{hom}_k(C, k)) \end{array}$$

Both rows are equalizers, and the vertical maps are iso if  $B, C \in {}_k\text{Mod}_k'$ .

The oplax monoidal structure for  $(-)^{\vee}$  is given by the canonical map

$$\begin{aligned} F^{\vee} {}_k\otimes_k G^{\vee} &= \text{Pro-Mod}_k(\mathcal{O}_F, k) \otimes_k \text{Pro-Mod}_k(\mathcal{O}_G, k) \\ &\rightarrow \text{Pro-Mod}_k(\mathcal{O}_F \otimes_k \mathcal{O}_G, k) = (F \otimes_k G)^{\vee}. \end{aligned}$$

By duality (Lemma 10.1), since  ${}^{\vee}(-)$  is strongly monoidal on  ${}_k\text{Mod}_k'$ , so is  $(-)^{\vee}$  on  ${}_k\widehat{\text{Mod}}_k'$ .  $\square$

We summarize the main results of this section in the following theorem.

**Theorem 10.6.** *Assume that  $k \otimes_K k$  is flat over  $k$ . Then the categories  ${}_k\widehat{\text{Mod}}_k' \subseteq {}_k\widehat{\text{Mod}}_k$  and  ${}_k\text{Mod}_k' \subseteq {}_k\text{Mod}_k$  are full 2-monoidal subcategories, and the functors  $(-)^{\vee}$  and  ${}^{\vee}(-)$  are inverse strong bimonoidal equivalences between them.*

*Proof.* The only point left to show is that  ${}_k\widehat{\text{Mod}}'_k$  and  ${}_k\text{Mod}'_k$  are closed under the two monoidal structures. This is obvious for  ${}_k\text{Mod}'_k$ : if  $B, C$  are flat as right  $k$ -modules then so are  $B \circ C$  and  $B {}_k\otimes_k C$ . It follows for  ${}_k\widehat{\text{Mod}}'_k$  by Thm. 10.5 and Lemma 10.3.  $\square$

## APPENDIX A. PRO-CATEGORIES AND LATTICES

In this appendix, I will review some background results about pro- and ind-categories.

Throughout, I will make use of ends and coends. We denote the end of a functor  $F: I^{\text{op}} \otimes I \rightarrow \mathcal{C}$  by

$$\int_i F(i, i) = \text{eq} \left( \prod_{i \in I} F(i, i) \rightrightarrows \prod_{i \rightarrow j \in I} F(i, j) \right)$$

and the coend of a functor  $G: I \otimes I^{\text{op}} \rightarrow \mathcal{C}$  by

$$\int^i F(i, i) = \text{coeq} \left( \coprod_{i \rightarrow j \in I} F(i, j) \rightrightarrows \coprod_{i \in I} F(i, i) \right).$$

Recall that a category  $I$  is cofiltered if each finite diagram  $X: F \rightarrow I$  has a cone, i.e. an object  $i$  together with a natural transformation  $\text{const}_i \rightarrow X$  in the category of functors from  $F$  to  $I$ . Let  $\text{Pro}$  denote the (2-) category of all small cofiltered categories. If  $\mathcal{C}$  is any category, the category  $\text{Pro-}\mathcal{C}$  has as objects pairs  $(I, X)$  where  $I \in \text{Pro}$  and  $X: I \rightarrow \mathcal{C}$  is a diagram; morphisms are defined by

$$\text{Pro-}\mathcal{C}((I, X), (J, Y)) = \lim_{j \in J} \text{colim}_{i \in I} \mathcal{C}(X(i), Y(j)).$$

It is easy to show (cf. for example [EH76, Thm. 2.1.6]) that  $\text{Pro-}\mathcal{C}$  is equivalent to the subcategory of objects indexed by cofiltered posets in which every ascending chain is finite. The dual of this property is usually called “cofinite”, but I will refrain from calling this property “cocofinite” or “finite” and use the term “noetherian” and the dual property “artinian.”

If  $I$  is a meet-semilattice (a poset with all finite limits), then  $I$  is in particular a cofiltered poset – the meet  $\lim F$  of a finite set  $F \subseteq X$  is a cone – but the converse is not true. Let  $\text{Lat}$  be the category of all noetherian meet-semilattices. It will be technically convenient to work with the full subcategory  $\text{Lat}(\mathcal{C})$  of  $\text{Pro-}\mathcal{C}$  generated by objects indexed by posets in  $\text{Lat}$ . The following lemma shows that this will usually not be a loss of generality.

