Yang-Mills connections of cohomogeneity one on $SO(n)$-bundles over Euclidean spheres

Andreas Gastel

1 Introduction

For many geometric variational problems or p.d.e., there is a construction of spherical solutions via *joins of spheres.* Very roughly, these constructions use the fact that S^{m+n+1} is the *join* of S^m and S^n, which means a set made up from curves, each one of which connects one point in S^m with one in S^n. These curves allow a common parametrization over $[0, \frac{\pi}{2}]$, say, and depending on this parameter t, one can try to construct all kinds of geometric objects on S^{m+n+1} from homogeneous objects of the same type on S^m and S^n. Homogeneity of the latter helps reducing the partial differential equations, which usually describe such objects, to *ordinary* differential equations. The symmetries described here often, but not always, correspond to some $SO(m+1) \times SO(n+1)$-invariance or -equivariance of the objects being constructed. This family of constructions has lead to examples of

- harmonic maps between spheres [Sm], [Di], [PR] (and also some variants like p-harmonic maps [Fa] and biharmonic maps [GZ]);
- constant mean curvature hypersurfaces in $\mathbb{R}^{n+1}$ [Hs1];
- non-equatorial minimal embeddings of S^n in S^{n+1} [Hs2], [Hs3];
- Einstein metrics on spheres [Bo].

A common feature of these constructions is that they all reduce the original p.d.e. to a (system of) o.d.e. with *singular boundary values.* They tend to work best in dimensions which are slightly above the "critical dimension" of the respective equation. An excellent source presenting the first three of the examples in a unified way is the book [ER2].

The aim of this paper is to establish a similar construction for Yang-Mills connections; more precisely for Yang-Mills $SO(n)$-connections over some S^m, $m \geq 5$. The join construction for such connections will exhibit features similar to the ones listed above. Note that, due to the supercritical dimension $m \geq 5$ and to the fact that we cannot work in a "Hermitian Yang-Mills" setting, there is currently no way to prove existence of such connections by variational methods. This is of course closely related to the lack of good gauges for connections in these dimensions. We make up for this by choosing a suitable equivariant ansatz which already is in a "good" gauge.

The methods we use are close to the methods for harmonic maps invented by Smith, and the conditions for solvability are very reminiscent of the "damping conditions" known from harmonic map theory. The most notable difference is that in our case the equivariant ansatz does not reduce the problem to a single o.d.e., but to a system of two o.d.e. This fact adds a little bit of the flavor of Böhm's construction of Einstein metrics to our considerations.

What we are going to construct are Yang-Mills connections of cohomogeneity one. It should be noted that such connections over manifolds of dimension four have been studied extensively by Urakawa [Ur], who also provides a very general reduction setting for o.d.e. in Yang-Mills theory. The typical degree three nonlinearities for the o.d.e.s found there and here probably appeared first in Parker's construction of non-minimizing Yang-Mills fields [Par]. Recently, Park and Urakawa [PU] have also studied completely homogeneous Yang-Mills connections, which in a special case we will also have to do in this paper.

The paper is organized as follows. In **Section 2**, we give a rather general short introduction to equivariant Yang-Mills connections. **Section 3** is devoted to a rather more special case of homogeneous pull-back bundles of TS^n under mappings $S^m \to S^n$. We will need the so-called "Yang-Mills eigenmaps" obtained from these consideration as the homogeneous "building blocks" for our join construction.

In **Section 4**, we observe that there is only a very restricted class of joins of *vector bundles* which are again smooth vector bundles. This justifies our reduction ansatz in **Section 5**, which otherwise would look a bit special at first glimpse. In this section, the reduction of the Yang-Mills equation to a system of two o.d.e. (equipped with singular boundary data) is performed.

The solvability of the singular o.d.e. boundary value problem thus obtained is discussed in some detail in **Section 6**. We get sufficient conditions for that, which (comparing to the harmonic map case where this is known) we expect to be also necessary. Finally, in **Section 7**, we apply the existence theorem to find nontrivial examples of smooth cohomogeneity one Yang-Mills connections over spheres. Among the examples we construct are

- one Yang-Mills connection on each of the countably many principal $SO(6)$-bundles over S^6,
- countably many Yang Mills connections on TS^n for $n \in \{5, \dots, 9\}$.

(Coincidentally, Böhm's join construction for Einstein metrics [Bo] produces nonhomogeneous Einstein metrics on S^n for exactly the same range of dimensions.)

Acknowledgment: This paper was finished while the author was visiting SFB Transregio 71 at Freiburg. He would like to thank for hospitality and support.

2 Equivariant connections

Let M be a compact Riemannian manifold, $\pi : E \to M$ be a G-vectorbundle of rank n for some compact Lie group $G \subseteq O(n)$; the latter means that we view E equipped with a bundle metric.

We assume that another compact Lie group K is acting on both E and M by isometries; we denote the action of $k \in K$ on M simply by $k : M \to M$, while on E we denote it by $\lambda_k : E \to E$. We assume that the K-actions are compatible with the projection, which means

$$\pi(\lambda_k v) = k\pi(v)$$

for all $k \in K$, $v \in E$. By $\Omega^0(E)$ we denote the set of smooth sections of E, and by $\Omega^\ell(E)$, $\ell \in \mathbb{N} \cup \{0, \infty\}$, the sections of $E \otimes \wedge^\ell T^*M$, i.e. the corresponding section-valued ℓ-forms.

The K-actions introduced above induce a natural K-action on $\Omega^0(E)$, with $\tau_k : \Omega^0(E) \to \Omega^0(E)$ given for $k \in K$ by

$$(\tau_k Y)(x) := \lambda_k Y(k^{-1}x).$$

A connection $D : \Omega^0(E) \to \Omega^1(E)$ is called K-*equivariant* if

$$D_{k_* u}(\tau_k Y) = \tau_k(D_u Y)$$

holds for all $k \in K$, $u \in \Omega^0(TM)$, and $Y \in \Omega^0(E)$. Here k_* means the derivative of $k : M \to M$.

Let us fix a K-equivariant reference connection ∇ of E. Then every G-connection of E is of the form $D = \nabla + A$ for some $A \in \Omega^1(\mathrm{ad}\, P)$, where P is the principal fiber bundle associated with E. We want to describe what equivariance of D (and ∇) means for A. For k, u, Y as above, we have

$$\begin{aligned}
\nabla_u Y(x) + A_u(x) Y(x) &= D_u Y(x) \\
&= \tau_k^{-1}(D_{k_* u}(\tau_k Y))(x) \\
&= \tau_k^{-1}(\nabla_{k_* u}(\tau_k Y))(x) + \tau_k^{-1}(A_{k_* u}\tau_k Y)(x) \\
&= \nabla_u Y(x) + \tau_k^{-1}(A_{k_* u}\tau_k Y)(x) \\
&= \nabla_u Y(x) + \lambda_k^{-1}(A_{k_* u}\tau_k Y)(kx) \\
&= \nabla_u Y(x) + \lambda_k^{-1} A_{k_* u}(kx)\lambda_k Y(x),
\end{aligned}$$

from which we read off that

$$A_u(x) = \lambda_k^{-1} A_{k_* u}(kx)\lambda_k$$

for all $x \in M$, $u \in E_x$, and $k \in K$. Similarly, we find the correct transformation of the curvature $F = F_A$ of D:

$$F_{uv}(x)Y(x) = (D_u D_v - D_v D_u)Y(x)$$

$$
\begin{aligned}
&= \tau_k^{-1}(D_{k_*u}D_{k_*v} - D_{k_*v}D_{k_*u})(\tau_k Y)(x) \\
&= \tau_k^{-1}(F_{k_*u,k_*v}\tau_k Y)(x) \\
&= \lambda_k^{-1}F_{k_*u,k_*v}(kx)\lambda_k Y(x),
\end{aligned}
$$

and hence

$$
F_{uv}(x) = \lambda_k^{-1}F_{k_*u,k_*v}(kx)\lambda_k
$$

for all $x \in M$, $u, v \in E_x$, $k \in K$.

A connection $D_A = \nabla + A$ is called a *Yang-Mills connection*, if it is a critical point of the Yang-Mills functional

$$
YM(A) = \frac{1}{2}\int_M |F_A|^2\, dx.
$$

A connection is Yang-Mills if and only if

$$
D_A^* F_A = 0,
$$

which for smooth A is equivalent to the weak formulation

$$
\int_M \langle F_A, D_A\varphi\rangle\, dx = 0 \qquad \text{for all } \varphi \in \Omega^1(\mathrm{ad}\, P).
$$

A first important observation about equivariant Yang-Mills maps is an instance of Palais' so-called *principle of symmetric criticality*, cf. [Pal] for the general philosophy.

Proposition 1 (symmetric criticality) *A smooth K-equivariant connection D_A on E is already Yang-Mills if it is only critical with respect to equivariant variations, i.e. if the first variation*

$$
\int_M \langle F_A, D_A\varphi\rangle\, dx = 0
$$

vanishes for those $\varphi \in \Omega^1(\mathrm{ad}\, P)$ satisfying

$$
\varphi_u(x) = \lambda_k^{-1}\varphi_{k_*u}(kx)\lambda_k
$$

for all $x \in M$, $u \in E_x$, $k \in K$.

