A spectral theory of linear operators on rigged Hilbert spaces under certain analyticity conditions

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Jul 29, 2011; last modified Aug 20, 2011

Abstract

A spectral theory of linear operators on rigged Hilbert spaces (Gelfand triplets) is developed under the assumptions that a linear operator T on a Hilbert space $\mathcal{H}$ is a perturbation of a selfadjoint operator, and the spectral measure of the selfadjoint operator has an analytic continuation near the real axis. It is shown that there exists a dense subspace X of $\mathcal{H}$ such that the resolvent $(\lambda - T)^{-1}\phi$ of the operator T has an analytic continuation from the lower half plane to the upper half plane for any $\phi \in X$, even when T has a continuous spectrum on $\mathbf{R}$, as an X'-valued holomorphic function, where X' is a dual space of X. The rigged Hilbert space consists of three spaces $X \subset \mathcal{H} \subset X'$. Basic tools of the usual spectral theory, such as spectra, resolvents and Riesz projections are extended to those defined on a rigged Hilbert space. They prove to have the same properties as those of the usual spectral theory. The results are applied to estimate exponential decays of the semigroups of linear operators.

Keywords: generalized eigenvalue; resonance pole; rigged Hilbert space; Gelfand triplet; generalized function

1 Introduction

A spectral theory of linear operators on topological vector spaces is one of the fundamental tools in functional analysis. Spectra of linear operators provide us with much information about the operators. However, there are phenomena that are not explained by spectra. For example, a solution $x(t)$ of a linear evolution equation $dx/dt = Tx$ on an infinite dimensional space can decay exponentially as t increases even if the linear operator T does not have spectrum on the left half plane. Such an exponential decay of a solution is known as Landau damping in plasma physics [5], and is often observed for Schrödinger operators [11, 19]. Now it is known that such an exponential decay can be induced by resonance poles or generalized eigenvalues.

In the literature, resonance poles are defined as follows: Let T be a selfadjoint operator (for simplicity) on a Hilbert space $\mathcal{H}$ with the inner product $(\,\cdot\,,\,\cdot\,)$. The spectrum $\sigma(T)$ of T lies on the real axis. By the definition of the spectrum, the resolvent $(\lambda - T)^{-1}$ diverges in norm when $\lambda \in \sigma(T)$. However, the matrix element $((\lambda - T)^{-1}\phi, \phi)$ for some "good"

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function $\phi \in \mathcal{H}$ may exist for $\lambda \in \sigma(T)$, and the function $f(\lambda) = ((\lambda - T)^{-1}\phi, \phi)$ may have an analytic continuation from the lower half plane to the upper half plane through an interval on $\sigma(T)$. Then, the analytic continuation may have poles on the upper half plane, which is called a resonance pole or a generalized eigenvalue. In general, the pole is not an eigenvalue of T, however, it is expected to have similar properties to eigenvalues.

The defects in such an approach to resonance poles are that it is not obvious that there exists an eigenvector associated with a resonance pole. Since an eigen-equation for a resonance pole is not defined, a corresponding eigenspace and an algebraic multiplicity are not defined. The purpose in this paper is to give a correct formulation of resonance poles in terms of operator theory on rigged Hilbert spaces (Gelfand triplets).

To explain our idea based on rigged Hilbert spaces, let us consider the multiplication operator $\mathcal{M} : \phi(\omega) \mapsto \omega\phi(\omega)$ on the Lebesgue space $L^2(\mathbf{R})$. The resolvent is given as

$$((\lambda - \mathcal{M})^{-1}\phi, \psi) = \int_{\mathbf{R}} \frac{1}{\lambda - \omega}\phi(\omega)\psi(\omega)d\omega.$$

In general, the integral in the right hand side diverges for $\lambda \in \mathbf{R}$. However, if ϕ and ψ have analytic continuations near the real axis, the quantity in the right hand side has an analytic continuation from the lower half plane to the upper half plane, which is given by

$$\int_{\mathbf{R}} \frac{1}{\lambda - \omega}\phi(\omega)\psi(\omega)d\omega + 2\pi\sqrt{-1}\phi(\lambda)\psi(\lambda).$$

Let X be a dense subspace of $L^2(\mathbf{R})$ consisting of functions having analytic continuations near the real axis. A mapping, which maps $\phi \in X$ to the above value, defines a continuous linear functional on X, that is, an element of the dual space X', if X is equipped with a suitable topology. Motivated by this idea, we define the linear operator $A(\lambda) : X \to X'$ to be

$$\langle A(\lambda)\psi \,|\, \phi \rangle = \begin{cases} \displaystyle\int_{\mathbf{R}} \frac{1}{\lambda - \omega}\psi(\omega)\phi(\omega)d\omega + 2\pi\sqrt{-1}\psi(\lambda)\phi(\lambda) & (\mathrm{Im}(\lambda) > 0), \\ \displaystyle\lim_{y \to -0} \int_{\mathbf{R}} \frac{1}{x + \sqrt{-1}y - \omega}\psi(\omega)\phi(\omega)d\omega & (x = \lambda \in \mathbf{R}), \\ \displaystyle\int_{\mathbf{R}} \frac{1}{\lambda - \omega}\psi(\omega)\phi(\omega)d\omega & (\mathrm{Im}(\lambda) < 0), \end{cases} \tag{1.1}$$

for $\psi, \phi \in X$, where $\langle \cdot | \cdot \rangle$ is a paring for (X', X). When $\mathrm{Im}(\lambda) < 0$, $A(\lambda) = (\lambda - \mathcal{M})^{-1}$, while when $\mathrm{Im}(\lambda) \geq 0$, $A(\lambda)\psi$ is not included in $L^2(\mathbf{R})$ but an element of X'. In this sense, $A(\lambda)$ is called the analytic continuation of the resolvent of $\mathcal{M}$ in the generalized sense. In this manner, the triplet $X \subset L^2(\mathbf{R}) \subset X'$, which is called the rigged Hilbert space or the Gelfand triplet [8, 16], is introduced.

In this paper, a spectral theory on a rigged Hilbert space is proposed for an operator of the form $T = H + K$, where H is a selfadjoint operator on a Hilbert space $\mathcal{H}$, whose spectral measure has an analytic continuation near the real axis, when the domain is restricted to some dense subspace X of $\mathcal{H}$, as above. K is an operator densely defined on X satisfying certain boundedness conditions. Our purpose is to investigate spectral properties of the operator $T = H + K$. At first, the analytic continuation $A(\lambda)$ of the resolvent

$(\lambda - H)^{-1} : \mathcal{H} \to \mathcal{H}$ is defined as an operator from X into X' in the same way as Eq.(1.1). In general, $A(\lambda) : X \to X'$ is defined on a nontrivial Riemann surface of λ so that when λ lies on the original complex plane, it coincides with the usual resolvent $(\lambda - H)^{-1}$. By using the operator $A(\lambda)$ and a rigged Hilbert space, resonance poles are defined in the following way : If the equation

$$(id - A(\lambda)K^{\times})\mu = 0 \tag{1.2}$$

has a nonzero solution μ in X', such a λ is called a generalized eigenvalue (resonance pole) and μ is called a generalized eigenfunction, where $K^{\times} : X' \to X'$ is a dual operator of K. When λ lies on the original complex plane, the above equation is reduced to the usual eigen-equation $(\lambda - T)\mu = 0$. In this manner, resonance poles and corresponding eigenfunctions are naturally obtained without using matrix elements.

Similarly, an analytic continuation of the resolvent of T in the generalized sense is defined to be

$$\mathcal{R}_\lambda = A(\lambda) \circ (id - K^{\times}A(\lambda)) : X \to X'. \tag{1.3}$$

When λ lies on the original complex plane, this is reduced to the usual resolvent $(\lambda - T)^{-1}$. With the aid of the generalized resolvent $\mathcal{R}_\lambda$, basic notions in the usual spectral theory, such as eigenspaces, algebraic multiplicities, point/continuous/residual spectra, Riesz projections are extended to those defined on a rigged Hilbert space. It is shown that they have the same properties as the usual theory. For example, the generalized Riesz projection Π_0 for an isolated resonance pole λ_0 is defined by the contour integral of the generalized resolvent.

$$\Pi_0 = \frac{1}{2\pi\sqrt{-1}} \int_\gamma \mathcal{R}_\lambda d\lambda : X \to X'. \tag{1.4}$$

Properties of the generalized Riesz projection Π_0 is investigated in detail. It is proved that the range of the generalized Riesz projection around an isolated resonance pole coincides with the generalized eigenspace of the resonance pole. Any function $\phi \in X$ proves to be uniquely decomposed as $\phi = \mu_1 + \mu_2$, where $\mu_1 \in \Pi_0 X$ and $\mu_2 = (id - \Pi_0)X$, both of which are elements of X'. These results play an important role when applying the theory to dynamical systems [4]. The generalized Riesz projection around a resonance pole λ_0 on the left half plane (resp. on the imaginary axis) defines a stable subspace (resp. a center subspace) in the generalized sense, both of which are subspaces of X'. The results in the present paper enable us to investigate the asymptotic behavior and bifurcations of an infinite dimensional dynamical system with the aid of the dual space X', even when a given dynamical system is defined on X. Such a dynamics induced by a resonance pole is not captured by the usual eigenvalues.

Many properties of the generalized spectrum (the set of singularities of $\mathcal{R}_\lambda$) will be shown. In general, the generalized spectrum consists of the generalized continuous spectrum, the generalized residual spectrum and the generalized point spectrum (the set of resonance poles). If the operator K satisfies a certain compactness condition, the Riesz-Schauder theory on a rigged Hilbert space applies to conclude that the generalized spectrum consists only of a countable number of resonance poles having finite multiplicities. It is remarkable that even if the operator T has the continuous spectrum (in the usual sense), the generalized spectrum consists only of a countable number of resonance poles when K satisfies the compactness condition. Since the topology on the dual space X' is weaker than that on the Hilbert space $\mathcal{H}$, the continuous spectrum of T disappears, while eigenvalues remain to exist as the generalized spectrum. This fact is useful to estimate embedded eigenvalues. Eigenvalues embedded in the continuous spectrum is no longer embedded in our spectral theory. Thus, the Riesz projection is applicable to obtain eigenspaces of them. Our theory is used to estimate an exponential decay of the semigroup $e^{\sqrt{-1}Tt}$ generated by $\sqrt{-1}T$. It is shown that resonance poles induce an exponential decay of the semigroup even if the operator $\sqrt{-1}T$ has no spectrum on the left half plane.

A spectral theory developed in this paper is applied to a bifurcation problem of infinite dimensional nonlinear dynamical systems in [4]. In [4], a bifurcation structure of an infinite dimensional coupled oscillators is investigated by means of rigged Hilbert spaces. It is shown that when a resonance pole, which is obtained by the linearization of the system around a steady state, gets across the imaginary axis as a parameter of the system varies, then a bifurcation occurs. Applications to Schrödinger operators will appear in a forthcoming paper.

Throughout this paper, $\mathrm{D}(\cdot)$ and $\mathrm{R}(\cdot)$ denote the domain and range of an operator, respectively.

2 Spectral theory on a Hilbert space

This section is devoted to a review of the spectral theory of a perturbed selfadjoint operator on a Hilbert space to compare the spectral theory on a rigged Hilbert space developed after Sec.3. Let $\mathcal{H}$ be a Hilbert space over $\mathbf{C}$. The inner product is defined so that

$$(a\phi, \psi) = (\phi, \overline{a}\psi) = a(\phi, \psi), \tag{2.1}$$

where $\overline{a}$ is the complex conjugate of $a \in \mathbf{C}$. Let us consider an operator $T := H + K$ defined on a dense subspace of $\mathcal{H}$, where H is a selfadjoint operator, and K is a compact operator on $\mathcal{H}$ which need not be selfadjoint. Let λ and $v = v_\lambda$ be an eigenvalue and an eigenfunction, respectively, of the operator T defined by the equation $\lambda v = Hv + Kv$. This is rearranged as

$$(\lambda - H)(id - (\lambda - H)^{-1}K)v = 0, \tag{2.2}$$

where id denotes the identity on $\mathcal{H}$. In particular, when λ is not an eigenvalue of H, it is an eigenvalue of T if and only if $id - (\lambda - H)^{-1}K$ is not injective in $\mathcal{H}$. Since the essential spectrum is stable under compact perturbations (see Kato [12]), the essential spectrum $\sigma_e(T)$ of T is the same as that of H, which lies on the real axis. In general, $\sigma_e(T)$ includes both of the continuous spectrum and eigenvalues. Since K is a compact perturbation, the spectrum outside the real axis consists of the discrete spectrum; for any $\delta > 0$, the number of eigenvalues satisfying $|\mathrm{Im}(\lambda)| \geq \delta$ is finite, and their algebraic multiplicities are finite. Eigenvalues may accumulate only on the real axis. To find eigenvalues embedded in the essential spectrum $\sigma_e(T)$ is a difficult and important problem. In this paper, a new spectral theory on rigged Hilbert spaces will be developed to obtain such embedded eigenvalues and corresponding eigenspaces.

Let $R_\lambda = (\lambda - T)^{-1}$ be the resolvent, which is given by

$$R_\lambda \phi = (\lambda - H)^{-1}\left(id - K(\lambda - H)^{-1}\right)^{-1}\phi, \quad \phi \in \mathcal{H}. \tag{2.3}$$

Let λ_j be an eigenvalue of T outside the real axis, and γ_j be a simple closed curve enclosing λ_j separated from the rest of the spectrum. The projection to the generalized eigenspace $V_j := \bigcup_{n\geq 1}\mathrm{Ker}(\lambda_j - T)^n$ is given by

$$\Pi_j = \frac{1}{2\pi\sqrt{-1}}\int_{\gamma_j} R_\lambda d\lambda. \tag{2.4}$$

Let us consider the semigroup $e^{\sqrt{-1}Tt}$ generated by $\sqrt{-1}T$. Since $\sqrt{-1}H$ generates the C^0-semigroup $e^{\sqrt{-1}Ht}$ and K is compact, $\sqrt{-1}T$ also generates the C^0-semigroup (see Kato [12]). It is known that $e^{\sqrt{-1}Tt}$ is obtained by the Laplace inversion formula (Hille and Phillips [10])

$$e^{\sqrt{-1}Tt} = \frac{1}{2\pi\sqrt{-1}}\lim_{x\to\infty}\int_{-x-\sqrt{-1}y}^{x-\sqrt{-1}y} e^{\sqrt{-1}\lambda t}(\lambda - T)^{-1}d\lambda, \quad x, y \in \mathbf{R}, \tag{2.5}$$

for $t > 0$, where $y > 0$ is chosen so that all eigenvalues λ of T satisfy $\mathrm{Im}(\lambda) > -y$. Thus the contour is the horizontal line on the lower half plane. Let $\varepsilon > 0$ be a small number and $\lambda_0, \cdots, \lambda_N$ eigenvalues of T satisfying $\mathrm{Im}(\lambda_j) \leq -\varepsilon$, $j = 0, \cdots, N$. The residue theorem provides

$$\begin{aligned} e^{\sqrt{-1}Tt} &= \frac{1}{2\pi\sqrt{-1}}\int_{\mathbf{R}} e^{\sqrt{-1}xt+\varepsilon t}(x - \sqrt{-1}\varepsilon - T)^{-1}dx \\ &\quad + \frac{1}{2\pi\sqrt{-1}}\sum_{j=0}^{N}\int_{\gamma_j} e^{\sqrt{-1}\lambda t}(\lambda - T)^{-1}d\lambda, \end{aligned}$$

where γ_j is a sufficiently small closed curve enclosing λ_j. Let M_j be the smallest integer such that $(\lambda_j - T)^{M_j}\Pi_j = 0$. This is less or equal to the algebraic multiplicity of λ_j. Then, $e^{\sqrt{-1}Tt}$ is calculated as

$$\begin{aligned} e^{\sqrt{-1}Tt} &= \frac{1}{2\pi\sqrt{-1}}\int_{\mathbf{R}} e^{\sqrt{-1}xt+\varepsilon t}(x - \sqrt{-1}\varepsilon - T)^{-1}dx \\ &\quad + \sum_{j=0}^{N} e^{\sqrt{-1}\lambda_j t}\sum_{k=0}^{M_j-1}\frac{(-\sqrt{-1}t)^k}{k!}(\lambda_j - T)^k\Pi_j. \end{aligned}$$

The second term above diverges as $t \to \infty$ because $\mathrm{Re}(\sqrt{-1}\lambda_j) \geq \varepsilon$. On the other hand, if there are no eigenvalues on the lower half plane, we obtain

$$e^{\sqrt{-1}Tt} = \frac{1}{2\pi\sqrt{-1}}\int_{\mathbf{R}} e^{\sqrt{-1}xt+\varepsilon t}(x - \sqrt{-1}\varepsilon - T)^{-1}dx,$$

for any small $\varepsilon > 0$. In such a case, the asymptotic behavior of $e^{\sqrt{-1}Tt}$ is quite nontrivial. One of the purposes in this paper is to give a further decomposition of the first term above under certain analyticity conditions to determine the dynamics of $e^{\sqrt{-1}Tt}$.

3 Spectral theory on a Gelfand triplet

In the previous section, we give the review of the spectral theory of the operator $T = H+K$ on $\mathcal{H}$. In this section, the notion of spectra, eigenfunctions, resolvents and projections are extended to a space of generalized functions by means of a rigged Hilbert space. It will be shown that they have similar properties to those on $\mathcal{H}$. They are used to estimate the asymptotic behavior of the semigroup $e^{\sqrt{-1}Tt}$ and to find embedded eigenvalues.

3.1 Rigged Hilbert spaces

Let X be a locally convex Hausdorff topological vector space over $\mathbf{C}$ and X' its dual space. X' is a set of continuous anti-linear functionals on X. For $\mu \in X'$ and $\phi \in X$, $\mu(\phi)$ is denoted by $\langle \mu \,|\, \phi \rangle$. For any $a, b \in \mathbf{C}$, $\phi, \psi \in X$ and $\mu, \xi \in X'$, the equalities

$$\langle \mu \,|\, a\phi + b\psi \rangle = \overline{a}\langle \mu \,|\, \phi \rangle + \overline{b}\langle \mu \,|\, \psi \rangle, \tag{3.1}$$

$$\langle a\mu + b\xi \,|\, \phi \rangle = a\langle \mu \,|\, \phi \rangle + b\langle \xi \,|\, \phi \rangle, \tag{3.2}$$

hold. In this paper, an element of X' is called a generalized function [7, 8]. Several topologies can be defined on the dual space X'. Two of the most usual topologies are the weak dual topology (weak * topology) and the strong dual topology (strong * topology). A sequence $\{\mu_j\} \subset X'$ is said to be weakly convergent to $\mu \in X'$ if $\langle \mu_j \,|\, \phi \rangle \to \langle \mu \,|\, \phi \rangle$ for each $\phi \in X$; a sequence $\{\mu_j\} \subset X'$ is said to be strongly convergent to $\mu \in X'$ if $\langle \mu_j \,|\, \phi \rangle \to \langle \mu \,|\, \phi \rangle$ uniformly on any bounded subset of X.

