Minimax lower bound for kink location estimators in a nonparametric regression model with long-range dependence

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Abstract

In this paper, a lower bound is determined in the minimax sense for change point estimators of the first derivative of a regression function in the fractional white noise model. Similar minimax results presented previously in the area focus on change points in the derivatives of a regression function in the white noise model or consider estimation of the regression function in the presence of correlated errors.

Keywords: nonparametric regression, long-range dependence, kink, minimax
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1. Introduction

Nonparametric estimation of a kink in a regression function has been considered for Gaussian white noise models by Cheng and Raimondo (2008); Goldenshluger, Juditsky, Tsybakov, and Zeevi (2008a,b). Recently, this was extended to the fractional Gaussian noise model by Wishart (2009). The fractional Gaussian noise model assumes the regression structure,

$$dY(x) = \mu(x)\,dx + \varepsilon^{\alpha}dB_H(x), \quad x \in \mathbb{R}, \tag{1}$$

where B_H is a fractional Brownian motion (fBm) and $\mu \colon \mathbb{R} \longrightarrow \mathbb{R}$ is the regression function. The level of error is controlled by $\varepsilon \asymp n^{-1/2}$ where the relation $a_n \asymp b_n$ means the ratio a_n/b_n is bounded above and below by constants. The level of dependence in the error is controlled by the Hurst parameter $H \in (1/2, 1)$ and $\alpha := 2 - 2H$, where the i.i.d. model corresponds to $\alpha = 1$. The fractional Gaussian noise model was used by Johnstone and Silverman (1997); Wishart (2009) among others to model regression problems with long-range dependent errors.

This paper is interested in the performance of estimators of a change-point in the first derivative of μ observed in model (1). This type of change point is called a kink and the location denoted by θ. Let $\widehat{\theta}_n$ denote an estimator of θ given n observations. A lower bound

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is established for the minimax rate of kink location estimation using the quadratic loss in the sense that,

$$\liminf_{n\to\infty} \inf_{\widehat{\theta}_n} \sup_{\mu\in\mathscr{F}_s(\theta)} \rho_n^{-2}\mathbb{E}\left|\widehat{\theta}_n - \theta\right|^2 \geq C \qquad \text{for some constant } C > 0. \qquad (2)$$

The main quantity of interest in this lower bound is the rate, ρ_n. In (2), $\inf_{\widehat{\theta}_n}$ denotes the infimum over all possible estimators of θ. The class of functions under consideration for μ is denoted $\mathscr{F}_s(\theta)$ and defined below.

Definition 1. *Let $s \geq 2$ be an integer and $a \in \mathbb{R} \setminus \{0\}$. Then, we say that $\mu \in \mathscr{F}_s(\theta)$ if,*

1. *The function μ has a kink at $\theta \in (0,1)$. That is,*

$$\lim_{x\downarrow\theta} \mu^{(1)}(x) - \lim_{x\uparrow\theta} \mu^{(1)}(x) = a \neq 0.$$

2. *The function $\mu \in \mathscr{L}_2(\mathbb{R}) \cap \mathscr{L}_1(\mathbb{R})$, and satisfies the following condition,*

$$\int_{\mathbb{R}} |\widetilde{\mu}(\omega)||\omega|^s \, d\omega < \infty, \qquad (3)$$

where $\widetilde{\mu}(\omega) := \int_{\mathbb{R}} e^{-2\pi i\omega x}\mu(x)\,dx$ is the Fourier transform of μ.

The minimax rate for the kink estimators has been discussed in the i.i.d. scenario by Cheng and Raimondo (2008); Goldenshluger et al. (2008a) and was shown to be $n^{-s/(2s+1)}$. An extension of the kink estimators to the long-range dependent scenario was considered in Wishart (2009) that built on the work of Cheng and Raimondo (2008). An estimator of kink locations was constructed by Wishart (2009) and achieved the rate in the probabilistic sense,

$$\left|\widehat{\theta}_n - \theta\right| = \mathcal{O}_p(n^{-\alpha s/(2s+\alpha)}), \qquad (4)$$

which includes the result of Cheng and Raimondo (2008) as a special case with the choice $\alpha = 1$. Both Cheng and Raimondo (2008) and Wishart (2009) considered a comparable model in the indirect framework and used the results of Goldenshluger, Tsybakov, and Zeevi (2006) to infer the minimax optimality of (4). However, the results of Cheng and Raimondo (2008) and Wishart (2009) require a slightly more restrictive functional class than $\mathscr{F}_s(\theta)$. The rate obtained by Cheng and Raimondo (2008) of $n^{-s/(2s+1)}$ was confirmed as the minimax rate by the work of Goldenshluger et al. (2008a) who used the i.i.d. framework and a functional class similar to $\mathscr{F}_s(\theta)$.

