A ONE-SIDED POWER SUM INEQUALITY

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Abstract. In this note we prove results of the following type. Let be given real numbers b_j and distinct complex numbers z_j satisfying the conditions $|z_j| = 1, z_j \neq 1$ for $j = 1, \ldots, n$ and for every z_j there exists an i such that $z_i = \overline{z_j}, b_i = b_j$. Then $\liminf_{k\to\infty} \sum_{j=1}^n b_j z_j^k \leq -\sum_{j=1}^n |b_j|/n$. Such results have applications in numerical analysis.

My colleague Marc N. Spijker asked the following question in view of an application in numerical analysis [3]:

Is it true that for given real numbers $b_j \geq 1$ and distinct complex numbers z_j satisfying the conditions $|z_j| = 1, z_j \neq 1$ for $j = 1, \ldots, n$ and

$$\textit{for every } z_j \textit{ there exists an } i \textit{ such that } z_i = \overline{z_j}, b_i = b_j$$

we have $\liminf_{k\to\infty} \sum_{j=1}^n b_j z_j^k \leq -1$?

Note that by the conjugate condition $\sum_{j=1}^n b_j z_j^k$ is real for all k. It follows from Dirichlet's Simultaneous Approximation Theorem ([1] p. 13) that under the above conditions $\limsup_{k\to\infty} \sum_{j=1}^n b_j z_j^k = \sum_{j=1}^n b_j$. Furthermore, Kronecker's Simultaneous Approximation Theorem ([1], p. 53) implies that if π and the arguments of the z_j's are maximally linearly independent over the rationals, then $\liminf_{k\to\infty} \sum_{j=1}^n b_j z_j^k = -\sum_{j=1}^n b_j$. Here maximally independent means that from every pair of conjugates z_i, z_j one is chosen. The only further results on the infimum I have found in the literature are Turán's one-sided inequalities ([4] Ch. 12). Some of them give explicitly an M depending on $z_1, \ldots, z_n$ and a function $f(r, M) < 0$ such that

$$\min_{k=r,r+1,\ldots,r+M-1} \sum_{j=1}^n b_j z_j^k < f(r, M),$$

provided that the numbers z_j are away from 1 by at least some prescribed amount. However, since $\lim_{r\to\infty} f(r, M) = 0$ it does not answer Spijker's question where a negative upper bound independent of r and of the distances from the numbers z_j to 1 is required.

2000 *Mathematics Subject Classification.* 11N30.

Key words and phrases. power sums, inequality, one-sided.

In this paper we answer Spijker's question in a slightly generalized and sharpened form. Spijker asked the question because of applications to numerical analysis. Linear multistep methods (LMMs) form a well-known class of numerical step-by-step methods for solving initial-value problems for certain systems of ordinary differential equations. In many applications of such methods it is essential that the LMM has specific stability properties. An important property of this kind is named *boundedness* and has recently been studied by Hundsdorfer, Mozartova and Spijker [2]. In that paper the *stepsize-coefficient* γ is a crucial parameter in the study of boundedness. In [3] Spijker attempts to single out all LMMs with a positive stepsize-coefficient γ for boundedness. By using Corollary 1 below he is able to nicely narrow the class of such LMMs.

Theorem 1. *Let n be a positive integer. Put $m = \lfloor n/2 \rfloor$. Let be given nonzero complex numbers b_j and distinct complex numbers z_j such that*

$$|z_j| = 1, z_j \neq 1 \ for \ j = 1, \dots, n,$$
$$b_n \in \mathbb{R}, z_n = -1 \ if \ n \ is \ odd \ and$$
$$b_{m+j} = \overline{b_j}, z_{m+j} = \overline{z_j} \ for \ j = 1, \dots, m.$$

Then $\sum_{j=1}^{n} b_j z_j^k \in \mathbb{R}$ for all k and $\liminf_{k\to\infty} \sum_{j=1}^{n} b_j z_j^k \leq -\frac{\sum_{j=1}^{n} |b_j|^2}{\sum_{j=1}^{n} |b_j|}$.

By applying the Cauchy-Schwarz inequality we immediately obtain the following consequence.