Let $\mathcal{H}$ be a Hilbert space with the inner product $(\cdot\,,\,\cdot)$ such that X is a dense subspace of $\mathcal{H}$. Since a Hilbert space is isomorphic to its dual space, we obtain $\mathcal{H} \subset X'$ through $\mathcal{H} \simeq \mathcal{H}'$.

Definition 3.1. If a locally convex Hausdorff topological vector space X is a dense subspace of a Hilbert space $\mathcal{H}$ and a topology of X is stronger than that of $\mathcal{H}$, the triplet

$$X \subset \mathcal{H} \subset X' \tag{3.3}$$

is called the *rigged Hilbert space* or the *Gelfand triplet*. The *canonical inclusion* $i : X \to X'$ is defined as follows; for $\psi \in X$, we denote $i(\psi)$ by $\langle \psi |$, which is defined to be

$$i(\psi)(\phi) = \langle \psi \,|\, \phi \rangle = (\psi, \phi), \tag{3.4}$$

for any $\phi \in X$. The inclusion from $\mathcal{H}$ into X' is also defined as above. It is easy to show that the canonical inclusion is injective if and only if X is a dense subspace of $\mathcal{H}$, and the canonical inclusion is continuous (for both of the weak dual topology and the strong dual topology) if and only if a topology of X is stronger than that of $\mathcal{H}$ (see Tréves [25]).

A topological vector space X is called Montel if it is barreled and every bounded set of X is relatively compact. A Montel space has a convenient property that on a bounded set A of a dual space of a Montel space, the weak dual topology coincides with the strong dual topology. In particular, a weakly convergent series in a dual of a Montel space also converges with respect to the strong dual topology (see Tréves [25]). Furthermore, a linear

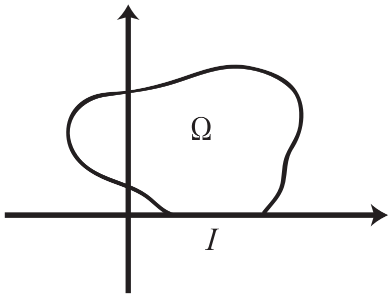

Fig. 1: A domain on which $E[\psi, \phi](\omega)$ is holomorphic.

map from a topological vector space to a Montel space is a compact operator if and only if it is a bounded operator. It is known that the theory of rigged Hilbert spaces works best when the space X is a Montel or a nuclear space [8]. See Grothendieck [9] and Komatsu [13] for sufficient conditions for a topological vector space to be a Montel space or a nuclear space.

3.2 Generalized eigenvalues and eigenfunctions

Let $\mathcal{H}$ be a Hilbert space over $\mathbf{C}$ and H a selfadjoint operator densely defined on $\mathcal{H}$ with the spectral measure $\{E(B)\}_{B\in\mathcal{B}}$; that is, H is expressed as $H = \int_{\mathbf{R}} \omega dE(\omega)$. Let K be some linear operator densely defined on $\mathcal{H}$. Our purpose is to investigate spectral properties of the operator $T := H + K$. Let $\Omega \subset \mathbf{C}$ be a simply connected open domain in the upper half plane such that the intersection of the real axis and the closure of Ω is a connected interval $\tilde{I}$. Let $I = \tilde{I} \backslash \partial \tilde{I}$ be an open interval (see Fig.1). For a given $T = H + K$, we suppose that there exists a locally convex Hausdorff vector space $X(\Omega)$ over $\mathbf{C}$ satisfying following conditions.

(X1) $X(\Omega)$ is a dense subspace of $\mathcal{H}$.
(X2) A topology on $X(\Omega)$ is stronger than that on $\mathcal{H}$.
(X3) $X(\Omega)$ is a quasi-complete barreled space.
(X4) For any $\phi \in X(\Omega)$, the spectral measure $(E(B)\phi, \phi)$ is absolutely continuous on the interval I. Its density function, denoted by $E[\phi, \phi](\omega)$, has an analytic continuation to $\Omega \cup I$.
(X5) For each $\lambda \in I \cup \Omega$, the bilinear form $E[\,\cdot\,, \cdot\,](\lambda) : X(\Omega) \times X(\Omega) \to \mathbf{C}$ is separately continuous.

Because of (X1) and (X2), the rigged Hilbert space $X(\Omega) \subset \mathcal{H} \subset X(\Omega)'$ is well defined, where $X(\Omega)'$ is a space of continuous *anti*-linear functionals and the canonical inclusion i is defined by Eq.(3.4). Sometimes we denote $i(\psi)$ by ψ for simplicity by identifying $iX(\Omega)$ with $X(\Omega)$. The assumption (X3) is used to define Pettis integrals and Taylor expansions of $X(\Omega)'$-valued holomorphic functions in Sec.3.5. For example, Montel spaces, Fréchet spaces, Banach spaces and Hilbert spaces are barreled. See Appendix A for the definitions of the Pettis integral and $X(\Omega)'$-valued holomorphic functions. Due to the assumption (X4) with the aid of the polarization identity, we can show that $(E(B)\phi, \psi)$ is absolutely continuous on I for any $\phi, \psi \in X(\Omega)$. Let $E[\phi, \psi](\omega)$ be the density function;

$$d(E(\omega)\phi, \psi) = E[\phi, \psi](\omega)d\omega, \quad \omega \in I. \tag{3.5}$$

Then, $E[\phi, \psi](\omega)$ is holomorphic in $\omega \in I \cup \Omega$. We will use the above notation for any $\omega \in \mathbf{R}$ for simplicity, although the absolute continuity is assumed only on I. Since $E[\phi, \psi](\omega)$ is absolutely continuous on I, H is assumed not to have eigenvalues on I. (X5) is used to prove the continuity of a certain operator (Prop.3.7).

Let A be a linear operator densely defined on $X(\Omega)$. Then, the dual operator A' is defined as follows: the domain $\mathrm{D}(A')$ is the set of elements $\mu \in X(\Omega)'$ such that the mapping $\phi \mapsto \langle \mu \,|\, A\phi \rangle$ from $\mathrm{D}(A)$ into $\mathbf{C}$ is continuous. Then, $A' : \mathrm{D}(A') \to X(\Omega)'$ is defined by

$$\langle A'\mu \,|\, \phi \rangle = \langle \mu \,|\, A\phi \rangle, \quad \phi \in \mathrm{D}(A),\ \mu \in \mathrm{D}(A'). \tag{3.6}$$

If A is continuous on $X(\Omega)$, then A' is continuous on $X(\Omega)'$ for both of the weak dual topology and the strong dual topology. The (Hilbert) adjoint A^* of A is defined through $(A\phi, \psi) = (\phi, A^*\psi)$ as usual when A is densely defined on $\mathcal{H}$.

Lemma 3.2. Let A be a linear operator densely defined on $\mathcal{H}$. Suppose that there exists a dense subspace Y of $X(\Omega)$ such that $A^*Y \subset X(\Omega)$ so that the dual $(A^*)'$ is defined. Then, $(A^*)'$ is an extension of A and $i \circ A = (A^*)' \circ i\,|_{\mathrm{D}(A)}$. In particular, $\mathrm{D}((A^*)') \supset i\mathrm{D}(A)$.

Proof. By the definition of the canonical inclusion i, we have

$$i(A\psi)(\phi) = (A\psi, \phi) = (\psi, A^*\phi) = \langle \psi \,|\, A^*\phi \rangle = \langle (A^*)'\psi \,|\, \phi \rangle, \tag{3.7}$$

for any $\psi \in \mathrm{D}(A)$ and $\phi \in Y$. ■

In what follows, we denote $(A^*)'$ by $A^\times$. Thus Eq.(3.7) means $i \circ A = A^\times \circ i\,|_{\mathrm{D}(A)}$. Note that $A^\times = A'$ when A is selfadjoint. For the operators H and K, we suppose that

(X6) there exists a dense subspace Y of $X(\Omega)$ such that $HY \subset X(\Omega)$.
(X7) K is H-bounded and $K^*Y \subset X(\Omega)$.
(X8) $K^\times A(\lambda) iX(\Omega) \subset iX(\Omega)$ for any $\lambda \in \{\mathrm{Im}(\lambda) < 0\} \cup I \cup \Omega$.

The operator $A(\lambda) : iX(\Omega) \to X(\Omega)'$ will be defined later. Recall that when K is H-bounded (relatively bounded with respect to H), $\mathrm{D}(T) = \mathrm{D}(H)$ and $K(\lambda - H)^{-1}$ is bounded on $\mathcal{H}$ for $\lambda \notin \mathbf{R}$. In some sense, (X8) is a "dual version" of this condition because $A(\lambda)$ proves to be an extension of $(\lambda - H)^{-1}$. In particular, we will show that $K^\times A(\lambda) i = i(K(\lambda - H)^{-1})$ when $\mathrm{Im}(\lambda) < 0$. Our purpose is to investigate the operator $T = H + K$ with these conditions. Due to (X6) and (X7), the dual operator $T^\times$ of $T^* = H + K^*$ is well defined. It follows that $\mathrm{D}(T^\times) = \mathrm{D}(H^\times) \cap \mathrm{D}(K^\times)$ and

$$\mathrm{D}(T^\times) \supset i\mathrm{D}(T) = i\mathrm{D}(H) \supset iY.$$

In particular, the domain of $T^\times$ is dense in $X(\Omega)'$.

Recall that the eigenfunction of T associated with an eigenvalue λ is given by $v = (\lambda - H)^{-1}(Kv)$. When $\lambda \in \mathbf{R}$, $v \notin \mathcal{H}$ in general because H may have spectrum on the real axis. However, we will show that $v = v_\lambda$ has an analytic continuation from the lower half plane to Ω with respect to λ as a generalized function. To see it, we need the next lemma.

Lemma 3.3. Suppose that a function $q(\omega)$ is integrable on $\mathbf{R}$ and holomorphic on $\Omega \cup I$. Then, the function

$$Q(\lambda) = \begin{cases} \displaystyle\int_{\mathbf{R}} \frac{1}{\lambda - \omega} q(\omega) d\omega & (\mathrm{Im}(\lambda) < 0), \\ \displaystyle\int_{\mathbf{R}} \frac{1}{\lambda - \omega} q(\omega) d\omega + 2\pi \sqrt{-1} q(\lambda) & (\lambda \in \Omega), \end{cases} \tag{3.8}$$

is holomorphic on $\{\lambda \,|\, \mathrm{Im}(\lambda) < 0\} \cup \Omega \cup I$.

Proof. Putting $\lambda = x + \sqrt{-1}y$ with $x, y \in \mathbf{R}$ yields

$$\int_{\mathbf{R}} \frac{1}{\lambda - \omega} q(\omega) d\omega = \int_{\mathbf{R}} \frac{x - \omega}{(x-\omega)^2 + y^2} q(\omega) d\omega - \sqrt{-1} \int_{\mathbf{R}} \frac{y}{(x-\omega)^2 + y^2} q(\omega) d\omega.$$

Due to the formula of the Poisson kernel, the equalities

$$\lim_{y \to +0} \int_{\mathbf{R}} \frac{y}{(x-\omega)^2 + y^2} q(\omega) d\omega = \pi q(x), \quad \lim_{y \to -0} \int_{\mathbf{R}} \frac{y}{(x-\omega)^2 + y^2} q(\omega) d\omega = -\pi q(x),$$

hold when q is continuous at $x \in I$ (Ahlfors [1]). Thus we obtain

$$\lim_{y \to -0} \int_{\mathbf{R}} \frac{1}{\lambda - \omega} q(\omega) d\omega = \lim_{y \to +0} \left(\int_{\mathbf{R}} \frac{1}{\lambda - \omega} q(\omega) d\omega + 2\pi \sqrt{-1} q(\lambda) \right) = \pi V(x) + \pi \sqrt{-1} q(x),$$

where

$$V(x) := \lim_{y \to 0} \frac{1}{\pi} \int_{\mathbf{R}} \frac{x - \omega}{(x-\omega)^2 + y^2} q(\omega) d\omega$$

is the Hilbert transform of q. It is known that $V(x)$ is Lipschitz continuous on I if $q(x)$ is (see Titchmarsh [24]). Therefore, two holomorphic functions in Eq.(3.8) coincide with one another on I and they are continuous on I. This proves that $Q(\lambda)$ is holomorphic on $\{\lambda \,|\, \mathrm{Im}(\lambda) < 0\} \cup \Omega \cup I$. ■

Put $u_\lambda = (\lambda - H)^{-1}\psi$ for $\psi \in \mathcal{H}$. In general, u_λ is not included in $\mathcal{H}$ when $\lambda \in I$ because of the continuous spectrum of H. Thus u_λ does not have an analytic continuation from the lower half plane to Ω with respect to λ as an $\mathcal{H}$-valued function. To define an analytic continuation of u_λ, we regard it as a generalized function in $X(\Omega)'$ by the canonical inclusion. Then, the action of $i((\lambda - H)^{-1}\psi)$ is given by

$$i((\lambda - H)^{-1}\psi)(\phi) = ((\lambda - H)^{-1}\psi, \phi) = \int_{\mathbf{R}} \frac{1}{\lambda - \omega} E[\psi, \phi](\omega) d\omega, \quad \mathrm{Im}(\lambda) < 0.$$

Because of the assumption (X4), this quantity has an analytic continuation to $\Omega \cup I$ as

$$\int_{\mathbf{R}} \frac{1}{\lambda - \omega} E[\psi, \phi](\omega) d\omega + 2\pi \sqrt{-1} E[\psi, \phi](\lambda), \quad \lambda \in \Omega.$$

Motivated by this observation, define the operator $A(\lambda) : iX(\Omega) \to X(\Omega)'$ to be

$$\langle A(\lambda)\psi \,|\, \phi \rangle = \begin{cases} \displaystyle\int_{\mathbf{R}} \frac{1}{\lambda - \omega} E[\psi, \phi](\omega) d\omega + 2\pi\sqrt{-1} E[\psi, \phi](\lambda) & (\lambda \in \Omega), \\ \displaystyle\lim_{y \to -0} \int_{\mathbf{R}} \frac{1}{x + \sqrt{-1}y - \omega} E[\psi, \phi](\omega) d\omega & (\lambda = x \in I), \\ \displaystyle\int_{\mathbf{R}} \frac{1}{\lambda - \omega} E[\psi, \phi](\omega) d\omega & (\mathrm{Im}(\lambda) < 0), \end{cases} \tag{3.9}$$

for any $\psi \in iX(\Omega)$, $\phi \in X(\Omega)$. Indeed, we can prove by using (X5) that $A(\lambda)\psi$ is a continuous functional. Due to Lemma 3.3, $\langle A(\lambda)\psi \,|\, \phi\rangle$ is holomorphic on $\{\mathrm{Im}(\lambda) < 0\} \cup \Omega \cup I$. When $\mathrm{Im}(\lambda) < 0$, we have $\langle A(\lambda)\psi \,|\, \phi\rangle = ((\lambda - H)^{-1}\psi, \phi)$. In this sense, the operator $A(\lambda)$ is called the analytic continuation of the resolvent $(\lambda - H)^{-1}$ as a generalized function. By using it, we extend the notion of eigenvalues and eigenfunctions.

Recall that the equation for eigenfunctions of T is given by $(id - (\lambda - H)^{-1}K)v = 0$. Since the analytic continuation of $(\lambda - H)^{-1}$ in $X(\Omega)'$ is $A(\lambda)$, we make the following definition.

Definition 3.4. Let $\mathrm{R}(A(\lambda))$ be the range of $A(\lambda)$. If the equation

$$(id - A(\lambda)K^{\times})\mu = 0 \tag{3.10}$$

has a nonzero solution μ in $\mathrm{R}(A(\lambda))$ for some $\lambda \in \Omega \cup I \cup \{\lambda \,|\, \mathrm{Im}(\lambda) < 0\}$, λ is called a *generalized eigenvalue* of T and μ is called a *generalized eigenfunction* associated with λ. A generalized eigenvalue on Ω is called a *resonance pole* (the word “resonance” originates from quantum mechanics [19]).

Note that the assumption (X8) is used to define $A(\lambda)K^{\times}\mu$ for $\mu \in \mathrm{R}(A(\lambda))$ because the domain of $A(\lambda)$ is $iX(\Omega)$. Applied by $K^{\times}$, Eq.(3.10) is rewritten as

$$(id - K^{\times}A(\lambda))K^{\times}\mu = 0. \tag{3.11}$$

If $K^{\times}\mu = 0$, Eq.(3.10) shows $\mu = 0$. Hence, λ is a generalized eigenvalue if and only if $id - K^{\times}A(\lambda)$ is not injective on $iX(\Omega)$.

Theorem 3.5. Let λ be a generalized eigenvalue of T and μ a generalized eigenfunction associated with λ. Then the equality

$$T^{\times}\mu = \lambda\mu \tag{3.12}$$

holds.

Proof. At first, let us show $\mathrm{D}(\lambda - H^{\times}) \supset \mathrm{R}(A(\lambda))$. By the operational calculus, we have $E[\psi, (\overline{\lambda} - H)\phi](\omega) = (\lambda - \omega)E[\psi, \phi](\omega)$. When $\lambda \in \Omega$, this gives

$$\begin{aligned} \langle A(\lambda)\psi \,|\, (\overline{\lambda} - H)\phi\rangle &= \int_{\mathbf{R}} \frac{1}{\lambda - \omega} E[\psi, (\overline{\lambda} - H)\phi](\omega) d\omega + 2\pi\sqrt{-1} E[\psi, (\overline{\lambda} - H)\phi](\lambda) \\ &= \int_{\mathbf{R}} E[\psi, \phi](\omega) d\omega + 2\pi\sqrt{-1}(\lambda - \omega)|_{\omega=\lambda} E[\psi, \phi](\lambda) \\ &= \langle \psi \,|\, \phi\rangle, \end{aligned}$$

for any $\psi \in iX(\Omega)$ and $\phi \in Y$. It is obvious that $\langle \psi \,|\, \phi \rangle$ is continuous in ϕ with respect to the topology of $X(\Omega)$. This proves that $\mathrm{D}(\lambda - H^\times) \supset \mathrm{R}(A(\lambda))$ and $(\lambda - H^\times)A(\lambda) = id : iX(\Omega) \to iX(\Omega)$. When μ is a generalized eigenfunction, $\mu \in \mathrm{D}(\lambda - H^\times)$ because $\mu = A(\lambda)K^\times \mu$. Then, Eq.(3.10) provides

$$(\lambda - H^\times)(id - A(\lambda)K^\times)\mu = (\lambda - H^\times - K^\times)\mu = (\lambda - T^\times)\mu = 0.$$

The proofs for the cases $\lambda \in I$ and $\mathrm{Im}(\lambda) < 0$ are done in the same way. ■

This theorem means that λ is indeed an eigenvalue of the dual operator $T^\times$. In general, the set of generalized eigenvalues is a proper subset of the set of eigenvalues of $T^\times$. Since the dual space $X(\Omega)'$ is "too large", typically every point on Ω is an eigenvalue of $T^\times$. In this sense, generalized eigenvalues are wider concept than eigenvalues of T, while narrower concept than eigenvalues of $T^\times$ (see Prop.3.17 for more details). In the literature, resonance poles are defined as poles of an analytic continuation of a matrix element of the resolvent [19]. Our definition is based on a straightforward extension of the usual eigen-equation and it is suitable for systematic studies of resonance poles.

