Injective hulls of certain discrete metric spaces and groups

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Abstract

Injective metric spaces, or absolute 1-Lipschitz retracts, share a number of properties with CAT(0) spaces. Isbell showed that every metric space X has an injective hull $\mathrm{E}(X)$. We prove that if X is the vertex set of a connected locally finite graph with a uniform stability property of intervals, then $\mathrm{E}(X)$ is a locally finite polyhedral complex with finitely many isometry types of n-cells, isometric to polytopes in l_∞^n, for each n. This applies to a class of finitely generated groups Γ, including word hyperbolic and abelian groups, among others. Then Γ acts properly on $\mathrm{E}(\Gamma)$ by cellular isometries, and the first barycentric subdivision of $\mathrm{E}(\Gamma)$ is a model for the classifying space $\underline{\mathrm{E}}\Gamma$ for proper actions. If Γ is word hyperbolic, $\mathrm{E}(\Gamma)$ is finite dimensional and the action is cocompact; the injective hull thus provides an alternative to the Rips complex, with some extra features.

1 Introduction

A metric space X is called *injective* if for every metric space B and every 1-Lipschitz map $f\colon A \to X$ defined on a set $A \subset B$ there exists a 1-Lipschitz extension $\overline{f}\colon B \to X$ of f. The terminology is in accordance with the notion of an injective object in category theory. Basic examples of injective metric spaces are the real line, all complete $\mathbb{R}$-trees, and $l_\infty(I)$ for an arbitrary index set I. Every injective metric space X is complete, geodesic, and contractible; in fact there is a map $\gamma\colon X \times X \times [0,1] \to X$ such that $\gamma_{xy} := \gamma(x, y, \cdot)$ is a constant speed geodesic from x to y and

$$d(\gamma_{xy}(t), \gamma_{x'y'}(t)) \le (1-t)d(x,x') + td(y,y'), \tag{1.1}$$

for all $x, y, x', y' \in X$ and $t \in [0,1]$. In the 1960es, Isbell [23] showed that every metric space X possesses an *injective hull* $(\mathrm{e}, \mathrm{E}(X))$; that is, $\mathrm{E}(X)$ is an injective metric space, $\mathrm{e}\colon X \to \mathrm{E}(X)$ is an isometric embedding, and every isometric embedding of X into some injective metric space Z factors through e. Furthermore, for any other injective hull (i, Y) of X there is a unique isometry $I\colon \mathrm{E}(X) \to Y$ with the property that

$I \circ \mathrm{e} = i$. If X is compact then so is $\mathrm{E}(X)$, and if X is finite then the injective hull is a finite polyhedral complex of dimension at most $\frac{1}{2}|X|$ whose n-cells are isometric to polytopes in $l_\infty^n = l_\infty(\{1, \dots, n\})$. If X is a normed real vector space, then $\mathrm{E}(X)$ agrees with the injective hull in the linear category, in particular $\mathrm{E}(X)$ has a Banach space structure. For references and a detailed review of injective metric spaces and hulls we refer to Sections 2 and 3.

Isbell's construction was rediscovered twenty years later by Dress [12] (and even another time by Chrobak and Larmore [9]). Due to this independent work and a characterization of injective metric spaces from [3], metric injective hulls are also called *tight spans* (Dress) or *hyperconvex hulls* in the literature, sometimes with a different connotation; furthermore 'hull' is often substituted by 'envelope'. Tight spans are widely known in discrete mathematics and have notably been used in phylogenetic analysis; see [16, 15] for some surveys. The purpose of the present article is to explore Isbell's construction further in the context of metric geometry and geometric group theory.