**Lemma A.1.** *Let  $\mathcal{C}$  be a category closed under finite limits. Then the natural inclusion  $\text{Lat}(\mathcal{C}) \hookrightarrow \text{Pro-}\mathcal{C}$  is an equivalence.*

*Proof.* We already know that the inclusion  $\text{Lat}(\mathcal{C}) \rightarrow \text{Pro-}\mathcal{C}$  is full and faithful, and it remains to show that every object  $X: I \rightarrow \mathcal{C}$  in  $\text{Pro-}\mathcal{C}$  is isomorphic to one in  $\text{Lat}(\mathcal{C})$ . We may assume that  $I$  is a noetherian cofiltered poset. We define a relation  $\leq$  on the set of finite subsets of  $I$  by  $F \leq F'$  iff  $F$  is cofinal in  $F'$ , i.e. for each  $x \in F'$  there is a  $y \in F$  such that  $y \leq x$ . This relation is transitive and reflexive and thus induces a partial order on the set  $\text{Fin}(I) = \{F \subseteq I \mid F \text{ finite}\} / \sim$ , where  $F \sim F' \iff F \leq F' \text{ and } F' \leq F$ . The poset  $\text{Fin}(I)$  is noetherian if  $I$  is. Furthermore,  $\text{Fin}(I)$  is closed under meets:  $F \wedge F' = F \cap F'$ .

There is a canonical inclusion functor  $\iota: I \rightarrow \text{Fin-}I$  given by singletons.

Now let  $X: I \rightarrow \mathcal{C}$  be a diagram representing an object in  $\text{Pro-}\mathcal{C}$ . Consider  $\text{RKan}_I X$  given by

$$(\text{RKan } X)(F) = \lim_F X,$$

which exists because  $\mathcal{C}$  was assumed to have finite limits. I claim that  $X$  and  $\text{RKan}_I X$  are isomorphic in  $\text{Pro-}\mathcal{C}$ . There is a canonical map  $\text{RKan}_I X \rightarrow X$  given by

$$X(\{i\}) = X(i).$$

To see this is an isomorphism in  $\text{Pro-}\mathcal{C}$ , we need to construct for each  $F \in \text{Fin-}I$  an object  $i' \in I$  and a map  $X(i') \rightarrow \lim_F X$  such that suitable diagrams commute. But since  $I$  is cofiltered, there is a cone  $i' \rightarrow F$  and compatible maps  $X(i') \rightarrow X(i)$  for each  $i \in F$ , thus a map  $X(i') \rightarrow \lim X \circ F$ .  $\square$

We need to simplify even further. If  $I, J$  are two noetherian semilattices, denote by  $I^J$  the set of all monotonic functions from  $J$  to  $I$ . There is a partial order on  $I^J$  given by  $f \leq f'$  iff  $f(j) \leq f'(j)$  for all  $j \in J$ . In fact,  $I^J$  is again a semilattice:  $(f \wedge g)(i) = f(i) \wedge g(i)$ . However,  $I^J$  is usually not noetherian.

Let  $\text{Lat}'(\mathcal{C})$  be the category with the same objects as  $\text{Lat}(\mathcal{C})$ , but where the morphisms are defined as follows: given two objects  $X: I \rightarrow \mathcal{C}, Y: J \rightarrow \mathcal{C}$ ,

$$\text{Lat}'(\mathcal{C})(X, Y) = \text{colim}_{f \in I^J} \int_{j \in J} \mathcal{C}(X(f(j)), Y(j)).$$

**Lemma A.2.** *The canonical functor  $\phi: \text{Lat}'(\mathcal{C}) \rightarrow \text{Lat}(\mathcal{C})$ , given by the identity on objects and on morphisms by*

$$(\alpha_j: X(f(j)) \rightarrow Y(j)) \in \int_j \mathcal{C}(X(f(j)), Y(j)) \mapsto \alpha_{j_0} \in \text{colim}_i \mathcal{C}(X(i), Y(j_0)),$$