3.3 Properties of the operator $A(\lambda)$

Before defining a multiplicity of a generalized eigenvalue, it is convenient to investigate properties of the operator $A(\lambda)$. For $n = 1, 2, \cdots$ let us define the linear operator $A^{(n)}(\lambda) : iX(\Omega) \to X(\Omega)'$ to be

$$\langle A^{(n)}(\lambda)\psi \,|\, \phi \rangle = \begin{cases} \displaystyle\int_{\mathbf{R}} \frac{1}{(\lambda - \omega)^n} E[\psi, \phi](\omega) d\omega + 2\pi \sqrt{-1} \frac{(-1)^{n-1}}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}}\Big|_{z=\lambda} E[\psi, \phi](z), & (\lambda \in \Omega), \\ \displaystyle\lim_{y \to -0} \int_{\mathbf{R}} \frac{1}{(x + \sqrt{-1}y - \omega)^n} E[\psi, \phi](\omega) d\omega, & (\lambda = x \in I), \\ \displaystyle\int_{\mathbf{R}} \frac{1}{(\lambda - \omega)^n} E[\psi, \phi](\omega) d\omega, & (\mathrm{Im}(\lambda) < 0). \end{cases} \tag{3.13}$$

It is easy to show by integration by parts that $\langle A^{(n)}(\lambda)\psi \,|\, \phi \rangle$ is an analytic continuation of $((\lambda - H)^{-n}\psi, \phi)$ from the lower half plane to Ω. $A^{(1)}(\lambda)$ is also denoted by $A(\lambda)$ as before. The next proposition will be often used to calculate the generalized resolvent and projections.

Proposition 3.6. For any integers $j \geq n \geq 0$. the operator $A^{(j)}(\lambda)$ satisfies

(i) $(\lambda - H^\times)^n A^{(j)}(\lambda) = A^{(j-n)}(\lambda)$, where $A^{(0)}(\lambda) := id$.

(ii) $A^{(j)}(\lambda)(\lambda - H^\times)^n|_{iX(\Omega)\cap \mathrm{D}(A^{(j)}(\lambda)(\lambda - H^\times)^n)} = A^{(j-n)}(\lambda)|_{iX(\Omega)\cap \mathrm{D}(A^{(j)}(\lambda)(\lambda - H^\times)^n)}$.
In particular, $A(\lambda)(\lambda - H^\times)\mu = \mu$ when $(\lambda - H^\times)\mu \in iX(\Omega)$.

(iii) $\displaystyle\frac{d^j}{d\lambda^j} \langle A(\lambda)\psi \,|\, \phi \rangle = (-1)^j j! \langle A^{(j+1)}(\lambda)\psi \,|\, \phi \rangle$, $j = 0, 1, \cdots$.

(iv) For each $\psi \in X(\Omega)$, $A(\lambda)\psi$ is expanded as

$$A(\lambda)\psi = \sum_{j=0}^{\infty} (\lambda_0 - \lambda)^j A^{(j+1)}(\lambda_0)\psi, \tag{3.14}$$

where the right hand side converges with respect to the strong dual topology.

Proof. (i) Let us show $(\lambda - H^{\times})A^{(j)}(\lambda) = A^{(j-1)}(\lambda)$. We have to prove that $\mathsf{D}(\lambda - H^{\times}) \supset \mathsf{R}(A^{(j)}(\lambda))$. For this purpose, put $\mu_\lambda(y) = \langle A^{(j)}(\lambda)\psi \,|\, (\overline{\lambda} - H)y\rangle$ $\psi \in iX(\Omega)$ and $y \in Y$. It is sufficient to show that the mapping $y \mapsto \mu_\lambda(y)$ from Y into $\mathbf{C}$ is continuous with respect to the topology on $X(\Omega)$. Suppose that $\mathrm{Im}(\lambda) > 0$. By the operational calculus, we obtain

$$\begin{aligned}\mu_\lambda(y) &= \int_{\mathbf{R}} \frac{1}{(\lambda-\omega)^j} E[\psi, (\overline{\lambda} - H)y](\omega)d\omega + 2\pi\sqrt{-1}\frac{(-1)^{j-1}}{(j-1)!}\frac{d^{j-1}}{dz^{j-1}}\Big|_{z=\lambda} E[\psi, (\overline{\lambda} - H)y](z) \\ &= \int_{\mathbf{R}} \frac{\lambda-\omega}{(\lambda-\omega)^j} E[\psi, y](\omega)d\omega + 2\pi\sqrt{-1}\frac{(-1)^{j-1}}{(j-1)!}\frac{d^{j-1}}{dz^{j-1}}\Big|_{z=\lambda} (\lambda - z)E[\psi, y](z) \\ &= ((\lambda - H)^{1-j}\psi, y) + 2\pi\sqrt{-1}\frac{(-1)^{j-2}}{(j-2)!}\frac{d^{j-2}}{dz^{j-2}}\Big|_{z=\lambda} E[\psi, y](z). \qquad (3.15)\end{aligned}$$

Since $E[\psi, y](z)$ is continuous in $y \in X(\Omega)$ (the assumption (X5)) and $E[\psi, y](z)$ is holomorphic in z, for any $\varepsilon > 0$, there exists a neighborhood U_1 of zero in $X(\Omega)$ such that $|(d^{j-2}/dz^{j-2})E[\psi, y](z)| < \varepsilon$ at $z = \lambda$ for $y \in U_1 \cap Y$. Let U_2 be a neighborhood of zero in $\mathcal{H}$ such that $||y||_{\mathcal{H}} < \varepsilon$ for $y \in U_2$. Since the topology on $X(\Omega)$ is stronger than that on $\mathcal{H}$, $U_2 \cap X(\Omega)$ is a neighborhood of zero in $X(\Omega)$. If $y \in U_1 \cap U_2 \cap Y$, we obtain

$$|\mu_\lambda(y)| \leq ||(\lambda - H)^{1-j}\psi||\varepsilon + 2\pi\sqrt{-1}\frac{(-1)^{j-2}}{(j-2)!}\varepsilon.$$

Note that $(\lambda - H)^{1-j}$ is bounded when $\lambda \notin \mathbf{R}$ and $1 - j \leq 0$ because H is selfadjoint. This proves that μ_λ is continuous, so that $\mu_\lambda = (\lambda - H^{\times})A^{(j)}(\lambda)\psi \in X(\Omega)'$. The proof of the continuity for the case $\mathrm{Im}(\lambda) < 0$ is done in the same way. When $\lambda \in I$, there exists a sequence $\{\lambda_j\}_{j=1}^{\infty}$ in the lower half plane such that $\mu_\lambda(y) = \lim_{j\to\infty} \mu_{\lambda_j}(y)$. Since $X(\Omega)$ is barreled, Banach-Steinhaus theorem is applicable to conclude that the limit μ_λ of continuous linear mappings is also continuous. This proves $\mathsf{D}(\lambda - H^{\times}) \supset \mathsf{R}(A^{(j)}(\lambda))$ and $(\lambda - H^{\times})A^{(j)}(\lambda)$ is well defined for any $\lambda \in \{\mathrm{Im}(\lambda) < 0\} \cup I \cup \Omega$. Then, the above calculation immediately shows that $(\lambda - H^{\times})A^{(j)}(\lambda) = A^{(j-1)}(\lambda)$. By the induction, we obtain (i).

(ii) is also proved by the operational calculus as above, and (iii) is easily obtained by induction.

For (iv), since $\langle A(\lambda)\psi \,|\, \phi\rangle$ is holomorphic, it is expanded in a Taylor series as

$$\begin{aligned}\langle A(\lambda)\psi \,|\, \phi\rangle &= \sum_{j=0}^{\infty} \frac{1}{j!}\frac{d^j}{d\lambda^j}\Big|_{\lambda=\lambda_0} \langle A(\lambda)\psi \,|\, \phi\rangle(\lambda - \lambda_0)^j \\ &= \sum_{j=0}^{\infty} (\lambda_0 - \lambda)^j \langle A^{(j+1)}(\lambda_0)\psi \,|\, \phi\rangle, \qquad (3.16)\end{aligned}$$

for each $\phi, \psi \in X(\Omega)$. This means that the functional $A(\lambda)\psi$ is weakly holomorphic in λ. Then, $A(\lambda)\psi$ turns out to be strongly holomorphic and expanded as Eq.(3.14) by Thm.A.3(iii) in Appendix, in which basic facts on $X(\Omega)'$-valued holomorphic functions are given. ■

Unfortunately, the operator $A(\lambda) : iX(\Omega) \to X(\Omega)'$ is not continuous if $iX(\Omega)$ is equipped with the relative topology from $X(\Omega)'$. Even if $\langle\psi| \to 0$ in $iX(\Omega) \subset X(\Omega)'$, the value $E[\psi,\phi](\lambda)$ does not tend to zero in general because the topology on $X(\Omega)'$ is weaker than that on $X(\Omega)$. However, $A(\lambda)$ proves to be continuous if $iX(\Omega)$ is equipped with the topology induced from $X(\Omega)$ by the canonical inclusion.

Proposition 3.7. $A(\lambda) \circ i : X(\Omega) \to X(\Omega)'$ is continuous if $X(\Omega)'$ is equipped with the weak dual topology.

Proof. Suppose $\lambda \in \Omega$ and fix $\phi \in X(\Omega)$. Because of the assumption (X5), for any $\varepsilon > 0$, there exists a neighborhood U_1 of zero in $X(\Omega)$ such that $|E[\psi,\phi](\lambda)| < \varepsilon$ for $\psi \in U_1$. Let U_2 be a neighborhood of zero in $\mathcal{H}$ such that $||\psi||_{\mathcal{H}} < \varepsilon$ for $\psi \in U_2$. Since the topology on $X(\Omega)$ is stronger than that on $\mathcal{H}$, $U_2 \cap X(\Omega)$ is a neighborhood of zero in $X(\Omega)$. If $\psi \in U := U_1 \cap U_2$,

$$\begin{aligned} |\langle A(\lambda)\psi \,|\, \phi\rangle| &\leq ||(\lambda - H)^{-1}||_{\mathcal{H}} \cdot ||\phi||_{\mathcal{H}} \cdot ||\psi||_{\mathcal{H}} + 2\pi\,|E[\psi,\phi](\lambda)| \\ &= \Big(||(\lambda - H)^{-1}||_{\mathcal{H}} \cdot ||\phi||_{\mathcal{H}} + 2\pi\Big)\,\varepsilon. \end{aligned}$$

This proves that $A(\lambda) \circ i$ is continuous in the weak dual topology. The proof for the case $\mathrm{Im}(\lambda) < 0$ is done in a similar manner. When $\lambda \in I$, there exists a sequence $\{\lambda_j\}_{j=1}^{\infty}$ in the lower half plane such that $A(\lambda) \circ i = \lim_{j\to\infty} A(\lambda_j) \circ i$. Since $X(\Omega)$ is barreled, Banach-Steinhaus theorem is applicable to conclude that the limit $A(\lambda) \circ i$ of continuous linear mappings is also continuous. ■

Now we are in a position to define an algebraic multiplicity and a generalized eigenspace of generalized eigenvalues. Usually, an eigenspace is defined as a set of solutions of the equation $(\lambda - T)^n v = 0$. For example, when $n = 2$, we rewrite it as

$$(\lambda - H - K)(\lambda - H - K)v = (\lambda - H)^2(id - (\lambda - H)^{-2}K(\lambda - H)) \circ (id - (\lambda - H)^{-1}K)v = 0.$$

Dividing by $(\lambda - H)^2$ yields

$$(id - (\lambda - H)^{-2}K(\lambda - H)) \circ (id - (\lambda - H)^{-1}K)v = 0.$$

Since the analytic continuation of $(\lambda - H)^{-n}$ in $X(\Omega)'$ is $A^{(n)}(\lambda)$, we consider the equation

$$(id - A^{(2)}(\lambda)K^{\times}(\lambda - H^{\times})) \circ (id - A(\lambda)K^{\times})\,\mu = 0.$$

Motivated by this observation, we define the operator $B^{(n)}(\lambda) : \mathrm{D}(B^{(n)}(\lambda)) \subset X(\Omega)' \to X(\Omega)'$ to be

$$B^{(n)}(\lambda) = id - A^{(n)}(\lambda)K^{\times}(\lambda - H^{\times})^{n-1}. \tag{3.17}$$

Then, the above equation is rewritten as $B^{(2)}(\lambda)B^{(1)}(\lambda)\mu = 0$. The domain of $B^{(n)}(\lambda)$ is the domain of $A^{(n)}(\lambda)K^{\times}(\lambda - H^{\times})^{n-1}$. The following equality is easily proved.

$$(\lambda - H^{\times})^k B^{(j)}(\lambda) = B^{(j-k)}(\lambda)(\lambda - H^{\times})^k|_{\mathrm{D}(B^{(j)}(\lambda))}, \quad j > k. \tag{3.18}$$

Definition 3.8. Let λ be a generalized eigenvalue of the operator T. The generalized eigenspace of λ is defined by

$$V_\lambda = \bigcup_{m\geq 1} \mathrm{Ker}\, B^{(m)}(\lambda) \circ B^{(m-1)}(\lambda) \circ \cdots \circ B^{(1)}(\lambda). \tag{3.19}$$

We call $\dim V_\lambda$ the algebraic multiplicity of the generalized eigenvalue λ.

Theorem 3.9. For any $\mu \in V_\lambda$, there exists an integer M such that $(\lambda - T^\times)^M \mu = 0$.

Proof. Suppose that $B^{(M)}(\lambda) \circ \cdots \circ B^{(1)}(\lambda)\mu = 0$. Put $\xi = B^{(M-1)}(\lambda) \circ \cdots \circ B^{(1)}(\lambda)\mu$. Eq.(3.18) shows

$$\begin{aligned} 0 &= (\lambda - H^\times)^{M-1} B^{(M)}(\lambda)\xi \\ &= B^{(1)}(\lambda)(\lambda - H^\times)^{M-1}\xi = (id - A(\lambda)K^\times)(\lambda - H^\times)^{M-1}\xi. \end{aligned}$$

Since $\mathsf{D}(\lambda - H^\times) \supset \mathsf{R}(A(\lambda))$, it turns out that $(\lambda - H^\times)^{M-1}\xi \in \mathsf{D}(\lambda - H^\times)$. Then, we obtain

$$\begin{aligned} 0 &= (\lambda - H^\times)(id - A(\lambda)K^\times)(\lambda - H^\times)^{M-1}\xi \\ &= (\lambda - H^\times - K^\times)(\lambda - H^\times)^{M-1}\xi = (\lambda - T^\times)(\lambda - H^\times)^{M-1}\xi. \end{aligned}$$

By induction, we obtain $(\lambda - T^\times)^M \mu = 0$. ■

In general, the space V_λ is a proper subspace of the usual eigenspace $\bigcup_{m\geq 1} \mathrm{Ker}\,(\lambda - T^\times)^m$ of $T^\times$. Typically $\bigcup_{m\geq 1} \mathrm{Ker}\,(\lambda - T^\times)^m$ becomes of infinite dimensional because the dual space $X(\Omega)'$ is "too large", however, V_λ is a finite dimensional space in many cases. In linear algebra, a solution of $(\lambda - T)^n v = 0$ for $n \geq 2$ is called a generalized eigenvector. In this paper, the word "generalized" is used for elements of the dual space. To avoid confusion, if $\mu \in V_\lambda$ satisfies $B^{(m)}(\lambda) \circ \cdots \circ B^{(1)}(\lambda)\mu = 0$ and $B^{(m-1)}(\lambda) \circ \cdots \circ B^{(1)}(\lambda)\mu \neq 0$, we call μ the generalized eigenfunction of multiplicity m.

3.4 Generalized resolvents

In this subsection, we define a generalized resolvent. As the usual theory, it will be used to construct projections and semigroups. Let $R_\lambda = (\lambda - T)^{-1}$ be the resolvent of T as an operator on $\mathcal{H}$. A simple calculation shows

$$R_\lambda \psi = (\lambda - H)^{-1} \left(id - K(\lambda - H)^{-1} \right)^{-1} \psi. \tag{3.20}$$

Since the analytic continuation of $(\lambda - H)^{-1}$ in the dual space is $A(\lambda)$, we make the following definition. In what follows, put $\hat{\Omega} = \Omega \cup I \cup \{\lambda \,|\, \mathrm{Im}(\lambda) < 0\}$.

Definition 3.10. If the inverse $(id - K^\times A(\lambda))^{-1}$ exists, define the generalized resolvent $\mathcal{R}_\lambda : iX(\Omega) \to X(\Omega)'$ to be

$$\mathcal{R}_\lambda = A(\lambda) \circ (id - K^\times A(\lambda))^{-1} = (id - A(\lambda)K^\times)^{-1} \circ A(\lambda), \quad \lambda \in \hat{\Omega}. \tag{3.21}$$

The second equality follows from $(id - A(\lambda)K^\times)A(\lambda) = A(\lambda)(id - K^\times A(\lambda))$. Recall that $id - K^\times A(\lambda)$ is injective on $iX(\Omega)$ if and only if $id - A(\lambda)K^\times$ is injective on $\mathsf{R}(A(\lambda))$.

Since $A(\lambda)$ is not continuous, $\mathcal{R}_\lambda$ is not a continuous operator in general. However, it is natural to ask whether $\mathcal{R}_\lambda \circ i : X(\Omega) \to X(\Omega)'$ is continuous or not because $A(\lambda) \circ i$ is continuous.