Proof. We abbreviate the right-hand side of the last equation by $(k^*\varphi)(x)$, and similarly for F. Let $\varphi \in \Omega^1(\mathrm{ad}\, P)$ be *any* form, not necessarily equivariant. Denoting the Haar measure of K by H_K, and using the fact that all K-actions are isometric and commute with D, we calculate

$$
\begin{aligned}
\int_M \langle F_A, D_A\varphi\rangle\, dx &= \int_M\int_K \langle (k^{-1})^* F_A, D_A\varphi\rangle\, dH_K\, dx \\
&= \int_M\int_K \langle F_A, k^*(D_A\varphi)\rangle\, dH_K\, dx \\
&= \int_M \left\langle F_A, D_A\int_K k^*\varphi\, dH_K\right\rangle dx \\
&= 0,
\end{aligned}
$$

where the first "=" holds because F_A is K-equivariant, and the last one because so is $\int_K k^*\varphi\, dH_K$. This proves that D_A is Yang-Mills. $\square$

3 Homogeneous connections over S^m

We start with some notation. For $a, b \in \mathbb{R}^n$, we denote by $a \otimes b : \mathbb{R}^n \to \mathbb{R}^n$ the linear mapping given by

$$(a \otimes b)(v) := \langle a, v\rangle b,$$

represented by the matrix

$$(a \otimes b)_{ij} = a_j b_i.$$

If M is a skew-symmetric $n \times n$-matrix, we have

$$(a \otimes b)_{ij} M_{jk} = a_j M_{jk} b_i = -M_{kj} a_j b_i$$

and hence

$$(a \otimes b)M = -(Ma) \otimes b.$$

Similarly,

$$M_{ij}(a \otimes b)_{jk} = M_{ij} a_k b_j,$$

which means

$$M(a \otimes b) = a \otimes (Mb).$$

We want to consider equivariant bundles over S^m on the pull-back bundle $E = h^*TS^n$ of some smooth map $h : S^m \to S^n$. We describe the bundle globally by identifying the fiber over $x \in S^m$ with $T_{h(x)}S^n$, i.e. we identify its total space with

$$E = \{(x, y) \in S^m \times \mathbb{R}^{n+1} : \langle h(x), y\rangle = 0\}.$$

The total space of the corresponding principal fiber bundle P can be identified with

$$\{(x, M) \in S^m \times SO(n + 1) : Mh(x) = h(x)\},$$

where the fiber over x is the isotropy subgroup of $h(x)$ in $O(n+1)$ acting on $\mathbb{R}^{n+1} \supset S^n$ in the standard way. Every connection on P (or equivalently on h^*TS^n) is of the form

$$D = \nabla + A,$$

where here A is a section in the adjoint vector bundle $\operatorname{ad} P$ with total space

$$\{(x, A) \in S^m \times \mathfrak{so}(n + 1) : Ah(x) = 0\},$$

and ∇ is the pull-back of the Levi-Civita connection on S^n. The latter means

$$\nabla_u Y = \partial_u Y - \langle h, \partial_u Y\rangle h = \partial_u Y + \langle \partial_u h, Y\rangle h$$

for all sections u of TS^m and Y of h^*TS^n.

Now we assume some homogeneous structure of h^*TS^n in the following way: We assume that $K = SO(m+1)$ is acting on both S^m and all $T_xS^m \subset \mathbb{R}^{m+1}$ in the standard way (and we do not distinguish between k and k_* here). On S^n we assume an operation of $SO(m+1)$ by some representation $\lambda : SO(m+1) \to SO(n+1)$ (and hence on vectors in E by the same matrices). Both operations of $SO(m+1)$ are isometric. Moreover, we assume that h is K-equivariant, which in our case means

$$h(kx) = \lambda_k h(x)$$

for all $k \in SO(m+1)$, $x \in S^m$.

Because of

$$\begin{aligned}
\partial_{ku}(Y(k^{-1}x)) &= (\partial_u Y)(k^{-1}x), \\
\partial_{ku}(\tau_k Y)(x) &= \lambda_k(\partial_u Y)(k^{-1}x), \\
\nabla_{ku}(\tau_k Y)(x) &= \lambda_k(\nabla_u Y)(k^{-1}x) \\
&= \tau_k \nabla_u Y,
\end{aligned}$$

∇ is $SO(m+1)$-equivariant, which means that ∇ can be used as reference connection as in the last section. We want to investigate for which A the connection $D = D_A = \nabla + A$ is Yang-Mills. We fix $x \in S^m$, $v \in T_xS^m$, and consider a path in $SO(m+1)$ given by

$$k(t) := \mathrm{id} + x \otimes \{(\cos t - 1)x + (\sin t)v\} + v \otimes \{(\cos t - 1)v - (\sin t)x\}.$$

We observe

$$\begin{aligned}
k(0) &= \mathrm{id}, \\
k'(0) &= x \otimes v - v \otimes x, \\
\lambda_k'(0) &= h(x) \otimes \partial_v h(x) - \partial_v h(x) \otimes h(x).
\end{aligned}$$

Differentiating the equivariance relation for A, we find

$$\begin{aligned}
0 &= \frac{d}{dt}_{|t=0}\Big(\lambda_{k(t)}^{-1} A_{k(t)u}(k(t)x)\lambda_{k(t)}\Big) \\
&= [A_u(x), \lambda_k'(0)] + A_{k'(0)u}(x) + A_u'(x)k'(0)x \\
&= [A_u(x), h(x) \otimes \partial_v h(x) - \partial_v h(x) \otimes h(x)] + A_{\langle x,u\rangle v - \langle v,u\rangle x}(x) + \partial_v A_u(x) \\
&= h(x) \otimes (A_u(x)\partial_v h(x)) - (A_u(x)\partial_v h(x)) \otimes h(x) + \partial_v A_u(x),
\end{aligned}$$

which means that all derivatives of A can be expressed by terms of order zero:

$$\partial_v A_u = (A_u \partial_v h) \otimes h - h \otimes (A_u \partial_v h).$$

This implies

$$(\partial_v A_u)Y = \langle A_u \partial_v h, Y\rangle h$$

for sections $Y \in \Omega^0(h^*TS^n)$. We use this to calculate further (with the first "=" being the definition of ∇ extended to forms)

$$\begin{aligned}
(\nabla_v A_u)Y &= \nabla_v(A_u Y) - A_u \nabla_v Y \\
&= \partial_v(A_u Y) + \langle \partial_v h, A_u Y\rangle h - A_u \partial_v Y - A_u(\langle \partial_v h, Y\rangle h) \\
&= (\partial_v A_u)Y + \langle \partial_v h, A_u Y\rangle h \\
&= \langle A_u \partial_v h, Y\rangle h + \langle \partial_v h, A_u Y\rangle h \\
&= 0
\end{aligned}$$

because $A_u \in \mathfrak{so}(n+1)$ is skew-symmetric. Therefore A_u is covariant constant with respect to ∇,

$$\nabla A = 0.$$

Knowing the curvature of ∇,

$$(F_0)_{uv} = \partial_v h \otimes \partial_u h - \partial_u h \otimes \partial_v h,$$

we infer that

$$\begin{aligned}
(F_A)_{uv} &= (F_0)_{uv} + \nabla_u A_v - \nabla_v A_u + [A_u, A_v] \\
&= \partial_v h \otimes \partial_u h - \partial_u h \otimes \partial_v h + [A_u, A_v].
\end{aligned}$$

Using the fact that $|F_A|^2$ is constant due to the transitivity of the $SO(m+1)$-action on S^m, we conclude that (without integration)

$$c\,YM(A) = \frac{1}{2}\sum_{u,v=1}^{m} \Big|\partial_v h(e) \otimes \partial_u h(e) - \partial_u h(e) \otimes \partial_v h(e) + [A_u(e), A_v(e)]\Big|^2$$

for some dimension-dependent constant $c > 0$, where we abbreviate $e = e_{m+1}$ and $\partial_u = \partial_{e_u}$. The first variation of this functional is (where from now on we omit the argument (e))

$$c\,\delta YM(A, \Phi) = \sum_{u,v=1}^{m} \Big\langle \partial_v h \otimes \partial_u h - \partial_u h \otimes \partial_v h + [A_u, A_v]\,,\, [\Phi_u, A_v] + [A_u, \Phi_v]\Big\rangle,$$

from which we read off its "Euler-Lagrange" equation, which in this case is just some system of algebraic equations:

$$\begin{aligned}
&\sum_{u,v=1}^{m}\sum_{i,j=1}^{n}\Big[\Big(\partial_v h^j \partial_u h^i - \partial_u h^j \partial_v h^i + \sum_{k=1}^{n}(A_u^{ik}A_v^{kj} - A_v^{ik}A_u^{kj})\Big) \\
&\quad \cdot \sum_{k=1}^{n}\Big(\Phi_u^{ik}A_v^{kj} - A_v^{ik}\Phi_u^{kj} + A_u^{ik}\Phi_v^{kj} - \Phi_v^{ik}A_u^{kj}\Big)\Big] \;=\; 0
\end{aligned}$$

for every choice of real numbers Φ_u^{ij} for $u \in \{1, \dots, m\}$ and $i, j \in \{1, \dots, n\}$ satisfying $\Phi_u^{ij} = -\Phi_u^{ji}$, where we have assumed w.l.o.g. that $h(e_{m+1}) = e_{n+1}$. Choosing $\Phi_z^{pq} = -\Phi_z^{qp} = 1$ for fixed $p < q$ and z, and $\Phi_u^{ij} = 0$ in all other cases, we can write this a s a system of $\frac{1}{2}mn(n-1)$ cubic equations for the same number of variables A_u^{ij} (cf. [PU] for another formulation of these cubic equations). Therefore, in principle, we know "all" Yang-Mills connections with the symmetries considered. But we will not have to go into any further detail, because all we want to know for the purpose of this paper is that $A \equiv 0$ is always a solution, which we easily read off from the equation above. To be more precise, we have proven that $\nabla = D_0$ is critical for YM with respect to equivariant variations (and maybe a few more). But by the "symmetric criticality" Proposition 1, this means that ∇ is actually Yang-Mills. Hence we have proven:

> *In the special homogeneous setting considered here, the pull-back ∇ of the Levi-Civita connection of TS^n via h is a Yang-Mills connection.*

This should have been well-known, and probably follows from the results in [PU] or even from Itoh's earlier paper [It], and we have given the proof mainly to introduce our setting and notation (which differ significantly from theirs).

Remark. It seems not to be true that the pull-back of the Levi-Civita connection of S^n via a homogeneous mapping $h : S^m \to S^n$ is always Yang-Mills. For example, the Hopf map $h : S^3 \to S^2$ is $U(2)$-equivariant with $U(2)$ acting transitively on S^3. However, the ∇ that we obtain from h is not Yang-Mills, as a direct calculation shows. The same probably applies for the Hopf maps $S^7 \to S^4$ and $S^{15} \to S^8$. The only point in the above proof that does not carry over is the choice of the path $k(t)$ in the symmetry group K. For the Hopf examples, this group is no longer $SO(m+1)$, and this is where the argument fails.

Examples. Nevertheless, there are enough examples of representations of $SO(m+1)$ to make the above considerations interesting for us:

(o) If $m = n$ and $SO(m+1)$ acts also on the target sphere in the standard way, then $h : S^m \to S^m$ is the identity and ∇ is the Levi-Civita connection of TS^m.

(i) If $m = n = 1$, we consider $e^{i\vartheta} \in SO(2)$ acting on the target circle as $e^{i\ell\vartheta}$ for some $\ell \in \mathbb{N}$. The connection ∇ is the flat one, but $h : S^1 \to S^1$ here is the mapping $z \mapsto z^\ell$.