The fBm concept is an extension of Brownian motion that can exhibit dependence among its increments which is typically controlled by the Hurst parameter, H (see Beran (1994); Doukhan, Oppenheim, and Taqqu (2003) for more detailed treatment on long-range dependence and fBm). The fBm process is defined below.

Definition 2. *The fractional Brownian motion $\{B_H(t)\}_{t\in\mathbb{R}}$ is a Gaussian process with mean zero and covariance structure,*

$$\mathbb{E}B_H(t)B_H(s) = \frac{1}{2}\left\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right\}.$$

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We assume throughout the paper that $H \in (1/2, 1)$, whereby the increments of B_H are positively correlated and are long-range dependent.

In this paper a lower bound for the minimax convergence rate of kink estimation using the quadratic loss function will be shown explicitly on model (1). This is a stronger result in terms of a lower bound than the simple probabilistic result in (4) given by Wishart (2009) and is applicable to a broader class of functions.

2. Lower bound

The aim of the paper is to establish the following result.

Theorem 1. *Suppose $\mu \in \mathscr{F}_s(\theta)$ is observed from the model* (1) *and $0 < \alpha < 1$. Then, there exists a positive constant $C < \infty$ that does not depend on n such that the lower rate of convergence for an estimator for the kink location θ with the square loss is of the form,*

$$\liminf_{n \to \infty} \inf_{\widehat{\theta}_n} \sup_{\mu \in \mathscr{F}_s(\theta)} n^{2\alpha s/(2s+\alpha)} \mathbb{E} \left| \widehat{\theta}_n - \theta \right|^2 \geq C.$$

From Theorem 1 one can see that the minimax rate for kink estimation in the i.i.d. case is recovered with the choice $\alpha = 1$ (see Goldenshluger et al., 2008a). Also unsurprisingly, the level of dependence is detrimental to the rate of convergence. For instance as the increments become more correlated, and $\alpha \to 0$, the rate of convergence diminishes.

As will become evident in the proof of Theorem 1 the Kullback-Leibler divergence is required between two measures involving modified fractional Brownian motions. To cater for this, some auxiliary definitions to precede the proof of Theorem 1 are given in the next section.

3. Preliminaries

In this paper, the functions under consideration are defined in the Fourier domain (see Definition 1). Among others, there are two representations for fBm that satisfy Definition 2 that are used in this paper. The first being the moving average representation of Mandelbrot and Van Ness (1968) in the time domain and second is the spectral representation given by Samorodnitsky and Taqqu (1994) in the Fourier domain. These both need to be considered since they are both used in the proof of the main result. Both representations have normalisation constants $C_{T,H}$ and $C_{F,H}$ for the time and spectral representations respectively to ensure the fBm satisfies Definition 2. Start with the time domain representation.

Definition 3. *The fractional Brownian motion $\{B_H(t)\}_{t \in \mathbb{R}}$ can be represented by,*

$$B_H(t) = \frac{1}{C_{T,H}} \int_{\mathbb{R}} \left((t-s)_+^{H-1/2} - (-s)_+^{H-1/2} \right) dB(s),$$

where $C_{T,H} = \Gamma(H+1/2)/\sqrt{2H \sin(\pi H)\Gamma(2H)}$ and $x_+ = x \mathbb{1}_{\{x>0\}}(x)$.

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For the spectral representation a complex Gaussian measure $\breve{B} := B^{[1]} + iB^{[2]}$ is used where $B^{[1]}$ and $B^{[2]}$ are independent Gaussian measures such that for $i = 1, 2$; $B^{[i]}(A) = B^{[i]}(-A)$ for any Borel set A of finite Lebesgue measure and $\mathbb{E}(B^{[i]}(A))^2 = \text{mesh}(A)/2$.