Definition 3.11. The generalized resolvent set $\hat{\varrho}(T)$ is defined to be the set of points $\lambda \in \hat{\Omega}$

satisfying following: there is a neighborhood $V_\lambda \subset \hat{\Omega}$ of λ such that for any $\lambda' \in V_\lambda$, $\mathcal{R}_{\lambda'} \circ i$ is a densely defined continuous operator from $X(\Omega)$ into $X(\Omega)'$, where $X(\Omega)'$ is equipped with the weak dual topology, and the set $\{\mathcal{R}_{\lambda'} \circ i(\psi)\}_{\lambda' \in V_\lambda}$ is bounded in $X(\Omega)'$ for each $\psi \in X(\Omega)$. The set $\hat{\sigma}(T) := \hat{\Omega} \backslash \hat{\varrho}(T)$ is called the *generalized spectrum* of T. The *generalized point spectrum* $\hat{\sigma}_p(T)$ is the set of points $\lambda \in \hat{\sigma}(T)$ at which $id - K^\times A(\lambda)$ is not injective (this is the set of generalized eigenvalues). The *generalized residual spectrum* $\hat{\sigma}_r(T)$ is the set of points $\lambda \in \hat{\sigma}(T)$ such that the domain of $\mathcal{R}_\lambda \circ i$ is not dense in $X(\Omega)$. The *generalized continuous spectrum* is defined to be $\hat{\sigma}_c(T) = \hat{\sigma}(T) \backslash (\hat{\sigma}_p(T) \cup \hat{\sigma}_r(T))$.

By the definition, $\hat{\varrho}(T)$ is an open set. The existence of the neighborhood V_λ in the above definition is introduced by Waelbroeck [26] (see also Maeda [15]) for the spectral theory on locally convex spaces. If $\hat{\varrho}(T)$ were simply defined to be the set of points such that $\mathcal{R}_\lambda \circ i$ is a densely defined continuous operator, $\hat{\varrho}(T)$ is not an open set in general. If $X(\Omega)$ is a Banach space and the operator $i^{-1}K^\times A(\lambda)i$ is continuous on $X(\Omega)$ for each $\lambda \in \hat{\Omega}$, we can show that $\lambda \in \hat{\varrho}(T)$ if and only if $id - i^{-1}K^\times A(\lambda)i$ has a continuous inverse on $X(\Omega)$ (Prop.3.18).

Theorem 3.12.
(i) For each $\psi \in X(\Omega)$, $\mathcal{R}_\lambda i\psi$ is an $X(\Omega)'$-valued holomorphic function in $\lambda \in \hat{\varrho}(T)$.
(ii) Suppose $\mathrm{Im}(\lambda) < 0$ and $\lambda \in \hat{\varrho}(T) \cap \varrho(T)$, where $\varrho(T)$ is the resolvent set of T in $\mathcal{H}$-sense. Then, $\langle \mathcal{R}_\lambda \psi \,|\, \phi \rangle = ((\lambda - T)^{-1}\psi, \phi)$ for any $\psi, \phi \in X(\Omega)$.

This theorem means that $\langle \mathcal{R}_\lambda \psi \,|\, \phi \rangle$ is an analytic continuation of $((\lambda - T)^{-1}\psi, \phi)$ from the lower half plane to $\hat{\varrho}(T)$ through the interval I. We always suppose that the domain of $\mathcal{R}_\lambda \circ i$ is continuously extended to the whole $X(\Omega)$ when $\lambda \notin \hat{\sigma}(T)$. The significant point to be emphasized is that to prove the *strong* holomorphy of $\mathcal{R}_\lambda \circ i(\psi)$, it is sufficient to assume that $\mathcal{R}_\lambda \circ i : X(\Omega) \to X(\Omega)'$ is continuous in the *weak* dual topology on $X(\Omega)'$.

Proof of Thm.3.12. Since $\hat{\varrho}(T)$ is open, when $\lambda \in \hat{\varrho}(T)$, $\mathcal{R}_{\lambda+h}$ exists for sufficiently small $h \in \mathbf{C}$. Put $\psi_\lambda = i^{-1}(id - K^\times A(\lambda))^{-1} i(\psi)$ for $\psi \in X(\Omega)$. It is easy to verify the equality

$$\mathcal{R}_{\lambda+h} i(\psi) - \mathcal{R}_\lambda i(\psi) = (A(\lambda+h) - A(\lambda)) i(\psi_\lambda) + \mathcal{R}_{\lambda+h} i \circ i^{-1} K^\times (A(\lambda+h) - A(\lambda)) i(\psi_\lambda).$$

Let us show that $i^{-1}K^\times A(\lambda) i(\psi) \in X(\Omega)$ is holomorphic in λ. For any $\psi, \phi \in X(\Omega)$, we obtain

$$\begin{aligned} & \langle \phi \,|\, i^{-1}K^\times A(\lambda) i\psi \rangle = (\phi, i^{-1}K^\times A(\lambda) i\psi) = \overline{(i^{-1}K^\times A(\lambda) i\psi, \phi)} \\ = \; & \overline{\langle K^\times A(\lambda) i\psi \,|\, \phi \rangle} = \overline{\langle A(\lambda) i\psi \,|\, K^* \phi \rangle}. \end{aligned}$$

From the definition of $A(\lambda)$, it follows that $\langle \phi \,|\, i^{-1}K^\times A(\lambda) i\psi \rangle$ is holomorphic in $\overline{\lambda}$. Since $X(\Omega)$ is dense in $X(\Omega)'$, $\langle \mu \,|\, i^{-1}K^\times A(\lambda) i\psi \rangle$ is holomorphic in $\overline{\lambda}$ for any $\mu \in X(\Omega)'$ by Montel's theorem. This means that $i^{-1}K^\times A(\lambda) i\psi$ is weakly holomorphic. Since $X(\Omega)$ is a quasi-complete locally convex space, any weakly holomorphic function is holomorphic with respect to the original topology (see Rudin [21]). This proves that $i^{-1}K^\times A(\lambda) i\psi$ is holomorphic in λ (note that the weak holomorphy in $\overline{\lambda}$ implies the strong holomorphy in λ because functionals in $X(\Omega)'$ are *anti*-linear).

Next, the definition of $\hat{\varrho}(T)$ implies that the family $\{\mathcal{R}_\mu \circ i\}_{\mu \in V_\lambda}$ of continuous operators is bounded in the pointwise convergence topology. Due to Banach-Steinhaus theorem

(Thm.33.1 of [25]), the family is equicontinuous. This and the holomorphy of $A(\lambda)$ and $i^{-1}K^{\times}A(\lambda)i(\psi)$ prove that $\mathcal{R}_{\lambda+h}i(\psi)$ converges to $\mathcal{R}_{\lambda}i(\psi)$ as $h \to 0$ with respect to the weak dual topology. In particular, we obtain

$$\lim_{h\to 0}\frac{\mathcal{R}_{\lambda+h}i-\mathcal{R}_{\lambda}i}{h}(\psi)=\frac{dA}{d\lambda}(\lambda)i(\psi_{\lambda})+\mathcal{R}_{\lambda}i\circ\frac{d}{d\lambda}(i^{-1}K^{\times}A(\lambda)i)(\psi_{\lambda}), \tag{3.22}$$

which proves that $\mathcal{R}_{\lambda}i(\psi)$ is holomorphic in λ with respect to the weak dual topology on $X(\Omega)'$. Since $X(\Omega)$ is barreled, the weak dual holomorphy implies the strong dual holomorphy (Thm.A.3 (iii)).

Let us prove (ii). Suppose $\mathrm{Im}(\lambda) < 0$. Note that $\mathcal{R}_{\lambda}\circ i$ is written as $\mathcal{R}_{\lambda}\circ i = A(\lambda)\circ(id - i^{-1}K^{\times}A(\lambda)i)^{-1}$. We can show the equality

$$(id - i^{-1}K^{\times}A(\lambda)i)f = (id - K(\lambda-H)^{-1})f \in X(\Omega). \tag{3.23}$$

Indeed, for any $f, \psi \in X(\Omega)$, we obtain

$$\begin{aligned}\langle (i-K^{\times}A(\lambda)i)f\,|\,\psi\rangle &= \langle if\,|\,\psi\rangle - \langle A(\lambda)if\,|\,K^{*}\psi\rangle\\ &= \langle if\,|\,\psi\rangle - \langle i\circ(\lambda-H)^{-1}f\,|\,K^{*}\psi\rangle\\ &= (f,\psi) - (K(\lambda-H)^{-1}f,\psi) = ((id-K(\lambda-H)^{-1})f,\psi).\end{aligned}$$

Thus, $\mathcal{R}_{\lambda}$ satisfies for $\phi = (id - i^{-1}K^{\times}A(\lambda)i)f$ that

$$\begin{aligned}\mathcal{R}_{\lambda}i\phi &= A(\lambda)i\circ(id-i^{-1}K^{\times}A(\lambda)i)^{-1}\phi\\ &= i(\lambda-H)^{-1}\circ(id-K(\lambda-H)^{-1})^{-1}\phi = i(\lambda-T)^{-1}\phi.\end{aligned}$$

Since $\lambda \in \hat{\rho}(T)$, $(id - i^{-1}K^{\times}A(\lambda)i)X(\Omega)$ is dense in $X(\Omega)$ and $\mathcal{R}_{\lambda}i : X(\Omega) \to X(\Omega)'$ is continuous. Since $\lambda \in \rho(T)$, $i(\lambda-T)^{-1} : \mathcal{H} \to X(\Omega)'$ is continuous. Therefore, taking the limit proves that $\mathcal{R}_{\lambda}i\phi = i(\lambda-T)^{-1}\phi$ holds for any $\phi \in X(\Omega)$. ■

Remark. Even when λ is in the continuous spectrum of T, Thm.3.12 (ii) holds as long as $(\lambda-T)^{-1}$ exists and $i\circ(\lambda-T)^{-1} : \mathcal{H} \to X(\Omega)'$ is continuous. In general, the continuous spectrum of T is not included in the generalized spectrum because the topology of $X(\Omega)'$ is weaker than that of $\mathcal{H}$.

Proposition 3.13. The generalized resolvent satisfies
(i) $(\lambda - T^{\times})\circ\mathcal{R}_{\lambda} = id|_{iX(\Omega)}$
(ii) If $\mu \in X(\Omega)'$ satisfies $(\lambda-T^{\times})\mu \in iX(\Omega)$, then $\mathcal{R}_{\lambda}\circ(\lambda-T^{\times})\mu = \mu$.
(iii) $T^{\times}\circ\mathcal{R}_{\lambda}|_{iY} = \mathcal{R}_{\lambda}\circ T^{\times}|_{iY}$.

Proof. Prop.3.6 (i) gives $id = (\lambda - H^{\times})A(\lambda) = (\lambda - T^{\times} + K^{\times})A(\lambda)$. This proves

$$\begin{aligned}&(\lambda-T^{\times})\circ A(\lambda) = id - K^{\times}A(\lambda)\\ \Rightarrow\ &(\lambda-T^{\times})\circ A(\lambda)\circ(id-K^{\times}A(\lambda))^{-1} = (\lambda-T^{\times})\circ\mathcal{R}_{\lambda} = id.\end{aligned}$$

Next, when $(\lambda-T^{\times})\mu \in iX(\Omega)$, $A(\lambda)(\lambda-T^{\times})\mu$ is well defined and Prop.3.6 (ii) gives

$$A(\lambda)(\lambda-T^{\times})\mu = A(\lambda)(\lambda-H^{\times}-K^{\times})\mu = (id - A(\lambda)K^{\times})\mu.$$

This proves $\mu = (id - A(\lambda)K^{\times})^{-1}A(\lambda)(\lambda-T^{\times})\mu = \mathcal{R}_{\lambda}(\lambda-T^{\times})\mu$. Finally, note that $(\lambda - T^{\times})iY = i(\lambda-T)Y \subset iX(\Omega)$ because of the assumptions (X6), (X7). Thus part (iii) of the proposition immediately follows from (i), (ii). ■

3.5 Generalized projections

Let $\Sigma \subset \hat{\sigma}(T)$ be a bounded subset of the generalized spectrum, which is separated from the rest of the spectrum by a simple closed curve $\gamma \subset \Omega \cup I \cup \{\lambda \,|\, \mathrm{Im}(\lambda) < 0\}$. Define the operator $\Pi_\Sigma : iX(\Omega) \to X(\Omega)'$ to be

$$\Pi_\Sigma \phi = \frac{1}{2\pi\sqrt{-1}} \int_\gamma \mathcal{R}_\lambda \phi \, d\lambda, \quad \phi \in iX(\Omega), \tag{3.24}$$

where the integral is defined as the Pettis integral. Since $X(\Omega)$ is assumed to be barreled by (X3), $X(\Omega)'$ is quasi-complete and satisfies the convex envelope property (see Appendix A). Since $\mathcal{R}_\lambda \phi$ is strongly holomorphic in λ (Thm.3.12), the Pettis integral of $\mathcal{R}_\lambda \phi$ exists by Thm.A.1. See Appendix A for the definition and the existence theorem of Pettis integrals. Since $\mathcal{R}_\lambda \circ i : X(\Omega) \to X(\Omega)'$ is continuous, Thm.A.1 (ii) proves that $\Pi_\Sigma \circ i$ is a continuous operator from $X(\Omega)$ into $X(\Omega)'$ equipped with the weak dual topology. Note that the equality

$$T^\times \int_\gamma \mathcal{R}_\lambda \phi \, d\lambda = \int_\gamma T^\times \mathcal{R}_\lambda \phi \, d\lambda, \tag{3.25}$$

holds. To see this, it is sufficient to show that the set $\{\langle T^\times \mathcal{R}_\lambda \phi \,|\, \psi \rangle\}_{\lambda \in \gamma}$ is bounded for each $\psi \in X(\Omega)$ due to Thm.A.1 (iii). Prop.3.13 (i) yields $T^\times \mathcal{R}_\lambda \phi = \lambda \mathcal{R}_\lambda \phi - \phi$. Since $\lambda \mathcal{R}_\lambda$ is holomorphic and γ is compact, $\{\langle T^\times \mathcal{R}_\lambda \phi \,|\, \psi \rangle\}_{\lambda \in \gamma}$ is bounded so that Eq.(3.25) holds.

Although $\Pi_\Sigma \circ \Pi_\Sigma$ is not defined, we call Π_Σ the *generalized Riesz projection* for Σ because of the next proposition.

Proposition 3.14. $\Pi_\Sigma(iX(\Omega)) \cap (id - \Pi_\Sigma)(iX(\Omega)) = \{0\}$ and the direct sum satisfies

$$iX(\Omega) \subset \Pi_\Sigma(iX(\Omega)) \oplus (id - \Pi_\Sigma)(iX(\Omega)) \subset X(\Omega)'. \tag{3.26}$$

In particular, for any $\phi \in X(\Omega)$, there exist μ_1, μ_2 such that ϕ is uniquely decomposed as

$$i(\phi) = \langle \phi | = \mu_1 + \mu_2, \quad \mu_1 \in \Pi_\Sigma(iX(\Omega)),\ \mu_2 \in (id - \Pi_\Sigma)(iX(\Omega)). \tag{3.27}$$

Proof. We simply denote $\langle \phi|$ as ϕ. It is sufficient to show that $\Pi_\Sigma(iX(\Omega)) \cap (id - \Pi_\Sigma)(iX(\Omega)) = \{0\}$. Suppose that there exist $\phi, \psi \in iX(\Omega)$ such that $\Pi_\Sigma \phi = \psi - \Pi_\Sigma \psi$. Since $\Pi_\Sigma(\phi + \psi) = \psi \in iX(\Omega)$, we can again apply the projection to the both sides as $\Pi_\Sigma \circ \Pi_\Sigma(\phi + \psi) = \Pi_\Sigma \psi$. Let γ' be a closed curve which is slightly larger than γ. Then,

$$\begin{aligned}
\Pi_\Sigma \circ \Pi_\Sigma(\phi + \psi) &= \left(\frac{1}{2\pi\sqrt{-1}}\right)^2 \int_{\gamma'} \mathcal{R}_{\lambda'} \left(\int_\gamma \mathcal{R}_\lambda (\phi + \psi) d\lambda \right) d\lambda' \\
&= \left(\frac{1}{2\pi\sqrt{-1}}\right)^2 \int_{\gamma'} \mathcal{R}_{\lambda'} \left(\int_\gamma \frac{(\lambda - \lambda') + (\lambda' - T^\times)}{\lambda - \lambda'} \mathcal{R}_\lambda (\phi + \psi) d\lambda \right) d\lambda' \\
&\quad - \left(\frac{1}{2\pi\sqrt{-1}}\right)^2 \int_{\gamma'} \mathcal{R}_{\lambda'} \left(\int_\gamma \frac{\lambda' - T^\times}{\lambda - \lambda'} \mathcal{R}_\lambda (\phi + \psi) d\lambda \right) d\lambda'.
\end{aligned}$$

Eq.(3.25) shows

$$\begin{aligned}\Pi_\Sigma \circ \Pi_\Sigma(\phi+\psi) &= \left(\frac{1}{2\pi\sqrt{-1}}\right)^2 \int_{\gamma'} \mathcal{R}_{\lambda'} \left(\int_\gamma \frac{\lambda - T^\times}{\lambda-\lambda'} \mathcal{R}_\lambda(\phi+\psi) d\lambda \right) d\lambda' \\ &\quad - \left(\frac{1}{2\pi\sqrt{-1}}\right)^2 \int_{\gamma'} \mathcal{R}_{\lambda'} \circ (\lambda' - T^\times) \left(\int_\gamma \frac{\mathcal{R}_\lambda}{\lambda-\lambda'} (\phi+\psi) d\lambda \right) d\lambda'.\end{aligned}$$

Prop.3.13 shows

$$\begin{aligned} &= \left(\frac{1}{2\pi\sqrt{-1}}\right)^2 \int_{\gamma'} \mathcal{R}_{\lambda'} \left(\int_\gamma \frac{\phi+\psi}{\lambda-\lambda'} d\lambda \right) d\lambda' - \left(\frac{1}{2\pi\sqrt{-1}}\right)^2 \int_{\gamma'} \left(\int_\gamma \frac{\mathcal{R}_\lambda}{\lambda-\lambda'} (\phi+\psi) d\lambda \right) d\lambda' \\ &= 0 - \left(\frac{1}{2\pi\sqrt{-1}}\right)^2 \int_\gamma \mathcal{R}_\lambda(\phi+\psi) \cdot \int_{\gamma'} \frac{1}{\lambda-\lambda'} d\lambda' \cdot d\lambda \\ &= \frac{1}{2\pi\sqrt{-1}} \int_\gamma \mathcal{R}_\lambda(\phi+\psi) d\lambda = \Pi_\Sigma(\phi+\psi).\end{aligned}$$

This proves that $\Pi_\Sigma \phi = 0$. ■

The above proof also shows that as long as $\Pi_\Sigma\phi \in iX(\Omega)$, $\Pi_\Sigma \circ \Pi_\Sigma$ is defined and $\Pi_\Sigma \circ \Pi_\Sigma \phi = \Pi_\Sigma \phi$.