To state the main results, we introduce some general notation used throughout the paper. Let X be a metric space with metric d. We generally assume X to be non-empty. For $x, y \in X$,

$$I(x, y) := \{v \in X : d(x, v) + d(v, y) = d(x, y)\}$$

denotes the *interval* between x and y (compare [30]), and for $x, v \in X$,

$$C(x, v) := \{y \in X : v \in I(x, y)\} \tag{1.2}$$

is the *cone* determined by the directed pair (x, v). Given a reference point $z \in X$, $d_z \colon X \to \mathbb{R}$ denotes the distance function to z, thus $d_z(x) = d(x, z)$. The metric space X is called *discretely geodesic* if the metric is integer valued and for every pair of points $x, y \in X$ there exists an isometric embedding $\gamma \colon \{0, 1, \dots, d(x, y)\} \to X$ such that $\gamma(0) = x$ and $\gamma(d(x, y)) = y$. We say that a discretely geodesic metric space X has *β-stable intervals*, for some constant $\beta \geq 0$, if for every triple of points $x, y, y' \in X$ with $d(y, y') = 1$ we have

$$d_{\mathrm{H}}(I(x, y), I(x, y')) \leq \beta, \tag{1.3}$$

where d_{H} denotes the Hausdorff distance in X. To verify this inequality it suffices, by symmetry, to show that for every $v \in I(x, y)$ there exists a $v' \in I(x, y')$ with $d(v, v') \leq \beta$; this means that some (but not necessarily every) discrete geodesic from x to y' passes close to v. We have the following result.

1.1 Theorem. *Let X be a discretely geodesic metric space such that all bounded subsets are finite. If X has β-stable intervals, then the injective hull $\mathrm{E}(X)$ is proper (that is, bounded closed subsets are compact) and has the structure of a locally finite polyhedral complex with only finitely many isometry types of n-cells, isometric to injective polytopes in l_∞^n, for every $n \geq 1$.*

In this article, polytopes are understood to be convex and compact. A detailed discussion of the polyhedral structure of $\mathrm{E}(X)$, under some weaker (but technical) assumption, is given in Section 4. Then, in Section 5, we employ the uniform stability of intervals to verify this assumption and to show that the structure is in fact locally finite. Inequality (1.3) is used through the following two consequences. First, for every fixed vertex $v \in X$ there are only finitely many distinct cones $C(x, v)$ as x ranges over X; the argument goes back to Cannon [7]. Second, for all $x, y, z \in X$ there exists $v \in I(x, y)$ such that $d_z(v) \le \beta(d_z(x) + d_z(y) - d(x, y))$ (this implies in turn that X has 2β-stable intervals). We derive upper bounds on the local dimension and complexity of $\mathrm{E}(X)$ in terms of the distance to a point $\mathrm{e}(z)$ in the image of the embedding $\mathrm{e}\colon X \to \mathrm{E}(X)$, the cardinality of balls centered at z, and the constant β. In particular, if there is a uniform bound on the number of points at distance one from any point in X, then every subcomplex of $\mathrm{E}(X)$ contained in a tubular neighborhood of $\mathrm{e}(X)$ is finite dimensional. Note that this applies to finitely generated groups, discussed below.

A metric space X is called *δ-hyperbolic*, for some constant $\delta \ge 0$, if

$$d(w, x) + d(y, z) \le \max\{d(w, y) + d(x, z), d(x, y) + d(w, z)\} + \delta \tag{1.4}$$

for all $w, x, y, z \in X$. It is easily seen that every discretely geodesic δ-hyperbolic metric space has $(\delta + 1)$-stable intervals. The following general embedding theorem holds (see [21, Proposition 6.4.D], [5, Theorem 4.1] for some results of similar type).

1.2 Theorem. *Let X be a δ-hyperbolic metric space. Then $\mathrm{E}(X)$ is δ-hyperbolic as well. If, in addition, X is geodesic or discretely geodesic, then $\mathrm{E}(X)$ is within distance δ or $\delta + \frac{1}{2}$, respectively, of the image of the embedding $\mathrm{e}\colon X \to \mathrm{E}(X)$.*

The first part of this result is mentioned in [16, Section 4.4], and the argument is given in [12, (4.1)] for the case $\delta = 0$. The second part (with a primarily different proof and a worse bound) served as the starting point for the present investigation and for the thesis [29], where also a weaker version of Theorem 1.3 below was shown.