*is an equivalence of categories.*

*Proof.* As described for instance in [DH76], a morphism in  $\text{Lat}(\mathcal{C})$  is given by a not necessarily monotonic function  $f: J \rightarrow I$  and compatible maps  $X(f(j)) \rightarrow Y(j)$  subject to the condition that for each arrow  $j \leq j'$  in  $J$ , there is an object  $i \in I$  such that  $i \leq f(j)$  and  $i \leq f(j')$ . Given such a nonmonotonic function, we can produce a monotonic function  $\tilde{f}: J \rightarrow I$  by

$$\tilde{f}(j) = \lim_{j' \geq j} f(j'),$$

using the noetherian condition on  $J$  (therefore the set  $\{j' \mid j' \geq j\}$  is finite) and the lattice condition on  $I$  (finite limits exist). We have that  $\tilde{f} \leq f$  and thus we get an induced compatible set of maps  $\tilde{\alpha}_j: X(\tilde{f}(j)) \rightarrow Y(j)$  representing the same map as  $\alpha$  in  $\text{Lat}(\mathcal{C})$ . Thus  $\phi$  is full. For faithfulness, let  $f, g: J \rightarrow I$  be two monotonic maps and  $\alpha_j: X(f(j)) \rightarrow Y(j)$  and  $\beta_j: X(g(j)) \rightarrow Y(j)$  two maps such that  $\phi(\alpha) = \phi(\beta)$ . This means that for every  $j$  there exists an element  $h(j) \leq f(j) \wedge g(j)$  such that  $\alpha_j|_{X(h(j))} = \beta_j|_{X(h(j))} =: \gamma_j$ . Again, by possibly choosing smaller  $h(j)$ , we can assume that  $h$  is monotonic. Thus  $h \leq f$  and  $h \leq g$ , and  $\gamma \leq \alpha, \gamma \leq \beta$ . Therefore,  $\alpha$  and  $\beta$  represent the same class of maps in  $\text{colim}_{f \in I^J} \int_{j \in J} \mathcal{C}(X(f(j)), Y(j))$ .  $\square$

## APPENDIX B. PRO- AND IND-CATEGORIES AND THEIR ENRICHMENTS

Let  $(\mathcal{V}, \otimes)$  be a 2-ring and  $\mathcal{C}$  a 2-bimodule over  $\mathcal{V}$ . In this section, we will study the structure of the categories  $\text{Ind } \mathcal{V}$  and  $\text{Pro } \mathcal{C}$  with respect to monoidality and enrichments. The examples we will use in this paper are:

- (1) For a graded commutative  $K$ -algebra  $k$ ,  $\mathcal{V} = \text{Mod}_K$  (graded  $K$ -modules) and  $\mathcal{C} = \text{Mod}_k$ ;
- (2)  $\mathcal{V} = \text{Set}$ , the category of graded sets, and  $\mathcal{C} = \text{Alg}_k$  for a graded commutative ring  $k$ .

**Lemma B.1.** *Let  $\mathcal{V}$  be a 2-ring. Then so is the category  $\text{Ind } \mathcal{V}$ , and the inclusion  $\mathcal{V} \rightarrow \text{Ind } \mathcal{V}$  as well as the colimit  $\text{Ind } \mathcal{V} \xrightarrow{\text{colim}} \mathcal{V}$  are strict monoidal functors.*

*Proof.* The fact that  $\text{Ind } \mathcal{V}$  is cocomplete follows from the fact that  $\text{Ind } \mathcal{V}$  always has filtered colimits, and that finite coproducts and coequalizers can be computed levelwise. By [Isa02],  $\text{Ind } \mathcal{V}$  is also complete. (The appendix of [AM69] is often cited for this fact, but it only proves it for small categories.)

Let  $I_i \xrightarrow{X_i} \mathcal{V}$  represent objects of  $\text{Ind } \mathcal{V}$  ( $i = 1, 2$ ), where  $I_i$  are filtered categories. Then the symmetric monoidal structure on  $\text{Ind } \mathcal{V}$  is defined by

$$X_1 \otimes X_2: I_1 \times I_2 \rightarrow \mathcal{V}, \quad (X_1 \otimes X_2)(i_1, i_2) = X_1(i_1) \otimes X_2(i_2).$$

Since  $\otimes$  is closed in  $\mathcal{V}$ , it commutes with all colimits in  $\mathcal{V}$ , and thus  $\text{colim}: \text{Ind } \mathcal{V} \rightarrow \mathcal{V}$  is strict monoidal.