Corollary 1. *Let be given real numbers b_j and distinct complex numbers z_j satisfying the conditions $|z_j| = 1, z_j \neq 1$ for $j = 1, \dots, n$ and*

$$for \ every \ z_j \ there \ exists \ an \ i \ such \ that \ b_i = b_j, z_i = \overline{z_j}.$$

Then $\liminf_{k\to\infty} \sum_{j=1}^{n} b_j z_j^k \leq -\frac{\sum_{j=1}^{n} |b_j|}{n}$.

Obviously this answers Spijker's question and shows that the upper bound -1 can only reached be reached if $b_j = 1$ for $j = 1, \dots, n$.

Another variant of the theorem is as follows.

Corollary 2. *Let be given nonzero complex numbers b_j and distinct complex numbers z_j satisfying the conditions $|z_j| = 1, z_j \neq 1$ for $j = 1, \dots, m$. Then $\liminf_{k\to\infty} \Re(\sum_{j=1}^{m} b_j z_j^k) \leq -\frac{\sum_{j=1}^{m} |b_j|^2}{2\sum_{j=1}^{m} |b_j|}$.*

The given bounds are the best possible. Let $z_1, \dots, z_n$ be the $(n+1)$-st roots of unity except for 1. Then $\sum_{j=1}^{n} z_j^k = n$ if k is a multiple of $n+1$ and -1 otherwise. This shows that the bounds in Theorem 1 and Corollary 1 are optimal. If n is even and we restrict our attention to the $(n+1)$-st roots of unity with positive imaginary part, then the real parts of the $\sum_{j=1}^n z_j^k$-values are halved because of symmetry and we attain the bound of Corollary 2.

Proof of Theorem 1. Put $s_k = b_1 z_1^k + \cdots + b_n z_n^k$. Note that $s_k \in \mathbb{R}$ for all k. Let K and N be positive integers. We have

$$\sum_{k=N}^{N+K-1} s_k = \sum_{j=1}^{n} \sum_{k=N}^{N+K-1} b_j z_j^k = \sum_{j=1}^{n} b_j \frac{z_j^N - z_j^{N+K}}{1 - z_j}.$$

Thus

$$|\sum_{k=N}^{N+K-1} s_k| \le \sum_{j=1}^{n} \frac{2|b_j|}{|1 - z_j|} =: C_1. \tag{1}$$

Furthermore,

$$\begin{aligned}\sum_{k=N}^{N+K-1} s_k^2 &= \sum_{k=N}^{N+K-1} \sum_{i=1}^{n} \sum_{j=1}^{n} b_i b_j z_i^k z_j^k \\ &= K \sum_{z_i = \overline{z_j}} b_i b_j + \sum_{z_i \ne \overline{z_j}} b_i b_j \sum_{k=N}^{N+K-1} (z_i z_j)^k \\ &= K \sum_{j=1}^{n} |b_j|^2 + \sum_{z_i \ne \overline{z_j}} b_i b_j \frac{(z_i z_j)^N - (z_i z_j)^{N+K}}{1 - z_i z_j}.\end{aligned}$$

Observe that

$$\left| \sum_{z_i \ne \overline{z_j}} b_i b_j \frac{(z_i z_j)^N - (z_i z_j)^{N+K}}{1 - z_i z_j} \right| \le \sum_{z_i \ne \overline{z_j}} |b_i b_j| \frac{2}{|1 - z_i z_j|} =: C_2.$$

Let the sum of all positive terms among s_k $(k = N, \ldots, N+K-1)$ equal $(L - \varepsilon) \sum_{j=1}^n |b_j|$ with $0 \le \varepsilon < 1$. Then we replace the positive terms among s_k $(k = N, \ldots, N+K-1)$ by $L - 1$ terms $\sum_{j=1}^n |b_j|$, one term $\sum_{j=1}^n |b_j| - \varepsilon$ and further terms 0 so that the total number of terms is still K. Denote the new terms by s_k^* for $k = N, \ldots, N+K-1$. Then

$$\sum_{k=N}^{N+K-1} s_k^* = \sum_{k=N}^{N+K-1} s_k \text{ and } \sum_{k=N}^{N+K-1} (s_k^*)^2 \ge \sum_{k=N}^{N+K-1} s_k^2.$$