Now let Γ be a group with a finite generating system S, equipped with the word metric d_S with respect to the alphabet $S \cup S^{-1}$. The isometric action of Γ by left multiplication on $\Gamma_S = (\Gamma, d_S)$ extends canonically to an isometric action on the injective hull $\mathrm{E}(\Gamma_S)$. If Γ_S has β-stable intervals (see Remark 5.9 for some sufficient conditions), Theorem 1.1 yields that $\mathrm{E}(\Gamma_S)$ is a locally finite polyhedral complex with finitely many isometry types of n-cells for every n.

1.3 Theorem. *Let $\Gamma_S = (\Gamma, d_S)$ be a finitely generated group. If Γ_S has β-stable intervals, then Γ acts properly by cellular isometries on the complex $\mathrm{E}(\Gamma_S)$. If Γ_S is δ-hyperbolic, then $\mathrm{E}(\Gamma_S)$ is finite dimensional and the action is cocompact in addition.*

Injective (or hyperconvex) metric spaces have some remarkable fixed point properties. For instance, every 1-Lipschitz map $L\colon X \to X$ of a bounded injective metric space X has an non-empty fixed point set which is itself injective and thus contractible; compare [19, Theorem 6.1] and the references there. Contrary to what one might expect, the boundedness condition cannot be relaxed to the assumption that the semigroup generated by the 1-Lipschitz map L has bounded orbits. Indeed Prus gave an example of an isometric embedding L of the Banach space l_∞ into itself such that L has bounded orbits but no fixed point; see [19, Remark 6.3]. However, the map L is not surjective and thus the example still leaves room for the following theorem. In the process of finishing this paper I became aware of the reference [14], where the result is shown for isometric actions of compact groups.

1.4 Theorem. *Let X be an injective metric space. If Λ is a subgroup of the isometry group of X with bounded orbits, then the fixed point set of Λ is itself injective, in particular it is non-empty and contractible.*

This should be compared with the analogous result for CAT(0) spaces (see [6, Corollary II.2.8]). As a consequence of Theorem 1.4 one obtains the following supplement of Theorem 1.3.

1.5 Theorem. *Let $\Gamma_S = (\Gamma, d_S)$ be a finitely generated group with β-stable intervals. Then the first barycentric subdivision of the complex $\mathrm{E}(\Gamma_S)$ is a model for the classifying space $\underline{\mathrm{E}}\Gamma$ for proper actions.*

The proofs of the above theorems are given in Section 6. If Γ_S is δ-hyperbolic, the barycentric subdivision of $\mathrm{E}(\Gamma_S)$ has only finitely many distinct Γ-orbits of cells and thus constitutes a (so-called) finite model for $\underline{\mathrm{E}}\Gamma$. A corresponding result is known for the Rips complex $P_R(\Gamma_S)$, provided the maximal simplex diameter R is chosen sufficiently large (see [28]). It should be noted, however, that the entire structure of $\mathrm{E}(\Gamma_S)$ is canonically determined by the choice of S and that $\mathrm{E}(\Gamma_S)$ comes with some features of nonpositive curvature such as (1.1). Furthermore, by this inequality and a result from [37], every injective metric space satisfies isoperimetric filling inequalities of Euclidean type for integral cycles in any dimension. In view of these properties it is natural to ask whether $\mathrm{E}(\Gamma_S)$ can be equipped with an honest equivariant CAT(0) metric. The answer to this question is positive, for instance, if $\mathrm{E}(\Gamma_S)$ has dimension two. This and further results on the structure of injective hulls of finitely generated groups will be discussed in a subsequent article.

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2 Injective metric spaces

In this section we discuss some basic examples, properties, and characterizations of injective metric spaces.

The set of all 1-Lipschitz maps from a metric space B into another metric space X will be denoted by $\mathrm{Lip}_1(B,X)$. Recall that X is *injective* if for every metric space B, every $A \subset B$, and every $f \in \mathrm{Lip}_1(A,X)$ there exists $\overline{f} \in \mathrm{Lip}_1(B,X)$ such that $\overline{f}|_A = f$. (Note that for $A = \emptyset \neq B$ this says that $X \neq \emptyset$.)

