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A UNIFIED GENERATING FUNCTION OF THE q-GENOCCHI POLYNOMIALS WITH THEIR INTERPOLATION FUNCTIONS

SERKAN ARACI, MEHMET AÇIKGÖZ, HASSAN JOLANY, AND JONG JIN SEO

Abstract. The purpose of this paper is to construct of the unification q-extension Genocchi polynomials. We give some interesting relations of this type of polynomials. Finally, we derive the q-extensions of Hurwitz-zeta type functions from the Mellin transformation of this generating function which interpolates the unification of q-extension of Genocchi polynomials.

1. Introduction, Definitions and Notations

Recently, many mathematician have studied to unification Bernoulli, Genocchi, Euler and Bernstein polynomials (see [20,21]). Ozden [20] introduced p-adic distribution of the unification of the Bernoulli, Euler and Genocchi polynomials and derived some properties of this type of unification polynomials.

In [20], Ozden constructed the following generating function:

$$\sum_{n=0}^{\infty} y_{n,\beta}(x;k,a,b)\frac{t^n}{n!}=\frac{2^{1-k}t^k e^{xt}}{\beta^b e^t-a^b},\quad \left|t+b\ln\left(\frac{\beta}{a}\right)\right|<2\pi \tag{1.1}$$

where $k\in\mathbb{N}=\{1,2,3,...\}$, $a,b\in\mathbb{R}$ and $\beta\in\mathbb{C}$. The polynomials $y_{n,\beta}(x;k,a,b)$ are the unification of the Bernoulli, Euler and Genocchi polynomials.

Ozden showed, $\beta=b=1$, $k=0$ and $a=-1$ into (1.1), we have

$$y_{n,1}(x;0,-1,1)=E_n(x),$$

where $E_n(x)$ denotes classical Euler polynomials, which are defined by the following generating function:

$$\frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!},\quad |t|<\pi, \tag{1.2}$$

In [5,20], classical Genocchi polynomials defined as follows:

$$\frac{2te^{xt}}{e^t+1}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!},\quad |t|<\pi, \tag{1.3}$$

From (1.2) and (1.3), we easily see,

$$G_n(x)=nE_{n-1}(x), \tag{1.4}$$

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For a fixed real number $|q|<1$, we use the notation of q-number as

$$[x]_q = \frac{1-q^x}{1-q}, \quad \text{(see [1-4,6-24])},$$

Thus, we note that $\lim_{q\to 1}[x]_q = x$.

In [1,7,8,10], q-extension of Genocchi polynomials are defined as follows:

$$G_{n+1,q}(x) = (n+1)[2]_q \sum_{l=0}^{\infty} (-1)^l q^l [x+l]_q^n .$$

In [2], (h,q)-extension of Genocchi polynomials are defined as follows:

$$G_{n+1,q}^{(h)}(x) = (n+1)[2]_q \sum_{l=0}^{\infty} (-1)^l q^{(h-1)l} [x+l]_q^n .$$

In this paper, we shall construct unification of q-extension of the Genocchi polynomials, however we shall give some interesting relationships. Moreover, we shall derive the q-extension of Hurwitz-zeta type functions from the Mellin transformation of this generating function which interpolates.

2. Novel Generating Functions of q-extension of Genocchi polynomials

Definition 1. *Let $a,b\in\mathbb{R}$, $\beta\in\mathbb{C}$ and $k\in\mathbb{N}=\{1,2,3,...\}$,Then the unification of q-extension of Genocchi polynomials defined as follows:*

$$F_{\beta,q}(t,x\,|k,a,b) = \sum_{n=0}^{\infty} S_{n,\beta,q}(x\,|k,a,b)\frac{t^n}{n!} \tag{2.1}$$

and

$$F_{\beta,q}(t,x\,|k,a,b) = -[2]_q^{1-k}\, t^k \sum_{m=0}^{\infty} \beta^{bm} a^{-bm-b} e^{[m+x]_q t}. \tag{2.2}$$

where into (2.1) substituting $x=0$, $S_{n,\beta,q}(0\,|k,a,b) = S_{n,\beta,q}(k,a,b)$ are called unification of q-extension of Genocchi numbers.

