POISSON-FURSTENBERG BOUNDARY AND GROWTH OF GROUPS

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Abstract. We study the Poisson-Furstenberg boundary of random walks on permutational wreath products. We give a sufficient condition for a group to admit a symmetric measure of finite first moment with non-trivial boundary, and show that this criterion is useful to establish exponential word growth of groups. We construct groups of exponential growth such that all finitely supported (not necessarily symmetric, possibly degenerate) random walks on these groups have trivial boundary. This gives a negative answer to a question of Kaimanovich and Vershik.

1. Introduction

Consider a set X with a basepoint ρ and a right action of a group G. A *random walk* on X is defined by a probability measure μ on G; the random walker starts at ρ and, at each step, moves from x to xg with probability $\mu(g)$. An important particular case is $X = G$, seen as a G-space under right-multiplication. A measure μ is *symmetric* if $\mu(g) = \mu(g^{-1})$ for all $g \in G$, and is *non-degenerate* if its support generates the group G.

The Poisson-Furstenberg boundary is the space of ergodic components of infinite trajectories of the walk. In the case of random walks on groups, there are several equivalent definitions of the Poisson-Furstenberg boundary, and we recall some of them in Section 2. For more information, see e.g. [21].

There is a strong relation between triviality/non-triviality of the boundary and other asymptotic properties of groups (see [21], and [13] for a more recent overview). For example, a result of Kaimanovich-Vershik and Rosenblatt [21, 27] states that a group is amenable if and only if it admits a measure with trivial boundary. One wondered whether exponential word growth can be characterized by non-triviality of the boundary for appropriate measures. Indeed, there are many manifestations of the analogy between non-triviality of the boundary and exponential growth, such as the Entropy Criterion of Derriennic and Kaimanovich-Vershik [8, 21]. It is known that if a group admits a finitely supported measure with non-trivial boundary, then its word growth is exponential.

Kaimanovich and Vershik ask in [21, page 466] the following question: "Does every group of exponential word growth admit a finitely supported measure μ such that the boundary of (G, μ) is non-trivial?"

In this paper, we give (in Section 5) a negative answer to the above mentioned question of Kaimanovich and Vershik:

Date: 7 July 2011.
The work is supported by the ERC starting grant GA 257110 "RaWG" and the Courant Research Centre "Higher Order Structures" of the University of Göttingen.

Theorem A. *There exists a finitely generated group G of exponential word growth, such that the boundary of (G,μ) is trivial for all finitely supported (possibly degenerate or non-symmetric) measures μ.*

There are many examples of groups of exponential growth (e.g. the wreath product of a non-trivial finite group with $\mathbb{Z}$ or $\mathbb{Z}^2$, or solvable Baumslag-Solitar groups), such that any symmetric finitely supported measure has trivial boundary. Nevertheless, these groups admit non-symmetric measures with non-trivial boundary. Likewise, on some groups (e.g. wreath products of a finite group with the infinite dihedral group [15]), any non-degenerate measure has trivial boundary, but these groups admit degenerate measures with non-trivial boundary.

Let us mention another question about possible characterisations of exponential growth, that remains open: Kaimanovich and Vershik conjecture in [21] that every group of exponential growth admits a *symmetric* (possibly infinitely supported) measure with non-trivial boundary.

The groups we construct to prove Theorem A are *permutational wreath products*, as defined in the next section: groups $W = A \wr_X G = \sum_X A \rtimes G$. More precisely, we consider a family of groups $W = A \wr_{X_1\times\cdots\times X_d} (G_1 \times \cdots \times G_d)$, in which each G_i is a copy of the first Grigorchuk group, each X_i is an orbital Schreier graph of the first Grigorchuk group, and A is a finite group. See Sections 2 and 4 for details of this construction. See also Section 6 for a larger family of examples.

Ordinary wreath products ($X = G$) have exponential growth as soon as A is non-trivial and G is infinite, but the situation is much more subtle for permutational wreath products, which may have intermediate growth. Indeed, it is shown in [5] that the W above has intermediate growth if $d = 1$. In this paper we consider the groups W with $d \geqslant 2$, and are in particular interested in the case $d = 2$. It turns out that, if $d \geqslant 2$ and G admits random walks with sublinear drift, bounded by n^α for some $\alpha < 1$, then W has exponential growth, see Corollary 4.7. For $d \geqslant 3$ one can show, moreover, that a simple random walk on such groups has non-trivial Poisson-Furstenberg boundary, see Example 4.9. For the proof of Theorem A we consider the case $d = 2$, which lies in some sense on a borderline between exponential and intermediate growth: on one hand, as we explain below, the growth is exponential. On the other hand, these groups are "close to groups of subexponential growth", in the sense that any finitely supported measure on them has trivial boundary.

In many known examples of groups, their exponential growth can be checked either by exhibiting a free semigroup (they exist in solvable groups of exponential growth, and more generally in elementarily amenable groups of exponential growth, see Chou [7]); or by proving the existence of an imbedded regular tree in the group's Cayley graph (as is the case for any non-amenable group [6]). Ordinary (non-permutational) wreath products of a non-trivial group by an infinite group also contain regular trees in their Cayley graph, and this class of groups contains interesting examples of torsion groups of exponential growth, see Grigorchuk [18].

Our understanding is that, for the groups we consider in this paper, it is not straightforward to check that their growth is exponential. To show that our examples have exponential growth we prove (in Section 4) the following criterion based on random walks:

Theorem B. *Let G be a group acting on a set X, and let μ be a finitely supported, symmetric, non-degenerate measure on G. Suppose that the drift of μ is sublinear, bounded by a function of type n^α, and that the probability of return of the induced* *random walk on X decays at least as $n^{-\delta}$, for some $\delta > \alpha$. Let A be a non-trivial group.*

Then $W := A \wr_X G$ admits a symmetric measure with finite first moment and non-trivial Poisson-Furstenberg boundary. In particular, the word growth of W is exponential.

In our situation, we consider $\delta = 1$ and some $\alpha < 1$. The assumption of Theorem B on the probability to return to the origin on X is a consequence of the fact that X is a product of two copies of infinite transitive Schreier graphs. For our main examples used for the proof of Theorem A, the condition on the drift in Theorem B follows from an upper bound on the growth of Grigorchuk groups. To get more examples of this kind, we consider groups for which the condition on the drift on G is ensured by a version of a self-similar-random-walk argument due to Bartholdi-Virag and Kaimanovich; see Example 6.5. We also give a torsion-free group with this property, see Example 6.6.

On the other hand, to prove that the random walks we consider have trivial Poisson-Furstenberg boundary, we use a criterion due to Kalpazidou and Mathieu ensuring recurrence of "centered" random walks, and a criterion for triviality of the boundary of random walks on permutational wreath products (Proposition 5.1). This criterion is, in a subtle manner, more complicated than in the case of ordinary wreath products, see the discussion at the beginning of Section 5.

The groups we construct in this paper admit a (symmetric, non-degenerate) finite first moment measure with non-trivial boundary. This leads us to ask the following question:

Question 1.1. Does there exist a group G of exponential word growth, such that all (not necessarily symmetric, not necessarily non-degenerate) measures with finite first moment have trivial Poisson-Furstenberg boundary?

2. Definitions and preliminaries

2.1. **Poisson-Furstenberg boundary, entropy and drift.** Consider two infinite trajectories $\boldsymbol{x}$ and $\boldsymbol{y}$. We say they are *equivalent* if they coincide after some instant, possibly up to the time shift: there exits $N \in \mathbb{N}, k \in \mathbb{Z}$ such that $\boldsymbol{x}_n = \boldsymbol{y}_{n+k}$ for all $n \geqslant N$. Consider the measurable hull of this equivalence relation in the space of infinite trajectories. The quotient by this equivalence relation is called the *Poisson-Furstenberg boundary.*

A function $F : G \to \mathbb{R}$ is called *μ-harmonic* if for all $g \in G$ we have $F(g) = \sum_{h \in G} F(gh)\mu(h)$. The Poisson-Furstenberg boundary is non-trivial if and only if G admits a bounded μ-harmonic function which is non-constant on the group generated by the support of μ.

The *entropy* of a probability measure μ is computed as $H(\mu) = -\sum_g \mu(g)\log(\mu(g))$. The *entropy of the random walk*, also called its *asymptotic entropy*, is the limit $h(\mu)$ of $H(\mu^{*n})/n$, as n tends to infinity. This limit is well-defined, since the function $H(n) := H(\mu^{*n})$ is subadditive. If $H(\mu) = \infty$, then $H(\mu^{*n}) = \infty$ for all n and in this case we put $h(\mu) = \infty$.

Fix a finite generating set S and consider on G the word metric $\|\cdot\|_S$ associated with S. Given a probability measure μ on G and $\beta > 0$, the *β-moment* of μ with respect to S is $L^\beta(\mu) := \sum_{g \in G} \mu(g)\|g\|_S^\beta$. Clearly, if the β-moment is finite with respect to some finite generating set S, then it is finite with respect to any other generating set. The *first moment* of μ is simply written $L(\mu)$.

The function $L(n) := L(\mu^{*n})$ is also subadditive, by the triangular inequality for $\|\cdot\|_S$. It expresses the mean distance to the origin in the word metric $\|\cdot\|_S$, after n steps of the random walk. The *drift*, also called *rate of escape*, of the random walk (G,μ) is the limit $\ell(\mu)$ of $L(n)/n$ as n tends to infinity; this limit is well-defined because $L(n)$ is subadditive. If the first moment of μ is finite, that is, if $L(1) < \infty$, then $\ell(\mu) \leqslant L(1) < \infty$.

The *entropy criterion* (Derriennic, Kaimanovich-Vershik [8,21]) states that if μ is a measure of finite entropy, then the boundary of (G,μ) is trivial if and only the entropy of the random walk $h(\mu)$ is zero. If $\ell(\mu) = 0$ then $h(\mu) = 0$. For symmetric measures, the converse is true: a symmetric measure μ of finite first moment has zero drift ($\ell(\mu) = 0$) if and only the entropy of the random walk (G,μ) is zero ($h(\mu) = 0$), see Karlsson-Ledrappier [24].

We say that a measure μ is *non-degenerate* if its support generates G. If the boundary of μ is trivial, then the group generated by the support of μ is amenable. Every amenable group admits a non-degenerate measure with trivial boundary (Kaimanovich-Vershik, Rosenblatt [21,27]); this measure can be chosen symmetric and with support equal to G.