Proposition 3.15. $\Pi_\Sigma|_{iY}$ is $T^\times$-invariant: $\Pi_\Sigma \circ T^\times|_{iY} = T^\times \circ \Pi_\Sigma|_{iY}$.

Proof. This follows from Prop.3.13 (iii) and Eq.(3.25). ■

Let λ_0 be an isolated generalized eigenvalue, which is separated from the rest of the generalized spectrum by a simple closed curve $\gamma_0 \subset \Omega \cup I \cup \{\lambda \,|\, \mathrm{Im}(\lambda) < 0\}$. Let

$$\Pi_0 = \frac{1}{2\pi\sqrt{-1}} \int_{\gamma_0} \mathcal{R}_\lambda d\lambda, \tag{3.28}$$

be a projection for λ_0 and $V_0 = \bigcup_{m\geq 1} \mathrm{Ker}\, B^{(m)}(\lambda_0) \circ \cdots \circ B^{(1)}(\lambda_0)$ a generalized eigenspace of λ_0. The main theorem in this paper is stated as follows:

Theorem 3.16. If $\Pi_0 iX(\Omega)$ is finite dimensional, then $\Pi_0 iX(\Omega) = V_0$.

In the usual spectral theory, this theorem is easily proved by using the resolvent equation. In our theory, the composition $\mathcal{R}_{\lambda'} \circ \mathcal{R}_\lambda$ is not defined because $\mathcal{R}_\lambda$ is an operator from $iX(\Omega)$ into $X(\Omega)'$. As a result, the resolvent equation does not hold and the proof of the above theorem is rather technical.

Proof. Let $\mathcal{R}_\lambda = \sum_{j=-\infty}^{\infty} (\lambda_0 - \lambda)^j E_j$ be a Laurent series of $\mathcal{R}_\lambda$, which converges in the strong dual topology (see Thm.A.3). Since

$$id = (\lambda - T^\times) \circ \mathcal{R}_\lambda = (\lambda_0 - T^\times - (\lambda_0 - \lambda)) \circ \sum_{j=-\infty}^{\infty} (\lambda_0 - \lambda)^j E_j,$$

we obtain $E_{-n-1} = (\lambda_0 - T^\times) E_{-n}$ for $n = 1, 2, \cdots$. Thus the equality

$$E_{-n-1} = (\lambda_0 - T^\times)^n E_{-1} \tag{3.29}$$

holds. Similarly, $id|_{iY} = \mathcal{R}_\lambda \circ (\lambda - T^\times)|_{iY}$ (Prop.3.13 (ii)) provides $E_{-n-1}|_{iY} = E_{-n} \circ (\lambda_0 - T^\times)|_{iY}$. Thus we obtain $\mathsf{R}(E_{-n-1}|_{iY}) \subseteq \mathsf{R}(E_{-n})$ for any $n \geq 1$. Since Y is dense in $X(\Omega)$ and the range of $E_{-1} = -\Pi_0$ is finite dimensional, it turns out that $\mathsf{R}(E_{-n}|_{iY}) = \mathsf{R}(E_{-n})$ and $\mathsf{R}(E_{-n-1}) \subseteq \mathsf{R}(E_{-n})$ for any $n \geq 1$. This implies that the principle part $\sum_{-\infty}^{-1}(\lambda_0 - \lambda)^j E_j$ of the Laurent series is a finite dimensional operator. Hence, there exists an integer $M \geq 1$ such that $E_{-M-1} = 0$. This means that λ_0 is a pole of $\mathcal{R}_\lambda$:

$$\mathcal{R}_\lambda = \sum_{j=-M}^{\infty} (\lambda_0 - \lambda)^j E_j. \tag{3.30}$$

Next, from the equality $(id - A(\lambda)K^\times) \circ \mathcal{R}_\lambda = A(\lambda)$, we have

$$\left(id - \sum_{k=0}^{\infty} (\lambda_0 - \lambda)^k A^{(k+1)}(\lambda_0) K^\times \right) \circ \sum_{j=-M}^{\infty} (\lambda_0 - \lambda)^j E_j = \sum_{k=0}^{\infty} (\lambda_0 - \lambda)^k A^{(k+1)}(\lambda_0).$$

Comparing the coefficients of $(\lambda_0 - \lambda)^{-1}$ on both sides, we obtain

$$(id - A(\lambda_0)K^\times)E_{-1} - \sum_{j=2}^{M} A^{(j)}(\lambda_0) K^\times E_{-j} = 0. \tag{3.31}$$

Substituting Eq.(3.29) and $E_{-1} = -\Pi_0$ provides

$$B^{(1)}(\lambda_0)\Pi_0 - \sum_{j=2}^{M} A^{(j)}(\lambda_0) K^\times (\lambda_0 - T^\times)^{j-1} \Pi_0 = 0. \tag{3.32}$$

In particular, this implies $\mathsf{R}(\Pi_0) \subset \mathsf{D}(B^{(1)}(\lambda_0))$. Hence, $(\lambda_0 - T^\times)\Pi_0$ can be rewritten as

$$(\lambda_0 - T^\times)\Pi_0 = (\lambda_0 - H^\times) \circ (id - A(\lambda_0)K^\times)\Pi_0 = (\lambda_0 - H^\times) B^{(1)}(\lambda_0)\Pi_0.$$

Then, by using the definition of $B^{(2)}(\lambda_0)$, Eq.(3.32) is rearranged as

$$B^{(2)}(\lambda_0) B^{(1)}(\lambda_0)\Pi_0 - \sum_{j=3}^{M} A^{(j)}(\lambda_0) K^\times (\lambda_0 - T^\times)^{j-1} \Pi_0 = 0.$$

Repeating similar calculations, we obtain

$$B^{(M)}(\lambda_0) \circ \cdots \circ B^{(1)}(\lambda_0)\Pi_0 = 0. \tag{3.33}$$

This proves $\Pi_0 iX(\Omega) \subset V_0$.

Let us show $\Pi_0 iX(\Omega) \supset V_0$. From the equality $\mathcal{R}_\lambda \circ (id - K^\times A(\lambda)) = A(\lambda)$, we have

$$\sum_{j=-M}^{\infty} (\lambda_0 - \lambda)^j E_j \circ \left(id - K^\times \sum_{k=0}^{\infty} (\lambda_0 - \lambda)^k A^{(k+1)}(\lambda_0) \right) = \sum_{k=0}^{\infty} (\lambda_0 - \lambda)^k A^{(k+1)}(\lambda_0). \tag{3.34}$$

Comparing the coefficients of $(\lambda_0 - \lambda)^k$ on both sides for $k = 1, 2, \cdots$, we obtain

$$E_k(id - K^{\times}A(\lambda_0))\phi - \sum_{j=1}^{\infty} E_{-j+k}K^{\times}A^{(j+1)}(\lambda_0)\phi = A^{(k+1)}(\lambda_0)\phi, \tag{3.35}$$

for any $\phi \in iX(\Omega)$, where the left hand side is a finite sum. Note that $K^{\times}A^{(j)}(\lambda_0)iX(\Omega) \subset iX(\Omega)$ for any $j = 1, 2, \cdots$ because $K^{\times}A(\lambda)iX(\Omega) \subset iX(\Omega)$ for any λ (the assumption (X8)).

Now suppose that $\mu \in V_0$ is a generalized eigenfunction satisfying $B^{(M)}(\lambda_0) \circ \cdots \circ B^{(1)}(\lambda_0)\mu = 0$. For this μ, we need the following lemma.

Lemma. For any $k = 0, 1, \cdots, M-1$,
(i) $(\lambda_0 - T^{\times})^k\mu = (\lambda_0 - H^{\times})^k B^{(k)}(\lambda_0) \circ \cdots \circ B^{(1)}(\lambda_0)\mu$.
(ii) $K^{\times}(\lambda_0 - T^{\times})^k\mu \in iX(\Omega)$.

Proof. Due to Thm.3.9, μ is included in the domain of $(\lambda_0 - T^{\times})^k$. Thus the left hand side of (i) indeed exists. Then, we have

$$\begin{aligned}(\lambda_0 - H^{\times})^k B^{(k)}(\lambda_0) &= (\lambda_0 - H^{\times})^k(id - A^{(k)}(\lambda_0)K^{\times}(\lambda_0 - H^{\times})^{k-1}) \\ &= (\lambda_0 - H^{\times} - K^{\times})(\lambda_0 - H^{\times})^{k-1} = (\lambda_0 - T^{\times})(\lambda_0 - H^{\times})^{k-1}.\end{aligned}$$

Repeating this procedure yields (i). To prove (ii), let us calculate

$$0 = K^{\times}(\lambda_0 - H^{\times})^k B^{(M)}(\lambda_0) \circ \cdots \circ B^{(1)}(\lambda_0)\mu.$$

Eq.(3.18) and the part (i) of this lemma give

$$\begin{aligned} 0 &= K^{\times}B^{(M-k)}(\lambda_0) \circ \cdots \circ B^{(k+1)}(\lambda_0) \circ (\lambda_0 - H^{\times})^k \circ B^{(k)}(\lambda_0) \circ \cdots \circ B^{(1)}(\lambda_0)\mu \\ &= K^{\times}B^{(M-k)}(\lambda_0) \circ \cdots \circ B^{(k+1)}(\lambda_0) \circ (\lambda_0 - T^{\times})^k\mu.\end{aligned}$$

For example, when $k = M - 1$, this is reduced to

$$0 = K^{\times}(id - A(\lambda_0)K^{\times}) \circ (\lambda_0 - T^{\times})^{M-1}\mu.$$

This proves $K^{\times}(\lambda_0 - T^{\times})^{M-1}\mu = K^{\times}A(\lambda_0)K^{\times}(\lambda_0 - T^{\times})^{M-1}\mu \in iX(\Omega)$. This is true for any $k = 0, 1, \cdots, M-1$; it follows from the definition of $B^{(j)}(\lambda_0)$'s that $K^{\times}(\lambda_0 - T^{\times})^k\mu$ is expressed as a linear combination of elements of the form $K^{\times}A^{(j)}(\lambda_0)\xi_j$, $\xi_j \in iX(\Omega)$. Since $K^{\times}A^{(j)}(\lambda_0)iX(\Omega) \subset iX(\Omega)$, we obtain $K^{\times}(\lambda_0 - T^{\times})^k\mu \in iX(\Omega)$. ■

Since $K^{\times}(\lambda_0 - T^{\times})^k\mu \in iX(\Omega)$, we can substitute $\phi = K^{\times}(\lambda_0 - T^{\times})^k\mu$ into Eq.(3.35). The resultant equation is rearranged as

$$\begin{aligned} &E_kK^{\times}(id - A(\lambda_0)K^{\times})(\lambda_0 - T^{\times})^k\mu - \left(id + \sum_{j=1}^{k} E_{-j+k}K^{\times}(\lambda_0 - H^{\times})^{k-j} \right) A^{(k+1)}(\lambda_0)K^{\times}(\lambda_0 - T^{\times})^k\mu \\ =\ & \sum_{j=k+1}^{\infty} E_{-j+k}K^{\times}A^{(j+1)}(\lambda_0)K^{\times}(\lambda_0 - T^{\times})^k\mu.\end{aligned}$$

Further, $(\lambda_0 - T^\times)^k = (\lambda_0 - H^\times)^k B^{(k)}(\lambda_0) \circ \cdots \circ B^{(1)}(\lambda_0)$ provides

$$\begin{aligned}
& E_k K^\times (\lambda_0 - H^\times)^k B^{(k+1)}(\lambda_0) \circ \cdots \circ B^{(1)}(\lambda_0)\mu \\
& \quad - \left(id + \sum_{j=1}^{k} E_{-j+k} K^\times (\lambda_0 - H^\times)^{k-j} \right) A^{(k+1)}(\lambda_0) K^\times (\lambda_0 - H^\times)^k B^{(k)}(\lambda_0) \circ \cdots \circ B^{(1)}(\lambda_0)\mu \\
= \quad & \sum_{j=k+1}^{\infty} E_{-j+k} K^\times A^{(j+1)}(\lambda_0) K^\times (\lambda_0 - T^\times)^k \mu. \qquad (3.36)
\end{aligned}$$

On the other hand, comparing the coefficients of $(\lambda_0 - \lambda)^0$ of Eq.(3.34) provides

$$E_0 (id - K^\times A(\lambda_0))\phi - \sum_{j=1}^{\infty} E_{-j} K^\times A^{(j+1)}(\lambda_0)\phi = A(\lambda_0)\phi,$$

for any $\phi \in iX(\Omega)$. Substituting $\phi = K^\times \mu \in iX(\Omega)$ provides

$$(id + E_0 K^\times) B^{(1)}(\lambda_0)\mu = \mu + \sum_{j=1}^{\infty} E_{-j} K^\times A^{(j+1)}(\lambda_0) K^\times \mu. \qquad (3.37)$$

By adding Eq.(3.37) to Eqs.(3.36) for $k = 1, \cdots, M-1$, we obtain

$$\begin{aligned}
& (id + E_0 K^\times) B^{(1)}(\lambda_0)\mu \\
& \quad - \sum_{k=1}^{M-1} \left(id + \sum_{j=1}^{k} E_{-j+k} K^\times (\lambda_0 - H^\times)^{k-j} \right) A^{(k+1)}(\lambda_0) K^\times (\lambda_0 - H^\times)^k B^{(k)}(\lambda_0) \circ \cdots \circ B^{(1)}(\lambda_0)\mu \\
& \quad + \sum_{k=1}^{M-1} E_k K^\times (\lambda_0 - H^\times)^k B^{(k+1)}(\lambda_0) \circ \cdots \circ B^{(1)}(\lambda_0)\mu \\
= \quad & \mu + \sum_{k=0}^{M-1} \sum_{j=1}^{\infty} E_{-j} K^\times A^{(j+k+1)}(\lambda_0) K^\times (\lambda_0 - T^\times)^k \mu. \qquad (3.38)
\end{aligned}$$

The left hand side above is rewritten as

$$\begin{aligned}
& (id + E_0 K^\times + E_1 K^\times (\lambda_0 - H^\times))\, B^{(2)}(\lambda_0) B^{(1)}(\lambda_0)\mu \\
& \quad - \sum_{k=2}^{M-1} \left(id + \sum_{j=1}^{k} E_{-j+k} K^\times (\lambda_0 - H^\times)^{k-j} \right) A^{(k+1)}(\lambda_0) K^\times (\lambda_0 - H^\times)^k B^{(k)}(\lambda_0) \circ \cdots \circ B^{(1)}(\lambda_0)\mu \\
& \quad + \sum_{k=2}^{M-1} E_k K^\times (\lambda_0 - H^\times)^k B^{(k+1)}(\lambda_0) \circ \cdots \circ B^{(1)}(\lambda_0)\mu.
\end{aligned}$$

Repeating similar calculations, we can verify that Eq.(3.38) is rewritten as

$$\begin{aligned}
& \left(id + \sum_{j=0}^{M-1} E_j K^\times (\lambda_0 - H^\times)^j \right) B^{(M)}(\lambda_0) \circ \cdots \circ B^{(1)}(\lambda_0)\mu \\
= \quad & \mu - \sum_{k=0}^{M-1} \sum_{j=1}^{\infty} E_{-j} K^\times A^{(j+k+1)}(\lambda_0) K^\times (\lambda_0 - T^\times)^k \mu. \qquad (3.39)
\end{aligned}$$

Since $B^{(M)}(\lambda_0) \circ \cdots \circ B^{(1)}(\lambda_0)\mu = 0$, we obtain

$$\mu = \sum_{k=0}^{M-1} \sum_{j=1}^{\infty} E_{-j} K^{\times} A^{(j+k+1)}(\lambda_0) K^{\times} (\lambda_0 - T^{\times})^k \mu.$$

Since $\mathsf{R}(E_{-j}) \subset \mathsf{R}(E_{-1}) = \mathsf{R}(\Pi_0)$, this proves $\Pi_0 iX(\Omega) \supset V_0$. Thus the proof of $\Pi_0 iX(\Omega) = V_0$ is completed. ■

3.6 Properties of the generalized spectrum

We show a few criteria to estimate the generalized spectrum. Recall that $\hat{\sigma}_p(T) \subset \sigma_p(T^{\times})$ because of Thm.3.5. The relation between $\hat{\sigma}(T)$ and $\sigma(T)$ is given as follows.

Proposition 3.17. Let $\mathbf{C}_- = \{\mathrm{Im}(\lambda) < 0\}$ be an open lower half plane. Let $\sigma_p(T)$ and $\sigma(T)$ be the point spectrum and the spectrum in the usual sense, respectively. Then, the following relations hold.
(i) $\hat{\sigma}(T) \cap \mathbf{C}_- \subset \sigma(T) \cap \mathbf{C}_-$. In particular, $\hat{\sigma}_p(T) \cap \mathbf{C}_- \subset \sigma_p(T) \cap \mathbf{C}_-$
(ii) Let $\Sigma \subset \mathbf{C}_-$ be a bounded subset of $\sigma(T)$ which is separated from the rest of the spectrum by a simple closed curve γ. Then, there exists a point of $\hat{\sigma}(T)$ inside γ. In particular, if $\lambda \in \mathbf{C}_-$ is an isolated point of $\sigma(T)$, then $\lambda \in \hat{\sigma}(T)$.

Proof. Note that when $\lambda \in \mathbf{C}_-$, the generalized resolvent satisfies $\mathcal{R}_\lambda \circ i = i \circ (\lambda - T)^{-1}$ due to Thm.3.12.

(i) Suppose that $\lambda \in \varrho(T) \cap \mathbf{C}_-$, where $\varrho(T)$ is the resolvent set of T in the usual sense. Since $\mathcal{H}$ is a Hilbert space, there is a neighborhood $V_\lambda \subset \varrho(T) \cap \mathbf{C}_-$ of λ such that $(\lambda' - T)^{-1}$ is continuous on $\mathcal{H}$ for any $\lambda' \in V_\lambda$ and the set $\{(\lambda' - T)^{-1}\psi\}_{\lambda' \in V_\lambda}$ is bounded in $\mathcal{H}$ for each $\psi \in X(\Omega)$. Since $i : \mathcal{H} \to X(\Omega)'$ is continuous and since the topology of $X(\Omega)$ is stronger than that of $\mathcal{H}$, $\mathcal{R}_{\lambda'} \circ i = i \circ (\lambda' - T)^{-1}$ is a continuous operator from $X(\Omega)$ into $X(\Omega)'$ for any $\lambda' \in V_\lambda$, and the set $\{\mathcal{R}_{\lambda'} \circ i\psi\}_{\lambda' \in V_\lambda}$ is bounded in $X(\Omega)'$. This proves that $\lambda \in \hat{\varrho}(T) \cap \mathbf{C}_-$.