We define an internal hom object  $\underline{\text{Ind}}(X_1, X_2)$  as follows: we may assume the indexing categories  $I_1, I_2$  are artinian join-semilattices (by the dual of Lemma A.1). Then the poset of monotonic maps from  $I_1$  to  $I_2$  is also a lattice. This will be the indexing set of  $\underline{\text{Ind}}(X_1, X_2)$ . For such an  $\alpha \in I_2^{I_1}$ ,  $\underline{\text{Ind}}(X_1, X_2)$  is given by

$$\underline{\text{Ind}}(X_1, X_2)(\alpha) = \int_{i_1} \text{map}(X_1(i_1), X_2(\alpha(i_1))).$$

To see that  $\otimes$  and  $\underline{\text{Ind}}$  are adjoint, we compute

$$\begin{aligned} \text{Ind } \mathcal{V}(X_1, \underline{\text{Ind}}(X_2, X_3)) &= \lim_{i_1} \text{colim}_{\alpha \in I_3^{I_2}} \mathcal{V}\left(X_1(i_1), \int_{i_2} \text{map}(X_2(i_2), X_3(\alpha(i_2)))\right) \\ &= \lim_{i_1} \text{colim}_{\alpha \in I_3^{I_2}} \int_{i_2} \mathcal{V}\left(X_1(i_1), \text{map}(X_2(i_2), X_3(\alpha(i_2)))\right) \\ &\stackrel{\text{Lemma A.2}}{=} \lim_{i_1} \lim_{i_2} \text{colim}_{i_3} \mathcal{V}\left(X_1(i_1), \text{map}(X_2(i_2), X_3(i_3))\right) \\ &= \lim_{i_1, i_2} \text{colim}_{i_3} \mathcal{V}(X_1(i_1) \otimes X_2(i_2), X_3(i_3)) \\ &= \text{Ind } \mathcal{V}(X_1 \otimes X_2, X_3). \end{aligned}$$

□

This construction leads us out of the category  $\text{Lat}(\mathcal{C})$  because  $I_2^{I_1}$  is not artinian, but we can always apply the equivalence  $\text{Ind } \mathcal{C} \rightarrow \text{Lat}(\mathcal{C})$  to get back an isomorphic internal hom object in  $\text{Lat}(\mathcal{C})$ .

Unfortunately, the category  $\text{Pro-}\mathcal{C}$  is not a 2-ring even if  $\mathcal{C}$  is. The analogous definition of a symmetric monoidal structure  $(X_1 \otimes X_2)(i_1, i_2) = X_1(i_1) \otimes X_2(i_2)$  on  $\text{Pro-}\mathcal{C}$  is unproblematic, but this structure is not closed. However,  $\text{Pro-}\mathcal{C}$  is a 2-bimodule over  $\text{Ind-}\mathcal{V}$ . We will first define the structure and then prove that it gives rise to an enrichment.

**Definition.**

- (1) Define a functor  $\overline{\otimes}: \text{Ind-}\mathcal{V} \times \text{Pro-}\mathcal{C} \rightarrow \text{Pro-}\mathcal{C}$  as follows. Let  $L \in \text{Ind-}\mathcal{V}$  be indexed by an artinian join-semilattice  $J$  and  $M \in \text{Pro-}\mathcal{C}$  be indexed by a noetherian meet-semilattice  $I$ . Then  $L \overline{\otimes} M$  is indexed by the meet-semilattice  $\alpha: J^{\text{op}} \rightarrow I$ , and

$$(L \overline{\otimes} M)(\alpha) = \int^{j \in J} L(j) \otimes M(\alpha(j)).$$

- (2) Define a functor  $\underline{\text{ind}}: \text{Pro-}\mathcal{C} \times \text{Pro-}\mathcal{C} \rightarrow \text{Ind-}\mathcal{V}$  as follows. Let  $M \in \text{Pro-}\mathcal{C}$  be indexed by a noetherian meet-semilattice  $I$  and  $N \in \text{Pro-}\mathcal{C}$  by a noetherian meet-semilattice  $J$ . The object  $\underline{\text{ind}}(M, N) \in \text{Ind-}\mathcal{V}$  is indexed by the opposite lattice of the meet-semilattice  $I^J$  and is given by

$$\underline{\text{ind}}(M, N)(\alpha) = \int_j \text{map}(M(\alpha(j)), N(j)).$$