(ii) Let $\ell \in \mathbb{N}$ and identify $\mathbb{R}^{n+1}$ with the space of ℓ-homogeneous harmonic polynomials on R^{m+1}, which implies $n = \frac{(2\ell+m-1)(\ell+m-2)!}{\ell!(m-1)!} - 1$. This identification is made via an orthonormal basis $\{b_i\}_{1 \le i \le n+1}$ of this space (with respect to the scalar product $\langle f, g \rangle = \int_{S^m} fg$). A representation $\lambda : SO(m+1) \to SO(n+1)$ is given by $\lambda_k f(x) := f(k^{-1}x)$, and the corresponding $h_{m,\ell} : S^m \to S^n$ is given by $h_{m,\ell}(x) = (b_1(x), \dots, b_{n+1}(x))$. This map has been considered by doCarmo and Wallach [dCW]; it is a harmonic mapping as well as (after rescaling the domain sphere suitably) a minimal immersion.

The geometries described in these examples have some more properties that will be important if we want to take them as "building blocks" for the join construction we will perform in this paper. We summarize what we need in the following definition. The term "Yang-Mills eigenmap" here is motivated to some extent by property (iii), but more by the fact that joining *harmonic maps* is based on a similar concept called "harmonic eigenmap". We write d_{LC} for Levi-Civita connections.

Definition (Yang-Mills eigenmap) *We call a map $h : S^m \to S^n$ a* Yang-Mills eigenmap, *if there exist numbers $\lambda > 0$, $\mu \geq 0$, such that*

$$\begin{array}{ll} (i) & h^* d_{LC} \text{ is a Yang-Mills connection,} \\ (ii) & |dh|^2 \equiv \lambda, \\ (iii) & \displaystyle\sum_v F(h^* d_{LC})_{uv} \partial_v h = \mu \partial_u h \qquad \forall u \in \{1, \ldots, m\}. \end{array}$$

Here, by $\sum_v$ and $\sum_V$, we mean summation over orthonormal bases of $T_x S^m$ or $T_{h(x)} S^n$, respectively.

An immediate consequence of (iii) is that $|F(h^* d_{LC})|^2$ is constant. Since

$$F(h^* d_{LC})_{uv} Y = \langle \partial_v h, Y \rangle \partial_u h - \langle \partial_u h, Y \rangle \partial_v h,$$

we have

$$\begin{aligned} |F(h^* d_{LC})|^2 &= \sum_{u,v} \sum_{U,V} \Big(\langle \partial_v h, Y \rangle \langle \partial_u h, Z \rangle - \langle \partial_u h, Y \rangle \langle \partial_v h, Z \rangle \Big)^2 \\ &= 2 \sum_{u,v} \Big(|\partial_u h|^2 |\partial_v h|^2 - \langle \partial_u h, \partial_v h \rangle^2 \Big) \\ &= 2 \sum_{u,v} \langle F(h^* d_{LC})_{uv} \partial_v h, \partial_u h \rangle \\ &\equiv 2\lambda\mu. \end{aligned} \tag{1}$$

Examples of Yang-Mills eigenmaps arise from the examples of homogeneous Yang-Mills connections above. We have

(o) the identities $\mathrm{id}_m : S^m \to S^m$, with $\lambda = m$, $\mu = m - 1$;

(i) the mappings $d_\ell : S^1 \to S^1$, $d_\ell(z) = z^\ell$ with $\lambda = \ell^2$, $\mu = 0$;

(ii) the standard immersions $h_{m,\ell} : S^m \to S^{\frac{(2\ell+m-1)(\ell+m-2)!}{\ell!(m-1)!} - 1}$ described above, with $\lambda = \ell(\ell + m - 1)$ and $\mu = \frac{m-1}{m} \ell(\ell + m - 1)$.

Strictly speaking, (o) and (i) are special cases of (ii); we list them separately because of their distinctive geometric features.

All these examples happen to be harmonic eigenmaps, too. Detailed accounts of harmonic eigenmaps and related concepts can be found in the books [Ba] and [ER1]. As we do not have any more examples than the ones listed here, we do not know whether every Yang-Mills eigenmap is automatically also a harmonic eigenmap. Nor do we know whether we should prepare for Yang-Mills eigenmaps which do not come from group representations (they do exist in the case of harmonic eigenmaps).

Remark 1 *In all of our examples of Yang-Mills eigenmaps, we observe $\mu = \frac{m-1}{m}\lambda$. We have not assumed that in the definition of eigenmaps, because we will not need it in our discussion of reduction of Yang-Mills to an o.d.e., nor is it needed for the sufficient conditions for solving that o.d.e. The only point where it might prove important is the question whether the conditions obtained are also necessary, cf. Remark 3 below.*

4 Topological motivation of our ansatz

Now we have to justify the special kind of ansatz we are going to make below. To this end, we write $S^{m_1+m_2+1}$ as the join $S^{m_1} * S^{m_2}$, that is the warped product

$$S^{m_1+m_2+1} \cong [0,\pi/2] \times_{\cos^2} S^{m_1} \times_{\sin^2} S^{m_2}$$

which closes smoothly across the endpoints of $[0,\pi/2]$. Assume we are given an $SO(n_1)$-vectorbundle $E_1 \to S^{m_1}$ and an $SO(n_2)$-vectorbundle $E_2 \to S^{m_2}$. We want to construct a *join* $E_1 * E_2$ of E_1 and E_2 as an $SO(n_1+n_2+1)$-vectorbundle over $S^{m_1} * S^{m_2}$ by roughly "connecting every point in a fiber of E_1 with every point in a fiber of E_2". To make this precise, we parametrize $S^{m_1+m_2+1}$ by three patches:

$$\begin{aligned}
&\varphi_1 : (0,\pi/2) \times S^{m_1} \times S^{m_2} \to S^{m_1+m_2+1},\\
&\qquad \varphi_1(t,x_1,x_2) := (x_1\cos t, x_2 \sin t);\\
&\varphi_2 : B^{m_1+1} \times S^{m_2} \to S^{m_1+m_2+1},\\
&\qquad \varphi_2(y_1,x_2) := \exp_{(0,x_2)}(y_1,0) = (\tfrac{y_1}{|y_1|}\sin|y_1|, x_2\cos|y_1|);\\
&\varphi_3 : S^{m_1} \times B^{m_2+1} \to S^{m_1+m_2+1},\\
&\qquad \varphi_3(x_1,y_2) := \exp_{(x_1,0)}(0,y_2) = (x_1\cos|y_2|, \tfrac{y_2}{|y_2|}\sin|y_2|).
\end{aligned}$$

On the image of φ_1 (which is all of $S^{m_1+m_2+1}$ except for two "singular spheres"), the join of E_1 and E_2 is easily described: It is simply the φ_1^{-1}-pullback of the product bundles $T \times E_1 \times E_2$, where T is the trivial $\mathbb{R}$-bundle over $(0,\pi/2)$. The question now is, under which condition this bundle closes smoothly across the singular spheres to give a smooth $SO(n_1+n_2+1)$-vectorbundle over $S^{m_1+m_2+1}$.

This is a topological condition on the bundles E_1 and E_2, and it can be formalized as follows. Denote by T now the trivial $\mathbb{R}$-bundle over $(0,1]$. Then the product bundle $T \times E_i \to (0,1] \times S^{m_i}$ must be the pull-back of an $SO(m_i+1)$-bundle $\widetilde{E}_i \to B^{m_i+1}$ via the map $(t, x_i) \mapsto t x_i$. Since every bundle over B^{m_i+1} is trivial, it is no restriction to assume $\widetilde{E}_i = B^{m_i+1} \times \mathbb{R}^{n_i+1}$. Moreover, for every $y_i \in \mathbb{R}^{m_i+1}$, the fiber $\mathbb{R}^{n_i+1}$ over y_i contains a well-defined direction which corresponds to the positive T-direction in the product bundle. This defines a mapping $h_i : S^{m_i} \to S^{n_i}$ for which $E = h_i^* TS^{n_i}$. Therefore the only bundles $E_i \to S^{m_i}$ ($i \in \{1, 2\}$) for which a smooth join can be defined are the pull-back bundles $h_i^* TS^{n_i}$ for a pair of maps $h_i : S^{m_i} \to S^{n_i}$.

Given such a pair, we still have to find out, for which connections on $h_i^* TS^{n_i}$ a suitable ansatz will reduce the Yang-Mills equation to an o.d.e. system. Of course, we must think of such connections as being Yang-Mills and "totally homogeneous" in a suitable sense. It turns out that suitable "building blocks" for our construction will be the pull-backs of the Levi-Civita-connections of TS^{n_i} under the Yang-Mills eigenmaps defined and discussed above.

5 Reduction

We consider the sphere $S^{m_1+m_2+1}$ represented (somewhat sloppy concerning the interval endpoints) as the doubly warped product

$$(M, \gamma) := [0, \pi/2] \times_{\cos^2} S^{m_1} \times_{\sin^2} S^{m_2}.$$

The Riemannian manifold (M, γ) is isometric to the sphere with the standard Euclidean metric.

As indicated above, we consider an $SO(n_1 + n_2 + 1)$-bundle over M which is given as follows: Let $h_1 : S^{m_1} \to S^{n_1}$ and $h_2 : S^{m_2} \to S^{n_2}$ be Yang-Mills eigenmaps. The bundle $\Phi^* TN \to M$ under consideration is the pull-back of the tangent bundle of the warped product

$$[0, \pi/2] \times_{\cos^2} S^{n_1} \times_{\sin^2} S^{n_2} =: (N, g) \cong S^{n_1+n_2+1}$$

via the map

$$\Phi := (\mathrm{id}, h_1, h_2) : M \to N,$$

where the parameters (λ, μ) from above are now denoted by (λ_i, μ_i) for h_i. As discussed in the previous section, this bundle can be viewed as a bundle on all of $S^{n_1+n_2+1}$, closing smoothly across the endpoints of $[0, \pi/2]$.

The h_i-pullback of the Levi-Civita connection of S^{n_i} will be denoted by ∇^i and its curvature by F^i. In particular, F^1 and F^2 are given by

$$\begin{aligned} F^1_{uv} U &= g(\partial_v h_1, U) \partial_u h_1 - g(\partial_u h_1, U) \partial_v h_1, \\ F^2_{wz} W &= g(\partial_z h_2, W) \partial_w h_2 - g(\partial_w h_2, W) \partial_z h_2. \end{aligned}$$

In what follows, we denote the variable in $[0, \pi/2]$ by t, and the vector field $\frac{\partial}{\partial t}$ by x if viewed as a vector field in TM, and by X when viewed as a vector field in $\Phi^* TN$.