As well as, from (2.1) and (2.2) Ozden's constructed the following generating function, namely, we obtain (1.1),

$$\lim_{q\to 1} F_{\beta,q}(t,x\,|k,a,b) = \frac{2^{1-k}t^k e^{xt}}{\beta^b e^t - a^b}.$$

By (2.2), we see readily,

$$\begin{aligned}
\sum_{n=0}^{\infty} S_{n,\beta,q}(x\,|k,a,b)\frac{t^n}{n!} &= -[2]_q^{1-k}\, t^k \sum_{m=0}^{\infty} \beta^{bm} a^{-bm-b} e^{[m+x]_q t} \\
&= \frac{e^{[x]_q t}}{q^{kx}} \left(-[2]_q^{1-k} (q^x t)^k \sum_{m=0}^{\infty} \beta^{bm} a^{-bm-b} e^{(q^x t)[m]_q} \right) \\
&= q^{-kx} \left(\sum_{n=0}^{\infty} [x]_q^n \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} q^{nx} S_{n,\beta,q}(k,a,b)\frac{t^n}{n!} \right) \qquad (2.3)
\end{aligned}$$

From (2.3) by using Cauchy product we get

$$\sum_{n=0}^{\infty} S_{n,\beta,q}\left(x\left|k,a,b\right.\right)\frac{t^n}{n!} = q^{-kx}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}q^{lx}S_{l,\beta,q}\left(k,a,b\right)[x]_q^{n-l}\right)\frac{t^n}{n!}$$

By comparing coefficients of $\frac{t^n}{n!}$ in the both sides of the above equation, we arrive at the following theorem:

Theorem 1. *For $a,b\in\mathbb{R}$, $\beta\in\mathbb{C}$ which k is* positive *integer. We obtain*

$$S_{n,\beta,q}\left(x\left|k,a,b\right.\right) = \sum_{l=0}^{n}\binom{n}{l}q^{lx}S_{l,\beta,q}\left(k,a,b\right)[x]_q^{n-l}$$

As well as, we obtain corollary 1:

Corollary 1. *For $a,b\in\mathbb{R}$, $\beta\in\mathbb{C}$ which k is* positive *integer. We obtain,*

$$S_{n,\beta,q}\left(x\left|k,a,b\right.\right) = \left(S_{\beta,q}\left(k,a,b\right)+[x]_q\right)^n \tag{2.4}$$

with usual the convention about replacing $\left(S_{\beta,q}\left(x\left|k,a,b\right.\right)\right)^n$ by $S_{n,\beta,q}\left(x\left|k,a,b\right.\right)$.

By applying the definition of generating function of $S_{n,\beta,q}\left(x\left|k,a,b\right.\right)$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} S_{n,\beta,q}\left(x\left|k,a,b\right.\right)\frac{t^n}{n!} &= -[2]_q^{1-k}t^k\sum_{m=0}^{\infty}\beta^{bm}a^{-bm-b}\left(\sum_{n=0}^{\infty}[m+x]_q^n\frac{t^n}{n!}\right)\\ &= \sum_{n=0}^{\infty}\left(-[2]_q^{1-k}\sum_{m=0}^{\infty}\beta^{bm}a^{-bm-b}[m+x]_q^n\right)\frac{t^{n+k}}{n!}\end{aligned}$$

So we derive the Theorem 2 which we state hear:

Theorem 2. *For $a,b\in\mathbb{R}$, $\beta\in\mathbb{C}$ which k is positive integer. We obtain,*

$$S_{n,\beta,q}\left(x\left|k,a,b\right.\right) = -\frac{n!\,[2]_q^{1-k}}{(n-k)!}\sum_{m=0}^{\infty}\beta^{bm}a^{-bm-b}[m+x]_q^{n-k} \tag{2.5}$$