2.2. **Random walks on permutational wreath products.** We consider groups A, G, such that G acts on a set X from the right. The *(permutational) wreath product* $W = A \wr_X G$ is the semidirect product of $\sum_X A$ with G. The *support* $\operatorname{supp}(f)$ of a function $f : X \to A$ consists in those $x \in X$ such that $f(x) \neq 1$. Elements of $\sum_X A$ can be viewed as finitely supported functions $X \to A$. The left action of G on $\sum_X A$ is then defined by $(g \cdot f)(x) = f(xg)$; observe that for g_1, g_2 in G

$$(g_1g_2 \cdot f)(y) = f(y(g_1g_2)) = f((yg_1)g_2) = (g_2 \cdot f)(yg_1) = (g_1 \cdot g_2 \cdot f)(y).$$

We have in particular $\operatorname{supp}(g^{-1} \cdot f) = \operatorname{supp}(f)g$.

If A and G are finitely generated and if the action of G on X is transitive, then the permutational wreath product is a finitely generated group. Indeed, fix finite generating sets S_A and S_G of A and G respectively, and fix a basepoint $\rho \in X$. The wreath product is generated by $S = S_A \cup S_G$. Here and in the sequel we identify G with its image under the imbedding $g \to (1,g)$ and identify A with its image under the imbedding $a \to (f_a, 1)$, where $f_a : X \to A$ is defined by $f_a(\rho) = a$ and $f(x) = 1$ for all $x \neq \rho$. We call S the *standard generating set* of W defined by S_A, S_G. Analogously, if the action of G on X has finitely many orbits, then W is finitely generated by $S_G \cup (S_A \times X/G)$. If the action has infinitely many orbits, then the permutational wreath product is not finitely generated.

The Cayley graph of the permutational wreath product with respect to the standard generating set S can be described as follows. Elements of $W = \sum_X A \rtimes G$ are written fg with $f \in \sum_X A$ and $g \in G$; multiplication is given by $(f_1g_1)(f_2g_2) = f_1(g_1 \cdot f_2)g_1g_2$.

Consider a word $v = s_1s_2\dots s_\ell$, with all $s_i \in S$, and write its value in W as f_vg_v. Set $u = f_ug_u = s_1s_2\dots s_{\ell-1}$. Here g_u, g_v belong to G, and f_u, f_v belong to $\sum_X A$.

We consider two cases, depending on whether $s_\ell \in S_A$ or $S_\ell \in S_G$. If $s_\ell \in S_A$, we have an edge of "A" type from u to v. The multiplication formula gives $g_v = g_u$ and $f_v(x) = f_u(x)$ for all $x \neq \rho g_u^{-1}$, while $f_v(\rho g_u^{-1}) = f_u(\rho g_u^{-1})s_\ell$.

If $s_\ell \in S_G$, we have an edge of "G" type from u to v. In that case, $f_v = f_u$, and $g_v = g_u s_\ell$.

We have begun to study asymptotic properties of permutational wreath products in [5]. The asymptotic geometry of these groups turns out to be much richer than in the particular case of ordinary wreath products (namely, for which $X = G$). It is easy to see that the word growth of $A \wr G$ is exponential whenever $X = G$ is infinite and A is non-trivial. However, among permutational wreath products there are groups of intermediate growth, see [5].

Given a probability measure μ on $W = A \wr_X G$, we say that the random walk is *translate-or-switch* if the support of μ belongs to the union of G and A; in other words, $\mu = p\mu_A + q\mu_G$, where $p, q \geqslant 0$, $p + q = 1$, the support of μ_A belongs to A, and the support of μ_G belongs to G.

We say that the random walk is *switch-translate-switch* if $\mu = \mu_A * \mu_G * \mu_A$, for measures μ_A, μ_G supported on A and G respectively. If $X = G$, the "switch-translate-switch" random walks are called "switch-walk-switch". In this case, we can view each step of the random walk as follows: we "switch" the value of the configuration at the point where the random walker stands, then make one step of the random walk on G and then switch the configuration at the point of the arrival. Note however that no such interpretation is valid for a general permutational wreath product, because translation and movement are in general genuinely different operations.

Let $w = g_1 \dots g_n$ be a word over G of length $n = |w|$, and let $\rho \in X$ be a base point. The *inverted orbit* of w is the set $\{\rho, \rho g_n, \rho g_{n-1} g_n, \dots, \rho g_1 \dots g_n\}$; the *inverted orbit growth* is the cardinality $\delta_\rho(w)$ of that set. In the sequel, ρ will be fixed, and we will usually omit it from the notation.

If w is a word corresponding to a length-n trajectory of a random walk, then we can consider $\delta_\rho(w)$ as a random variable with values in $\{1, \dots, n+1\}$.

3. Criteria for non-triviality of the boundary

We characterize in this section groups with a non-trivial Poisson-Furstenberg boundary, with the goal of applying it to permutational wreath products. For ordinary wreath products, a well-known criterion by Kaimanovich and Vershik [21, Proposition 6.4] states that, for $A \neq 1$ finite and finitely supported measures, $A \wr G$ has trivial boundary if and only if the projection of the random walk to G is recurrent. We extend this criterion to permutational wreath products.

Lemma 3.1. *Let the group G act on X and let μ be a probability measure on G. Let $\rho \in X$ be a basepoint. Then the induced random walk on X starting at ρ is recurrent if and only if the expectancy of the inverted orbit growth $\mathbf{E}[\delta_\rho(w)]$ is sublinear in $|w|$.*

Proof. Consider the random variable $A_{i,n}$ which equals 1 if the i-th point on the inverted orbit of the trajectory of the random walk is distinct from any points from 1 to $i-1$, and equals 0 otherwise. Consider a word $w = g_1 \dots g_n$. Since G acts by permutations on X, we have

$$\begin{aligned} w \in A_{i,n} &\Leftrightarrow \rho g_i \dots g_n \notin \{\rho g_{i-1} \dots g_n, \rho g_{i-2} \dots g_n, \dots, \rho g_1 \dots g_n\} \\ &\Leftrightarrow \rho \notin \{\rho g_{i-1}, \rho g_{i-2} g_{i-1}, \dots, \rho g_1 \dots g_{i-1}\} \\ &\Leftrightarrow \rho \notin \{\rho g_{i-1}^{-1}, \rho g_{i-1}^{-1} g_{i-2}^{-1}, \dots, \rho g_{i-1}^{-1} g_{i-2}^{-1} \dots g_1^{-1}\}. \end{aligned}$$

Therefore,

$$\mathbf{E}[A_i] = \mathbf{P}[\rho \neq \rho g_{i-1}^{-1} \text{ and } \rho \neq \rho g_{i-1}^{-1} g_{i-2}^{-1} \dots \text{ and } \rho \neq \rho g_{i-1}^{-1} g_{i-2}^{-1} \dots g_1^{-1}];$$

observe that this is the probability p_i that the random walk on X induced by $(G, \check{\mu})$ with $\check{\mu}(g) = \mu(g^{-1})$, starting from ρ, never returns to this base point ρ during time moments between 1 and $i-1$. Note that for each i we have $p_i \geqslant p_{i+1}$. If the random walk on X induced by $(G, \check{\mu})$ is recurrent, then $p_i \to 0$, while if the random walk on X induced by$(G, \check{\mu})$ is transient, then there exists $p > 0$ such that $p_i \geqslant p$ for all i.

Next, observe that the expectation of $\delta(w)$ is

$$\mathbf{E}[\delta(w)] = \mathbf{E}[A_{1,n} + A_{2,n} + \dots + A_{n,n}] = \mathbf{E}[A_{1,n}] + \mathbf{E}[A_{2,n}] + \dots + \mathbf{E}[A_{n,n}].$$

Therefore, the expectation of $\delta(w)$ grows linearly (at least pn) if the random walk is transient, while $\delta(w)/|w|$ tends to zero if the random walk is recurrent.

Finally, observe that $(G, \check{\mu})$ induces a transient random walk on X if and only if the stabilizer G_ρ of $\rho \in X$ is a transient set for the random walk $(G, \check{\mu})$; because $G_\rho = G_\rho^{-1}$, this happens if and only if (G, μ) induces a transient random walk on X. □

We continue now with two propositions, each giving a sufficient condition for non-triviality of the boundary. One uses finiteness of the entropy of μ, and the other a weak restriction on the support of μ. The proof of Lemma 3.2 appears further below.

Lemma 3.2. *Let μ be a non-degenerate measure on W with finite entropy $H(\mu)$, and suppose that the inverted orbits of the random walk defined by the projection of μ to G grow linearly. Then then entropy $h(\mu)$ of the random walk (W, μ) is positive; so the Poisson-Furstenberg boundary of (G, μ) is non-trivial.*

Combining Lemma 3.1 and Lemma 3.2, we get the following

Proposition 3.3. *Let μ be a non-degenerate random walk on W with finite entropy and transient projection to X. Then the Poisson-Furstenberg boundary of the random walk (G, μ) is non-trivial.*

In order to prove Lemma 3.2, we will use the following elementary combinatorial lemma; we include its proof for the reader's convenience.

Lemma 3.4. *Let $k \leqslant n$ elements $i_1, \dots, i_k$ be chosen uniformly independently in $\{1, \dots, n\}$. Then*

$$\mathbf{E}[\#\{i_1, \dots, i_k\}] \geqslant (1 - e^{-1})k.$$

Proof. Let the cardinality of $\{i_1, \dots, i_k\}$ be ℓ. This happens when ℓ ordered numbers in $\{1, \dots, n\}$ are chosen, and $\{1, \dots, k\}$ is partitioned in ℓ parts, each part corresponding to a subset of the $i_1, \dots, i_k$ taking the same value. This occurs in $\binom{n}{\ell}\ell!\left\{{k \atop \ell}\right\}$ manners; here and below $\left\{{k \atop \ell}\right\}$ denotes the Stirling number of the second kind. The expected size of $\{i_1, \dots, i_\ell\}$ is then

$$\mathbf{E}[\#\{i_1, \dots, i_k\}] = \sum_{\ell=0}^{n} \ell \binom{n}{\ell} \ell! \left\{{k \atop \ell}\right\} n^{-k} =: E.$$

We now use the classical formula $\ell!\left\{{k \atop \ell}\right\} = \sum_{j=0}^{\ell}(-1)^{\ell-j}\binom{\ell}{j}j^k$, to obtain

$$E = \sum_{\ell=0}^{n}\ell\binom{n}{\ell}\sum_{j=0}^{\ell}(-1)^{\ell-j}\binom{\ell}{j}(j/n)^k.$$

We rewrite this sum as

$$\sum_{\ell,j}\ell\binom{n}{j}(-1)^{\ell-j}\binom{n-j}{n-\ell}(j/n)^k,$$

and set $m = n-\ell$ to obtain

$$E = n - \sum_{j}\binom{n}{j}(-1)^{n-j}(j/n)^k\sum_{m}(-1)^m m\binom{n-j}{m};$$

now $\sum_m(-1)^m m\binom{n-j}{m} = 0$ unless $n-j=1$, in which case the sum equals -1; so

$$E = n - \binom{n}{n-1}((n-1)/n)^k = n(1-(1-1/n)^k) \geqslant n(1-e^{-k/n});$$

since $1-e^{-k/n} \geqslant (1-e^{-1})k/n$, we have proved the lemma. □

Proof of Lemma 3.2. The argument is similar to that of [11, Theorem 3.1].