Next, suppose that $\lambda \in \mathbf{C}_-$ is a generalized eigenvalue satisfying $(id - K^{\times}A(\lambda))i(\psi) = 0$ for $\psi \in X(\Omega)$. Since $\lambda - H$ is invertible on $\mathcal{H}$ when $\lambda \in \mathbf{C}_-$, putting $\phi = (\lambda - H)^{-1}\psi$ provides

$$(id - K^{\times}A(\lambda))i(\lambda - H)\phi = (i(\lambda - H) - K^{\times}i)\phi = i(\lambda - T)\phi = 0,$$

and thus $\lambda \in \sigma_p(T)$.

(ii) Let $\mathcal{P}$ be the Riesz projection for $\Sigma \subset \sigma(T) \cap \mathbf{C}_-$, which is defined as $\mathcal{P} = (2\pi\sqrt{-1})^{-1} \int_\gamma (\lambda - T)^{-1} d\lambda$. Since γ encloses a point of $\sigma(T)$, $\mathcal{P}\mathcal{H} \neq \emptyset$. Since $X(\Omega)$ is dense in $\mathcal{H}$, $\mathcal{P}X(\Omega) \neq \emptyset$. This fact and $\mathcal{R}_\lambda \circ i = i \circ (\lambda - T)^{-1}$ prove that the range of the generalized Riesz projection defined by Eq.(3.24) is not zero. Hence, the closed curve γ encloses a point of $\hat{\sigma}(T)$. ■

A few remarks are in order. If the spectrum of T on the lower half plane consists of discrete eigenvalues, (i) and (ii) show that $\sigma_p(T) \cap \mathbf{C}_- = \sigma(T) \cap \mathbf{C}_- = \hat{\sigma}(T) \cap \mathbf{C}_-$. However, it is possible that a generalized eigenvalue on I is not an eigenvalue in the usual sense. See [4] for such an example. In most cases, the continuous spectrum on the lower half plane is not included in the generalized spectrum because the topology on $X(\Omega)'$ is weaker than that on $\mathcal{H}$, although the point spectrum and the residual spectrum may remain to exist as the generalized spectrum. Note that the continuous spectrum on the interval I also disappears; for the resolvent $(\lambda - T)^{-1} = (\lambda - H)^{-1}(id - K(\lambda - H)^{-1})^{-1}$ in the usual sense, the factor $(\lambda - H)^{-1}$ induces the continuous spectrum on the real axis because H is selfadjoint. For the generalized resolvent, $(\lambda - H)^{-1}$ is replaced by $A(\lambda)$, which has no singularities. This suggests that obstructions when calculating the Laplace inversion formula by using the residue theorem may disappear.

Recall that a linear operator L from a topological vector space X_1 to another topological vector space X_2 is said to be bounded if there exists a neighborhood $U \subset X_1$ such that $LU \subset X_2$ is a bounded set. When $L = L(\lambda)$ is parameterized by λ, it is said to be bounded uniformly in λ if such a neighborhood U is independent of λ. When the domain X_1 is a Banach space, $L(\lambda)$ is bounded uniformly in λ if and only if $L(\lambda)$ is continuous for each λ (U is taken to be the unit sphere). Similarly, L is called compact if there exists a neighborhood $U \subset X_1$ such that $LU \subset X_2$ is relatively compact. When $L = L(\lambda)$ is parameterized by λ, it is said to be compact uniformly in λ if such a neighborhood U is independent of λ. When the domain X_1 is a Banach space, $L(\lambda)$ is compact uniformly in λ if and only if $L(\lambda)$ is compact for each λ. When the range X_2 is a Montel space, a (uniformly) bounded operator is (uniformly) compact because every bounded set in a Montel space is relatively compact. Put $\hat{\Omega} := \{\mathrm{Im}(\lambda) < 0\} \cup I \cup \Omega$ as before. In many applications, $i^{-1}K^{\times}A(\lambda)i$ is a bounded operator. In such a case, the following proposition is useful to estimate the generalized spectrum.

Proposition 3.18. Suppose that for $\lambda \in \hat{\Omega}$, there exists a neighborhood $U_\lambda \subset \hat{\Omega}$ of λ such that $i^{-1}K^{\times}A(\lambda')i : X(\Omega) \to X(\Omega)$ is a bounded operator uniformly in $\lambda' \in U_\lambda$. If $id - i^{-1}K^{\times}A(\lambda)i$ has a continuous inverse on $X(\Omega)$, then $\lambda \notin \hat{\sigma}(T)$.

Proof. Note that $\mathcal{R}_\lambda \circ i$ is rewritten as $\mathcal{R}_\lambda \circ i = A(\lambda) \circ i \circ (id - i^{-1}K^{\times}A(\lambda)i)^{-1}$. Since $A(\lambda) \circ i$ is continuous, it is sufficient to prove that there exists a neighborhood V_λ of λ such that the set $\{(id - i^{-1}K^{\times}A(\lambda')i)^{-1}\psi\}_{\lambda' \in V_\lambda}$ is bounded in $X(\Omega)$ for each $\psi \in X(\Omega)$. For this purpose, it is sufficient to prove that the mapping $\lambda' \mapsto (id - i^{-1}K^{\times}A(\lambda')i)^{-1}\psi$ is continuous in $\lambda' \in V_\lambda$. Since $i^{-1}K^{\times}A(\lambda)i$ is holomorphic (see the proof of Thm.3.12), there is an operator $D(\lambda, h)$ on $X(\Omega)$ such that

$$
\begin{aligned}
id - i^{-1}K^{\times}A(\lambda + h)i &= id - i^{-1}K^{\times}A(\lambda)i - hD(\lambda, h) \\
&= \left(id - hD(\lambda, h)(id - i^{-1}K^{\times}A(\lambda)i)^{-1}\right) \circ (id - i^{-1}K^{\times}A(\lambda)i).
\end{aligned}
$$

Since $i^{-1}K^{\times}A(\lambda)i$ is a bounded operator uniformly in $\lambda \in U_\lambda$, $D(\lambda, h)$ is a bounded operator when h is sufficiently small. Since $(id - i^{-1}K^{\times}A(\lambda)i)^{-1}$ is continuous by the assumption, $D(\lambda, h)(id - i^{-1}K^{\times}A(\lambda)i)^{-1}$ is a bounded operator. Then, Bruyn's theorem [3] shows that $id - hD(\lambda, h)(id - i^{-1}K^{\times}A(\lambda)i)^{-1}$ has a continuous inverse for sufficiently small h and the inverse is continuous in h (when $X(\Omega)$ is a Banach space, Bruyn's theorem is reduced to the existence of the Neumann series). This proves that $(id - i^{-1}K^{\times}A(\lambda + h)i)^{-1}\psi$ exists and continuous in h for each ψ. ■

As a corollary, if $X(\Omega)$ is a Banach space and $i^{-1}K^{\times}A(\lambda)i$ is a continuous operator on $X(\Omega)$ for each λ, then $\lambda \in \hat{\varrho}(T)$ if and only if $id - i^{-1}K^{\times}A(\lambda)i$ has a continuous inverse on $X(\Omega)$. Because of this proposition, we can apply the spectral theory on locally convex spaces (for example, [2, 6, 17, 18, 20, 22]) to the operator $id - i^{-1}K^{\times}A(\lambda)i$ to estimate the generalized spectrum. In particular, like as Riesz-Schauder theory in Banach spaces, we can prove the next theorem.

Theorem 3.19. In addition to (X1) to (X8), suppose that $i^{-1}K^{\times}A(\lambda)i : X(\Omega) \to X(\Omega)$ is a compact operator uniformly in $\lambda \in \hat{\Omega} := \{\mathrm{Im}(\lambda) < 0\} \cup I \cup \Omega$. Then, the following statements are true.
(i) For any compact set $D \subset \hat{\Omega}$, the number of generalized eigenvalues in D is finite (thus $\hat{\sigma}_p(T)$ consists of a countable number of generalized eigenvalues and they may accumulate only on the boundary of $\hat{\Omega}$ or infinity).
(ii) For each $\lambda_0 \in \hat{\sigma}_p(T)$, the generalized eigenspace V_0 is of finite dimensional and $\Pi_0 iX(\Omega) = V_0$.
(iii) $\hat{\sigma}_c(T) = \hat{\sigma}_r(T) = \emptyset$.

If $X(\Omega)$ is a Banach space, the above theorem follows from well known Riesz-Schauder theory. Even if $X(\Omega)$ is not a Banach space, we can prove the same result (see below). Thm.3.19 is useful to find embedded eigenvalues of T:

Corollary 3.20. Suppose that T is selfadjoint. Under the assumptions in Thm.3.19, the number of eigenvalues of $T = H + K$ (in $\mathcal{H}$-sense) in any compact set $D \subset I$ is finite. Their algebraic multiplicities $\dim \mathrm{Ker}\,(\lambda - T)$ are finite.

Proof. Let $\lambda_0 \in I$ be an eigenvalue of T. It is known that the projection $\mathcal{P}_0$ to the corresponding eigenspace is given by

$$\mathcal{P}_0\phi = \lim_{\varepsilon \to -0} \sqrt{-1}\varepsilon \cdot (\lambda_0 + \sqrt{-1}\varepsilon - T)^{-1}\phi, \quad \phi \in \mathcal{H}, \tag{3.40}$$

where the limit is taken with respect to the topology on $\mathcal{H}$. When $\mathrm{Im}(\lambda) < 0$, we have $\mathcal{R}_\lambda i(\phi) = i(\lambda - T)^{-1}\phi$ for $\phi \in X(\Omega)$. This shows

$$i \circ \mathcal{P}_0\phi = \lim_{\varepsilon \to -0} \sqrt{-1}\varepsilon \cdot \mathcal{R}_{\lambda_0 + \sqrt{-1}\varepsilon} \circ i(\phi), \quad \phi \in X(\Omega).$$

Let $\mathcal{R}_\lambda = \sum_{j=-\infty}^{\infty}(\lambda_0 - \lambda)^j E_j$ be the Laurent expansion of $\mathcal{R}_\lambda$, which converges around λ_0. This provides

$$i \circ \mathcal{P}_0 = \lim_{\varepsilon \to -0} \sqrt{-1}\varepsilon \sum_{j=-\infty}^{\infty} (-\sqrt{-1}\varepsilon)^j E_j \circ i.$$

Since the right hand side converges with respect to the topology on $X(\Omega)'$, we obtain

$$i \circ \mathcal{P}_0 = -E_{-1} \circ i = \Pi_0 \circ i, \quad E_{-2} = E_{-3} = \cdots = 0, \tag{3.41}$$

where Π_0 is the generalized Riesz projection for λ_0. Since λ_0 is an eigenvalue, $\mathcal{P}_0\mathcal{H} \neq \emptyset$. Since $X(\Omega)$ is a dense subspace of $\mathcal{H}$, $\mathcal{P}_0 X(\Omega) \neq \emptyset$. Hence, we obtain $\Pi_0 iX(\Omega) \neq \emptyset$, which implies that λ_0 is a generalized eigenvalue; $\sigma_p(T) \subset \hat{\sigma}_p(T)$. Since $\hat{\sigma}_p(T)$ is countable, so is $\sigma_p(T)$. Since $\Pi_0 iX(\Omega)$ is a finite dimensional space, so is $\mathcal{P}_0 X(\Omega)$. Then, $\mathcal{P}_0\mathcal{H} = \mathcal{P}_0 X(\Omega)$

proves to be finite dimensional because $\mathcal{P}_0\mathcal{H}$ is the closure of $\mathcal{P}_0 X(\Omega)$. ■

Our results are also useful to calculate eigenvectors for embedded eigenvalues. In the usual Hilbert space theory, if an eigenvalue λ is embedded in the continuous spectrum of T, we can not apply the Riesz projection for λ because there are no closed curves in **C** which separate λ from the rest of the spectrum. In our theory, $\hat{\sigma}_c(T) = \hat{\sigma}_r(T) = \emptyset$. Hence, the generalized eigenvalues are indeed isolated and the Riesz projection Π_0 is applied to yield $\Pi_0 iX(\Omega) = V_0$. Then, the eigenspace in $\mathcal{H}$-sense is obtained as $V_0 \cap \mathrm{D}(T)$.

Proof of Thm.3.19. The theorem follows from Riesz-Schauder theory on locally convex spaces developed in Ringrose [20]. Here, we give a simple review of the argument in [20]. We denote $X(\Omega) = X$ and $i^{-1}K^{\times}A(\lambda)i = C(\lambda)$ for simplicity. A pairing for (X', X) is denoted by $\langle \cdot \mid \cdot \rangle_X$.

Since $C(\lambda) : X \to X$ is compact uniformly in λ, there exists a neighborhood V of zero in X, which is independent of λ, such that $C(\lambda)V \subset X$ is relatively compact. Put $p(x) = \inf\{|\lambda|; x \in \lambda V\}$. Then, p is a continuous semi-norm on X and $V = \{x \,|\, p(x) < 1\}$. Define a closed subspace M in X to be

$$M = \{x \in X \,|\, p(x) = 0\} \subset V. \tag{3.42}$$

Let us consider the quotient space X/M, whose elements are denoted by $[x]$. The semi-norm p induces a norm P on X/M by $P([x]) = p(x)$. If X/M is equipped with the norm topology induced by P, we denote the space as $\mathcal{B}$. The completion of $\mathcal{B}$, which is a Banach space, is denoted by $\mathcal{B}_0$. The dual space $\mathcal{B}_0'$ of $\mathcal{B}_0$ is a Banach space with the norm

$$||\mu||_{\mathcal{B}_0'} := \sup_{P([x])<1} |\langle \mu \,|\, [x] \rangle_{\mathcal{B}_0}|, \tag{3.43}$$

where $\langle \cdot \mid \cdot \rangle_{\mathcal{B}_0}$ is a pairing for $(\mathcal{B}_0', \mathcal{B}_0)$. Define a subspace $S \subset X'$ to be

$$S = \{\mu \in X' \mid \sup_{x \in V} |\langle \mu \,|\, x \rangle_X| < \infty\}. \tag{3.44}$$

The linear mapping $\hat{}\, : S \to \mathcal{B}_0'$ $(\mu \mapsto \hat{\mu})$ defined through $\langle \hat{\mu} \,|\, [x] \rangle_{\mathcal{B}_0} = \langle \mu \,|\, x \rangle_X$ is bijective. Define the operator $Q(\lambda) : \mathcal{B} \to \mathcal{B}$ to be $Q(\lambda)[x] = [C(\lambda)x]$. Then, the equality

$$\langle \hat{\mu} \,|\, Q(\lambda)[x] \rangle_{\mathcal{B}_0} = \langle \mu \,|\, C(\lambda)x \rangle_X \tag{3.45}$$

holds for $\mu \in S$ and $x \in X$. Let $Q_0(\lambda) : \mathcal{B}_0 \to \mathcal{B}_0$ be a continuous extension of $Q(\lambda)$. Then, $Q_0(\lambda)$ is a compact operator on a Banach space, and thus the usual Riesz-Schauder theory is applied. By using Eq.(3.45), it is proved that $z \in \mathbf{C}$ is an eigenvalue of $C(\lambda)$ if and only if it is an eigenvalue of $Q_0(\lambda)$. In this manner, we can prove that

Theorem 3.21 [20]. The number of eigenvalues of the operator $C(\lambda) : X \to X$ is at most countable, which can accumulate only at the origin. The eigenspaces $\bigcup_{m\geq 1} \mathrm{Ker}\,(z - C(\lambda))^m$ of nonzero eigenvalues z are finite dimensional. If $z \neq 0$ is not an eigenvalue, $z - C(\lambda)$ has a continuous inverse on X. See [20] for the complete proof.

Now we are in a position to prove Thm.3.19. Suppose that λ is not a generalized eigenvalue. Then, 1 is not an eigenvalue of $C(\lambda) = i^{-1}K^{\times}A(\lambda)i$. The above theorem concludes that $id - C(\lambda)$ has a continuous inverse on $X(\Omega)$. Since $C(\lambda)$ is compact uniformly in λ, Prop.3.18 implies $\lambda \notin \hat{\sigma}(T)$. This proves the part (iii) of Thm.3.19.

Let us show the part (i) of the theorem. Let $z = z(\lambda)$ be an eigenvalue of $C(\lambda)$. We suppose that $z(\lambda_0) = 1$ so that λ_0 is a generalized eigenvalue. As was proved in the proof of Thm.3.12, $\langle \mu \,|\, C(\lambda)x\rangle_X$ is holomorphic in λ. Eq.(3.45) shows that $\langle \hat{\mu} \,|\, Q(\lambda)[x]\rangle_{\mathcal{B}_0}$ is holomorphic for any $\hat{\mu} \in \mathcal{B}_0'$ and $[x] \in \mathcal{B}$. Since $\mathcal{B}_0$ is a Banach space and $\mathcal{B}$ is dense in $\mathcal{B}_0$, $Q_0(\lambda)$ is a holomorphic family of operators. Recall that the eigenvalue $z(\lambda)$ of $C(\lambda)$ is also an eigenvalue of $Q_0(\lambda)$ satisfying $z(\lambda_0) = 1$. Then, the analytic perturbation theory of operators (see Chapter VII of Kato [12]) shows that there exists a natural number p such that $z(\lambda)$ is holomorphic as a function of $(\lambda-\lambda_0)^{1/p}$. Let us show that $z(\lambda)$ is not a constant function. If $z(\lambda) \equiv 1$, every point in $\hat{\Omega}$ is a generalized eigenvalue. Due to Prop.3.17, the open lower half plane is included in the point spectrum of T. Hence, there exists $f = f_\lambda$ in $\mathcal{H}$ such that $f = K(\lambda - H)^{-1} f$ for any $\lambda \in \mathbf{C}_-$. However, since K is H-bounded, there exist nonnegative numbers a and b such that

$$||K(\lambda - H)^{-1}|| \leq a||(\lambda - H)^{-1}|| + b||H(\lambda - H)^{-1}|| = a||(\lambda - H)^{-1}|| + b||\lambda(\lambda - H)^{-1} - id||,$$

which tends to zero as $|\lambda| \to \infty$ outside the real axis. Therefore, $||f|| \leq ||K(\lambda-H)^{-1}||\cdot||f|| \to 0$, which contradicts with the assumption. Since $z(\lambda)$ is not a constant, there exists a neighborhood $U \subset \mathbf{C}$ of λ_0 such that $z(\lambda) \neq 1$ when $\lambda \in U$ and $\lambda \neq \lambda_0$. This implies that $\lambda \in U\backslash\{\lambda_0\}$ is not a generalized eigenvalue and the part (i) of Thm.3.19 is proved.