- (3) Define a functor  $\text{hom}: \text{Ind-}\mathcal{V} \times \text{Pro-}\mathcal{C} \rightarrow \text{Pro-}\mathcal{C}$  by assigning to  $L: I \rightarrow \mathcal{C}$  and  $M: J \rightarrow \mathcal{C}$ , where  $I$  is cofiltered and  $J$  is filtered, the object indexed by  $I^{\text{op}} \times J$  and given by

$$\text{hom}(L, M)(i, j) = \text{hom}(L(i), M(j)).$$

**Lemma B.2.** *With the structure given above, the category  $\text{Pro-}\mathcal{C}$  is a 2-bimodule over  $\text{Ind-}\mathcal{V}$ .*

*Proof.* Let  $M, N \in \text{Pro-}\mathcal{C}$  and  $L, H \in \text{Ind-}\mathcal{V}$ . We need to see:

- (1)  $(H \otimes L) \overline{\otimes} M \cong H \overline{\otimes} (L \overline{\otimes} M)$ .
- (2)  $\text{Pro-}\mathcal{C}(L \overline{\otimes} M, N) \cong \text{Ind-}\mathcal{V}(L, \underline{\text{ind}}(M, N))$
- (3)  $\text{Pro-}\mathcal{C}(L \overline{\otimes} M, N) \cong \text{Pro-}\mathcal{C}(M, \text{hom}(L, N))$

(1): Let  $I \xrightarrow{M} \mathcal{C}$ ,  $J_1 \xrightarrow{H} \mathcal{V}$ , and  $J_2 \xrightarrow{L} \mathcal{V}$  be representations. Then we have for  $\alpha: J_1^{\text{op}} \times J_2^{\text{op}} \rightarrow I$ :

$$\begin{aligned} ((H \otimes L) \overline{\otimes} M)(\alpha) &= \int_{\substack{j_1 \in J_1 \\ j_2 \in J_2}} H(j_1) \otimes L(j_2) \otimes M(\alpha(j_1, j_2)) \\ &= \int^{j_2} \left( H(j_1) \otimes \int^{j_1} L(j_2) \otimes M(\alpha(j_1, j_2)) \right) \\ &= (H \overline{\otimes} (L \overline{\otimes} M))(\alpha^\#), \end{aligned}$$

where  $\alpha^\#: J_2^{\text{op}} \rightarrow I^{J_1}$  is the adjoint of  $\alpha$ .

(2) As before, pick representatives  $I_1 \xrightarrow{M} \mathcal{C}$ ,  $I_2 \xrightarrow{N} \mathcal{C}$ ,  $J \xrightarrow{L} \mathcal{V}$ . We compute

$$\begin{aligned}
\text{Pro-}\mathcal{C}(L \overline{\otimes} M, N) &= \lim_{i_2} \text{colim}_{\alpha: J^{\text{op}} \rightarrow I_1} \mathcal{C} \left( \int^{j \in J} L(j) \otimes M(\alpha(j)), N(i_2) \right) \\
&\cong \lim_{i_2} \text{colim}_{\alpha: J^{\text{op}} \rightarrow I_1} \int_j \mathcal{C}(L(j) \otimes M(\alpha(j)), N(i_2)) \\
(*) \quad &= \lim_{i_2} \lim_j \text{colim}_{i_1} \mathcal{C}(L(j) \otimes M(i_1), N(i_2)) \\
&= \lim_{i_2, j} \text{colim}_{i_1} \mathcal{V}(L(j), \text{map}(M(i_1), N(i_2))) \\
&= \lim_j \text{colim}_{\beta: I_2 \rightarrow I_1} \int_{i_2} \mathcal{V} \left( L(j), \text{map}(M(\beta(i_2)), N(i_2)) \right) \\
&= \lim_j \text{colim}_{\beta: I_2 \rightarrow I_1} \mathcal{V}(L(j), \underline{\text{Ind}}(M, N)(\beta)) \\
&= \underline{\text{Ind}}-\mathcal{V}(L, \underline{\text{Ind}}(M, N)).
\end{aligned}$$