By u,v we mean vector fields in TM tangential to S^{m_1} and by w,z tangential to S^{m_2}. Similarly, U,V denote vector fields in Φ^*TN tangential to S^{n_1} and W,Z vector fields tangential to S^{n_2}.

The pull-back ∇ of the Levi-Civita connection of $N \cong S^{n_1+n_2+1}$ by Φ is characterized by

$$\begin{aligned}
\nabla_x X &= 0, \\
\nabla_u V &= \cos(t)^{-1}\nabla^1_u V + \tan(t) g(\partial_u h_1, V)X, \\
\nabla_w Z &= \sin(t)^{-1}\nabla^2_w Z - \cot(t) g(\partial_w h_2, Z)X, \\
\nabla_u X &= -\tan(t)\partial_u h_1, \\
\nabla_w X &= \cot(t)\partial_w h_2, \\
\nabla_x V &= -\tan(t)V, \\
\nabla_x Z &= \cot(t)Z, \\
\nabla_u Z &= 0, \\
\nabla_w V &= 0.
\end{aligned}$$

The ansatz we make for our connection D on Φ^*TN differs only slightly from that, in an "equivariant" way:

$$\begin{aligned}
D_x X &= 0, \\
D_u V &= \frac{1}{\cos(t)}\left(\nabla^1_u V + \alpha(t) g(\partial_u h_1, V)X\right), \\
D_w Z &= \frac{1}{\sin(t)}\left(\nabla^2_w Z - \beta(t) g(\partial_w h_2, Z)X\right), \\
D_u X &= -\frac{\alpha(t)}{\cos(t)}\partial_u h_1, \\
D_w X &= \frac{\beta(t)}{\sin(t)}\partial_w h_2, \\
D_x V &= -\tan(t)V, \\
D_x Z &= \cot(t)Z, \\
D_u Z &= 0, \\
D_w V &= 0.
\end{aligned}$$

This connection is still metric with respect to g. The basic idea for finding α and β for which D is Yang-Mills will be minimizing the Yang-Mills functional over $SO(m_1+1)\times SO(m_2+1)$-equivariant connections (which D is).

Now we are ready to calculate the curvature of D, which we denote by F. Since we know that F is a tensor, i.e. a differential operator of order 0, we can assume we are calculating everything in a point where $\nabla^1_u U = 0$ etc. and $\nabla^1_u \partial_v h_1 = \nabla^1_v \partial_u h_1$ etc.:

$$F_{uv}U = (D_u D_v - D_v D_u)U$$

$$
\begin{array}{rcl}
 & = & D_u(\frac{1}{\cos}\nabla^1_v U + \frac{\alpha}{\cos} g(\partial_v h_1, U)X) - D_v(\frac{1}{\cos}\nabla^1_u U + \frac{\alpha}{\cos} g(\partial_u h_1, U)X) \\
 & = & \frac{1}{\cos^2}(\nabla^1_u\nabla^1_v U - \nabla^1_v\nabla^1_u U) \\
 & & + \frac{\alpha}{\cos}(g(\nabla^1_u\partial_v h_1 - \nabla^1_v\partial_u h_1, U) + g(\partial_v h_1, \nabla^1_u U) - g(\partial_u h_1, \nabla^1_v U))X \\
 & & + \frac{\alpha^2}{\cos^2}(g(\partial_u h_1, \partial_v h_1)g(X,U) - g(\partial_v h_1, \partial_u h_1)g(X,U))X \\
 & & + \frac{\alpha^2}{\cos^2}(g(\partial_u h_1, U)g(\partial_v h_1, X) - g(\partial_v h_1, U)g(\partial_u h_1, X))X \\
 & & + \frac{\alpha^2}{\cos^2}(-g(\partial_v h_1, U)\partial_u h_1 + g(\partial_u h_1, U)\partial_v h_1) \\
 & = & \frac{1}{\cos^2}F^1_{uv}U + \frac{\alpha^2}{\cos^2}(g(\partial_u h_1, U)\partial_v h_1 - g(\partial_v h_1, U)\partial_u h_1) \\
 & = & \frac{\alpha^2-1}{\cos^2}(g(\partial_u h_1, U)\partial_v h_1 - g(\partial_v h_1, U)\partial_u h_1), \\
F_{uv}W & = & 0, \\
F_{uv}X & = & -D_u(\frac{\alpha}{\cos}\partial_v h_1) + D_v(\frac{\alpha}{\cos}\partial_u h_1) \\
 & = & -\nabla^1_u(\frac{\alpha}{\cos^2}\partial_v h_1) + \nabla^1_v(\frac{\alpha}{\cos^2}\partial_u h_1) - \frac{\alpha^2}{\cos^2}(g(\partial_u h_1, \partial_v h_1) - g(\partial_v h_1, \partial_u h_1))X \\
 & = & 0, \\
F_{wz}W & = & \frac{\beta^2-1}{\sin^2}(g(\partial_w h_1, W)\partial_z h_1 - g(\partial_z h_1, W)\partial_w h_1), \\
F_{wz}U & = & 0, \\
F_{wz}X & = & 0, \\
F_{xu}U & = & D_x(\frac{1}{\cos}\nabla^1_u U + \frac{\alpha}{\cos} g(\partial_u h_1, U)X) - D_u(-\tan U) \\
 & = & \frac{\alpha'+\alpha\tan}{\cos} g(\partial_u h_1, U)X - 2\frac{\alpha\tan}{\cos} g(\partial_u h_1, U)X + \frac{\alpha\tan}{\cos} g(\partial_u h_1, U)X \\
 & = & \frac{\alpha'}{\cos} g(\partial_u h_1, U)X \\
F_{xu}W & = & D_x(D_u W) - D_u(\cot W) \\
 & = & 0, \\
F_{xu}X & = & D_x D_u X \\
 & = & -D_x(\frac{\alpha}{\cos}\partial_u h_1) \\
 & = & -(\frac{\alpha'+\alpha\tan}{\cos} - \frac{\alpha\tan}{\cos})\partial_u h_1 \\
 & = & -\frac{\alpha'}{\cos}\partial_u h_1, \\
F_{xw}U & = & 0, \\
F_{xw}W & = & -\frac{\beta'}{\sin} g(\partial_w h_2, W)X, \\
F_{xw}X & = & -\frac{\beta'}{\sin}\partial_w h_2, \\
F_{uw}U & = & -D_w D_u U \\
 & = & -D_w(\frac{1}{\cos}\nabla^1_u U + \frac{\alpha}{\cos} g(\partial_u h_1, U)X) \\
 & = & -\frac{\alpha}{\cos} g(\partial_u h_1, U)D_w X \\
 & = & -\frac{\alpha\beta}{\cos\sin} g(\partial_u h_1, U)\partial_w h_2, \\
F_{uw}W & = & \frac{\alpha\beta}{\cos\sin} g(\partial_w h_2, W)\partial_u h_1,
\end{array}
$$

$$
\begin{aligned}
F_{uw}X &= D_u(\cot\partial_w h_2) - D_w(\tan\partial_u h_1)\\
&= 0.
\end{aligned}
$$

Summing over γ-orthonormal bases u,v,w,z and g-orthonormal bases U,W, we infer

$$
|F|^2 = 4\lambda_1\,\frac{\alpha'^2}{\cos^2} + 4\lambda_2\frac{\beta'^2}{\sin^2} + 2\lambda_1\mu_1\,\frac{(\alpha^2-1)^2}{\cos^4} + 2\lambda_2\mu_2\,\frac{(\beta^2-1)^2}{\sin^4} + 4\lambda_1\lambda_2\frac{\alpha^2\beta^2}{\cos^2\sin^2}.
$$

Therefore, up to a constant depending only on m_1, m_2, the Yang-Mills functional of D equals

$$
\begin{aligned}
J(\alpha,\beta) \;:=\; & \int_0^{\pi/2}\Big\{\frac{2\lambda_1}{\cos^2}\,\alpha'^2 + \frac{2\lambda_2}{\sin^2}\,\beta'^2 + \frac{2\lambda_1\lambda_2}{\cos^2\sin^2}\,\alpha^2\beta^2\\
&\quad + \frac{\lambda_1\mu_1}{\cos^4}(\alpha^2-1)^2 + \frac{\lambda_2\mu_2}{\sin^4}(\beta^2-1)^2\Big\}\cos^{m_1}\sin^{m_2}\,dt. \qquad (2)
\end{aligned}
$$

The Euler-Lagrange equations of J are

$$
\alpha'' + (m_2\cot - (m_1-2)\tan)\alpha' - \frac{\mu_1}{\cos^2}(\alpha^3-\alpha) - \frac{\lambda_2}{\sin^2}\,\alpha\beta^2 = 0, \qquad (3)
$$

$$
\beta'' + ((m_2-2)\cot - m_1\tan)\beta' - \frac{\mu_2}{\sin^2}(\beta^3-\beta) - \frac{\lambda_1}{\cos^2}\,\alpha^2\beta = 0. \qquad (4)
$$

The reduction setting is made in such a way that stationary points of the reduced functional J represent Yang-Mills fields:

Proposition 2 (reduction theorem) *The connection D is a smooth Yang-Mills connection on Φ^*TN if and only if the functions $\alpha,\beta : [0,\pi/2]\to\mathbb{R}$ are solutions of (3), (4) with the boundary values*

$$
\alpha(0)=0, \qquad \alpha(\pi/2)=1, \qquad \beta(0)=1, \qquad \beta(\pi/2)=0. \qquad (5)
$$

Proof. To calculate the Yang-Mills equations for our setting, we have to differentiate F. In the following calculations, we make the same assumptions on the vector fields as above. We also use the fact that F^1, F^2 are Yang-Mills connections, and assume summation if an "index" is repeated.