With regard to (2.5), we see after some calculations

$$\begin{aligned}S_{n,\beta,q}\left(x\left|k,a,b\right.\right) &= -\frac{n!\,[2]_q^{1-k}}{a^b(n-k)!}\left(\frac{1}{1-q}\right)^{n-k}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l q^{lx}\sum_{m=0}^{\infty}\left(\frac{\beta^b}{a^b}\right)^m q^{lm}\\ &= \frac{n!\,[2]_q^{1-k}}{a^b(n-k)!}\left(\frac{1}{1-q}\right)^{n-k}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l q^{lx}\frac{1}{\beta^b q^l-a^b}\\ &= \frac{k!\,[2]_q^{1-k}}{a^b}\left(\frac{1}{1-q}\right)^{n-k}\sum_{l=0}^{n-k}\binom{n}{k}\binom{n-k}{l}(-1)^l q^{lx}\frac{1}{\beta^b q^l-a^b} \qquad (2.6)\end{aligned}$$

From (2.6) and well known identity $\left[\binom{n}{k}\binom{n-k}{l}=\binom{n}{k+l}\binom{k+l}{k}\right]$, we obtain the following theorem:

Theorem 3. *For $a,b\in\mathbb{R}$, $\beta\in\mathbb{C}$ which k is* positive *integer. We obtain*

$$S_{n,\beta,q}\left(x\left|k,a,b\right.\right) = \frac{k!\,[2]_q^{1-k}}{a^b}\left(\frac{1}{1-q}\right)^{n-k}\sum_{l=k}^{n}\binom{n}{l}\binom{l}{k}(-1)^{l-k}q^{(l-k)x}\frac{1}{\beta^b q^{l-k}-a^b} \tag{2.7}$$

We put $x \to 1-x, \beta \to \beta^{-1}, q \to q^{-1}$ and $a \to a^{-1}$ into (2.7), namely,

$$\begin{aligned}
& S_{n,\beta^{-1},q^{-1}}\left(1-x\left|k,a^{-1},b\right.\right) \\
= & \frac{k!\,[2]_{q^{-1}}^{1-k}}{a^{-b}}\left(\frac{1}{1-q^{-1}}\right)^{n-k}\sum_{l=k}^{n}\binom{n}{l}\binom{l}{k}(-1)^{l-k}q^{-(l-k)(1-x)}\frac{1}{\beta^{-b}q^{-(l-k)}-a^{-b}} \\
= & (-1)^{n-k}q^{k-1}q^{n-k}k!\,[2]_q^{1-k}a^{b}\left(\frac{1}{1-q}\right)^{n-k}\sum_{l=k}^{n}\binom{n}{l}\binom{l}{k}(-1)^{l-k}q^{k-l}q^{(l-k)x}\frac{\beta^{b}q^{l-k}a^{b}}{a^{b}-\beta^{b}q^{l-k}} \\
= & (-1)^{n-k-1}q^{n-1}a^{3b}\beta^{b}\left(\frac{k!\,[2]_q^{1-k}}{a^{b}}\left(\frac{1}{1-q}\right)^{n-k}\sum_{l=k}^{n}\binom{n}{l}\binom{l}{k}(-1)^{l-k}q^{x(l-k)}\frac{1}{\beta^{b}q^{l-k}-a^{b}}\right) \\
= & (-1)^{n-k-1}q^{n-1}a^{3b}\beta^{b}S_{n,\beta,q}\left(x\left|k,a,b\right.\right)
\end{aligned}$$

So, we obtain symmetric properties of $S_{n,\beta,q}\left(x\left|k,a,b\right.\right)$ as follows:

Theorem 4. *For $a,b\in\mathbb{R}$, $\beta\in\mathbb{C}$ which k is* positive *integer. We obtain*

$$S_{n,\beta^{-1},q^{-1}}\left(1-x\left|k,a^{-1},b\right.\right)=(-1)^{n-k-1}q^{n-1}a^{3b}\beta^{b}S_{n,\beta,q}\left(x\left|k,a,b\right.\right).$$