Let $\rho \in X$ be any basepoint. First, there exists $N \in \mathbb{N}$ and $f \neq f' \in \sum_X A$ both supported on $\{\rho\}$ such that $f, f' \in \operatorname{supp}(\mu^{*N})$, because μ is non-degenerate. Recall that we identify A with those functions $f \in \sum_X A$ that are supported on $\{\rho\}$. Since $h(\mu^{*N}) = Nh(\mu)$, it suffices to prove $h(\mu^{*N}) > 0$; up to replacing μ^{*N} by μ, we suppose, from now on, that there are at least two elements in $A \cap \operatorname{supp}(\mu)$; in particular, $\mu(A) > 0$. If at time instant n the increment of the random walk belongs to A, we say that at this instant there is an 'A' *step* of the random walk. Define the normalized measure $\nu : A \to \mathbb{R}$ by $\nu(a) = \mu(a)/\mu(A)$; by assumption, $H(\nu) > 0$.

Let $\overline{\mu}$ denote the image of μ on G. Observe that, for every $n \in \mathbb{N}$ large enough, for every $\epsilon > 0$, and for every $d < \liminf_{n\to\infty}\mathbf{E}[\delta(w)/n \mid n = |w|]$, the inverted orbit at ρ of the projected random walk $(X, \overline{\mu})$ visits at least dn different values with probability $> 1-\epsilon$.

Observe also that, for every $n \in \mathbb{N}$ large enough, for every $\epsilon > 0$, and for every $q < \mu(A)$, the random walk (W, μ) does an 'A' step at least qn times with probability $> 1-\epsilon$.

Therefore, with probability at least $1-2\epsilon$, the above two conditions hold. Consider a random trajectory $w = w_1w_2\cdots w_n \in W^n$, with steps w_i, satisfying both conditions, and let $w_{i_1}, \dots, w_{i_s}$ be the 'A' steps in w, with $s \geqslant qn$. Let $\overline{w}$ denote the trajectory obtained by removing the letters $w_{i_1}, \dots, w_{i_s}$ from w.

We compute the conditional entropy $H(\mu^{*n} \mid \overline{w})$, namely the entropy of the restriction of μ^{*n} to those trajectories $w' \in W^n$ with $\overline{w'} = \overline{w}$. If the word $\overline{w}$ is known, then the possible w's giving that $\overline{w}$ are obtained by inserting arbitrarily $s = n - |\overline{w}|$ elements $w_{i_1}, \dots, w_{i_s} \in A$ into the word w. Let $x_1, \dots, x_t \in X$ be the different points on the inverted orbit of $\overline{w}$; recall $t \geqslant dn$. Each of the insertions of $w_{i_1}, \dots, w_{i_s}$ will modify the evaluation of w at some coördinate $\in \{x_1, \dots, x_t\}$, and these s insertions are independent and controlled by the normalized measure ν.

By Lemma 3.4, there is a constant $r > 0$ such that the expected number c of coördinates in $\{x_1, \dots, x_t\}$ that are modified is at least rn; so

$$H(\mu^{*n} \mid \overline{w}) \geqslant \mathbf{E}[c]H(\nu) \geqslant rnH(\nu);$$

we have $H(\mu^{*n}) \geqslant H(\mu^{*n} \mid \overline{w})$, because entropy is not less than the mean conditional entropy, see e.g. [26, §5], so

$$h(\mu) = \lim \frac{1}{n} H(\mu^{*n}) \geqslant (1-2\epsilon) \lim \frac{1}{n} H(\mu^{*n} \mid \overline{w}) \geqslant (1-2\epsilon) r H(\nu) > 0. \qquad \square$$

We consider now a different sufficient condition for non-triviality of the boundary, requiring only a very weak form of non-degeneracy of the random walk:

Proposition 3.5. *Let μ be "switch-translate-switch" random walk on W, and assume that there exist $n \in \mathbb{N}$ and two elements in the support of μ^{*n} with equal projection to G. Assume also that the projected random walk $(X, \overline{\mu})$ is transient.*

Then the Poisson-Furstenberg boundary of the random walk (W, μ) is non-trivial.

Note, in particular, that the first condition holds as soon as μ is non-degenerate and $A \neq 1$. Note also that the 'translation' part of μ is allowed to be infinitely supported on X. Indeed, we will later apply Proposition 3.5 to infinitely supported measures.

Proof. Consider a trajectory $\Theta = (1, f_1 g_1, f_2 g_2, \dots)$ of the random walk on W, with $f_i \in \sum_X A$ and $g_i \in G$, and $(f_i g_i)^{-1} f_{i+1} g_{i+1} \sim \mu$.

By assumption, there exists $n \in \mathbb{N}$ and $u, v \in \sum_X A$, $g \in G$ such that two elements $ug \neq vg \in W$ are reached with positive probability at time n of the walk. Choose a coördinate $\sigma \in X$ in which u and v differ, say $v(\sigma) = au(\sigma)$ for some $a \neq 1$ in A.

Consider $f_i(\sigma)$. Since the random walk we consider is "switch-translate-switch", for all i the elements f_i and f_{i+1} differ in at most two places. More precisely, $f_i(\sigma) \neq f_{i+1}(\sigma)$ only when $\sigma \in \{\rho g_i^{-1}, \rho g_{i+1}^{-1}\}$.

Since $\overline{\mu}$ is transient, there is a bound $R \in \mathbb{N}$ such that, almost surely, we have $\sigma \in \{\rho g_i^{-1}, \rho g_{i+1}^{-1}\}$ in at most R instants i. It follows that

$$\phi_\sigma(\Theta) := \lim_{i \to \infty} f_i(\sigma)$$

almost surely exists, and defines a measurable function on the space of trajectories.

For any $\epsilon > 0$, at least $1 - \epsilon$ of the mass of μ is concentrated on a finite set $W_\epsilon \subset W$; there exists a finite subset $A_\epsilon \subseteq A$ such that $f \in \sum_X A_\epsilon$ whenever $fg \in W_\epsilon$; so, with probability $1 - R\epsilon$, the limit ϕ_σ belongs to the finite set A_ϵ^R. Take now ϵ small enough so that $R\epsilon < \mu^{*n}(ug)$. Then there exists $b \in A_\epsilon^R$ such that with positive probability the trajectory Θ visits ug at time n and satisfies $\phi_\sigma(\Theta) = b$.

For each such trajectory, replace the initial n steps (reaching ug) with an n-step random walk reaching vg. This produces a positive-measure set of trajectories that visit vg at time n and satisfy $\phi_\sigma(\Theta) = v(\sigma)u(\sigma)^{-1}b = ab \neq b$. Therefore, ϕ_σ is not constant.

We have exhibited a non-constant continuous function on the space of exits of the random walk, so its boundary is not trivial. $\square$

Alternatively, as a replacement for the last three paragraphs of the proof, let the element $a = v(\sigma)u(\sigma)^{-1} \in A$ have order $m \in \mathbb{N} \cup \{\infty\}$. Let T be a right transversal of $\langle a \rangle$ in A; that is, $A = T \sqcup aT \sqcup \cdots$. If $m = \infty$, set $A_0 = \bigsqcup_{n \in \mathbb{Z}} a^{2n} T$ and

$A_1 = \bigsqcup_{n\in\mathbb{Z}} a^{2n+1}T$, while if $m < \infty$, set $A_n = a^nT$ for all $n \in \{0, \dots, m-1\}$. Then the function $\chi : \Theta \mapsto (n \text{ if } \phi_\sigma(\Theta) \in A_n)$ is measurable, takes finitely many values, and takes value n with positive probability if and only if it takes value $n-1$ (mod m) or 2 with positive probability, so is not constant.

4. Groups admitting finite first moment measures with non-trivial Poisson-Furstenberg boundary

Our aim, in this section, is to prove that most wreath products of the form $W = A \wr_{X_1\times X_2} (G_1 \times G_2)$ have a non-trivial boundary for an appropriate measure. This measure will be an infinite convex combination of the convolutions powers of some symmetric finitely supported measure on W. Our main task is to chose the coëfficients in the convex combinations in such a way that they decay not to fast; on the other hand, they must decay fast enough that the measure we construct has finite first moment.

To show that the measure has non-trivial boundary, we use the results of the previous section (Propositions 3.3 and 3.5). At the end of this section, we give applications of Theorem B, and construct groups of exponential growth that we will later use to prove Theorem A.

Theorem 4.1 (= Theorem B)**.** *Let G be a group acting on a set X, and let μ be a finitely supported, symmetric, non-degenerate measure on G. Suppose that its drift satisfies $L_{\mu_i}(n) \leqslant Dn^\alpha$ for all $n \in \mathbb{N}$, for some constants D and $\alpha < 1$. Suppose also that, for every $\rho \in X$, the probability of return to ρ satisfies $\mu^{*n}(\mathrm{stab}_G(\rho)) \leqslant C/n^\delta$ for all $n \in \mathbb{N}$, for some constants C and $\delta > \alpha$. Let A be a non-trivial group.*

Then $W := A \wr_X G$ admits a symmetric measure with finite first moment and non-trivial Poisson-Furstenberg boundary. In particular, the word growth of W is exponential.

The idea of the proof is to construct a measure μ, with finite first moment, such that the induced random walk on X is transient; and then to use Proposition 3.3 or Proposition 3.5 to conclude that the boundary of the random walk (W, μ) is non-trivial.