$$
\begin{aligned}
(D_uF_{uv})U &= D_u(F_{uv}U) - F_{uv}D_uU\\
&= \tfrac{\alpha^2-1}{\cos^2}D_u(g(\partial_u h_1,U)\partial_v h_1 - g(\partial_v h_1,U)\partial_u h_1)\\
&\quad - \tfrac{1}{\cos}F_{uv}\nabla^1_u U - \tfrac{\alpha}{\cos}g(\partial_u h_1,U)F_{uv}X\\
&= (\nabla^{1*}F^1)_v U + \tfrac{\alpha^3-\alpha}{\cos^3}(g(\partial_u h_1,U)g(\partial_u h_1,\partial_v h_1)\\
&\quad - g(\partial_v h_1,U)g(\partial_u h_1,\partial_u h_1))X,\\
&= \tfrac{\alpha^3-\alpha}{\cos^3}(g(\partial_u h_1,U)g(\partial_u h_1,\partial_v h_1) - g(\partial_v h_1,U)g(\partial_u h_1,\partial_u h_1))X,
\end{aligned}
$$

$$
\begin{aligned}
(D_u F_{uv})W &= 0, \\
(D_u F_{uv})X &= D_u(F_{uv}X) + F_{uv}(\tfrac{\alpha}{\cos}\partial_u h_1) \\
&= \tfrac{\alpha^3-\alpha}{\cos^3}(g(\partial_u h_1, \partial_u h_1)\partial_v h_1 - g(\partial_u h_1, \partial_v h_1)\partial_u h_1), \\
(D_w F_{wv})U &= D_w(\tfrac{\alpha\beta}{\cos\sin} g(\partial_v h_1, U)\partial_w h_2) \\
&= -\tfrac{\alpha\beta^2}{\cos\sin^2} g(\partial_v h_1, U) g(\partial_w h_2, \partial_w h_2)X, \\
(D_w F_{wv})W &= -D_w(\tfrac{\alpha\beta}{\sin\cos} g(\partial_w h_2, W)\partial_v h_1) \\
&= 0, \\
(D_w F_{wv})X &= -F_{wv} D_w X \\
&= -\tfrac{\beta}{\sin} F_{wv}\partial_w h_2 \\
&= \tfrac{\alpha\beta^2}{\cos\sin^2} g(\partial_w h_2, \partial_w h_2)\partial_v h_1, \\
(D_x F_{xv})U &= D_x(\tfrac{\alpha'}{\cos} g(\partial_v h_1, U)X) - F_{xv}(-\tan U) \\
&= (\tfrac{\alpha''+\alpha'\tan}{\cos} - 2\tfrac{\alpha'\tan}{\cos} + \tfrac{\alpha'\tan}{\cos}) g(\partial_v h_1, U)X \\
&= \tfrac{\alpha''}{\cos} g(\partial_v h_1, U)X, \\
(D_x F_{xv})W &= 0, \\
(D_x F_{xv})X &= -D_x(\tfrac{\alpha'}{\cos}\partial_v h_1) \\
&= (-\tfrac{\alpha''+\alpha'\tan}{\cos} + \tfrac{\alpha'\tan}{\cos})\partial_v h_1 \\
&= -\tfrac{\alpha''}{\cos}\partial_v h_1; \\
D_u F_{ux} &= 0;
\end{aligned}
$$

and similar terms for S^{n_2}-components.

The next thing we have to check is what D^* looks like in our coordinates. There are induced metrics from g for 1-forms and 2-forms, which we again denote by g. By the definition of D^* and partial integration, we find for every 2-form G and every 1-form φ

$$
\begin{aligned}
&-\int_0^{2\pi}\int_{S^{m_1}}\int_{S^{m_2}} g(D^*G, \varphi)\cos^{m_1}\sin^{m_2}\ d\mathrm{vol}_2\, d\mathrm{vol}_1\, dt \\
&\quad= -\int_0^{2\pi}\int_{S^{m_1}}\int_{S^{m_2}} g(G, D\varphi)\cos^{m_1}\sin^{m_2}\ d\mathrm{vol}_2\, d\mathrm{vol}_1\, dt \\
&\quad= \int_0^{2\pi}\int_{S^{m_1}}\int_{S^{m_2}} g(D\cdot G, \varphi)\cos^{m_1}\sin^{m_2}\ d\mathrm{vol}_2\, d\mathrm{vol}_1\, dt \\
&\qquad+ \int_0^{2\pi}\int_{S^{m_1}}\int_{S^{m_2}} g(G(x,\,\cdot\,), \varphi)(m_2\cos^{m_1+1}\sin^{m_2-1} - m_1\cos^{m_1-1}\sin^{m_2+1})\, d\mathrm{vol}_2\, d\mathrm{vol}_1\, dt \\
&\qquad+ \int_0^{2\pi}\int_{S^{m_1}}\int_{S^{m_2}} (\partial_t g)(G(x,\,\cdot\,), \varphi)\cos^{m_1}\sin^{m_2}\ d\mathrm{vol}_2\, d\mathrm{vol}_1\, dt.
\end{aligned}
$$

Knowing that

$$\partial_t g = 2\,\mathrm{diag}(-(\tan t)\mathrm{id}_{m_1}, (\cot t)\mathrm{id}_{m_2}, 0)\, g,$$

we can read off from the previous equation how D^* operates. This is combined with the calculation above and (ii), (iii) to give

$$\begin{aligned}
-(D^*F)_v U &= (D\cdot F)_v U + (m_2\cot - (m_1-2)\tan)F_{xv}U \\
&= \Big\{ -\frac{\mu_1}{\cos^3}(\alpha^3-\alpha) - \frac{\lambda_2}{\cos\sin^2}\alpha\beta^2 + \frac{\alpha''}{\cos} \\
&\qquad + \frac{\alpha'}{\cos}(m_2\cot - (m_1-2)\tan)\Big\} g(\partial_v h_1, U)X, \\
-(D^*F)_v X &= (D\cdot F)_v X + (m_2\cot - (m_1-2)\tan)F_{xv}X \\
&= \Big\{ \frac{\mu_1}{\cos^3}(\alpha^3-\alpha) + \frac{\lambda_2}{\cos\sin^2}\alpha\beta^2 - \frac{\alpha''}{\cos} \\
&\qquad - \frac{\alpha'}{\cos}(m_2\cot - (m_1-2)\tan)\Big\} \partial_v h_1,
\end{aligned}$$

and the corresponding equations for the S^{n_2} components. (Some components always vanish.) This proves that D is Yang-Mills away from $t \in \{0, \pi/2\}$ if and only if (3) and (4) are satisfied.

Now we turn to the boundary conditions. Because $\alpha = \sin$ and $\beta = \cos$ correspond to the pullback of the Levi-Civita connection of $TS^{n_1+n_2+1}$, the boundary conditions (5) make sure that the connection is continuous even across the singular orbits $\{t = 0\}$ and $\{t = \pi/2\}$. For D to be of class C^1, α and β also have to satisfy

$$\alpha'(0) = 1, \qquad \alpha'(\pi/2) = 0, \qquad \beta'(0) = 0, \qquad \beta'(\pi/2) = -1. \tag{6}$$

But this is easily seen to hold for any solution of the boundary value problem made of (3), (4), (5). Once this is checked, the parity of the differential equations (3) and (4) implies that for any solution with the boundary values (5) and (6) the function α is odd with respect to $t = 0$ and even with respect to $t = \pi/2$, while for β the opposite holds. But those are exactly the conditions to ensure that D is smooth across the singular orbits. This proves the reduction theorem. $\square$

From the harmonic map analogon of our problem, we know that the substitution

$$\alpha(t) = A(\log(\tan t)), \qquad \beta(t) = B(\log(\tan t))$$

is useful. With $s = \log(\tan t)$ we calculate

$$\begin{aligned}
\alpha'(t) &= (e^s + e^{-s})A'(s), \\
\alpha''(t) &= (e^s + e^{-s})^2 A''(s) + (e^{2s} - e^{-2s})A'(s)
\end{aligned}$$

etc., which transforms (3)–(5) to give

$$\begin{aligned}
A'' - \frac{(m_1-3)e^s - (m_2-1)e^{-s}}{e^s+e^{-s}}A' - \frac{\mu_1 e^s}{e^s+e^{-s}}(A^3 - A) - \frac{\lambda_2 e^{-s}}{e^s+e^{-s}}AB^2 &= 0, \quad (7)\\
B'' - \frac{(m_1-1)e^s - (m_2-3)e^{-s}}{e^s+e^{-s}}B' - \frac{\mu_2 e^{-s}}{e^s+e^{-s}}(B^3 - B) - \frac{\lambda_1 e^s}{e^s+e^{-s}}A^2B &= 0. \quad (8)
\end{aligned}$$

with the boundary conditions

$$A(-\infty) = 0, \qquad A(\infty) = 1, \qquad B(-\infty) = 1, \qquad B(\infty) = 0. \tag{9}$$

6 Existence of solutions

6.1 The case $\mu_1, \mu_2 > 0$

The first case we consider is the case where none of the eigenconnections which are joined is flat. This is the case $\mu_1, \mu_2 > 0$ which is only possible if $m_1, m_2 \geq 2$. Then we find a minimizer of J by the direct method of the calculus of variations.

Lemma 1 (existence of minimizers) *Assume $m_1, m_2 \geq 2$ and $\mu_1, \mu_2 > 0$. Then there is a solution (a, b) of (3), (4) on $(0, \pi/2)$ which minimizes J among all $(\alpha, \beta) \in C^1((0, \pi/2))^2$. It satisfies $0 \leq a \leq 1$ and $0 \leq b \leq 1$.*

Proof. First we write $D = \nabla + \omega$ with a matrix-valued one-form ω which is given by

$$\begin{aligned}
\omega(u) &= \frac{\alpha - \sin}{\cos}(\partial_u h_1 \otimes X - X \otimes \partial_u h_1), \\
\omega(w) &= \frac{\cos - \beta}{\sin}(\partial_w h_2 \otimes X - X \otimes \partial_w h_2), \\
\omega(x) &= 0.
\end{aligned}$$

We find

$$\begin{aligned}
|\omega|^2 &= \frac{2\lambda_1}{\cos^2}(\alpha - \sin)^2 + \frac{2\lambda_2}{\sin^2}(\beta - \cos)^2, \\
|\nabla\omega|^2 &= \frac{2\lambda_1^2}{\cos^4}\alpha^2(\alpha - \sin)^2 + \frac{2\lambda_2^2}{\sin^4}\beta^2(\beta - \cos)^2 + \frac{2\lambda_1}{\cos^2}(\alpha' - \cos)^2 + \frac{2\lambda_2}{\sin^2}(\beta' + \sin)^2.
\end{aligned}$$

It is easily read off from this that

$$\int_M (|\omega|^4 + |\nabla\omega|^2) \leq c(1 + J(\alpha, \beta)) \tag{10}$$

with a constant c depending on $\lambda_1, \lambda_2, \mu_1, \mu_2$. Now we consider a minimizing sequence of connections D with $\omega \in W^{1,2} \cap L^4$ of the special form considered in this paper for the Yang-Mills functional YM. Since YM is lower semi-continuous on $W^{1,2} \cap L^4$, and since we have just checked that the $W^{1,2}$ and L^4 norms of ω stay bounded for such a sequence, there is a connection minimizing YM of the form considered in $W^{1,2} \cap L^4$, which must be continuous in the orbits over $(0, \pi/2)$. This connection is represented by a minimizer (a, b) of J. Since minimizers of J satisfy its Euler-Lagrange equations, (a, b) must be smooth on $(0, \pi/2)$.