By (2.2), we see

$$\begin{aligned}
& \frac{\beta}{a}F_{\beta,q}\left(t,1\left|k,a,b\right.\right)-F_{\beta,q}\left(t,0\left|k,a,b\right.\right) \\
= & \sum_{n=0}^{\infty}\left(\left(\frac{\beta}{a}\right)S_{n,\beta,q}\left(1\left|k,a,b\right.\right)-S_{n,\beta,q}\left(k,a,b\right)\right)\frac{t^n}{n!}=-\frac{[2]_q^{1-k}}{a^{b}}t^{k}
\end{aligned} \tag{2.8}$$

From (2.8), we obtain the following theorem:

Theorem 5. *For $a,b\in\mathbb{R}$, $\beta\in\mathbb{C}$ which k is* positive *integer. We obtain*

$$S_{n,\beta,q}\left(k,a,b\right)-\left(\frac{\beta}{a}\right)S_{n,\beta,q}\left(1\left|k,a,b\right.\right)=\begin{cases}0, & n\neq k\\ \frac{[2]_q^{1-k}}{a^{b}}k!, & n=k\end{cases} \tag{2.9}$$

From (2.4) and (2.9), we obtain corollary as follows:

Corollary 2. *For $a,b\in\mathbb{R}$, $\beta\in\mathbb{C}$, which is k positive integer. We get*

$$S_{n,\beta,q}\left(k,a,b\right)-\frac{\beta}{aq^{k}}\left(qS_{\beta,q}\left(k,a,b\right)+1\right)^{n}=\begin{cases}0, n\neq k\\ \frac{[2]_q^{1-k}}{a^{b}}k!, n=k\end{cases}$$

with the usual convention about replacing $\left(S_{\beta,q}\left(k,a,b\right)\right)^{n}$ by $S_{n,\beta,q}\left(k,a,b\right)$.

From (9), now, we shall obtain distribution relation for unification q-extension of Genocchi polynomials, after some calculations, namely,

$$\begin{aligned} S_{n,\beta,q}\left(x\left|k,a,b\right.\right) &= -\frac{n!\left[2\right]_q^{1-k}}{a^b\left(n-k\right)!}\sum_{m=0}^{\infty}\left(\frac{\beta}{a}\right)^{bm}\left[m+x\right]_q^{n-k} \\ &= -\frac{n!\left[2\right]_q^{1-k}}{a^b\left(n-k\right)!}\sum_{m=0}^{\infty}\sum_{l=0}^{d-1}\left(\frac{\beta}{a}\right)^{b(l+md)}\left[l+md+x\right]_q^{n-k} \\ &= \left[d\right]_q^{n-k}\sum_{l=0}^{d-1}\left(\frac{\beta}{a}\right)^{bl}\left(-\frac{n!\left[2\right]_q^{1-k}}{a^b\left(n-k\right)!}\sum_{m=0}^{\infty}\left(\frac{\beta^d}{a^d}\right)^{bm}\left[m+\frac{x+l}{d}\right]_{q^d}^{n-k}\right) \\ &= \frac{\left[2\right]_q^{1-k}}{\left[2\right]_{q^d}^{1-k}}\left[d\right]_q^{n-k}\sum_{l=0}^{d-1}\left(\frac{\beta}{a}\right)^{bl}S_{n,\beta^d,q^d}\left(\frac{x+l}{d}\left|k,a^d,b\right.\right) \end{aligned}$$

Therefore, we obtain the following theorem:

Theorem 6. ($distribution\ \ formula\ for\ S_{n,\beta,q}\left(x\left|k,a,b\right.\right)$)*For* $a,b\in\mathbb{R}$, $\beta\in\mathbb{C}$ *which* k *is* positive *integer. We obtain,*

$$S_{n,\beta,q}\left(x\left|k,a,b\right.\right)=\frac{\left[2\right]_q^{1-k}}{\left[2\right]_{q^d}^{1-k}}\left[d\right]_q^{n-k}\sum_{l=0}^{d-1}\left(\frac{\beta}{a}\right)^{bl}S_{n,\beta^d,q^d}\left(\frac{x+l}{d}\left|k,a^d,b\right.\right)$$

3. Interpolation function of the polynomials $S_{n,\beta,q}\left(x\left|k,a,b\right.\right)$

In this section, we give interpolation function of the generating functions of $S_{n,\beta,q}\left(x\left|k,a,b\right.\right)$ however, this function is meromorphic function. This function interpolates $S_{n,\beta,q}\left(x\left|k,a,b\right.\right)$ at negative integers.