Definition 4. *The fractional Brownian motion* $\{B_H(t)\}_{t \in \mathbb{R}}$ *can be represented by,*

$$B_H(t) = \frac{1}{C_{F,H}} \int_{\mathbb{R}} \frac{e^{ist} - 1}{is} |s|^{-(H-1/2)} d\breve{B}(s),$$

where $C_{F,H} = \sqrt{\pi/(2H\sin(\pi H)\Gamma(2H))}$.

As will become evident in Section 4, to obtain the lower bound result for the minimax rate, it is crucial to know which functional class to consider for $\mu : \mathbb{R} \longrightarrow \mathbb{R}$ such that the process $\int_{\mathbb{R}} \mu(x)\, dB_H(x)$ is a well defined random variable with finite variance. Two such classes of functions will be considered, $\mathcal{H}$ and $\widetilde{\mathcal{H}}$, which correspond to the time and spectral versions of fBm respectively. Begin with the moving average representation.

Definition 5. *Let* $H \in (1/2, 1)$ *be constant. Then the class* $\mathcal{H}$ *is defined by,*

$$\mathcal{H} = \left\{ \mu : \mathbb{R} \longrightarrow \mathbb{R} \,\middle|\, \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \mu(x)\mu(y)|x-y|^{-\alpha}\, dy\, dx \right| < \infty \right\}.$$

Simlar to (5) there is an inner product on the space $\mathcal{H}$ that satisfies the following. For all $f, g \in \mathcal{H}$,

$$\mathbb{E}\left\{ \int_{\mathbb{R}} f(x)\, dB_H(x) \int_{\mathbb{R}} g(y)\, dB_H(y) \right\} = C_\alpha \int_{\mathbb{R}} \int_{\mathbb{R}} f(x)g(y)|x-y|^{-\alpha}\, dy\, dx =: \langle f, g \rangle_{\mathcal{H}},$$

where the constant $C_\alpha = \frac{1}{2}(1-\alpha)(2-\alpha)$. The other functional class for the spectral representation is denoted by $\widetilde{\mathcal{H}}$ and defined below.

Definition 6. *Let* $H \in (1/2, 1)$ *be constant. Then the class* $\widetilde{\mathcal{H}}$ *is defined by,*

$$\widetilde{\mathcal{H}} = \left\{ \mu : \mathbb{R} \longrightarrow \mathbb{R} \,\middle|\, \int_{\mathbb{R}} |\widetilde{\mu}(\omega)|^2 |\omega|^{-(1-\alpha)}\, d\omega < \infty \right\}.$$

On the space $\widetilde{\mathcal{H}}$, the stochastic integrals with respect to fBm are well defined and satisfy the following. For all $f, g \in \widetilde{\mathcal{H}}$,

$$\mathbb{E}\left\{ \int_{\mathbb{R}} f(x)\, dB_H(x) \int_{\mathbb{R}} g(y)\, dB_H(y) \right\} = \frac{1}{C_{F,H}^2} \int_{\mathbb{R}} \widetilde{f}(\omega)\overline{\widetilde{g}(\omega)}|\omega|^{-(1-\alpha)}\, d\omega =: \langle f, g \rangle_{\widetilde{\mathcal{H}}}, \quad (5)$$

where $\overline{\widetilde{g}}$ denotes the complex conjugate of $\widetilde{g}$.

These two classes of integrands were considered extensively in Pipiras and Taqqu (2000). In this context of this paper the inner products can be used interchangeably because if $\mu \in \mathscr{F}_s(\theta)$ then $\mu \in \mathscr{L}_1(\mathbb{R}) \cap \mathscr{L}_2(\mathbb{R})$ and by Pipiras and Taqqu (2000, Proposition 3.1) then $\mu \in \mathcal{H}$. Also, using Pipiras and Taqqu (2000, Proposition 3.2) with the isometry Biagini et al. (2008, Lemma 3.1.2) and Parseval's Theorem then $\mu \in \widetilde{\mathcal{H}}$ and consequently $\mu \in \mathcal{H} \cap \widetilde{\mathcal{H}}$.