4.1. Reminder: properties of Stable Laws.

We start by recalling classical results on stable laws from [19], which we restrict to measures on $\mathbb{R}_+$. A measure μ on $\mathbb{R}_+$ is *stable* if for any $a_1, a_2 > 0$ there are $a > 0, b$ such that $\mu(a_1 \cdot x) * \mu(a_2 \cdot x) = \mu(a \cdot x + b)$; in particular, if μ is the law of independent random variables X_1, X_2, then the law of $X_1 + X_2$ is an affine transformation of μ.

Let $X_1, X_2, \dots$ be independent random variables with law μ'. We say μ' is in the *domain of attraction* of a non-degenerate stable law μ if there are constants A_n, B_n such that the law μ_n of $(X_1 + \cdots + X_n - A_n)/B_n$ converges weakly to μ; namely, if $\mu_n(M) \to \mu(M)$ for all Borel subsets $M \subseteq \mathbb{R}$ whose boundary is μ-negligible.

The *distribution* of a measure μ on $\mathbb{R}_+$ is the function $F(x) = \mu([0, x])$. Attraction towards a stable law can be checked by estimating the regularity of the tails of the distribution:

Theorem 4.2 (Part of [19, Theorem 2.6.1])**.** *A measure belongs to the domain of attraction of a stable law if and only if its distribution F satisfies*

$$F(x) = 1 - \frac{h(x)}{x^\alpha} \text{ as } x \to \infty,$$

for a function h that varies slowly in the sense of Karamata (meaning $h(tx)/h(x) \to 1$ for all $t > 0$) and a parameter $\alpha \in (0,2)$. This parameter is called the exponent *of the measure.* □

We will also use a local limit theorem due to Gnedenko: again, we only quote a subcase of the general result. A stable measure is continuous, even if it is the limit of discrete measures. A continuous measure μ has a *density* g if $\mu(M) = \int_M g(x)dx$ for all measurable $M \subset \mathbb{R}$. All stable measures have a density, which furthermore may be supposed to be continuous, and therefore bounded (because summable); indeed, if μ is stable of exponent α then there exist a real function b and a constant c such that

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(ib(t) - |ct|^\alpha)dt \leqslant \frac{2}{2\pi}\int_0^\infty \exp(-|c|^\alpha t^\alpha)dt,$$

and g is continuous as the absolutely convergent integral of a continuous function. Note that, in general, g does not admit an analytic description.

Theorem 4.3 (Part of [19, Theorem 4.2.1])**.** *Suppose that μ' is supported on $\mathbb{N}$, but not on $h\mathbb{N}$ for any $h > 1$, and suppose that μ' is in the domain of attraction of a stable law with density g. Then*

$$\lim_{n\to\infty} \sup_{k\in\mathbb{N}} \left|B_n(\mu')^{*n}(k) - g((k - A_n)/B_n)\right| = 0.$$

4.2. **Proof of Theorem 4.1.**

Proof. Consider a parameter $\gamma \in (1,2)$, to be fixed later. For $i \in \{1,2,\dots\}$, set $\alpha_i = C_\gamma/i^\gamma$ for a constant C_γ defined such that $\sum_{i=1}^\infty \alpha_i = C_\gamma \sum_{i=1}^\infty i^{-\gamma} = 1$. Define measures ν_γ on $\mathbb{N}$ and λ_γ on G by

$$\nu_\gamma(i) = \alpha_i, \qquad \lambda_\gamma = \sum_{i=1}^\infty \alpha_i \mu^{*i}.$$

By the definition of C_γ, both ν_γ and λ_γ are probability measures. The following estimate on negative moments of ν_γ^{*n} will be needed later:

Lemma 4.4. *For all $\delta > 0$ there exists a constant C such that, for all $n \geqslant 2$,*

$$\sum_{i=1}^\infty \nu_\gamma^{*n}(i)/i^\delta \leqslant \begin{cases} C/n^{1/(\gamma-1)}\log(n) & \text{if } \delta = 1,\\ C/n^{\delta/(\gamma-1)} & \text{if } \delta \neq 1.\end{cases}$$

Proof. We first show that ν_γ is in the domain of attraction of a stable law, by checking the hypotheses of Theorem 4.2. Its distribution satisfies $1 - F(x) = \sum_{x<i\in\mathbb{N}} C_\gamma i^{-\gamma}$, so

$$\frac{C_\gamma}{\gamma-1}x^{1-\gamma} = \int_x^\infty C_\gamma t^{-\gamma}dt \leqslant 1 - F(x) \leqslant \int_{x+1}^\infty C_\gamma t^{-\gamma}dt = \frac{C_\gamma}{\gamma-1}(x+1)^{1-\gamma},$$

so $C_\gamma/(\gamma-1) \leqslant h(x) \leqslant C_\gamma/(\gamma-1)(1+1/x)^{1-\gamma}$ and h is slowly varying. Therefore, ν_γ is in the domain of attraction of a stable law of exponent $\alpha = \gamma - 1$.

It then follows from [19, Theorem 2.1.1] that $B_n = n^{1/\alpha}h(n)$ for another function h that slowly varies in the sense of Karamata.

Let g be the density of the stable law towards which ν_γ converges. By Theorem 4.3, $\sup_k B_n(\nu_\gamma)^{*n}(k) - g((k - A_n)/B_n)$ converges to 0 as $n \to \infty$, and g is bounded, so $\sup_k B_n\nu_\gamma^{*n}(k)$ is bounded. Therefore, there exists a constant C' such that $(\nu_\gamma)^{*n}(k) \leqslant C'n^{-1/(\gamma-1)}$ for all $k \in \mathbb{N}$.

We are now ready to prove the lemma. Set $a_n = n^{1/(\gamma-1)}$, and split the sum as

$$\sum_{i=1}^{\infty} \nu_\gamma^{*n}(i)/i^\delta = \sum_{i=1}^{a_n} \nu_\gamma^{*n}(i)/i^\delta + \sum_{i=a_n+1}^{\infty} \nu_\gamma^{*n}(i)/i^\delta.$$

In the first summand, we use $\nu_\gamma^{*n}(i) \leqslant C'/n^{1/(\gamma-1)}$ for all i, so

$$\begin{aligned}\sum_{i=1}^{a_n} \nu_\gamma^{*n}(i)/i^\delta &\leqslant C'/n^{1/(\gamma-1)} \sum_{i=1}^{a_n} 1/i^\delta \\ &\leqslant C'/n^{1/(\gamma-1)} \begin{cases} \log a_n \leqslant C''/n^{1/(\gamma-1)} \log(n) & \text{if } \delta = 1, \\ a_n^{1-\delta}/(1-\delta) \leqslant C''/n^{\delta/(\gamma-1)} & \text{if } \delta \neq 1, \end{cases}\end{aligned}$$

for some constant C''. For the second summand, we use the coarse estimate

$$\sum_{i=a_n+1}^{\infty} \nu_\gamma^{*n}(i)/i^\delta \leqslant 1/a_n^\delta \sum_{i=a_n+1}^{\infty} \nu_\gamma^{*n}(i) \leqslant 1/a_n^\delta = 1/n^{\delta/(\gamma-1)}$$

and we are done, setting $C = C'' + 1$. □

Let us next find out for which γ the random walk on X defined by λ_γ is transient. The argument is close to that of [12, Lemma 3.1]. In that lemma, it was shown that for any transitive action of G on an infinite set X, the measures λ_γ define a transient random walk on X as soon as $\gamma \in (1, 3/2)$. For the proof of Theorem 4.1, however, it is not sufficient to work with γ between 1 and 3/2, because the theorem's assumptions do not imply that λ_γ has finite first moment for some $\gamma < 3/2$. Indeed, we will use in an essential manner the additional assumption on the action to weaken the condition on γ.

Proposition 4.5. *Suppose that the probability of return to the origin $\rho \in X$ for the random walk on X induced by the measure μ satisfies $\mu^{*n}(\mathrm{stab}_G(\rho)) \leqslant C/n^\delta$ for some $\delta > 0$. Then, for all $\gamma \in (1, 1+\delta)$, the random walk (λ_γ, X) is transient.*

Proof. For any $H \subset G$, we have

$$\lambda_\gamma^{*n}(H) = \sum_{i \geqslant 0} \nu_\gamma^{*n}(i) \mu^{*i}(H).$$

In particular, this holds with H the stabilizer of $\rho \in X$:

$$\lambda_\gamma^{*n}(\mathrm{stab}_G(\rho)) = \sum_{i \geqslant 1} \nu_\gamma^{*n}(i) \mu^{*i}(\mathrm{stab}_G(\rho)).$$

By Lemma 4.4 we know that for any δ

$$\sum_{i=1}^{\infty} \nu_\gamma^{*n}(i)/i^\delta \leqslant C/n^{\delta/(\gamma-1)} \log(n).$$

Therefore, for all $n \geqslant 2$ we have

$$\begin{aligned}\lambda_\gamma^{*n}(\mathrm{stab}_G(\rho)) &= \sum_{i=1}^{\infty} \nu_\gamma^{*n}(i) \mu^{*i}(\mathrm{stab}_G(\rho)) \\ &\leqslant \sum_{i=1}^{\infty} \nu_\gamma^{*n}(i) C/i^\delta \leqslant C/n^{\delta/(\gamma-1)} \log(n).\end{aligned}$$

Since $\gamma < 1 + \delta$, we have $\delta/(\gamma - 1) > 1$, so

$$\sum_{n=0}^{\infty} \lambda_\gamma^{*n}(\mathrm{stab}_G(\rho)) < \infty.$$

This means that $\mathrm{stab}_G(\rho)$ is a transient subgroup for the measure λ_γ, or, in other words, that the random walk on X induced by the measure λ_γ is transient. □

We can now finish the proof Theorem 4.1. It is time to use the assumptions on the first moments of μ_1, μ_2.

Lemma 4.6. *For $\gamma > 1 + \alpha$, the first moment of the measure λ_γ is finite.*

Proof. Recall that $L_\mu(i)$ denotes the first moment of the measure μ^{*i}. By our assumptions, there exists a constant D such that $L_\mu(i) < Di^\alpha$ for all $i \in \mathbb{N}$. The first moment of λ_γ is therefore equal to

$$\sum_{i=1}^{\infty} \nu_\gamma(i) L_\mu(i) \leqslant CD \sum_{i=1}^{\infty} C_\gamma i^{-\gamma} i^\alpha = CDC_\gamma \sum_{i=1}^{\infty} i^{-(\gamma-\alpha)} < \infty,$$

if $\gamma - \alpha > 1$. □

Now fix $\gamma \in (1+\alpha, 1+\delta)$. By Proposition 4.5, the random walk λ_γ is transient, while by Lemma 4.6 the first moment of λ_γ is finite.