Finally, let us prove the part (ii) of Thm.3.19. Put $\tilde{C}(z) = (z-1)\cdot id + C(z)$ and $\tilde{Q}(z) = (z-1)\cdot id + Q(z)$. They satisfy $\langle \hat{\mu} \,|\, \tilde{Q}(\lambda)[x]\rangle_{\mathcal{B}_0} = \langle \mu \,|\, \tilde{C}(z)x\rangle_X$ and

$$\langle \hat{\mu} \,|\, (\lambda - \tilde{Q}(z))^{-1}[x]\rangle_{\mathcal{B}_0} = \langle \mu \,|\, (\lambda - \tilde{C}(z))^{-1}x\rangle_X.$$

Since an eigenspace of $Q(z)$ is finite dimensional, an eigenspace of $\tilde{Q}(z)$ is also finite dimensional. Thus the resolvent $(\lambda - \tilde{Q}(z))^{-1}$ is meromorphic in $\lambda \in \hat{\Omega}$. Since $\tilde{Q}(z)$ is holomorphic, $(\lambda - \tilde{Q}(\lambda))^{-1}$ is also meromorphic. The above equality shows that $\langle \mu \,|\, (\lambda - \tilde{C}(\lambda))^{-1}x\rangle_X$ is meromorphic for any $\mu \in S$. Since S is dense in X', it turns out that $(\lambda-\tilde{C}(\lambda))^{-1}x$ is meromorphic with respect to the topology on X. Therefore, the generalized resolvent

$$\mathcal{R}_\lambda \circ i = A(\lambda) \circ i \circ (id - i^{-1}K^{\times}A(\lambda)i)^{-1} = A(\lambda) \circ i \circ (\lambda - \tilde{C}(\lambda))^{-1} \tag{3.46}$$

is meromorphic on $\hat{\Omega}$. Now we have shown that the Laurent expansion of $\mathcal{R}_\lambda$ is of the form (3.30) for some $M \geq 0$. Then, we can prove Eq.(3.33) by the same way as the proof of Thm.3.16. To prove that $\Pi_0 iX(\Omega)$ is of finite dimensional, we need the next lemma.

Lemma 3.22. $\dim \mathrm{Ker}\, B^{(n)}(\lambda) \leq \dim \mathrm{Ker}\,(id - K^{\times}A(\lambda))$ for any $n \geq 1$.

Proof. Suppose that $B^{(n)}(\lambda)\mu = 0$ with $\mu \neq 0$. Then, we have

$$\begin{aligned} K^{\times}(\lambda - H^{\times})^{n-1}B^{(n)}(\lambda)\mu &= K^{\times}(\lambda - H^{\times})^{n-1}(id - A^{(n)}(\lambda)K^{\times}(\lambda - H^{\times})^{n-1})\mu \\ &= (id - K^{\times}A(\lambda)) \circ K^{\times}(\lambda - H^{\times})^{n-1}\mu = 0. \end{aligned}$$

If $K^{\times}(\lambda - H^{\times})^{n-1}\mu = 0$, $B^{(n)}(\lambda)\mu = 0$ yields $\mu = A^{(n)}(\lambda)K^{\times}(\lambda - H^{\times})^{n-1}\mu = 0$, which contradicts with the assumption $\mu \neq 0$. Thus we obtain $K^{\times}(\lambda - H^{\times})^{n-1}\mu \in \mathrm{Ker}\,(id - K^{\times}A(\lambda))$

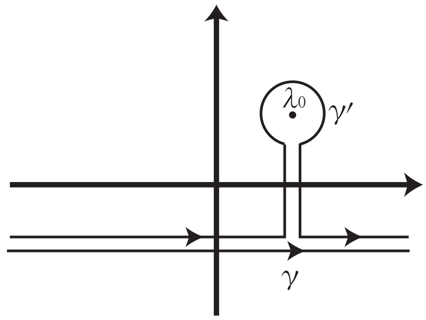

Fig. 2: Deformation of the contour.

and the mapping $\mu \mapsto K^{\times}(\lambda - H^{\times})^{n-1}\mu$ is one-to-one. ■

Due to Thm.3.21, $\mathrm{Ker}\,(id - K^{\times}A(\lambda))$ is of finite dimensional. Hence, $\mathrm{Ker}\,B^{(n)}(\lambda)$ is also finite dimensional for any $n \geq 1$. This and Eq.(3.33) prove that $\Pi_0 iX(\Omega)$ is a finite dimensional space. By Thm.3.16, $\Pi_0 iX(\Omega) = V_0$, which completes the proof of Thm.3.19 (ii). ■

3.7 Semigroups

In this subsection, we suppose that
(S1) The operator $\sqrt{-1}T = \sqrt{-1}(H + K)$ generates a C^0-semigroup $e^{\sqrt{-1}Tt}$ on $\mathcal{H}$.
For example, this is true when K is bounded on $\mathcal{H}$ or T is selfadjoint. By the Laplace inversion formula (2.5), the semigroup is given as

$$(e^{\sqrt{-1}Tt}\psi, \phi) = \frac{1}{2\pi\sqrt{-1}} \lim_{x\to\infty} \int_{-x-\sqrt{-1}y}^{x-\sqrt{-1}y} e^{\sqrt{-1}\lambda t}((\lambda - T)^{-1}\psi, \phi)d\lambda, \quad x, y \in \mathbf{R}, \tag{3.47}$$

where the contour is a horizontal line in the lower half plane below the spectrum of T. In Sec.2, we have shown that if there is an eigenvalue of T on the lower half plane, $e^{\sqrt{-1}Tt}$ diverges as $t \to \infty$, while if there are no eigenvalues, to investigate the asymptotic behavior of $e^{\sqrt{-1}Tt}$ is difficult in general. Let us show that resonance poles induce an exponential decay of the semigroup.

We use the residue theorem to calculate Eq.(3.47). Let $\lambda_0 \in \Omega$ be an isolated resonance pole of finite multiplicity. Suppose that the contour γ is deformed to the contour γ', which lies above λ_0, without passing the generalized spectrum $\hat{\sigma}(T)$ except for λ_0, see Fig.2. For example, it is possible under the assumptions of Thm.3.19. Recall that if $\psi, \phi \in X(\Omega)$, $((\lambda - T)^{-1}\psi, \phi)$ defined on the lower half plane has an analytic continuation $\langle \mathcal{R}_\lambda \psi \,|\, \phi \rangle$ defined on $\Omega \cup I \cup \{\lambda \,|\, \mathrm{Im}(\lambda) < 0\}$ (Thm.3.12). Thus we obtain

$$(e^{\sqrt{-1}Tt}\psi, \phi) = \frac{1}{2\pi\sqrt{-1}} \int_{\gamma'} e^{\sqrt{-1}\lambda t}\langle \mathcal{R}_\lambda \psi \,|\, \phi \rangle d\lambda - \frac{1}{2\pi\sqrt{-1}} \int_{\gamma_0} e^{\sqrt{-1}\lambda t}\langle \mathcal{R}_\lambda \psi \,|\, \phi \rangle d\lambda, \tag{3.48}$$

where γ_0 is a sufficiently small simple closed curve enclosing λ_0. Let $\mathcal{R}_\lambda = \sum_{j=-M}^{\infty}(\lambda_0 - \lambda)^j E_j$ be a Laurent series of $\mathcal{R}_\lambda$ as the proof of Thm.3.16. Due to Eq.(3.29) and $E_{-1} = -\Pi_0$, we obtain

$$\frac{1}{2\pi\sqrt{-1}}\int_{\gamma_0} e^{\sqrt{-1}\lambda t}\langle \mathcal{R}_\lambda\psi \,|\, \phi\rangle d\lambda = \sum_{k=0}^{M-1} e^{\sqrt{-1}\lambda_0 t}\frac{(-\sqrt{-1}t)^k}{k!}\langle (\lambda_0 - T^\times)^k \Pi_0\psi \,|\, \phi\rangle,$$

where Π_0 is the generalized projection to the generalized eigenspace of λ_0. Since $\mathrm{Im}(\lambda_0) > 0$, this proves that the second term in the right hand side of Eq.(3.48) decays to zero as $t \to \infty$. Such an exponential decay (of a part of) the semigroup induced by resonance poles is known as Landau damping in plasma physics [5], and is often observed for Schrödinger operators [19]. A similar calculation is possible without defining the generalized resolvent and the generalized spectrum as long as the quantity $((\lambda - T)^{-1}\psi, \phi)$ has an analytic continuation for some ψ and ϕ. Indeed, this has been done in the literature.

Let us reformulate it by using the dual space to find a decaying state corresponding to λ_0. For this purpose, we suppose that

(S2) the semigroup $\{(e^{\sqrt{-1}Tt})^*\}_{t\geq 0}$ is an equicontinuous C_0 semigroup on $X(\Omega)$.

Then, by the theorem in IX-13 of Yosida [27], the dual semigroup $(e^{\sqrt{-1}Tt})^\times = ((e^{\sqrt{-1}Tt})^*)'$ is also an equicontinuous C_0 semigroup generated by $\sqrt{-1}T^\times$. A convenient sufficient condition for (S2) is that:

(S2)' $K^*|_{X(\Omega)}$ is bounded and $\{e^{\sqrt{-1}Ht}\}_{t\geq 0}$ is an equicontinuous C_0 semigroup on $X(\Omega)$.

Indeed, the perturbation theory of equicontinuous C_0 semigroups [23] shows that (S2)' implies (S2). By using the dual semigroup, Eq.(3.47) is rewritten as

$$(e^{\sqrt{-1}Tt})^\times\psi = \frac{1}{2\pi\sqrt{-1}}\lim_{x\to\infty}\int_{-x-\sqrt{-1}y}^{x-\sqrt{-1}y} e^{\sqrt{-1}\lambda t}\mathcal{R}_\lambda\psi d\lambda. \tag{3.49}$$

for any $\psi \in iX(\Omega)$. Similarly, Eq.(3.48) yields

$$(e^{\sqrt{-1}Tt})^\times\psi = \frac{1}{2\pi\sqrt{-1}}\int_{\gamma'} e^{\sqrt{-1}\lambda t}\mathcal{R}_\lambda\psi d\lambda - \sum_{k=0}^{M-1} e^{\sqrt{-1}\lambda_0 t}\frac{(\sqrt{-1}t)^k}{k!}(\lambda_0 - T^\times)^k\Pi_0\psi, \tag{3.50}$$

when λ_0 is a generalized eigenvalue of finite multiplicity. For the dual semigroup, the following statements hold.

Proposition 3.23. Suppose (S1) and (S2).
(i) A solution of the initial value problem

$$\frac{d}{dt}\xi = \sqrt{-1}T^\times\xi, \quad \xi(0) = \mu \in \mathrm{D}(T^\times), \tag{3.51}$$

in $X(\Omega)'$ is uniquely given by $\xi(t) = (e^{\sqrt{-1}Tt})^\times\mu$.
(ii) Let λ_0 be a generalized eigenvalue and μ_0 a corresponding generalized eigenfunction. Then, $(e^{\sqrt{-1}Tt})^\times\mu_0 = e^{\sqrt{-1}\lambda_0 t}\mu_0$.
(iii) Let Π_0 be a generalized projection for λ_0. The space $\Pi_0 iX(\Omega)$ is $(e^{\sqrt{-1}Tt})^\times$-invariant: $(e^{\sqrt{-1}Tt})^\times\Pi_0 = \Pi_0(e^{\sqrt{-1}Tt})^\times|_{iX(\Omega)}$.

Proof. Since $\{(e^{\sqrt{-1}Tt})^{\times}\}_{t\geq 0}$ is an equicontinuous C_0 semigroup generated by $\sqrt{-1}T^{\times}$, (i) follows from the usual semigroup theory [27]. Because of Thm.3.5, we have $\sqrt{-1}T^{\times}\mu_0 = \sqrt{-1}\,\lambda_0\mu_0$. Then,

$$\frac{d}{dt}e^{\sqrt{-1}\,\lambda_0 t}\mu_0 = \sqrt{-1}\,\lambda_0 e^{\sqrt{-1}\,\lambda_0 t}\mu_0 = \sqrt{-1}T^{\times}(e^{\sqrt{-1}\,\lambda_0 t}\mu_0).$$

Thus $\xi(t) = e^{\sqrt{-1}\,\lambda_0 t}\mu_0$ is a solution of the equation (3.51). By the uniqueness of a solution, we obtain (ii). Because of Prop.3.13 (iii), we have

$$\begin{aligned} &\frac{d}{dt}(e^{\sqrt{-1}Tt})^{\times}\mathcal{R}_\lambda = \sqrt{-1}T^{\times}\left((e^{\sqrt{-1}Tt})^{\times}\mathcal{R}_\lambda\right), \\ &\frac{d}{dt}\mathcal{R}_\lambda(e^{\sqrt{-1}Tt})^{\times}|_{iY} = \mathcal{R}_\lambda\cdot(e^{\sqrt{-1}Tt})^{\times}\sqrt{-1}T^{\times}|_{iY} = \sqrt{-1}T^{\times}\left(\mathcal{R}_\lambda(e^{\sqrt{-1}Tt})^{\times}\right)|_{iY}. \end{aligned}$$

Hence, both of $(e^{\sqrt{-1}Tt})^{\times}\mathcal{R}_\lambda$ and $\mathcal{R}_\lambda(e^{\sqrt{-1}Tt})^{\times}$ are solutions of the equation (3.51). By the uniqueness, we obtain $(e^{\sqrt{-1}Tt})^{\times}\mathcal{R}_\lambda|_{iY} = \mathcal{R}_\lambda(e^{\sqrt{-1}Tt})^{\times}|_{iY}$. Then, the definition of the projection Π_0 proves $(e^{\sqrt{-1}Tt})^{\times}\Pi_0|_{iY} = \Pi_0(e^{\sqrt{-1}Tt})^{\times}|_{iY}$ with the aid of Eq.(3.25). Since Y is dense in $X(\Omega)$ and both operators $(e^{\sqrt{-1}Tt})^{\times}\Pi_0\circ i$ and $\Pi_0(e^{\sqrt{-1}Tt})^{\times}\circ i = \Pi_0\circ i\circ e^{\sqrt{-1}Tt}$ are continuous on $X(\Omega)$, the equality is true on $iX(\Omega)$. ■

By Prop.3.14, any usual function $\phi\in X(\Omega)$ is decomposed as $\langle\phi| = \mu_1+\mu_2$ with $\mu_1\in\Pi_0 iX(\Omega)$ and $\mu_2\in(id-\Pi_0)iX(\Omega)$ in the dual space. Due to Prop.3.23 (iii) above, this decomposition is $(e^{\sqrt{-1}Tt})^{\times}$-invariant. When $\lambda_0\in\Omega$, $(e^{\sqrt{-1}Tt})^{\times}\mu_1\in\Pi_0 iX(\Omega)$ decays to zero exponentially as $t\to\infty$. Eq.(3.50) gives the decomposition explicitly. Such an exponential decay can be well observed if we choose a function, which is sufficiently close to the generalized eigenfunction μ_0, as an initial state. Since $X(\Omega)$ is dense in $X(\Omega)'$ and since $(e^{\sqrt{-1}Tt})^{\times}$ is continuous, for any $T>0$ and $\varepsilon>0$, there exists a function ϕ_0 in $X(\Omega)$ such that

$$|\langle(e^{\sqrt{-1}Tt})^{\times}\phi_0\,|\,\psi\rangle - \langle(e^{\sqrt{-1}Tt})^{\times}\mu_0\,|\,\psi\rangle| < \varepsilon,$$

for $0\leq t\leq T$ and $\psi\in X(\Omega)$. This implies that

$$(e^{\sqrt{-1}Tt}\phi_0,\psi)\sim\langle(e^{\sqrt{-1}Tt})^{\times}\mu_0\,|\,\psi\rangle = e^{\sqrt{-1}\lambda_0 t}\langle\mu_0\,|\,\psi\rangle, \tag{3.52}$$

for the interval $0\leq t\leq T$. Thus generalized eigenvalues describe the transient behavior of solutions.

An interesting situation occurs when $I=\mathbf{R}$ and Ω includes a strip $\{\lambda\,|\,0<\mathrm{Im}(\lambda)<a\}$ for some $a>0$. Suppose that the generalized spectrum $\hat{\sigma}(T)$ in the region $\{\lambda\,|\,\mathrm{Im}(\lambda)<a\}$ consists of a finite number of generalized eigenvalues of finite multiplicities. Let λ_j, M_j and Π_j $(j=0,\cdots,N=N(a))$ be generalized eigenvalues in $\{\lambda\,|\,\mathrm{Im}(\lambda)<a\}$, their multiplicities and projections, respectively. In this case, the residue theorem proves that the semigroup is given by

$$\begin{aligned}(e^{\sqrt{-1}Tt}\psi,\phi) \;=\;& \frac{1}{2\pi\sqrt{-1}}\lim_{x\to\infty}\int_{-x+\sqrt{-1}a}^{x+\sqrt{-1}a} e^{\sqrt{-1}\lambda t}\langle\mathcal{R}_\lambda\psi\,|\,\phi\rangle d\lambda \\ & -\sum_{j=0}^{N}\sum_{k=0}^{M_j-1} e^{\sqrt{-1}\lambda_j t}\frac{(-\sqrt{-1}t)^k}{k!}\langle(\lambda_j-T^{\times})^k\Pi_j\psi\,|\,\phi\rangle,\end{aligned}$$

for $\phi, \psi \in X(\Omega)$ as long as the integral in the first term converges (for the convergence, we need an additional assumption for the growth rate of $E[\psi, \phi](\omega)$). Since the first term in the right hand side above decays with the order $O(e^{-at})$ as $t \to \infty$, $(e^{\sqrt{-1}Tt}\psi, \phi)$ decays to zero exponentially if there are no generalized eigenvalues on the real axis and the lower half plane.

A Pettis integrals and vector valued holomorphic functions on the dual space

The purpose in this Appendix is to give the definition and the existence theorem of Pettis integrals. After that, a few results on vector-valued holomorphic functions are given. For the existence of Pettis integrals, the following property
(CE) for any compact set K, the closed convex hull of K is compact,
which is sometimes called the convex envelope property, is essentially used. For the convenience of the reader, sufficient conditions for the property are listed below. We also give conditions for X to be barreled because it is assumed in (X3). Let X be a locally convex Hausdorff vector space, and X' its dual space.

- The closed convex hull $\overline{co}(K)$ of a compact set K in X is compact if and only if $\overline{co}(K)$ is complete in the Mackey topology on X (Krein's theorem, see Köthe [14], §24.5).
- X has the convex envelope property if X is quasi-complete.
- If X is bornological, the strong dual X' is complete. In particular, the strong dual of a metrizable space is complete.
- If X is barreled, the strong dual X' is quasi-complete. In particular, X' has the convex envelope property.
- Montel spaces, Fréchet spaces, Banach spaces and Hilbert spaces are barreled.
- The product, quotient, direct sum, (strict) inductive limit, completion of barreled spaces are barreled.

See Tréves [25] for the proofs.