4. Proof of Theorem 1

The lower bound for the minimax rate is constructed by adapting the results of Goldenshluger et al. (2006) to our framework. This requires obtaining the Kullback-Leibler divergence of two suitably chosen functions μ_0 and μ_1 from the functional class $\mathscr{F}_s(\theta)$. The main hurdle in determining the Kullback-Leibler divergence is the long-range dependent structure in the fBm increments. A summary of Girsanov type theorems for fBm have been established by Biagini, Hu, Øksendal, and Zhang (2008, Theorem 3.2.4). Here however, the Radon-Nikodym derivative is the main focus. Once that is determined, the Kullback-Leibler divergence is linked to the lower rate of convergence using Tsybakov (2009, Theorem 2.2 (iii)). Lastly, before proceeding to the proof, the quantity $C > 0$ denotes a generic constant that could possibly change from line to line.

Without loss of generality, consider a function $\mu_0 \in \mathscr{F}_s(\theta_0)$ where $\theta_0 \in (0, 1/2]$ and define $\theta_1 = \theta_0 + \delta$ where $\delta \in (0, 1/2)$ (a symmetric argument can be setup to accommodate the case when $\theta_0 \in [1/2, 1)$). Define the functions $v : \mathbb{R} \longrightarrow \mathbb{R}$ and $v_N : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$v(x) := a((\theta_1 \wedge x) - \theta_0) \mathbb{1}_{(\theta_0, 1]}(x), \qquad v_N(x) := \int_{-N}^{N} \widetilde{v}(\omega) e^{2\pi i x \omega} \, d\omega,$$

where a is the size of the jump given in Definition 1 and $\widetilde{v}$ is the Fourier transform of v. Note that, $v_N(x)$ is close to $v(x)$ in the sense that it is the inverse Fourier transform of $\widetilde{v}(\omega) \mathbb{1}_{|\omega| \leq N}$ and $\widetilde{v}_N(\omega) = \widetilde{v}(\omega) \mathbb{1}_{|\omega| \leq N}$. With these definitions, the derivative takes the form, $v^{(1)}(x) = a \mathbb{1}_{[\theta_0, \theta_1]}(x)$ and the function $(\mu_0 - v)$ has a single kink at θ_1. Then define $\mu_1 := \mu_0 - (v - v_N)$. The function v_N is infinitely differentiable across the whole real line and smooth for finite N, which implies that $\mu_1 = \mu_0 - (v - v_N)$ has a single kink at θ_1. It can be shown that,

$$|\widetilde{v}(\omega)| \leq a\delta/(2\pi |\omega|)^{-1}. \tag{6}$$

Further, if N is chosen to be $N = (s\pi C/(a\delta))^{1/s}$ then $\int_{\mathbb{R}} |\widetilde{v_N}(\omega)||\omega|^s \, d\omega < \infty$ and consequently $\mu_1 \in \mathscr{F}_s(\theta_1)$.

To be able to determine the Radon-Nikodym derivative, define $\Delta := \mu_0 - \mu_1 = v - v_N$ and note that $\Delta : \mathbb{R} \longrightarrow \mathbb{R}$. The Radon-Nikodym derivative also needs a paired function $\underline{\Delta} : \mathbb{R} \longrightarrow \mathbb{R}$. Define such a function with a singular integral operator with

$$\Delta(x) := \varepsilon^{-\alpha} C_\alpha \int_{\mathbb{R}} |x - y|^{-\alpha} \underline{\Delta}(y) \, dy = \frac{\Gamma(3 - \alpha)}{2} \left(\mathcal{D}_-^{-(1-\alpha)} \underline{\Delta}(x) + \mathcal{D}_+^{-(1-\alpha)} \underline{\Delta}(x) \right), \tag{7}$$

where, for $\nu \in (0, 1)$, $\mathcal{D}_-^{-\nu}$ and $\mathcal{D}_+^{-\nu}$ are the left and right fractional Liouville integral operators defined by,

$$\mathcal{D}_-^{-\nu} f(x) := \frac{1}{\Gamma(\nu)} \int_{-\infty}^{x} (x - y)^{\nu - 1} f(y) \, dy \qquad \mathcal{D}_+^{-\nu} f(x) := \frac{1}{\Gamma(\nu)} \int_{x}^{\infty} (y - x)^{\nu - 1} f(y) \, dy$$

This function $\underline{\Delta}$ has representation in the Fourier domain with

$$\widetilde{\underline{\Delta}}(\omega) \asymp \varepsilon^{-\alpha} |\omega|^{1-\alpha} \widetilde{\Delta}(\omega).$$