Put $c = c(N, K) = \min_{k=N,\ldots,N+K-1} s_k$. We have, by (1),

$$C_1 \ge \sum_{k=N}^{N+K-1} s_k = \sum_{k=N}^{N+K-1} s_k^* \ge (L - \varepsilon) \sum_{j=1}^{n} |b_j| + (K - L)c.$$

Hence

$$C_1 \geq L(\sum_{j=1}^{n} |b_j| - c) - \left(\varepsilon \sum_{j=1}^{n} |b_j| - Kc \right). \tag{2}$$

On the other hand,

$$K \sum_{j=1}^{n} |b_j|^2 - C_2 \leq \sum_{k=N}^{N+K-1} s_k^2 \leq (K-L)c^2 + L(\sum_{j=1}^{n} |b_j|)^2$$

$$= Kc^2 + L\left((\sum_{j=1}^{n} |b_j|)^2 - c \right).$$

If $c = -\sum_{j=1}^{n} |b_j|$ occurs for arbritrarily large K, then the theorem follows from the inequality $(\sum_{j=1}^{n} |b_j|)^2 \geq \sum_{j=1}^{n} |b_j|^2$. Otherwise $c^2 < (\sum_{j=1}^{n} |b_j|)^2$ for sufficiently large K and we obtain

$$L \geq \frac{K \sum_{j=1}^{n} |b_j|^2 - Kc^2 - C_2}{(\sum_{j=1}^{n} |b_j|)^2 - c^2}.$$

Substituting this into (2) we find

$$C_1 \geq \frac{K \sum_{j=1}^{n} |b_j|^2 - Kc^2 - C_2}{(\sum_{j=1}^{n} |b_j|)^2 - c^2} (\sum_{j=1}^{n} |b_j| - c) - \left(\varepsilon \sum_{j=1}^{n} |b_j| - Kc \right)$$

$$\geq \frac{K \left(\sum_{j=1}^{n} |b_j|^2 + c \sum_{j=1}^{n} |b_j| \right) - C_2 - \varepsilon (\sum_{j=1}^{n} |b_j|)^2 + \varepsilon c \sum_{j=1}^{n} |b_j|}{\sum_{j=1}^{n} |b_j| + c}. \tag{3}$$

Recall that $c = c(N,K)$. Put $c(N) = \inf_{K>0} c(N,K)$. It is easy to check that the right-hand side of (3) is monotonically increasing in c. Hence the inequality remains valid if we replace c by $c(N)$ in (3). This yields an inequality of the form

$$C_1 \geq C_3 K \left(\sum_{j=1}^{n} |b_j|^2 + c(N) \sum_{j=1}^{n} |b_j| \right) + C_4,$$

where $C_1 \geq 0, C_3 > 0, C_4 \geq 0$ are independent of K. Let $K \to \infty$. We conclude that

$$c(N) \leq -\frac{\sum_{j=1}^{n} |b_j|^2}{\sum_{j=1}^{n} |b_j|}.$$

Since N is arbitrary, this provides the claimed upper bound for $\liminf_{k\to\infty} \sum_{j=1}^{n} b_j z_j^k = \lim_{N\to\infty} c(N)$. □

Proof of Corollary 2. Define $b_{m+j} = \overline{b_j}, z_{m+j} = \overline{z_j}$ for $j = 1, \ldots, m$. Put $n = 2m$ and apply Theorem 1. Now the assertion follows from the relations $\Re(\sum_{j=1}^{n} b_j z_j^k) = 2\Re(\sum_{j=1}^{m} b_j z_j^k), \sum_{j=1}^{n} |b_j|^2 = 2\sum_{j=1}^{m} |b_j|^2, \sum_{j=1}^{n} |b_j| = 2\sum_{j=1}^{m} |b_j|$. □

Acknowledgement I thank Marc Spijker for asking me the mentioned question, making valuable comments on drafts of this paper and providing the information about the application in numerical analysis.

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