For $s\in\mathbb{C}$, by applying the Mellin transformation to (2.2), we obtain

$$\begin{aligned} \mathfrak{S}_{\beta,q}\left(s;x,a,b\right) &= \frac{\left(-1\right)^{k+1}}{\Gamma\left(s\right)}\oint t^{s-k-1}F_{\beta,q}\left(-t,x\left|k,a,b\right.\right)dt \\ &= \left[2\right]_q^{1-k}\sum_{m=0}^{\infty}\beta^{bm}a^{-bm-b}\frac{1}{\Gamma\left(s\right)}\int_0^{\infty}t^{s-1}e^{-t\left[m+x\right]_q t} \end{aligned}$$

So, we have

$$\mathfrak{S}_{\beta,q}\left(s;x,a,b\right)=\left[2\right]_q^{1-k}\sum_{m=0}^{\infty}\frac{\beta^{bm}a^{-bm-b}}{\left[m+x\right]_q^s}$$

We define $q-$extension Hurwitz-zeta type function as follows theorem:

Theorem 7. *For* $a,b\in\mathbb{R}$, $\beta,s\in\mathbb{C}$ *which* k *is* positive *integer. We obtain,*

$$\mathfrak{S}_{\beta,q}\left(s;x,a,b\right)=\left[2\right]_q^{1-k}\sum_{m=0}^{\infty}\frac{\beta^{bm}a^{-bm-b}}{\left[m+x\right]_q^s} \tag{3.1}$$

for all $s\in\mathbb{C}$. *We note that* $\mathfrak{S}_{\beta,q}\left(s;x,a,b\right)$ *is analytic function in the whole complex* s*-plane.*

Theorem 8. *For* $a,b\in\mathbb{R}$, $\beta\in\mathbb{C}$ *which* k *is* positive *integer. We obtain,*

$$\mathfrak{S}_{\beta,q}\left(-n;x,a,b\right)=-\frac{\left(n-k\right)!}{n!}S_{n,\beta,q}\left(x\left|k,a,b\right.\right).$$

Proof. Let $a, b \in \mathbb{R}$, $\beta \in \mathbb{C}$ and $k \in \mathbb{N}$ with $k \in \mathbb{N} = \{1, 2, 3, ...\}$. $\Gamma(s)$, has simple poles at $z = -n = 0, -1, -2, -3, \cdots$. The residue of $\Gamma(s)$ is

$$\operatorname{Re} s\left(\Gamma(s), -n\right) = \frac{(-1)^n}{n!}.$$

We put $s \to -n$ into (3.1) and using the above relations, the desired result can be obtained. □

References

[1] Araci, S., Erdal, D., and Kang, D-J., Some New Properties on the q-Genocchi numbers and Polynomials associated with q-Bernstein polynomials, Honam Mathematical J. 33 (2011) no.2, pp. 261-270.

[2] Araci, S., Seo, J.J., Erdal, D., Different Approach On The (h, q) Genocchi Numbers and Polynomials Associated with q-Bernstein Polynomials, (Submitted)

[3] Araci, S., Seo, J.J., Erdal, D., New Construction weighted (h, q)-Genocchi numbers and Polynomials Related to Zeta Type Functions, Discrete Dynamics in Nature and Society(in press).

[4] Araci, S., Erdal D., and Seo, J-J., A study on the fermionic p-adic q-integral representation on $\mathbb{Z}_p$ Associated with weighted q-Bernstein and q-Genocchi polynomials, Abstract and Applied Analysis(in press).

[5] Jolany, H., and Sahrifi, H., Some Results for the Apostol-Genocchi polynomials of Higher Order, Accepted in Bulletin of the Malaysian Mathematical Sciences Society, arXiv:1104.1501v1 [math. NT].