Furthermore, $\underline{\Delta} \in \mathcal{H} \cap \widetilde{\mathcal{H}}$. Indeed, by definition, $\Delta = \mu_0 - \mu_1$ with $\mu_0 \in \mathscr{F}_s(\theta_0)$ and $\mu_1 \in \mathscr{F}_s(\theta_1)$ which implies that $\Delta \in \mathscr{L}_1(\mathbb{R}) \cap \mathscr{L}_2(\mathbb{R})$ and $\widetilde{\Delta}(\omega) = o(\omega^{-s})$ due to (3). First, it will be shown that, $\underline{\Delta} \in \widetilde{\mathcal{H}}$.

$$
\begin{aligned}
\langle \underline{\Delta}, \underline{\Delta} \rangle_{\widetilde{\mathcal{H}}} &\asymp \int_{\mathbb{R}} |\widetilde{\Delta}(\omega)|^2 |\omega|^{1-\alpha} \, d\omega \\
&\leq C \left\{ \|\Delta\|_1^2 \int_{|\omega| \leq 1} |\omega|^{1-\alpha} \, d\omega + \int_{|\omega| \geq 1} |\widetilde{\Delta}(\omega)|^2 |\omega|^{1-\alpha} \, d\omega \right\},
\end{aligned} \tag{8}
$$

where $C > 0$ is some constant and $\|\Delta\|_1 = \int_{\mathbb{R}} |\Delta(x)| \, dx$. In (8), the first integral is finite since $\alpha \in (0, 1)$ and the last integral is finite since $\widetilde{\Delta}(\omega) = o(\omega^{-s})$ for $s \geq 2$, proving $\underline{\Delta} \in \widetilde{\mathcal{H}}$. Then apply the isometry in Biagini et al. (2008, Lemma 3.1.2) with Plancherel and (8), it follows that $\underline{\Delta} \in \mathcal{H}$.

Now let P_0 and P_1 be the probability measures associated with model (1) with $\mu = \mu_0$ and $\mu = \mu_1$ respectively. Define, $\mathring{B}_H(x) := \varepsilon^{-\alpha} \int_0^x \Delta(x) \, dx + B_H(x)$. Then under the P_0 measure,

$$
dY_0(x) = \mu_0(x) \, dx + \varepsilon^\alpha \, dB_H(x) = \mu_1(x) \, dx + \varepsilon^\alpha \, d\mathring{B}_H(x).
$$

The Radon-Nikodym derivative between these measures takes the form,

$$
\frac{dP_1}{dP_0} := \exp \left\{ -\int_{\mathbb{R}} \underline{\Delta}(x) \, dB_H(x) - \frac{1}{2} \mathbb{E}_{P_0} \left(\int_{\mathbb{R}} \underline{\Delta}(x) \, dB_H(x) \right)^2 \right\}. \tag{9}
$$

Indeed to show (9) is valid, for $\underline{\Delta} \in \mathcal{H}$ and $\psi \in \mathcal{H}$, use (7) and apply Biagini et al. (2008, Lemma 3.2.1) with the change of measure formula in (9) to yield,

$$
\mathbb{E}_{P_1} \left[\psi(\mathring{B}_H(x)) \right] = \mathbb{E}_{P_0} \left[\psi(\mathring{B}_H(x)) \frac{dP_1}{dP_0} \right] = \mathbb{E}_{P_0} \left[\psi(B_H(x)) \right]. \tag{10}
$$

So, using (5) in (9), the Kullback-Leibler divergence between the two models can be evaluated,

$$
\mathcal{K}(P_0, P_1) := \mathbb{E} \ln \frac{dP_0}{dP_1} = \frac{1}{2} \langle \underline{\Delta}, \underline{\Delta} \rangle_{\widetilde{\mathcal{H}}}. \tag{11}
$$

To evaluate (11), obtain a finer bound on $|\widetilde{\underline{\Delta}}(\omega)|^2$ by recalling that $\Delta = v - v_N$ and using (6),

$$
|\widetilde{\underline{\Delta}}(\omega)|^2 \asymp \varepsilon^{-2\alpha} |\widetilde{v}(\omega)|^2 \mathbb{1}_{\{|\omega| \geq N\}} |\omega|^{2-2\alpha} \leq \frac{C^2 a^2 \delta^2}{4\pi^2} \varepsilon^{-2\alpha} |\omega|^{-2\alpha} \mathbb{1}_{\{|\omega| \geq N\}}. \tag{12}
$$