Take a measure μ_A on A with finite first moment, whose support contains 1 and generates A. Set $\lambda = \mu_A * \lambda_\gamma * \mu_A$. Observe that λ is a non-degenerate random walk with finite first moment, and that the induced random walk on X is transient. Therefore, by Proposition 3.3, the boundary of (W, λ) is non-trivial. This completes the proof of Theorem 4.1.

Alternatively, note that λ is a "switch-translate-switch" random walk, so that Proposition 3.5 applies. □

4.3. **Consequences of Theorem 4.1.**

Corollary 4.7. *Let $\alpha < 1$ be given; for each $i = 1, 2$, let G_i act transitively on an infinite set X_i, and let μ_i be a finitely supported, symmetric, non-degenerate, whose drift satisfies $L_{\mu_i}(n) \leqslant Dn^\alpha$ for a constant D and all $n \in \mathbb{N}$.*

Set $G = G_1 \times G_2$ and $X = X_1 \times X_2$, on which G acts coördinatewise. Let A be a non-trivial group.

Then $W := A \wr_X G$ admits a symmetric measure with finite first moment and non-trivial Poisson-Furstenberg boundary. In particular, the word growth of W is exponential.

Proof. Set $\mu = \mu_1 \times \mu_2$; it is the random walk on X that walks independently on X_1 and X_2. Choose a basepoint $\rho = (\rho_1, \rho_2) \in X$. Observe $\mathrm{stab}_G(\rho) \cap G_1 = \mathrm{stab}_{G_1}(\rho_1)$ and $\mathrm{stab}_G(\rho) \cap G_2 = \mathrm{stab}_{G_2}(\rho_2)$. For all $n \geqslant 0$ we have $\mu^{*n}(\mathrm{stab}_G(\rho)) = \mu_1^{*n}(\mathrm{stab}_{G_1}(\rho_1))\mu_2^{*n}(\mathrm{stab}_{G_2}(\rho_2))$.

We say that a symmetric random walk on a connected locally finite graph is a *nearest neighbour* random walk, if it is a symmetric random walk which walks along the edges of the graph with probability bounded away from zero: $p_1(x,y) = p_1(y,x)$, $p_1(x,y) > 0$ implies x, y is an edge, and there exists $p > 0$ such that $p_1(x,y) \geqslant p$ whenever x and y are joined by an edge.

For a nearest-neighbour symmetric random walk on a connected infinite locally finite graph, the n-step transition probabilities satisfy $p_n(x,y) \leqslant C'/\sqrt{n}$ for some

$C' > 0$ and all $n \geqslant 1$, and, in particular, the probability to return to the origin satisfies $p_n(x,x) \leqslant C'/\sqrt{n}$ for all x and all $n \geqslant 1$; see Woess [28, Corollary 14.6].

This implies $\mu_1^{*n}(\mathrm{stab}_{G_1}(\rho_1)) \leqslant C_1/\sqrt{n}$ and $\mu_2^{*n}(\mathrm{stab}_{G_2}(\rho_2)) \leqslant C_2/\sqrt{n}$ for some constant C_1, C_2 depending on X_1, X_2 and all $n \geqslant 1$. Therefore, $\mu^{*n}(\mathrm{stab}_G(\rho)) \leqslant C/n$, for $C = C_1C_2$ and all $n \geqslant 1$.

Consider $S = S_1 \cup S_2$. Clearly, S is a generating set of $G = G_1 \times G_2$ whenever S_1 and S_2 are generating sets of G_1 and G_2 respectively, and $\|(g_1,g_2)\|_S = \|g_1\|_{S_1} + \|g_2\|_{S_2}$ for all $g_1 \in G_1$, $g_2 \in G_2$. For all $n \geqslant 0$, we have $L_{\mu,G,S}(n) = L_{\mu_1,G_1,S_1}(n) + L_{\mu_2,G_2,S_2}(n)$; so, by the assumptions of the corollary, $L_\mu(n) = L_{\mu,G,S}(n) \leqslant Cn^\alpha$.

We may therefore apply Theorem 4.1 with $\delta = 1$. □

Groups satisfying the assumption of the theorem admit a symmetric measure of finite first moment whose boundary is non-trivial. However, there are groups of exponential growth, such as for example wreath products of a finite group with $\mathbb{Z}$, on which any symmetric finite first moment measure has trivial boundary.

Remark 4.8. The assumption that X is a direct product is important, and is used to bound from above the return probabilities to the origin. There are examples of wreath products with infinite X, such as the group $A \wr_{X_1} G_{012}$ studied in [5], that have intermediate word growth and therefore trivial boundary for all measures of finite first moment.

Example 4.9. Consider $G = G_1 \times G_2 \times G_3$ and $X = X_1 \times X_2 \times X_3$ with all X_i infinite, transitive G_i-spaces. Then all permutational wreath products $A \wr_X G$ have exponential word growth, without any assumption on the μ_i. Indeed, all simple random walks on these groups have a non-trivial boundary, as follows from Proposition 3.3.

Remark 4.10. Let G be a group with word growth $v(n)$ at most $\exp(n^\beta)$ for some $\beta < 1$, and let μ be a finitely supported measure on G. Then $L_{G,\mu}(n) \leqslant Cn^{(1+\beta)/2}$.

Proof. For any symmetric finitely supported random walk on a group G, there exists $K > 0$ such that $L(n) \leqslant K\sqrt{n \log v(n) + \log(n)}$ for all n, see [10, Lemma 7.(ii)]. □

Example 4.11. Consider G_1 and G_2 both equal to the first Grigorchuk group G_{012}, and X_1 and X_2 some orbits for the action on the boundary of the rooted tree. Recall that G_{012} has subexponential word growth, and more precisely by [16] has growth at most $\exp(n^\beta)$ for some $\beta < 1$. The best known upper bound is $\beta = \log(2)/\log(2/\eta) \cong 0.7674$ with $\eta^3 + \eta^2 + \eta = 2$, see [2]. In view of Remark 4.10, the assumptions of Corollary 4.7 are satisfied for $\alpha = (1+\beta)/2$, so $A \wr_{X_1 \times X_2} (G_1 \times G_2)$ has exponential growth as soon as A is not trivial.

Among the Grigorchuk groups, there are groups with growth arbitrary close to exponential along a subsequence [17], and in particular not bounded from above by any function of the form $\exp(n^\alpha)$. We cannot use Remark 4.10 to estimate the drift of simple random walks on such groups. However, every Grigorchuk group admits a finitely supported random walk whose drift function is bounded from above by Cn^α for $\alpha = 3/4$, see Corollary 6.3.

5. A sufficient condition for triviality of the Poisson-Furstenberg boundary

It is well known that the triviality of the boundary of an ordinary wreath product of $H \wr G$ is related to the recurrence of G, see Kaimanovich and Vershik [21, Proposition 6.4]. However, their argument does not seem to provide information about the triviality of the boundary in the case of a permutational wreath $H \wr_X G$, in which the action of G on X is recurrent. Indeed, let $W = H \wr_X G$ be a permutational wreath product, let ρ be a point of X and let W' be the subgroup of W that projects to the stabilizer of ρ in G. Starting with a random walk on W which induces a recurrent random walk on X, we can claim (by renormalizing the random walk at stop times in the stabilizer of ρ) that the boundary of this random walk is equivalent to the boundary of some (in general, infinitely supported) random walk on W'; however, in contrast with the ordinary wreath product case the group W' may be large, even if H is small.

Another approach to criteria for triviality of the boundary in case of ordinary wreath products $H \wr G$, in which the induced random walk on G is recurrent, is to estimate the entropy of the random walk [9]. The proposition below is an analogue of such a criterion, but now in the case of permutational wreath products. The main difficulty of the proof of this proposition, which does not appear in the case of ordinary wreath products, is in the estimation of the number of choices of the inverted orbit, see Remark 5.3.

Proposition 5.1. *Let A, G be groups of subexponential word growth, and set $W := A \wr_X G$. Let μ be finitely supported probability measure on W.*

If the expected inverted orbit growth of the projected random walk $(X, \overline{\mu})$ grows sublinearly, then $h(\mu) = 0$; so the random walk on (W, μ) has trivial Poisson-Furstenberg boundary.

Lemma 5.2. *Let G be a group of subexponential word growth, and let $\delta : \mathbb{N} \to \mathbb{N}$ be a sublinear function. Then the function*

$$v^+(n) := \#\{(g_1, \dots, g_k) \in G^k \mid k \leqslant \delta(n), \|g_1\| + \dots + \|g_k\| \leqslant n\}$$

grows subexponentially.

Proof. Let $v(n)$ denote the growth function of G; then by hypothesis, for every $\epsilon > 0$, there exists C such that $\log v(n) \leqslant \epsilon n + C$. We then estimate

$$\begin{aligned} v^+(n) &= \sum_{0 \leqslant m \leqslant n} \sum_{0 \leqslant k \leqslant \delta(n)} \sum_{n_1 + \dots + n_k = m} v(n_1) \cdots v(n_k) \\ &\leqslant n\delta(n) \binom{n + \delta(n) - 1}{\delta(n)} \max_{n_1 + \dots + n_{\delta(n)} = n} v(n_1) \cdots v(n_{\delta(n)}). \end{aligned}$$

Let us show that the binomial coëfficient $\binom{n+\delta(n)-1}{\delta(n)}$ is subexponential when δ is sublinear. We use the following simple approximation for binomial coëfficients, which comes from Stirling's formula for $n!$:

$$\binom{n}{k} \approx \sqrt{\frac{2\pi n}{k(n-k)}} \left(\frac{k}{n}\right)^{-k} \left(\frac{n-k}{n}\right)^{k-n},$$

in the sense that the quotient tends to 1 as $n,k\to\infty$. In particular, if $k\leqslant n/2$ then $\frac{1}{n}\log\binom{n}{k}\leqslant 2\frac{-k}{n}\log(k/n)+\frac{-\log(2\pi)}{n}\log(k/n)\to 0$ as $k/n\to 0$. Therefore,

$$\lim_{n\to\infty}\frac{1}{n}\log v^+(n)\leqslant\lim_{n\to\infty}\frac{1}{n}\log\binom{n+\delta(n)-1}{\delta(n)}+\lim_{n\to\infty}\frac{1}{n}\big(\epsilon n+C\delta(n)\big)=\epsilon.$$

Since this holds for all $\epsilon>0$, we have $\lim_{n\to\infty}\frac{1}{n}\log v^+(n)=0$. □

Proof of Proposition 5.1. We will show, for every n, that with positive probability a length-n random walk lands in a subset of W of subexponential size in n.