(3) Pick representatives as in (2). We pick up the computation at (\*):

$$\begin{aligned}
\lim_{i_2} \lim_j \text{colim}_{i_1} \mathcal{C}(L(j) \otimes M(i_1), N(i_2)) &= \lim_{i_2, j} \text{colim}_{i_1} \mathcal{C}(M(i_1), \text{hom}(L(j), N(i_2))) \\
&= \text{Pro-}\mathcal{C}(M, \text{hom}(L, N)).
\end{aligned}$$

□

Now assume that  $(\mathcal{C}, \boxtimes, J)$  is a 2-algebra over  $\mathcal{V}$ . We define a symmetric monoidal structure  $\boxtimes$  on  $\text{Pro-}\mathcal{C}$  by  $(M \boxtimes N)(i, j) = M(i) \boxtimes N(j)$ . The unit is the constant object  $J$ . Furthermore, there is a homomorphism object  $\underline{\text{Pro}}(M, N)$  indexed by  $J$  and given by

$$\underline{\text{Pro}}(M, N)(j) = \text{colim}_i \text{Hom}(M(i), N(j)).$$

**Lemma B.3.** *The category  $\text{Pro-}\mathcal{C}$  with the structure above is a 2-algebra over  $\underline{\text{Ind}}-\mathcal{V}$ . More precisely,*

- (1) *Let  $M, N \in \text{Pro-}\mathcal{C}$  and  $L \in \underline{\text{Ind}}-\mathcal{V}$ . Then there is a natural map  $L \overline{\otimes} (M \otimes N) \rightarrow (L \overline{\otimes} M) \otimes N$  which is an isomorphism if  $N$  is pro-constant.*
- (2) *There is a natural morphism*

$$\text{Pro-}\mathcal{C}(X \otimes Y, Z) \rightarrow \text{Pro-}\mathcal{C}(X, \underline{\text{Pro}}(Y, Z))$$

*which is an isomorphism if  $X$  consists of small objects, i.e. if  $\mathcal{C}(X(i), -)$  commutes with directed colimits for all  $i$ , or if  $Y$  is pro-constant.*

*Proof.* (1): Pick representatives  $I_1 \xrightarrow{M} \mathcal{C}$ ,  $I_2 \xrightarrow{N} \mathcal{C}$ ,  $J \xrightarrow{L} \text{Mod}_K$ . For  $\alpha: J^{\text{op}} \rightarrow I_1$ ,  $i_2 \in I_2$ , we have

$$\begin{aligned}
((L \overline{\otimes} M) \otimes N)(\alpha, i_2) &= \left( \int^{j \in J} L(j) \otimes M(\alpha(j)) \right) \otimes N(i_2) \\
&= \int^{j \in J} L(j) \otimes M(\alpha(j)) \otimes N(i_2) \\
&= (L \overline{\otimes} (M \otimes N))(\alpha \times \text{const}_{i_2}).
\end{aligned}$$



Since  $L \overline{\otimes} (M \otimes N)$  is indexed by the larger category of all functors  $J^{\text{op}} \rightarrow I_1 \times I_2$ , this only defines a natural pro-map  $L \overline{\otimes} (M \otimes N) \rightarrow (L \overline{\otimes} M) \otimes N$ , which is an isomorphism if  $I_2 = \{i_2\}$ .

(2): The natural map

$$\begin{aligned} \text{Pro-}\mathcal{C}(X \otimes Y, Z) &= \lim_k \text{colim}_{i,j} \mathcal{C}(X(i) \otimes Y(j), Z(k)) \\ &= \lim_k \text{colim}_{i,j} \mathcal{C}(X(i), \text{map}(Y(j), Z(k))) \\ &\rightarrow \lim_k \text{colim}_i \mathcal{C}(X(i), \text{colim}_j \mathcal{C}(Y(j), Z(k))) \\ &= \text{Pro-}\mathcal{C}(X, \underline{\text{Pro}}(Y, Z)). \end{aligned}$$

is an isomorphism if  $X(i)$  is small or  $J = \{j_0\}$ . □

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