The next step is to prove that there are minimizers with values in $[0,1]$. To this end, consider $f:[0,\pi/2]\to\mathbb{R}$ and define

$$\tilde{f}(x) := \begin{cases} |f(x)| & \text{if } |f(x)|\leq 1, \\ \frac{1}{|f(x)|} & \text{if } |f(x)|>1. \end{cases}$$

Then $\tilde{f}'^2 \leq f'^2$ and $(\tilde{f}^2-1)^2 \leq (f^2-1)^2$. Hence, if (a,b) is minimizing, so is $(\tilde{a},\tilde{b})$, which means we have found the minimizing solution stated in the lemma. □

Remark 2 *Assume $m_1,m_2\geq 2$ and $\mu_1,\mu_2>0$. There are exactly three constant solutions of (3), (4) with values in $[0,1]$, namely $(\alpha,\beta)\equiv(0,0)$, $\equiv(0,1)$ or $\equiv(1,0)$. The constant solution $(0,0)$ is never minimizing because of $J(0,0)>J(0,1)$ and $J(0,0)>J(1,0)$.*

Lemma 2 (nonconstant minimizers I) *Assume $m_1,m_2\geq 2$ and $\mu_1,\mu_2>0$. No nonconstant J-minimizing solution (α,β) of (3), (4) with values in $[0,1]$ assumes the values $\alpha(t)\in\{0,1\}$ or $\beta(t)\in\{0,1\}$ at any $t\in(0,\pi/2)$.*

Proof. Assume $\alpha(t)=1$, then we have $\alpha'(t)=0$, which by way of (3) implies $\alpha''(t)=\lambda_2\sin^{-2}(t)\beta(t)^2\geq 0$. Since α has a maximum at t, this can be true only if $\alpha''(t)=0$ and $\beta(t)=0$. The latter implies $\beta'(t)=0$. By uniqueness of the solution of the boundary value problem with $\alpha,\alpha',\beta,\beta'$ prescribed at t, we would have $(\alpha,\beta)\equiv(1,0)$.

By the same reasoning, $\beta(t)=1$ implies $(\alpha,\beta)\equiv(0,1)$.

Now assume $\alpha(t)=0$, then $\alpha'(t)=0$, and (3) implies $\alpha''(t)=0$. Differentiate (3) and find $\alpha^{(k)}(t)=0$ for all $k\in\mathbb{N}$. By the analyticity of α, we find $\alpha\equiv 0$. Once we have this, we find

$$J(\alpha,\beta)=\int_0^{\pi/2}\left\{\frac{2\lambda_2}{\sin^2}\beta'^2+\frac{\lambda_2\mu_2}{\sin^4}(\beta^2-1)^2+\frac{\lambda_1\mu_1}{\cos^4}\right\}\cos^{m_1}\sin^{m_2}\,dt,$$

which is infinity in case $m_1\leq 3$ (a contradiction) or easily seen to be minimized by $\beta\equiv 1$.

By the same reasoning, $\beta(t)=0$ gives a contradiction or $(\alpha,\beta)\equiv(1,0)$. □

Lemma 3 (nonconstant minimizers II) *Assume $m_1,m_2\geq 2$ and $\mu_1,\mu_2>0$. If neither of the constant solutions $(0,1)$ and $(1,0)$ is minimizing, there is a solution of the boundary value problem (3)–(5).*

Proof. Since $(0,0)$ is never minimizing, the assumption implies that there has to be a nonconstant minimizer (α,β) of J with values in $[0,1]$.

In case $m_1\in\{2,3\}$ we see that $J(\alpha,\beta)<\infty$ only if $\alpha(\pi/2)=1$. But then (4) implies that $\beta(\pi/2)=0$ (using the fact that β' cannot explode like $\frac{1}{t-\pi/2}$ if J is finite).

Similarly, $m_2 \in \{2,3\}$ implies $\beta(0)=1$ and $\alpha(0)=0$.

No we consider $m_2 \geq 4$. Observe that the only boundary values for $\beta(0)$ that the equations (3), (4) allow are -1, 0 or 1. We have already ruled out -1. If $\beta(0)$ was 0, the corresponding solution B of (7), (8) would asymptotically (as $s \to -\infty$) satisfy the linearized version of (7), $B'' + (m_2-3)B' + \mu_2 B = 0$. But a fundamental system for this linearized equation consists of $\exp((-\frac{m_2-3}{2} \pm \frac{1}{2}\sqrt{(m_2-3)^2 - 4\mu_2})s)$ neither of which is bounded at $-\infty$. Hence $B(-\infty)=0$ is not possible. This proves $\beta(0)=1$, and $\alpha(0)=0$ follows as in the case $m_2 \leq 3$.

The same way we see that $\alpha(\pi/2)=1$ and $\beta(\pi/2)=0$ also in the case $m_1 \geq 4$. $\square$

The lemma shows that it helps to know if the constant solutions are minimizing. Now they are clearly not minimizing if they are unstable (in the sense of negative directions for the second variation) or have infinite J-energy.

Lemma 4 (unstable constant solution) *Assume* $m_1, m_2 \geq 2$, $\mu_1, \mu_2 > 0$. *The constant solution* $(\alpha, \beta) \equiv (0,1)$ *is unstable or has infinite* J*-energy iff*

$$\begin{array}{ll} & m_1 \in \{2,3\} \\ \textit{or} & (m_1-3)^2 < 4\mu_1 \\ \textit{or} & \sqrt{(m_2-1)^2 + 4\lambda_2} + \sqrt{(m_1-3)^2 - 4\mu_1} < m_1 + m_2 - 4. \end{array}$$

Proof. The case $m_1 \in \{2,3\}$ is trivial. For $m_1 \geq 4$, we calculate

$$\begin{aligned} \frac{d^2}{ds^2}_{|s=0} J(s\varphi, 1+s\psi) \quad = \quad & \int_0^{\pi/2} \Big\{ \frac{4\lambda_1}{\cos^2}\varphi'^2 + \frac{4\lambda_2}{\sin^2}\psi'^2 + \frac{4\lambda_1\lambda_2}{\cos^2\sin^2}\varphi^2 \\ & - \frac{4\lambda_1\mu_1}{\cos^4}\varphi^2 + \frac{8\lambda_2\mu_2}{\sin^4}\psi^2 \Big\} \cos^{m_1}\sin^{m_2}\, dt. \end{aligned}$$

This means that $(0,1)$ is unstable iff the quadratic form

$$H(\varphi) := \int_0^{\pi/2} \Big\{ \varphi'^2 + \Big(\frac{\lambda_2}{\sin^2} - \frac{\mu_1}{\cos^2}\Big)\varphi^2 \Big\} \cos^{m_1-2}\sin^{m_2}\, dt$$

becomes negative for some (bounded) function φ. This has been discussed by Ding in [Di] for the same function H that arises with different constants in the construction of harmonic maps as joins of harmonic eigenmaps. A detailed discussion can be found in [ER2, IX (4.4)–(4.16)]. It shows that $H(\varphi)$ attains negative values if and only if one of the three assumptions of the lemma is fulfilled. $\square$

Combining the four Lemmas from this section and the analogon of Lemma (4) for the constant solution $(1,0)$, we get our main theorem:

Theorem 1 *Assume $m_1, m_2 \geq 2$ and $\mu_1, \mu_2 > 0$. There is a Yang-Mills connection of $\Phi^* TN$ corresponding to a solution (α, β) of the boundary value problem (3)–(5) if the following conditions hold:*

$$
\begin{array}{lll}
(D1) & & m_1 \in \{2,3\} \\
& or & (m_1-3)^2 < 4\mu_1 \\
& or & \sqrt{(m_2-1)^2+4\lambda_2} + \sqrt{(m_1-3)^2-4\mu_1} < m_1+m_2-4
\end{array}
$$

and

$$
\begin{array}{lll}
(D2) & & m_2 \in \{2,3\} \\
& or & (m_2-3)^2 < 4\mu_2 \\
& or & \sqrt{(m_1-1)^2+4\lambda_1} + \sqrt{(m_2-3)^2-4\mu_2} < m_1+m_2-4.
\end{array}
$$

Remark 3 *If $m_1 = m_2$, $\lambda_1 = \lambda_2$, and $\mu_1 = \mu_2$, the boundary value problem (3)–(5) has always a solution. This can be seen by modifying the proof in such a way that one only minimizes over (α, β) satisfying $\alpha(\frac{\pi}{2} - t) = \beta(t)$ for all $t \in (0, \frac{\pi}{2})$. Therefore, the conditions $(D1)$ and $(D2)$ can only be sharp if they are automatically satisfied in the case $m_1 = m_2$, $\lambda_1 = \lambda_2$, $\mu_1 = \mu_2$. Unfortunately, this is not the case if μ_1 is small compared to λ_1. Therefore, in the general setting considered here, the condition of Theorem 1 is sufficient, but not necessary.*

However, in our examples we always have $\mu_i = \frac{m_i-1}{m_i}\lambda_i$; and maybe this is so for all Yang-Mills eigenmaps. Under this additional assumption (which is the only case relevant for our construction at the moment), $(D1)$ and $(D2)$ are always fulfilled if $m_1 = m_2$, $\lambda_1 = \lambda_2$, and it is still possible that Theorem 1 is sharp. We tend to expect this to hold, because of the close analogy to the harmonic map case, where very similar conditions have been proven to be sharp [Di] [PR].