Since μ is finitely supported, there exists a finite set $Y\subseteq X$ and a finite set $S\subset A$, which we may assume is generating, such that $\operatorname{supp}(\mu)\subseteq\sum_Y S\times G$; namely, the random walk modifies only positions in Y, and does at most a step in S at these positions. Let $\delta(n)$ be the expectation of the inverted orbit growth of (G,X), starting at all positions in Y. By assumption, δ grows sublinearly.

We restrict ourselves to length-n trajectories Ω_n whose inverted orbit visits less than $2\delta(n)$ points. These describe a subset of trajectories of measure at least 1/2: indeed, $\mathbf{E}[\delta(w)]=\delta(n)\leqslant(1-\mu(\Omega_n))2\delta(n)$ whence $\mu(\Omega_n)\geqslant\frac{1}{2}$.

Let $\boldsymbol{w}=w_1\dots w_n\in\Omega_n$ be a trajectory. Considering simultaneously all $y\in Y$, the inverted orbit of $\boldsymbol{w}$ visits (a subset of) $\mathcal{O}=\{yw_{i(1)}\cdots w_n,\dots,yw_{i(k)}\cdots w_n\colon y\in Y\}\subseteq X$, say for definiteness at lexicographically minimal times $i(1),\dots,i(k)$; and $k\leqslant 2\delta(n)$. This inverted orbit is determined by the sequence of group elements

$$(w_{i(1)}\cdots w_{i(2)-1},w_{i(2)}\cdots w_{i(3)-1},\dots,w_{i(k)}\cdots w_n)\in G^k.$$

By Lemma 5.2, there is a subexponential number $v^+(n)$ of possibilities for $\mathcal{O}$ that may occur.

Once a subset $\mathcal{O}$ of X is chosen, let us consider the endpoint $\boldsymbol{w}=(f,g)$ of the trajectory, with $g\in G$ and $f\in\sum_X A$. The support of f is contained in $\mathcal{O}$, and the random walk did a total of at most $|Y|n$ steps at positions in $\mathcal{O}$. Let $u(n)$ denote the growth function of A, by assumption subexponential. Assume the random walk did n_x steps at each $x\in\mathcal{O}$, with $\sum_{x\in\mathcal{O}}n_x\leqslant|Y|n$. Then $f\in\prod_{x\in\mathcal{O}}B_A(n_x)$, which is a subset of $\sum_X A$ of subexponential growth, again by Lemma 5.2.

Since a product of subexponential functions is again subexponential, $\boldsymbol{w}$ belongs to a set of subexponential growth, when $\mathcal{O}$ ranges over all possible inverse orbits of trajectories in Ω_n.

Finally, to estimate the asymptotic entropy of μ, it suffices to compute it on a subset of trajectories of positive measure. Indeed, consider $\epsilon>0$ and subsets $\Theta_n\subset W^n$ with $\mu^n(\Theta_n)\geqslant\epsilon$. If $h(\mu)=h>0$, then $\lim\frac{-1}{n}\log\mu^{*n}(\boldsymbol{w})=h$ for almost every trajectory $\boldsymbol{w}\in G^\infty$, by [21, Theorem 2.1]; so

$$-\sum_{\substack{g\in W\\ g=g_1\dots g_n\\ (g_i)\in\Theta_n}}\mu^{*n}(g)\log\mu^{*n}(g)=h\epsilon>0.$$

It therefore suffices, as we have done, to show that the asymptotic entropy of μ vanishes on a subset of positive measure. □

The proposition implies that the symmetric finitely supported random walk on W from Example 4.11 has trivial boundary. Indeed, any nearest neighbour random walk on $\mathbb{Z}_+^2$ or $\mathbb{Z}^2$ is recurrent [1]; this property depends only on the graph, not the random walk, see the remark before Corollary 5.7. Note also that a subgraph of a recurrent graph is also recurrent (see again [1], or [28, Corollary 2.15]), so we need not worry whether the random walk is degenerate or intransitive.

Remark 5.3. Implicit in the application of Lemma 5.2 is the following function $v_i(n,k)$ that deserves further study: for a group G, with generating set S, acting on a set X with basepoint ρ, write

$$\begin{aligned} v_i(n,k) = \#\{&Y \subset X \mid \#Y = k \\ &\text{and } Y \text{ is the inverted orbit of an } S\text{-path of length } n \text{ starting at } \rho\}. \end{aligned}$$

Indeed, the lemma was used to show that, if G is a group of subexponential growth with sublinear inverted orbit growth $\delta(n)$, then $v_i(n,\delta(n))$ is subexponential.

By comparison, consider the corresponding function for direct orbits:

$$\begin{aligned} v_d(n,k) = \#\{&Y \subset X \mid \#Y = k \\ &\text{and } Y \text{ is the direct orbit of an } S\text{-path of length } n \text{ starting at } \rho\}. \end{aligned}$$

Each directed orbit Y is a connected subset of X containing ρ; and a connected subset of cardinality k can be traversed by a path of length $2k$, so we have the simple bound $v_d(n,k) \leqslant (\#S)^{2k}$, which implies that $v_d(n,k)$ is subexponential in n as soon as k is sublinear in n.

In contrast with the direct orbit case, it is not possible in general to bound $v_i(n,k)$ by a function of k only. For example, consider the first Grigorchuk group G acting on a ray X. The stabilizer of ρ is infinite; let S contain the generating set of an infinite subgroup of it. If $s_2,\dots,s_n$ fix ρ but s_1 does not, then the inverted orbit of $s_1\dots s_n$ contains only two points $\{\rho,\rho'\}$, and ρ' is arbitrary under the condition $d(\rho,\rho') \leqslant n$; so $v_i(n,2) \sim n$.

More generally, we have the obvious bound $v_i(n,k) \leqslant \#B_X(n)^k$. This bound is never tight enough for our purposes.

We now show that in this example (or, more generally, in any torsion Grigorchuk group) the assumption that the random walk is symmetric can be dropped, see Corollary 5.7.

5.1. **Centered Markov chains.**

There is a class of non-symmetric (and not necessarily reversible) Markov chains that resembles in many aspects symmetric ones. These are chains that admit a certain "decompositions into cycles", see [22]. In particular, it is shown by Kalpazidou in [23] that under some conditions the recurrence of such random walks does not depend on the choice of the random walk. We will use a version of this statement which is due to Mathieu.

Definition 5.4 ([25, Definition 2.1])**.** Let V be an oriented graph, possibly with loops and multiple edges. A *centered Markov chain* on V is defined as follows. There is a collection of $\{\gamma_i\}$ of oriented cycles on V, which we assume edge-self-avoiding but not necessarily vertex-self-avoiding. Each cycle has a weight q_i. Each edge must belong to exactly one cycle (but remember, we allow multiple edges!). For any vertex x in V, the sum of the weights of all cycles passing through x (counted with multiplicity, if the cycle passes several times through x) is equal to one.

The Markov chain has the vertex set of V as set of states. The probability of moving in one step of the Markov chain from vertex x to vertex y is given as follows:

choose a cycle containing x according to the weights q_i; then move to the successor of x along that cycle. We write the transition kernel as follows:

$$q(x,y) = \sum_{i:(x,y)\in\gamma_i} q_i.$$

(The definition above is a particular case of [25, Definition 2.1], and is slightly more general than [25, Example 2.4]. Indeed, observe that under our assumption $q_i \leqslant 1$ and, in the notation of [25], we can consider $m(x) = 1$ for all $x \in V$).

Centered Markov chains are generalizations of symmetric Markov chains: indeed, in any non-oriented graph, replace each edge by two oriented edges that form a cycle of length two; set the weight of that cycle to be the weight of the original edge. In fact, the general definition of centered Markov chains in [25, Definition 2.1] is a generalization of reversible Markov chains.

Remark 5.5. (i) If μ is a finitely supported measure on a group G and all elements of the support of μ are torsion, then the random walk (G,μ) is a centered Markov chain on the Cayley graph of G with generating set $\mathrm{supp}(\mu)$. This is used in [25] to prove Carne-Varopoulos estimates for random walks on torsion groups.

(ii) More generally, if μ is a finitely supported measure on a group G and all elements of the support of μ are torsion, and G acts on a set X, then the random walk on X is a centered random walk on the Schreier graph of (G,X).

Proof. For each $g \in \mathrm{supp}(\mu)$ there exits a minimal $m \geqslant 1$ such that $g^m = 1$. For each such g, consider all the cycles of the form $(x, xg, \dots xg^{m-1})$, and define the weight of this cycle to be $\mu(g)$. The random walk on X induced by the measure μ is the same as the centered Markov chain defined by these weighted cycles. □

Lemma 5.6 (Mathieu, [25, Proposition 2.13(iii)])**.** *Let V be a connected locally finite graph, and let q be a centered Markov chain on V. Let q_0 be the associated symmetric Markov chain: $q_0(x,y) = \frac{1}{2}(q(x,y) + q(y,x))$.*

Then the chain q is recurrent if and only if q_0 is recurrent.

Recall that a random walk is *uniformly irreducible* if its one-step transition probabilities are uniformly bounded from below. It is well known (see e.g. [28, Corollary 3.5]) that if V^0 is a non-oriented graph, the recurrence/transience of uniformly irreducible symmetric random walks on V^0 does not depend on the probability measure, but only on the graph. Thus, by the lemma, if V is recurrent considered as a non-oriented graph, then all centered Markov chains on X are recurrent.

The following example (statements (i) and (ii) of the corollary below) gives a negative answer to the question of Kaimanovich and Vershik from [21]:

Corollary 5.7 (= Theorem A)**.** *Let G be any Grigorchuk torsion group, for example G_{012}, and let X be an orbit of G on the tree's boundary. Let A be a non-trivial finite group, and set $W = A \wr_{X\times X} (G \times G)$. Then*

(i) *W has exponential word growth;*
(ii) *any finitely supported random walk on W has trivial Poisson-Furstenberg boundary;*
(iii) *any finitely supported random walk on W has zero drift.*

Proof. (i) follows from Theorem 4.1.