Apply the bound in (12) to (11) with the chosen $N = (s\pi C/(a\delta))^{1/s}$,

$$
\begin{aligned}
\mathcal{K}(P_0, P_1) &= \frac{1}{2} \int_{\mathbb{R}} |\widetilde{\underline{\Delta}}(\omega)|^2 |\omega|^{-(1-\alpha)} \, d\omega \\
&\leq \frac{C a^2 \delta^2}{4\pi^2} \varepsilon^{-2\alpha} \int_{|\omega| \geq N} |\omega|^{-\alpha-1} \, d\omega \\
&= C a^2 \delta^2 \varepsilon^{-2\alpha} \left(s/(a\delta) \right)^{-\alpha/s} \\
&\asymp \delta^{(2s+\alpha)/s} \varepsilon^{-2\alpha}.
\end{aligned}
$$

Now choose $\delta \asymp \varepsilon^{2\alpha s/(2s+\alpha)}$ which guarantees that $\mathcal{K}(P_0, P_1) \leq K < \infty$ for some finite positive constant K. Then by Tsybakov (2009, Theorem 2.2 (iii)) combined with the fact that $\varepsilon \asymp n^{-1/2}$ it follows that the lower rate of convergence for the minimax risk is $\varepsilon^{2\alpha s/(2s+\alpha)} \asymp n^{-\alpha s/(2s+\alpha)}$. $\qquad\square$

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References

Beran, J., 1994. Statistics for long-memory processes. Vol. 61 of Monographs on Statistics and Applied Probability. Chapman and Hall, New York.

Biagini, F., Hu, Y., Øksendal, B., Zhang, T., 2008. Stochastic Calculus for Fractional Brownian Motion and Applications. Probability and its Applications (New York). Springer-Verlag London Ltd., London.

Cheng, M.-Y., Raimondo, M., 2008. Kernel methods for optimal change-points estimation in derivatives. J. Comput. Graph. Statist. 17 (1), 56–75.
URL http://dx.doi.org/10.1198/106186008X289164

Doukhan, P., Oppenheim, G., Taqqu, M. S. (Eds.), 2003. Theory and applications of long-range dependence. Birkhäuser Boston Inc., Boston, MA.

Goldenshluger, A., Juditsky, A., Tsybakov, A. B., Zeevi, A., 2008a. Change-point estimation from indirect observations. I. Minimax complexity. Ann. Inst. Henri Poincaré Probab. Stat. 44 (5), 787–818.
URL http://dx.doi.org/10.1214/07-AIHP110

Goldenshluger, A., Juditsky, A., Tsybakov, A., Zeevi, A., 2008b. Change-point estimation from indirect observations. II. Adaptation. Ann. Inst. Henri Poincaré Probab. Stat. 44 (5), 819–836.
URL http://dx.doi.org/10.1214/07-AIHP144

Goldenshluger, A., Tsybakov, A., Zeevi, A., 2006. Optimal change-point estimation from indirect observations. Ann. Statist. 34 (1), 350–372.
URL http://dx.doi.org/10.1214/009053605000000750

Johnstone, I. M., Silverman, B. W., 1997. Wavelet threshold estimators for data with correlated noise. J. Roy. Statist. Soc. Ser. B 59 (2), 319–351.
URL http://dx.doi.org/10.1111/1467-9868.00071

Mandelbrot, B. B., Van Ness, J. W., 1968. Fractional Brownian motions, fractional noises and applications. SIAM Rev. 10, 422–437.
URL http://dx.doi.org/10.1137/1010093

Pipiras, V., Taqqu, M. S., 2000. Integration questions related to fractional Brownian motion. Probab. Theory Related Fields 118 (2), 251–291.
URL http://dx.doi.org/10.1007/s440-000-8016-7

Samorodnitsky, G., Taqqu, M. S., 1994. Stable non-Gaussian random processes. Stochastic Modeling. Chapman & Hall, New York, stochastic models with infinite variance.

Tsybakov, A. B., 2009. Introduction to Nonparametric Estimation. Springer Publishing Company, Incorporated.

Wishart, J., 2009. Kink estimation with correlated noise. Journal of the Korean Statistical Society 38 (2), 131–143.
URL http://dx.doi.org/10.1016/j.jkss.2008.08.001