6.2 The case $m_2 = 1$

An interesting case is $m_2 = 1$, $h_2 : S^1 \to S^1$ with $h_2(z) = z^k$ for some $k \in \mathbb{Z} \setminus \{0\}$. The function h_2 is a Yang-Mills eigenmap, but we have $\mu_2 = 0$, which means the techniques of the previous section do not apply. However, in this particular case, we can modify the proof of Theorem 1 to make it still work (which we cannot do if ∇^2 is a flat connection on a higher-dimensional sphere). The existence theorem here reads as follows (with $\lambda_2 = k^2$)

Theorem 2 *Assume $m_1 \geq 2$, $\mu_1 > 0$, $m_2 = 1$, $\mu_2 = 0$, $\lambda_2 = k^2 \neq 0$. There is a Yang-Mills connection of $\Phi^* TN$ corresponding to a solution (α, β) of the boundary value problem (3)–(5) if (D1) holds, which now reads*

$$
\begin{array}{ll}
& m_1 \in \{2,3\} \\
or & (m_1-3)^2 < 4\mu_1 \\
or & 2\,|k| + \sqrt{(m_1-3)^2-4\mu_1} < m_1 - 3.
\end{array}
$$

Proof. Few changes are to be made compared to the proof of Theorem 1. The main problem is that $J(\alpha,\beta)$ does no longer contain a $(\beta^2-1)^2$-term. This means that $\beta(0)\in\{-1,0,1\}$ is no longer needed to make J finite. Now $\beta(0)$ can take any value in $\mathbb{R}$ (probably), and it will not be true that a minimizer of J will more or less automatically satisfy $\beta(0)=1$. But here we can impose the boundary value $\beta(0)=1$ and minimize under this condition.

To prove this assertion, let $(\alpha_n,\beta_n)_{n\in\mathbb{N}}$ be a minimizing sequence for J under the additional hypothesis $\beta(0)=1$. Again we may assume that the images of α_n and β_n are contained in $[0,1]$. As above, we assume that the minimizing sequence belongs to a form $\omega\in W^{1,2}\cap L^4$. Finiteness of the norms implies that $\beta_n'(0)=-1$ for all $n\in\mathbb{N}$.

In the proof of Lemma (1), it is not immediately clear that (10) still holds, because this time the $(\beta-\cos)^4$-term and the $\beta^2(\beta-\cos)^2$-Term cannot be estimated easily by the $(\beta^2-1)^2$-term which is no longer in J. But in the case $m_2=1$, it can be estimated by the β'^2-term of J instead. This can be seen as follows. We assume that the image of β is contained in $[0,1]$ and that $\beta(0)=1$, $\beta'(0)=0$. We combine

$$\begin{aligned}
\int_0^1(\beta(t)-\cos(t))^2\cos(t)^{m_1}\sin(t)^{-3}\,dt &\le c+c\int_0^1(\beta(t)-1)^2t^{-3}\,dt\\
&\le c+c\int_0^1\Big(\int_0^t\beta'(\tau)\,d\tau\Big)^2t^{-3}\,dt\\
&\le c+c\int_0^1\int_0^t\beta'(\tau)^2\,d\tau\,t^{-2}\,dt\\
&\le c+c\int_0^1\beta'(t)^2\int_t^1\tau^{-2}\,d\tau\,dt\\
&= c+\frac{c}{2}\int_0^1\beta'(t)^2(t^{-1}-1)\,dt\\
&\le c+c\int_0^1\beta'(t)^2\cos(t)^{m_1}\sin(t)^{-1}\,dt
\end{aligned}$$

with

$$\begin{aligned}
\int_1^{\pi/2}(\beta-\cos)^2\cos^{m_1}\sin^{-3}\,dt &\le c\int_1^{\pi/2}(\beta^2+\cos^2)\cos^{m_1}\sin^{-3}\,dt\\
&\le c+c\int_1^{\pi/2}\beta^2\cos^{m_1}\sin^{-3}\,dt\\
&\le c+c\int_1^{\pi/2}\beta'^2\cos^{m_1}\sin^{-1}\,dt
\end{aligned}$$

to find

$$\int_0^{\pi/2}(\beta-\cos)^2\cos^{m_1}\sin^{-3}\,dt\le c+c\int_0^{\pi/2}\beta'^2\cos^{m_1}\sin^{-1}\,dt. \tag{11}$$

Here, we have used $\beta \in [0,1]$ several times, and the Sobolev inequality for the manifold $([0,\pi/2], \cos^{m_1} \sin^{-3})$ in the second estimate. All integrals are finite because of $\beta(0) = 1$ and $\beta'(0) = 0$. Using again $\beta \in [0,1]$, we see that (11) implies

$$\int_0^{\pi/2} \{\beta^2(\beta - \cos)^2 + (\beta - \cos)^4\} \cos^{m_1} \sin^{-3}\, dt \leq \int_0^{\pi/2} \beta'^2 \cos^{m_1} \sin^{-1}\, dt,$$

which is exactly what was missing in the proof that (10) still holds.

Once we have (10), we know that our minimizing sequence $(\alpha_n, \beta_n)_{n\in\mathbb{N}}$ stays bounded in $W^{1,2} \cap L^4$ and hence has a weakly convergent subsequence. Again we use lower semi-continuity of YM to conclude convergence of (α_n, β_n) to a minimizer (α, β) of J under the additional condition $\beta(0) = 1$. The only thing that remains to be checked is that the latter condition is actually preserved in the limit. Assume it is not. Then there is $\varepsilon > 0$ and a subsequence of (β_n), again denoted by (β_n), such that $\min_{[0,1/n]} \beta_n(x) \leq 1 - \varepsilon$. Since also $\beta_n(0) = 1$, this would imply $\int_0^{1/n} \beta_n'^2\, dt \geq \frac{1}{n}(n\varepsilon)^2 = n\varepsilon^2$, which would mean $J(\alpha_n, \beta_n) \to \infty$, a contradiction.

We have now proved, that there exist a minimizer (α, β) of J under the additional condition $\beta(0) = 1$. From here, we proceed as in the proof of Theorem 1 to prove Theorem 2. $\square$

6.3 The case $m_2 = 0$: Suspensions

The case $m_2 = 0$ makes sense, not only formally. Remembering that $S^0 = \{-1, 1\}$, we see that the join of S^{m_1} and S^0 is nothing else than the suspension S^{m_1+1} of S^{m_1}. Consequently, we speak of Yang-Mills suspensions here rather than of joins. This corresponds to a (simply) warped product $M = [-\pi/2, \pi/2] \times_{\cos^2} S^{m_1} \cong S^{m_1+1}$ where we are looking for connections on some bundle Φ^*TN of the form

$$\begin{aligned} D_x X &= 0, \\ D_u V &= \frac{1}{\cos(t)} (\nabla_u^1 V + \alpha(t) g(\partial_u h_1, V) X), \\ D_u X &= -\frac{\alpha(t)}{\cos(t)}\, \partial_u h_1, \\ D_x V &= -\tan(t) V. \end{aligned}$$

All notation that is not declared has a similar meaning as before.

The reduced Yang-Mills functional is

$$J(\alpha) := \int_{-\pi/2}^{\pi/2} \left\{ \frac{2\lambda_1}{\cos^2}\, \alpha'^2 + \frac{\lambda_1 \mu_1}{\cos^4} (\alpha^2 - 1)^2 \right\} \cos^{m_1}\, dx$$

and the system of Euler-Lagrange equations reduces to a single equation

$$\alpha''(t) - (m_1 - 2)\tan(t)\alpha'(t) - \frac{\mu_1}{\cos(t)^2}\, (\alpha(t)^3 - \alpha(t)) = 0, \tag{12}$$

with the natural boundary conditions

$$\alpha(-\pi/2) = -1, \qquad \alpha(\pi/2) = 1. \tag{13}$$

A very similar boundary value problem has been solved for harmonic suspensions by Eells and Ratto [ER1]. Since here we are very close to the harmonic map case, we can omit details of the proof.

Theorem 3 *Assume $\mu_1 > 0$.*

(i) If $m_1 \geq 4$, there is a minimizing solution of the boundary value problem (12), (13) if and only if $\mu_1 > m_1 - 3$. If $m_1 \in \{2, 3\}$, there is always such a solution.

(ii) If $m_1 \geq 4$ and $\mu_1 > (m_1 - 3)^2/4$, there are even countably many $\alpha : [-\pi/2, \pi/2] \to [-1, 1]$ solving (12), (13) and representing smooth Yang-Mills connections on $\Phi^* TN$, none of which are gauge equivalent.

Proof. (i) Minimizing among all α with $\alpha(-t) = -\alpha(t)$, we find a minimizing solution of the boundary value problem (12), (13) if and only if the constant solution $\alpha \equiv 0$ is unstable or $J(0) = \infty$; that is (cf. [ER1, section 9]) if $\mu_1 > m_1 - 3$ or $m_1 \in \{2, 3\}$.

(ii) The proof is a very minor modification of the proof of the theorem in [BC], where the same is proved for harmonic suspensions $S^n \to S^n$ $(3 \leq n \leq 6)$ of the identity. □

7 Examples of Yang-Mills joins

Let us see what we can get out of the existence theorems.

Example 0. The Levi-Civita connection of $S^{m_1+m_2+1}$ is trivially Yang-Mills and is the special case $\alpha(t) = \sin t$ and $\beta(t) = \cos t$ (if β is needed) that Theorem 1, Theorem 2, and Theorem 3 allow if h_1 and h_2 are identities.

Example 1. Nevertheless, Theorem 3 also produces nontrivial solutions when applied to $h = \mathrm{id}_{m_1}$ with $4 \leq m_1 \leq 8$. And it is easy to prove that solutions of the o.d.e. with different functions α cannot be gauge equivalent to each other. This means that the theorem implies: *On every TS^m for $5 \leq m \leq 9$, there are countably many Yang-Mills connections that are mutually not gauge equivalent.*

Example 2. Now we try to join $h_{m_1,\ell}$ $(m_1 \geq 2)$ with id_{m_2}. It is easily checked that the conditions of the existence theorems are satisfied if $0 \leq m_2 \leq 8$. We can formulate that in the following way: *Each of the pulled back Levi-Civta connections on $h^*_{m,\ell} TS^{\frac{(2\ell+m-1)(\ell+m-2)!}{\ell!(m-1)!}-1}$ can be suspended as Yang-Mills connections 9 times.* This corresponds to Smith's observation [Sm] that every harmonic eigenmap can be suspended harmonically 6 times.

Example 3. The same applies for the case $m_1 = 1$, that is every $d_\ell : S^1 \to S^1$ can be suspended as a Yang-Mills connection 9 times (even for $\ell \in \mathbb{Z}$, once we know this for $\ell \in \mathbb{N}$). This is geometrically interesting for the following reason: We can interpret the joined bundles as $f_\ell^* TS^n$ for every $\ell \in \mathbb{Z}$ and every $n \in \{2, \dots, 10\}$, with f_ℓ being a map $S^n \to S^n$ of Brouwer degree ℓ. Depending on n, these may be many bundles, maybe even all $SO(n)$-bundles over S^n.