For (ii), take a finitely supported measure μ on $G \times G$ and consider the induced random walk on X^2. Since W is a torsion group, the random walk on X^2 is a centered Markov chain. The graph X is $\mathbb{Z}_+$ or $\mathbb{Z}$, so the random walk on X^2 is a centered Markov chain on a graph which is a subgraph of $\mathbb{Z}^2$. By Lemma 5.6, this random walk is recurrent. Therefore, we can apply Proposition 5.1, and conclude that the random walk (W, μ) has trivial boundary.

For (iii), take a finitely supported measure μ on W. Any finitely supported measure has finite entropy. We have shown in (ii) that the random walk (W, μ) has trivial boundary. Therefore, by the entropy criterion, $h(\mu) = 0$.

Mathieu has proven in [25] that Carne-Varopoulos estimates hold for centered Markov chains. In particular, he has shown that, for centered random walks on groups, $h(\mu) = 0$ if and only if $\ell(\mu) = 0$. We conclude that $\ell(\mu) = 0$. □

6. Further examples. Drift estimates for self-similar random walks

Self-similar groups are groups G endowed with a homomorphism $\phi : G \to G \wr \mathfrak{S}_d$, with $\mathfrak{S}_d$ the symmetric group on $\{1, \dots, d\}$. By iterating the map ϕ, every self-similar group acts on sets of cardinality d^n, for all $n \in \mathbb{N}$; these sets form the levels of a d-regular rooted tree. If we write $\phi(g) = \langle\!\langle g_1, \dots, g_d \rangle\!\rangle \pi$, then the permutation $\pi \in \mathfrak{S}_d$ describes the action of g on the neighbours of the root, while $g_1, \dots, g_d$ describe recursively the action of g on the subtrees attached to the root.

A fundamental example is the first Grigorchuk group G_{012}. It is the self-similar group characterized as follows: it is generated by four elements a, b, c, d; it acts faithfully on the 2-regular rooted tree; and ϕ is given on the generators by

$$\phi(a) = \langle\!\langle 1, 1 \rangle\!\rangle (1, 2), \quad \phi(b) = \langle\!\langle a, c \rangle\!\rangle, \quad \phi(c) = \langle\!\langle a, d \rangle\!\rangle, \quad \phi(d) = \langle\!\langle 1, b \rangle\!\rangle.$$

Self-similar random walks were introduced by the first author and Virag in [3]; see below for the definition. In that paper, they show that the so-called "Basilica group" admits a self-similar random walk, and then this self-similar measure is used to show that this random walk has zero drift with respect to some metric (which is not a word metric, in contrast with usual definition of the drift).

Kaimanovich uses a similar idea in [20], but works with the entropy of the random walk $h(\mu)$ instead. The main idea of these papers is to use the self-similarity of the random walk to prove that its asymptotic entropy vanishes. In a similar way one can use self-similar measures in order to estimate $H_\mu(n)$, see [4, Proposition 4.11]. The following lemma is similar to that proposition.

Definition 6.1. A *self-similar sequence of groups* is a sequence $(G_1, G_2, \dots)$ of groups, with homomorphisms $\phi_i : G_i \to G_{i+1} \wr \mathfrak{S}_d$.

Let μ_i be a measure on G_i. It defines a random walk on $G_{i+1} \times \{1, \dots, d\}$, via ϕ_i: if $\phi_i(g) = \langle\!\langle g_1, \dots, g_d \rangle\!\rangle \pi$, then the walk moves from (h, i) to $(hg_i, \pi(i))$ with probability $\mu_i(g)$. The *renormalization* of μ_i is the measure μ_i' on G_{i+1} defined by running μ_i on $(1, 1)$ till it reaches $G_{i+1} \times 1$; in formulas,

$$\mu_i'(g) = \sum_{h_1, \dots, h_n \in G_{i+1}}{}' \mu_i(h_1) \cdots \mu_i(h_n),$$

where the sum extends over all n-tuples $(h_1, \dots, h_n)$ such that $\phi_i(h_1 \cdots h_n) \in \{g\} \times G_{i+1}^{d-1} \times \operatorname{stab}_{\mathfrak{S}_d}(1)$ and $\phi_i(h_1 \cdots h_j) \notin G_{i+1}^d \times \operatorname{stab}_{\mathfrak{S}_d}(1)$ for all $j < n$.

A *self-similar sequence of measures* on a sequence (G_i) of groups is a sequence $(\mu_1, \mu_2, \dots)$ of measures, each μ_i a measure on G_i, and a sequence of numbers $(\alpha_1, \alpha_2, \dots)$ in $[0, 1]$, such that $\mu_i' = (1 - \alpha_i)\delta_1 + \alpha_i \mu_{i+1}$, namely μ_i' is a convex combination of μ_{i+1} and the Dirac measure at $1 \in G_{i+1}$. It is a *lazy* random walk, with *laziness* α_i.

The following lemma generalizes [4, Proposition 4.11]:

Lemma 6.2. *Let (G_i) be a self-similar sequence of groups, and let (μ_i) be a self-similar sequence of measures on (G_i), with laziness (α_i). Assume $\sup_i H(\mu_i) < \infty$. Then there exists a constant K such that*

$$H_{G_1,\mu_1}(n) \leqslant Kn^\beta \text{ for all } n, \text{ with } \beta = \frac{\log d}{\log d - \log(\sup \alpha_i)}.$$

Proof. A random variable in G_i is determined by its projection to $\mathfrak{S}_d$ and by its d renormalizations in G_{i+1}. Say an n-step walk starting at 1 visits n_i times point i, for all $i \in \{1, \dots, d\}$. Then

$$H_{\mu_i}(n) \leqslant \mathbf{E}\Big[\sum_{j=1}^d H_{\mu_i'}(n_i) \mid n_1 + n_2 + \cdots + n_d \leqslant n\Big] + d\log d.$$

For ν a measure, we extend $H_\nu(n)$ to real arguments $n \in \mathbb{R}$ by interpolating linearly: $H_\nu((1-\theta)n + \theta(n+1)) = (1-\theta)H_\mu(n) + \theta H_\mu(n+1)$. By [21, Proposition 1.3], for $n \in \mathbb{N}$ the numbers $H_\nu(n+1) - H_\nu(n)$ decrease monotonically to $h(\nu)$; so the affine extension $H_\nu : \mathbb{R} \to \mathbb{R}$ is a concave function. Therefore,

$$H_{\mu_i}(n) \leqslant dH_{\mu_i'}(n/d) + d\log d.$$

Next, for all $m \in \mathbb{N}$,

$$H_{\mu_i'}(m) = \sum_{k=0}^m \binom{m}{k} \alpha_i^k (1-\alpha_i)^{m-k} H_{\mu_{i+1}}(k);$$

the binomial distribution has mean $\alpha_i m$, so again by concavity

$$H_{\mu_i'}(m) \leqslant H_{\mu_{i+1}}(\alpha_i m).$$

Therefore,

$$H_{\mu_i}(n) \leqslant dH_{\mu_{i+1}}(n\alpha_i/d) + d\log d.$$

We then iterate this relation, to obtain

$$H_{\mu_1}(d^k/\alpha_1 \cdots \alpha_k) \leqslant d\log d + \cdots + d^k \log d + d^k H(\mu_{k+1}) \leqslant Kd^k$$

for a constant K, and we are done. □

For any finitely supported random walk on a finitely generated group G, there exist constants $C, D > 0$ such that

$$C\left(\frac{L(n)}{n}\right)^2 \leqslant \frac{H(n)}{n} \leqslant D\frac{L(n)}{n} \tag{1}$$

for all n; the first inequality follows from Varopoulos's long range estimates, see e.g. [10, page 1201]. These inequalities hold, more generally, for any random walk with a finite second moment, see [14, Corollary 9.(ii)].

Kaimanovich observes in [20] that the first Grigorchuk group admits a self-similar measure μ with laziness $1/2$. An example of such a measure is μ defined by $\mu(1) = 5/12$, $\mu(a) = 1/3$, $\mu(b) = \mu(c) = \mu(d) = 1/12$: for this measure one has $\mu' = 1/2\delta_1 + 1/2\mu$. We can therefore apply Lemma 6.2 with $d = 2$ and $\alpha = 1/2$ and conclude that the entropy function $H_\mu(n)$ of this random walk satisfies $H(n) \leqslant Kn^{1/2}$.

Now take a sequence $\omega = (v_1, v_2, \dots) \in \{\mathbf{0},\mathbf{1},\mathbf{2}\}^\infty$, and define $\omega_i = (v_i, v_{i+1}, \dots)$ its shift; consider the corresponding sequence of Grigorchuk groups $G_i = G_{\omega_i}$, which form a similar sequence. The standard generators of G_i are still written a, b, c, d. On each G_i define a probability measure μ_i by $\mu_i(1) = 5/12$, $\mu_i(a) = 1/3$, $\mu_i(b) = \mu_i(c) = \mu_i(d) = 1/12$. These form a self-similar sequence of measures on (G_i), and, as in [20], one has $\mu_i' = 1/2\delta_1 + 1/2\mu_{i+1}$. Combining this with Lemma 6.2 and (1), we get the

Corollary 6.3. *On every Grigorchuk group G_ω, there exists a symmetric non-degenerate finitely supported measure μ and a constant C such that $H(n) \leqslant Cn^{1/2}$ and $L(n) \leqslant Cn^{3/4}$ for all $n \in \mathbb{N}$.*

Remark 6.4. Examples of Grigorchuk groups above stress the importance of the fact that [3] works with drift with respect to a special non-word metric, and [20] works with entropy of random walks, and not with drift: although Grigorchuk groups admit self-similar measure sequences with laziness 1/2, it is not true that on these groups one has $L(n) \leqslant Cn^{1/2}$. Indeed, it is shown in [12, Corollary 1] that any simple random walk on the first Grigorchuk group satisfies $L(n) \geqslant n^\kappa$ for some $\kappa > 1/2$ and infinitely many n's.

Example 6.5. Let G_1, G_2 be two Grigorchuk groups. Let respectively X_1, X_2 be orbits for their action on the boundary of the rooted tree. By Corollary 6.3, the assumption of Theorem 4.1 is satisfied. Therefore, for any non-trivial group A, the wreath product $W = A \wr_{X_1 \times X_2} G_1 \times G_2$ has exponential word growth.

If G_1 and G_2 are torsion groups, then every finitely supported measure on W has trivial boundary, so these are other negative answers to the Kaimanovich-Vershik question.