Let X be a topological vector space over $\mathbf{C}$ and (S, μ) a measure space. Let $f : S \to X$ be a measurable X-valued function. If there exists a unique $I_f \in X$ such that $\langle \xi \,|\, I_f \rangle = \int_S \langle \xi \,|\, f \rangle d\mu$ for any $\xi \in X'$, I_f is called the *Pettis integral* of f. It is known that if X is a locally convex Hausdorff vector space with the convex envelope property, S is a compact Hausdorff space with a finite Borel measure μ, and if $f : S \to X$ is continuous, then the Pettis integral of f exists (see Rudin [21]). In Sec.3.5, we have defined the integral of the form $\int_\gamma \mathcal{R}_\lambda \phi d\lambda$, where $\mathcal{R}_\lambda \phi$ is an element of the dual $X(\Omega)'$. Thus our purpose here is to define a "dual version" of Pettis integrals.

In what follows, let X be a locally convex Hausdorff vector space over $\mathbf{C}$, X' a strong dual with the convex envelope property, and let S be a compact Hausdorff space with a finite Borel measure μ. For our purpose in Sec.3.5, S is always a closed path on the complex plane. Let $f : S \to X'$ be a continuous function with respect to the strong dual topology on X'.

Theorem A.1. (i) Under the assumptions above, there exists a unique $I(f) \in X'$ such that

$$\langle I(f) \,|\, x\rangle = \int_S \langle f \,|\, x\rangle d\mu \tag{A.1}$$

for any $x \in X$. $I(f)$ is denoted by $I(f) = \int_S f d\mu$ and called the Pettis integral of f.

(ii) The mapping $f \mapsto I(f)$ is continuous in the following sense; for any neighborhood U of zero in X' equipped with the weak dual topology, there exists a neighborhood V of zero in X' such that if $f(s) \in V$ for any $s \in S$, then $I(f) \in U$.

(iii) Furthermore, suppose that X is a barreled space. Let T be a linear operator densely defined on X and T' its dual operator with the domain $\mathrm{D}(T') \subset X'$. If $f(S) \subset \mathrm{D}(T')$ and the set $\{\langle T' f(s) \,|\, x\rangle\}_{s\in S}$ is bounded for each $x \in X$, then, $I(f) \in \mathrm{D}(T')$ and $T' I(f) = I(T' f)$ holds; that is,

$$T' \int_S f d\mu = \int_S T' f d\mu \tag{A.2}$$

holds.

The proof of (i) is done in a similar manner to that of the existence of Pettis integrals on X [21]. Note that T is not assumed to be continuous for the part (iii). When T is continuous, the set $\{\langle T' f(s) \,|\, x\rangle\}_{s\in S}$ is bounded because T' and f are continuous.

Proof. At first, note that the mapping $\langle \cdot \,|\, x\rangle : X' \to \mathbf{C}$ is continuous because X can be canonically embedded into the dual of the strong dual X'. Thus $\langle f(\cdot) \,|\, x\rangle : S \to \mathbf{C}$ is continuous and it is integrable on the compact set S with respect to the Borel measure.

Let us show the uniqueness. If there are two elements $I_1(f), I_2(f) \in X'$ satisfying Eq.(A.1), we have $\langle I_1(f) \,|\, x\rangle = \langle I_2(f) \,|\, x\rangle$ for any $x \in X$. By the definition of X', it follows $I_1(f) = I_2(f)$.

Let us show the existence. We can assume without loss of generality that X is a vector space over $\mathbf{R}$ and μ is a probability measure. Let $L \subset X$ be a finite set and put

$$V_L(f) = V_L := \{x' \in X' \,|\, \langle x' \,|\, x\rangle = \int_S \langle f \,|\, x\rangle d\mu, \ \forall x \in L\}. \tag{A.3}$$

Since $\langle \cdot \,|\, x\rangle$ is a continuous mapping, V_L is closed. Since f is continuous, $f(S)$ is compact in X'. Due to the convex envelope property, the closed convex hull $\overline{co}(f(S))$ is compact. Hence, $W_L := V_L \cap \overline{co}(f(S))$ is also compact. By the definition, it is obvious that $W_{L_1} \cap W_{L_2} = W_{L_1 \cup L_2}$. Thus if we can prove that W_L is not empty for any finite set L, a family $\{W_L\}_{L\in\{\text{finite set}\}}$ has the finite intersection property. Then, $\bigcap_L W_L$ is not empty because $\overline{co}(f(S))$ is compact. This implies that there exists $I(f) \in \bigcap_L W_L$ such that $\langle I(f) \,|\, x\rangle = \int_S \langle f \,|\, x\rangle d\mu$ for any $x \in X$.

Let us prove that W_L is not empty for any finite set $L = \{x_1, \cdots, x_n\} \subset X$. Define the mapping $\mathcal{L} : X' \to \mathbf{R}^n$ to be

$$\mathcal{L}(x') = (\langle x' \,|\, x_1\rangle, \cdots, \langle x' \,|\, x_n\rangle).$$

This is continuous and $\mathcal{L}(f(S))$ is compact in $\mathbf{R}^n$. Let us show that the element

$$y := \left(\int_S \langle f \,|\, x_1 \rangle d\mu \,, \cdots \,, \int_S \langle f \,|\, x_n \rangle d\mu \right) \tag{A.4}$$

is included in the convex hull $co(\mathcal{L}(f(S)))$ of $\mathcal{L}(f(S))$. If otherwise, there exist real numbers $c_1, \cdots, c_n$ such that for any $(z_1, \cdots, z_n) \in co(\mathcal{L}(f(S)))$, the inequality

$$\sum_{i=1}^{n} c_i z_i < \sum_{i=1}^{n} c_i y_i, \quad y = (y_1, \cdots, y_n)$$

holds (this is a consequence of Hahn-Banach theorem for $\mathbf{R}^n$). In particular, since $\mathcal{L}(f(S)) \subset co(\mathcal{L}(f(S)))$,

$$\sum_{i=1}^{n} c_i \langle f \,|\, x_i \rangle < \sum_{i=1}^{n} c_i y_i .$$

Integrating both sides (in the usual sense) yields $\sum_{i=1}^{n} c_i y_i < \sum_{i=1}^{n} c_i y_i$. This is a contradiction, and therefore $y \in co(\mathcal{L}(f(S)))$. Since $\mathcal{L}$ is linear, there exists $v \in co(f(S))$ such that $y = \mathcal{L}(v)$. This implies that $v \in V_L \cap co(f(S))$, and thus W_L is not empty. By the uniqueness, $\bigcap_L W_L = \{I(f)\}$. Part (ii) of the theorem immediately follows from Eq.(A.1) and properties of the usual integral.

Next, let us show Eq.(A.2). When X is a barreled space, $I(f)$ is included in $\mathsf{D}(T')$ so that $T'I(f)$ is well defined. To prove this, it is sufficient to show that the mapping

$$x \mapsto \langle I(f) \,|\, Tx \rangle = \int_S \langle f \,|\, Tx \rangle d\mu = \int_S \langle T' f \,|\, x \rangle d\mu$$

from $\mathsf{D}(T) \subset X$ into $\mathbf{C}$ is continuous. By the assumption, the set $\{\langle T' f(s) \,|\, x \rangle\}_{s \in S}$ is bounded for each $x \in X$. Then, Banach-Steinhaus theorem implies that the family $\{T' f(s)\}_{s \in S}$ of continuous linear functionals are equicontinuous. Hence, for any $\varepsilon > 0$, there exists a neighborhood U of zero in X such that $|\langle T' f(s) \,|\, x \rangle| < \varepsilon$ for any $s \in S$ and $x \in U$. This proves that the above mapping is continuous, so that $I(f) \in \mathsf{D}(T')$ and $T'I(f) = T' \bigcap_L W_L$.

For a finite set $L \subset X$, put

$$V_L(T' f) = \{x' \in X' \,|\langle x' \,|\, x \rangle = \int_S \langle T' f \,|\, x \rangle d\mu, \ \forall x \in L\},$$

$$T' V_{TL}(f) = \{T' x' \in X' \,|\, x' \in \mathsf{D}(T'), \ \langle x' \,|\, x \rangle = \int_S \langle f \,|\, x \rangle d\mu, \ \forall x \in TL\}.$$

Put $W_L(f) = V_L(f) \cap \overline{co}(f(S))$ as before. It is obvious that $\bigcap_L W_L(f) \subset \bigcap_L W_{TL}(f)$. Therefore,

$$\begin{aligned} \{T'I(f)\} &= T' \bigcap_L W_L(f) \subset T' \bigcap_L W_{TL}(f) \cap \mathsf{D}(T') \\ &\subset T' \bigcap_L (V_{TL}(f) \cap \overline{co}(f(S)) \cap \mathsf{D}(T')) \\ &\subset \bigcap_L (T' V_{TL}(f) \cap T' \overline{co}(f(S)) \cap \mathsf{R}(T')) . \end{aligned}$$

On the other hand, if $y' \in T'V_{TL}(f)$, there exists $x' \in X'$ such that $y' = T'x'$ and $\langle x' \,|\, x\rangle = \int_S \langle f \,|\, x\rangle d\mu$ for any $x \in TL$. Then, for any $x \in L \cap \mathrm{D}(T)$,

$$\langle y' \,|\, x\rangle = \langle T'x' \,|\, x\rangle = \langle x' \,|\, Tx\rangle = \int_S \langle f \,|\, Tx\rangle d\mu = \int_S \langle T'f \,|\, x\rangle d\mu.$$

This implies that $y' \in V_{L\cap \mathrm{D}(T)}(T'f)$, and thus $T'V_{TL}(f) \subset V_{L\cap \mathrm{D}(T)}(T'f)$. Hence, we obtain

$$\{T'I(f)\} \quad \subset \quad \bigcap_L V_{L\cap \mathrm{D}(T)}(T'f) \cap \overline{co}(T'f(S)) = \bigcap_L W_{L\cap \mathrm{D}(T)}(T'f).$$

If $\langle x' \,|\, x\rangle = \int_S \langle f \,|\, x\rangle d\mu$ for dense subset of X, then it holds for any $x \in X$. Hence, we have

$$\{I(T'f)\} = \bigcap_L W_L(T'f) = \bigcap_L W_{L\cap \mathrm{D}(T)}(T'f) \supset \{T'I(f)\}. \tag{A.5}$$

which proves $T'I(f) = I(T'f)$. ■

Now that we can define the Pettis integral on the dual space, we can develop the "dual version" of the theory of holomorphic functions. Let X and X' be as in Thm.A.1. Let $f : D \to X'$ be an X'-valued function on an open set $D \subset \mathbf{C}$.

Definition A.2. (i) f is called weakly holomorphic if $\langle f \,|\, x\rangle$ is holomorphic on D in the classical sense for any $x \in X$ (more exactly, it should be called weak-dual-holomorphic).
(ii) f is called strongly holomorphic if

$$\lim_{z_0 \to z} \frac{1}{z_0 - z} (f(z_0) - f(z)), \quad \text{(the strong dual limit)} \tag{A.6}$$

exists in X' for any $z \in D$ (more exactly, it should be called strong-dual-holomorphic).

Theorem A.3. Suppose that the strong dual X' satisfies the convex envelope property and $f : D \to X'$ is weakly holomorphic.

(i) If f is strongly continuous, Cauchy integral formula and Cauchy integral theorem hold:

$$f(z) = \frac{1}{2\pi\sqrt{-1}} \int_\gamma \frac{f(z_0)}{z_0 - z} dz_0, \qquad \int_\gamma f(z_0) dz_0 = 0,$$

where $\gamma \subset D$ is a closed curve enclosing $z \in D$.
(ii) If f is strongly continuous and if X' is quasi-complete, f is strongly holomorphic and is of C^∞ class.
(iii) If X is barreled, the weak holomorphy implies the strong continuity. Thus (i) and (ii) above hold; f is strongly holomorphic and is expanded in a Taylor series as

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n, \quad \text{(strong dual convergence)}, \tag{A.7}$$

near $a \in D$. Similarly, a Laurent expansion and the residue theorem hold if f has an isolated singularity.

Proof. (i) Since f is continuous with respect to the strong dual topology, the Pettis integral

$$I(z) = \frac{1}{2\pi\sqrt{-1}} \int_\gamma \frac{f(z_0)}{z_0 - z} dz_0$$

exists. By the definition of the integral,

$$\langle I(z)\,|\,x\rangle = \frac{1}{2\pi\sqrt{-1}} \int_\gamma \frac{\langle f(z_0)\,|\,x\rangle}{z_0 - z} dz_0$$

for any $x \in X$. Since $\langle f(z)\,|\,x\rangle$ is holomorphic in the usual sense, the right hand side above is equal to $\langle f(z)\,|\,x\rangle$. Thus we obtain $I(z) = f(z)$, which gives the Cauchy formula. The Cauchy theorem also follows from the classical one.

(ii) Let us prove that f is strongly holomorphic at z_0. Suppose that $z_0 = 0$ and $f(z_0) = 0$ for simplicity. By the same way as above, we can verify that

$$\begin{aligned} \frac{f(z)}{z} &= \frac{1}{2\pi\sqrt{-1}} \int_\gamma \frac{f(z_0)}{z_0(z_0 - z)} dz_0 \\ &= \frac{1}{2\pi\sqrt{-1}} \int_\gamma \frac{f(z_0)}{z_0^2} dz_0 + \frac{z}{2\pi\sqrt{-1}} \int_\gamma \frac{f(z_0)}{z_0^2(z_0 - z)} dz_0. \end{aligned}$$

Since X' is quasi-complete, the above converges as $z \to 0$ to yield

$$f'(0) := \lim_{z\to 0} \frac{f(z)}{z} = \frac{1}{2\pi\sqrt{-1}} \int_\gamma \frac{f(z_0)}{z_0^2} dz_0.$$

In a similar manner, we can verify that

$$f^{(n)}(z) := \frac{d^n}{dz^n} f(z) = \frac{n!}{2\pi\sqrt{-1}} \int_\gamma \frac{f(z_0)}{(z_0 - z)^{n+1}} dz_0 \tag{A.8}$$

exists for any $n = 0, 1, 2, \cdots$.

(iii) If X is barreled, weakly bounded sets in X' are strongly bounded (see Thm.33.2 of Tréves [25]). By using it, let us prove that a weakly holomorphic f is strongly continuous. Suppose that $f(0) = 0$ for simplicity. Since $\langle f(z)\,|\,x\rangle$ is holomorphic in the usual sense, Cauchy formula provides

$$\frac{\langle f(z)\,|\,x\rangle}{z} = \frac{1}{2\pi\sqrt{-1}} \int_\gamma \frac{1}{z_0 - z} \frac{\langle f(z_0)\,|\,x\rangle}{z_0} dz_0.$$

Suppose that $|z| < \delta$ and γ is a circle of radius 2δ centered at the origin. Since $\langle f(\cdot)\,|\,x\rangle$ is holomorphic, there exists a positive number M such that $|\langle f(z_0)\,|\,x\rangle| < M$ for any $z_0 \in \gamma$. Then,

$$\left| \frac{\langle f(z)\,|\,x\rangle}{z} \right| \leq \frac{1}{2\pi} \cdot \frac{1}{\delta} \cdot \frac{M}{2\delta} \cdot 4\pi\delta = \frac{M}{\delta}.$$

This shows that the set $B := \{f(z)/z \mid |z| < \delta\}$ is weakly bounded in X'. Since X is barreled, B is strongly bounded. By the definition of bounded sets, for any convex balanced neighborhood U of zero in X' equipped with the strong dual, there is a number $t > 0$ such that $tB \subset U$. This proves that

$$f(z) - f(0) = f(z) \in \frac{z}{t} U \subset \frac{\delta}{t} U$$

for $|z - 0| < \delta$, which implies the continuity of f with resect to the strong dual topology.

If X is barreled, X' is quasi-complete and has the convex envelope property. Thus the results in (i) and (ii) hold.

Finally, let us show that $f(z)$ is expanded in a Taylor series around $a \in D$. Suppose $a = 0$ for simplicity. Let us prove that

$$S_m = \sum_{n=0}^{m} \frac{1}{n!} \frac{d^n f}{dz^n}(0) z^n$$

forms a Cauchy sequence with respect to the strong dual topology. It follows from (A.8) that

$$\frac{1}{n!} \langle f^{(n)}(0) \,|\, x \rangle = \frac{1}{2\pi \sqrt{-1}} \int_{\gamma} \frac{\langle f(z_0) \,|\, x \rangle}{z_0^{n+1}} dz_0$$

for any $x \in X$. Suppose that γ is a circle of radius 2δ centered at the origin. There exists a constant $M_x > 0$ such that $|\langle f(z_0) \,|\, x \rangle| < M_x$ for any $z_0 \in \gamma$, which implies that the set $\{ f(z_0) \,|\, z_0 \in \gamma \}$ is weakly bounded. Because X is barreled, it is strongly bounded. Therefore, for any bounded set $B \subset X$, there is a positive number M_B such that $|\langle f(z_0) \,|\, x \rangle| < M_B$ for $x \in B$ and $z_0 \in \gamma$. Then, we obtain

$$\left| \frac{1}{n!} \langle f^{(n)}(0) \,|\, x \rangle \right| \leq \frac{1}{2\pi} \cdot \frac{M_B}{(2\delta)^{n+1}} \cdot 4\pi\delta = \frac{M_B}{(2\delta)^n}.$$

By using this, it is easy to verify that $\{ \langle S_m \,|\, x \rangle \}_{m=0}^{\infty}$ is a Cauchy sequence uniformly in $x \in B$ when $|z| < \delta$. Since X' is quasi-complete, S_m converges as $m \to \infty$ in the strong dual topology. By the Taylor expansion in the classical sense, we obtain

$$\langle f(z) \,|\, x \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n}{dz_0^n}\Big|_{z_0 = 0} \langle f(z_0) \,|\, x \rangle z^n = \sum_{n=0}^{\infty} \frac{1}{n!} \langle f^{(n)}(0) \,|\, x \rangle z^n.$$

Since $\lim_{m \to \infty} S_m$ exists and $\langle \cdot \,|\, x \rangle : X' \to \mathbf{C}$ is continuous, we have

$$\langle f(z) \,|\, x \rangle = \langle \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) z^n \,|\, x \rangle,$$

for any $x \in X$. This proves Eq.(A.7) for $a = 0$. The proof of a Laurent expansion, when f has an isolated singularity, is done in the same way. Then, the proof of the residue theorem immediately follows from the classical one. ■

Remark. In a well known theory of Pettis integrals on a space X [21], not a dual X', we need not assume that X is barreled because every locally convex space X has the property that any weakly bounded set is bounded with respect to the original topology. Since the dual X' does not have this property, we have to assume that X is barreled so that any weakly bounded set in X' is strongly bounded.

Acknowledgements.
This work was supported by Grant-in-Aid for Young Scientists (B), No.22740069 from MEXT Japan.

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