To be more precise, the $SO(n)$-bundles over S^n are classified rather easily. Since $SO(n)$ can be covered by just two coordinate patches overlapping on an annular region around the equator, they are classified by the homotopy class of the one transition function that is used to patch the two trivial bundles together; clearly this homotopy class can be seen as an element of $\pi_{n-1}(SO(n))$. But which element of $\pi_{n-1}(SO(n))$ corresponds to the bundles $f_\ell^* TS^n$ mentioned above? Since $\pi_{n-1}(SO(n))$ depends on n in a seemingly unpredictable way, there may be no simple answer. We can, however, make use of the fact that there is a homomorphism $e : \pi_{n-1}(SO(n)) \to \pi_{n-1}(S^{n-1}) \cong \mathbb{Z}$, and that the latter is simply parametrized by Brouwer degree, hence well-understood. The homeomorphism e is induced by simply evaluating every matrix-valued $A : S^{n-1} \to SO(n)$ at some fixed vector $x_0 \in S^n$ to give a mapping $Ax_0 : S^{n-1} \to S^{n-1}$.

Our first step now is to calculate which element of $\pi_{n-1}(S^{n-1})$ here corresponds to the tangent bundle TS^n. To this end, we observe that TS^n is parametrized by two coordinate patches $f_\pm : \overline{B^n} \times \mathbb{R}^n \to TS^n$ given by

$$\begin{aligned} f_+(x,v) &:= \Big(x + \sqrt{1-|x|^2}\, e_{n+1}\,,\, v - \frac{v\cdot x}{|x|^2}\, x + \frac{v\cdot x}{|x|^2}\Big(\sqrt{1-|x|^2}\,\frac{x}{|x|} - |x| e_{n+1}\Big)\Big), \\ f_-(x,v) &:= \Big(\bar{x} - \sqrt{1-|x|^2}\, e_{n+1}\,,\, v - \frac{v\cdot \bar{x}}{|x|^2}\, \bar{x} + \frac{v\cdot \bar{x}}{|x|^2}\Big(\sqrt{1-|x|^2}\,\frac{\bar{x}}{|x|} + |x| e_{n+1}\Big)\Big), \end{aligned}$$

where $\bar{x} = (-x_1, x_2, x_3, \dots, x_n)$. The images $f_+(S^{n-1})$ and $f_-(S^{n-1})$ overlap and both parametrize the bundle restricted to the equator of S^n. On S^{n-1}, f_+ and f_- simplify an read

$$\begin{aligned} f_+(x,v) &= (x, v - (v\cdot x)x - (v\cdot x)e_{n+1}), \\ f_-(x,v) &= (\bar{x}, v - (v\cdot \bar{x})\bar{x} + (v\cdot \bar{x})e_{n+1}). \end{aligned}$$

To get TS^n, we must find one transition map, and we observe

$$f_+(x,v) = f_-(\bar{x}, v - 2(v\cdot x)x) =: f_-(\bar{x}, \Phi(\bar{x})(v)),$$

where

$$\Phi : S^{n-1} \to SO(n), \qquad \Phi(x) := \mathrm{id} - 2\bar{x}\otimes\bar{x}$$

defines the transition map we have been looking for. Now

$$\Phi(x)(e_1) = e_1 + 2x_1\bar{x},$$

and it is easily calculated that this mapping $S^{n-1} \to S^{n-1}$ represents $\pm 2 \in \mathbb{Z} \cong \pi_{n-1}(S^{n-1})$ if n is even, and 0 if n is odd. Similarly, every f^*TS^{n-1} for continuous $f : S^n \to S^n$ can be assigned an element in $\pi_{n-1}(S^{n-1})$ this way, and this gives a homomorphism $\pi_n(S^n) \to \pi_{n-1}(S^{n-1})$. Therefore $f_\ell^*TS^n$ represents a bundle classified by an element of $\pi_{n-1}(SO(n))$ that e maps to $\pm 2\ell \in \pi_{n-1}(S^{n-1})$ if n is even, and 0 if n is odd. Hence we restrict to even n in our search for topologically nontrivial examples of our Yang-Mills join construction.

Recall that for $n \in \{2, 4, 6, 8, 10\}$, we were able to find a Yang-Mills connection on every $f_\ell^*TS^n$ for every $\ell \in \mathbb{Z}$, and the bundles correspond to $2\ell \in \mathbb{Z} \cong \pi_{n-1}(S^{n-1})$ under e. The groups $\pi_{n-1}(SO(n))$ and $\pi_{n-1}(S^{n-1})$ are related by the long exact sequence of the homogeneous space $S^{n-1} = SO(n)/SO(n-1)$ which reads

$$\ldots \xrightarrow{e} \pi_{k+1}(S^{n-1}) \to \pi_k(SO(n-1)) \to \pi_k(SO(n)) \xrightarrow{e} \pi_k(S^{n-1}) \to \pi_{k-1}(SO(n-1)) \to \ldots$$

where e is as before. To illustrate how we can use this, let us first consider the case $n = 6$. Here is a piece of the exact sequence:

$$\begin{array}{ccccccccc} \pi_5(SO(5)) & \xrightarrow{0} & \pi_5(SO(6)) & \xhookrightarrow{\cdot 2} & \pi_5(S^5) & \twoheadrightarrow & \pi_4(SO(5)) & \to & \pi_4(SO(6)). \\ \mathbb{Z}_2 & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z}_2 & & 0 \end{array}$$

It shows that $\pi_5(SO(6))$ maps injectively to $\pi_5(S^5)$, which means that every $SO(6)$-bundle over S^6 (represented by $j \in \pi_5(SO(6))$) can be written as $f_j^*TS^6$, and on those we find Yang-Mills connections. Hence

> *we have constructed Yang-Mills connections on each of the countably many (principal) $SO(6)$-bundles over S^6.*

Of course, we always mean S^6 equipped with its standard metric.

The same works for $SO(2)$-bundles over S^2, but the result is trivial because of the sub-critical domain dimension.

Here are the details for the remaining dimensions. For $n \in \{4, 8\}$, we have

$$\begin{array}{ccccc} \pi_{n-1}(SO(n)) & \twoheadrightarrow & \pi_{n-1}(S^{n-1}) & \to & \pi_{n-2}(SO(n-1)), \\ \mathbb{Z}^2 & & \mathbb{Z} & & 0 \end{array}$$

hence we find Yang-Mills connections on infinitely many $SO(4)$-bundles over S^4 or $SO(8)$-bundles over S^8, but not on all of them. For $n = 10$,

$$\begin{array}{ccccccccc} \pi_9(SO(10)) & \xrightarrow{(\cdot 2, 0)} & \pi_9(S^9) & \to & \pi_8(SO(9)) & \twoheadrightarrow & \pi_8(SO(10)) & \to & \pi_8(S^9), \\ \mathbb{Z} \oplus \mathbb{Z}_2 & & \mathbb{Z} & & \mathbb{Z}_2^2 & & \mathbb{Z}_2 & & 0 \end{array}$$

which shows that we can construct Yang-Mills connections on “half of” the countably many $SO(10)$-bundles over S^{10}.

References

[Ba] P. Baird: *Harmonic maps with symmetry, harmonic morphisms and deformations of metrics.* Research Notes in Mathematics, 87. Pitman (Advanced Publishing Program), Boston, MA, 1983.

[BC] P. Bizoń, T. Chmaj: *Harmonic maps between spheres.* Proc. Royal Soc. Lond., Ser. A **453** (1997), 403–415.

[Bo] C. Böhm: *Inhomogeneous Einstein metrics on low-dimensional spheres and other low-dimensional spaces.* Invent. Math. **134** (1998), 145–176.

[dCW] M.P. do Carmo, N.R. Wallach: *Minimal immersions of spheres into spheres.* Ann. of Math. (2) **93** (1971), 43–62.

[Di] W.Y. Ding: *Symmetric harmonic maps between spheres.* Comm. Math. Phys. **118** (1988), 641–649.

[ER1] J. Eells, A. Ratto: *Harmonic maps between spheres and ellipsoids.* Inter. J. Math. **1** (1990), 1–27.

[ER2] J. Eells, A. Ratto: *Harmonic maps and minimal immersions with symmetries.* Annals of Math. Studies **130**. Princeton University Press 1993.

[Fa] A. Fardoun: *On equivariant p-harmonic maps.* Ann. Inst. H. Poincaré Anal. Non Linéaire **15** (1998), 25–72.

[GZ] A. Gastel, F. Zorn: *Biharmonic maps of cohomogeneity one between spheres.* Submitted.

[Hs1] W.-Y. Hsiang: *Generalized rotational hypersurfaces of constant mean curvature in Euclidean spaces.* J. Diff. Geom. **17** (1982), 337–356.

[Hs2] W.-Y. Hsiang: *Minimal cones and the spherical Bernstein problem I.* Ann. Math. **118** (1983), 61–73.

[Hs3] W.-Y. Hsiang: *Minimal cones and the spherical Bernstein problem II.* Invent. Math. **74** (1983), 351–369.

[It] M. Itoh: *Invariant connections and Yang-Mills solutions.* Trans. Amer. Math. Soc. **267** (1981), 229–236.

[Pal] R. Palais: *The principle of symmetric criticality.* Comm. Math. Phys. **69** (1979), 19–30.

[Par] T.H. Parker: *Non-minimal Yang-Mills fields and dynamics.* Invent. Math. **107** (1992), 397–420.

[PU] J.S. Park, H. Urakawa: *Yang-Mills connections in homogeneous principal fibre bundles.* Int. J. Pure Appl. Math. **10** (2004), 79–90.

[PR] V. Pettinati, A. Ratto: *Existence and nonexistence results for harmonic maps between spheres.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **17** (1990), 273–282.

[Sm] R.T. Smith: *Harmonic mappings of spheres.* Amer. J. Math. **97** (1975), 364–385.

[St] N. Steenrod: *The topology of fibre bundles.* Princeton University Press 1951.

[Ur] H. Urakawa: *Equivariant theory of Yang-Mills connections over Riemannian manifolds of cohomogeneity one.* Indiana Univ. Math. J. **37** (1988), 753–788.

Andreas Gastel
Universität Duisburg-Essen
Fakultät für Mathematik, Campus Duisburg
D-47048 Duisburg, Germany
andreas.gastel@uni-due.de