Example 6.6. Let $G_1 = G_2 = H$ be the Grigorchuk torsion-free group of subexponential growth from [17]; recall that H maps onto G_{012}, and therefore acts on an orbit X of the Grigorchuk group on the boundary of the rooted tree. Consider the wreath product $W = \mathbb{Z} \wr_{X \times X} (H \times H)$. Then W is a torsion-free group of exponential growth, such that every finitely supported measure on W has trivial Poisson-Furstenberg boundary.

Proof. Clearly W is torsion-free, as an extension of torsion-free groups. Since the action of $H \times H$ on $X \times X$ actually comes from the action of $G_{012} \times G_{012}$, the random walk μ on $X \times X$ induced by $H \times H$ is the same as a random walk induced by a measure on $G_{012} \times G_{012}$. Therefore, μ defines a centered random walk on a subgraph of $\mathbb{Z}^2$. Applying Lemma 5.6 as we did in the proof of Corollary 5.7(ii), we conclude that μ induces a recurrent random walk. By Lemma 3.1, the expected inverted orbit growth is sublinear. Since both $\mathbb{Z}$ and $H \times H$ have subexponential growth, Proposition 5.1 gives that every finitely supported measure on W has trivial boundary. On the other hand, W has exponential word growth since, by Theorem 4.1, its quotient $\mathbb{Z} \wr_{X \times X} (G_{012} \times G_{012})$ already has exponential growth. □

7. Lipschitz imbeddings of regular trees

We gave, in Theorem 4.1, a general criterion for a permutational wreath product of a product of two groups to have exponential word growth. For most of the examples we produce, it does not seem at all straightforward to check without using random walks that they have exponential growth.

Below is one example in which we prove more directly that the growth of inverted orbits of (G, X) is linear (and hence that the word growth of the corresponding wreath product $A \wr_X G$ is exponential). We consider $G = G_{012}$, acting diagonally on $X = X_1 \times X_2$, where X_1 and X_2 are orbits under G_{012} of the rays $\rho_1 = (01)^\infty$ and $\rho_2 = (10)^\infty$ respectively (regarded as points of the boundary of the rooted tree on which G_{012} acts).

Proposition 7.1. *Let w_n be the word over $\{a, b, c, d\}$ of length $\sim (2/\eta)^n$ constructed as follows. Write $\Omega' = \{ab, ac, ad\}^* \subset \Omega = \{a, b, c, d\}^*$, consider the substitution $\zeta : \Omega' \to \Omega'$ given by*

$$\zeta : ab \mapsto abadac, \quad ac \mapsto abab, \quad ad \mapsto acac,$$

and consider the word $w_n = \zeta^n(ad)$.

Define as before $\delta(w) = \#\{(\rho_1, \rho_2)w_i \cdots w_n \mid 0 \leqslant i \leqslant n\}$ for $\rho_1 = (01)^\infty$ and $\rho_2 = (10)^\infty$. Then $\delta(w_n) = |w_n| + 1$; namely, all points on the inverse orbit of w_n are distinct.

The words w_n in the statement of the lemma above were used in [5, Proposition 4.7] to estimate the growth of the permutational wreath product of the first Grigorchuk group.

Proof. Write $w_n = g_1 \cdots g_\ell$ and $\rho = (\rho_1, \rho_2)$. We are to show that for all $i < j$ we have $\rho g_i \cdots g_\ell \neq \rho g_j \cdots g_\ell$; or, equivalently, that $\rho g_i \cdots g_{j-1} \neq \rho$, namely, no subword of w_n fixes ρ.

Let $H \subset G$ denote the stabilizer of ρ. Let $\Omega'' \subset \Omega$ denote those words alternating in 'a' and 'b, c, d' letters. We easily check that, if $w \in \Omega'$ is non-trivial and represents an element of H, then either $w = yaxadaxay$ for some $x \in \{b, c\}$ and $y \in \{1, c, d\}$, or w contains at least 6 letters among $\{b, c, d\}$. This is done by a tedious but straightforward enumeration. Note that no word of the form $yaxadaxay$ belongs to the image of ζ, because if $w = \zeta(v)$ then all 'd' letters are preceded by a 'b' and followed by a 'c'.

Assume now for contradiction that the subword $g_i \cdots g_{j-1}$ fixes ρ. Recall that $\phi(w_n) = \langle\!\langle a^{-1} w_{n-1} a, w_{n-1} \rangle\!\rangle$; so $\phi(g_i \cdots g_{j-1}) = \epsilon^s \langle\!\langle u', u'' \rangle\!\rangle$ for words u', u'' which, up to pre- and post-multiplication by a letter in $\{a, b, c, d\}$, are equal, and equal to a subword $g'_{i'} \cdots g'_{j'-1}$ of w_{n-1}. Furthermore, $g'_{i'} \cdots g'_{j'-1}$ also fixes ρ, and has length half that of $g_i \cdots g_{j-1}$.

Continuing in this manner, we get for every $m < n$ a word u_m of the form $s'_m v_m s''_m$ for words s'_m, s''_m of length $\leqslant 2$ and a subword v_m of w_m. Let m be maximal such that v_m is trivial. Then v_{m+1} is a non-trivial subword of w_{m+1} which fixes ρ, and has length at most 7; this contradicts the assertion that the shortest word in Ω'' fixing ρ has length $\geqslant 11$. $\square$

Let us tentatively introduce the following notion. Consider a group G acting transitively on a set X, and fix $\rho \in X$. Say that the growth of inverted orbits of G on (X, ρ) is *strongly linear*, if there exits a finite generating set S of G such that for each $n \in \mathbb{N}$ there exits a word w_n of length n over elements of S such that the inverted orbit of w_n has exactly $n + 1$ points (recall that this is the maximal value it may assume).

Proposition 7.1 shows that (G, X) has strongly linear growth. Observe the following consequence of strongly linear growth of inverted orbits:

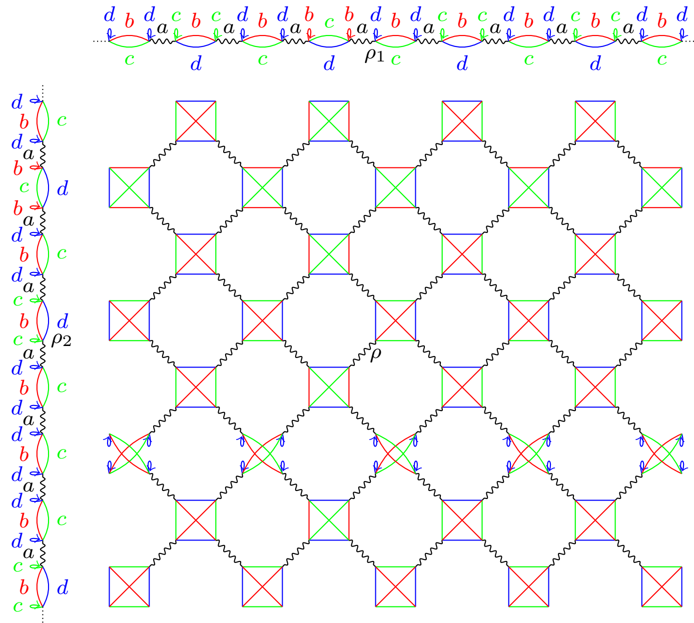

FIGURE 1. The Schreier graph of G_{012} on $G_{012}/\mathrm{stab}(\rho_1) \times G_{012}/\mathrm{stab}(\rho_2)$. Edges are indicated by colours: black for a, red/green/blue for $b/c/d$.

Lemma 7.2. *If G has strongly linear inverted orbit growth on X and A is non-trivial, then some Cayley graph of $A \wr_X G$ contains an imbedded copy of the infinite binary rooted tree.*

Proof. Let $a_0 \neq a_1$ be two elements of A. Let S' be a generating set of G for which the inverted orbits grow strongly linearly. Let S be a generating set of $W := A \wr_X G$ containing $\{a_0, a_1\} \times S'$. For $n \in \mathbb{N}$, let $w_n = g_1 \cdots g_n$ be a word of length n visiting n points in X, and consider all words of the form $a_{i_m} g_m \cdots a_{i_n} g_n$ for all $m \in \{1, \cdots, n+1\}$ and all $i_m, \dots, i_n \in \{0, 1\}$. We claim that these are the vertices of the height-n binary rooted tree in the Cayley graph of W.

First, these elements are all distinct: consider $a_{i_m} \cdots g_n$ and $a_{i'_{m'}} \cdots g_n$. If $m \neq m'$ then their projections to G are distinct; while if $m = m'$ then, because the inverted walk $g_m \cdots g_n$ visits $n - m + 1$ distinct positions, the elements are distinct as soon as $i_j \neq i'_j$ for some $j \in \{m, \dots, n\}$.

Because all $a_{i_m} g_m$ belong to S, there is an edge in the Cayley graph from $a_{i_m} g_m \cdots a_{i_n} g_n$ to $a_{i_{m+1}} g_{m+1} \cdots a_{i_n} g_n$; these edges form a binary tree, rooted at 1.

Since n was arbitrary, we obtain for all n a binary tree of height n and rooted at 1. A classical diagonal argument then extracts from this sequence an infinite binary rooted tree. $\square$

Corollary 7.3. *The wreath product $W = A \wr_{X_1 \times X_2} G_{012}$ has exponential word growth, for X_1 the orbit of $\rho_1 = (01)^\infty$ and X_2 is the orbit of $\rho_2 = (10)^\infty$. Moreover, some Cayley graph of W contains an infinite binary rooted tree.*

It also follows from 4.1 that W has exponential growth; indeed, G_{012} and $G_{012} \times G_{012}$ are commensurable, so we are, up to finite index, in the situation of a product of groups $G_1 \times G_2$ acting on $X_1 \times X_2$. We have elected to give a direct proof that W has exponential growth, because we also deduce along the way that W contains trees in its Cayley graph.

A classical question of Rosenblatt [27] asks whether every group of exponential growth admits a Lipschitz imbedding of the infinite binary rooted tree. A result of Benjamini and Schramm [6] implies that every non-amenable graph contains the image a regular tree by a Lipschitz imbedding; so it is sufficient, to answer positively Rosenblatt's question, to exhibit a non-amenable subgraph. Rosenblatt's question is answered positively for virtually soluble groups (the group contains a free subsemigroup) and non-amenable groups (since their Cayley graph is non-amenable), but is open in general. The group W we construct in this article also contains Lipschitzly imbedded infinite binary rooted trees (by Lemma 7.2), though for a different reason than those mentioned above.

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