[6] Ryoo. C-S., Kim, T., and Lee, B., q-Bernoulli numbers and polynomials revisited, Accepted in Advances in Difference Equations.

[7] Kim, T., On the q-extension of Euler and Genocchi numbers, J. Math. Anal. Appl. 326 (2007) 1458-1465.

[8] Kim, T., On the multiple q-Genocchi and Euler numbers, Russian J. Math. Phys. 15 (4) (2008) 481-486. arXiv:0801.0978v1 [math.NT]

[9] Kim, T., On the weighted q-Bernoulli numbers and polynomials, Advanced Studies in Contemporary Mathematics 21(2011), no.2, p. 207-215, http://arxiv.org/abs/1011.5305.

[10] Kim, T., A Note on the q-Genocchi Numbers and Polynomials, Journal of Inequalities and Applications 2007 (2007) doi:10.1155/2007/71452. Article ID 71452, 8 pages.

[11] Kim, T., q-Volkenborn integration, Russ. J. Math. phys. 9(2002) , 288-299.

[12] Kim, T., An invariant p-adic q-integrals on $\mathbb{Z}_p$, Applied Mathematics Letters, vol. 21, pp. 105-108,2008.

[13] Kim, T., q-Euler numbers and polynomials associated with p-adic q-integrals, J. Nonlinear Math. Phys., 14 (2007), no. 1, 15–27.

[14] Kim, T., New approach to q-Euler polynomials of higher order, Russ. J. Math. Phys., 17 (2010), no. 2, 218–225.

[15] Kim, T., Some identities on the q-Euler polynomials of higher order and q-Stirling numbers by the fermionic p-adic integral on $\mathbb{Z}_p$, Russ. J. Math. Phys., 16 (2009), no.4,484–491.

[16] Kim, T. and Rim, S.-H., On the twisted q-Euler numbers and polynomials associated with basic q-l-functions, Journal of Mathematical Analysis and Applications, vol. 336, no. 1, pp. 738–744, 2007.

[17] Kim, T., On p-adic q-l-functions and sums of powers, J. Math. Anal. Appl. (2006), doi:10.1016/j.jmaa.2006.07.071.

[18] Jang. L.-C., On a q-analogue of the p-adic generalized twisted L-functions and p-adic q-integrals, Journal of the Korean Mathematical Society, vol. 44, no. 1, pp. 1–10, 2007.

[19] Ryoo. C. S., A note on the weighted q-Euler numbers and polynomials, Advan. Stud. Contemp. Math. 21(2011), 47-54.

[20] Ozden, H., Unification of generating function of the Bernoulli, Euler and Genocchi polynomials, Numerical Analysis and Applied mathematics (2010), Amer. Inst. Phys. Conf. Proc., vol. 1281, 2010, pp.1125–1128.

[21] Simsek, Y., Construction a new generating function of Bernstein type polynomials, Applied Mathematics and Computation, arXiv:1010.3711v1 [math.CA].

[22] Y. Simsek, Theorems on twisted L-function and twisted Bernoulli numbers, Advan. Stud. Contemp. Math., 11(2005), 205-218.
[23] Y. Simsek, Twisted (h, q)-Bernoulli numbers and polynomials related to twisted $(h, q)-$zeta function and L-function, J. Math. Anal. Appl., 324(2006), 790-804.
[24] Y. Simsek, On p-Adic Twisted q-L-Functions Related to Generalized Twisted Bernoulli Numbers, Russian J. Math. Phys., 13(3)(2006), 340-348.

University of Gaziantep, Faculty of Science and Arts, Department of Mathematics, 27310 Gaziantep, TURKEY
E-mail address: mtsrkn@hotmail.com

University of Gaziantep, Faculty of Science and Arts, Department of Mathematics, 27310 Gaziantep, TURKEY
E-mail address: acikgoz@gantep.edu.tr

School of Mathematics, Statistics and Computer Science, University of Tehran, Iran
E-mail address: hassan.jolany@khayam.ut.ac.ir

Department of Applied Mathematics, Pukyong National University, Busan, 608-737,Korea
E-mail address: seo2011@pknu.ac.kr