The most basic examples of injective metric spaces are the real line $\mathbb{R}$ and all non-empty closed subintervals, with the usual metric. For instance, if $f \in \mathrm{Lip}_1(A,\mathbb{R})$, where $A \neq \emptyset$ is a subset of a metric space B, then

$$\overline{f}(b) := \sup_{a \in A}(f(a) - d(a,b)) \tag{2.1}$$

defines the least possible extension $\overline{f} \in \mathrm{Lip}_1(B,\mathbb{R})$ of f.

It follows easily from the definition that every injective metric space X is complete and geodesic. Indeed, if $\bar{X}$ denotes the completion, then the identity map on X extends to a 1-Lipschitz retraction $\pi \colon \bar{X} \to X$ which turns out to be an isometry as X is dense in $\bar{X}$. Furthermore, given $x, y \in X$, the map that sends $0 \in \mathbb{R}$ to x and $l := d(x,y)$ to y extends to a 1-Lipschitz map $\gamma \colon [0,l] \to X$ which, due to the triangle inequality, is in fact an isometric embedding.

Another basic property is that for every triple of points x, y, z in an injective metric space X there is a (not necessarily unique) median point $v \in X$, that is, a point in $I(x,y) \cap I(y,z) \cap I(z,x)$. This is shown by extending the isometric inclusion $\{x,y,z\} \to X$ to a 1-Lipschitz map from $Q := (\{x,y,z,u\}, \bar{d})$ to X, where the metric $\bar{d}$ is determined by the requirement that it agrees with d on $\{x,y,z\}$ and that the additional point u is a median point of x, y, z in Q (thus $\bar{d}(u,z) = (x \,|\, y)_z$, etc., see (5.4)). As above, this 1-Lipschitz extension is in fact an isometric embedding, and the image of u is the desired median point $v \in Y$. One may choose geodesic segments $[v,x], [v,y], [v,z]$ to produce a geodesic tripod spanned by x, y, z (thus $[v,x] \cup [v,y]$ is a geodesic segment from x to y, etc.). Checking the existence of median points is a simple and useful first test for injectivity. Furthermore, it follows that if X is an injective metric space with the property that every pair of points x, y is connected by a unique geodesic segment $[x,y]$, then every geodesic triangle in X is a tripod and so X is an $\mathbb{R}$-tree. The converse is a well-known fact:

2.1 Proposition. *Every complete $\mathbb{R}$-tree X is injective.*

Most proofs in the literature proceed via pointwise extensions and transfinite induction (compare Proposition 2.3 below). The following direct argument, extracted from a more general construction in [26], adapts (2.1) to trees.

Proof. Fix a base point $z \in X$. Let $f \in \mathrm{Lip}_1(A, X)$, where $\emptyset \neq A \subset B$. For every pair $(a, b) \in A \times B$, define

$$\varrho(a, b) := \max\{0, d_z(f(a)) - d(a, b)\}$$

and let $x(a, b)$ be the point on the segment $[z, f(a)]$ at distance $\varrho(a, b)$ from z. For two such pairs $(a, b), (a', b')$, consider the tripod spanned by $z, f(a), f(a')$. Depending on the positions of $x(a, b)$ and $x(a', b')$ on the tripod, $d(x(a, b), x(a', b'))$ equals either $|\varrho(a, b) - \varrho(a', b')|$ or $d(f(a), f(a')) - d(a, b) - d(a', b')$. Since $d(f(a), f(a')) \leq d(a, a')$, it follows that

$$d(x(a, b), x(a', b')) \leq \max\{|\varrho(a, b) - \varrho(a', b')|, d(b, b')\}. \tag{2.2}$$

To define the extension at $b \in B$, choose a sequence (a_i) in A such that

$$\lim_{i \to \infty} \varrho(a_i, b) = \bar{\varrho}(b) := \sup_{a \in A} \varrho(a, b).$$

The corresponding sequence $(x(a_i, b))$ in X is Cauchy by (2.2), and $\overline{f}(b)$ is defined as its limit, which is independent of the choice of (a_i). Note that $\bar{\varrho}\colon B \to \mathbb{R}$ is the least nonnegative 1-Lipschitz extension of $d_z \circ f$. Now it follows from (2.2) that $\overline{f} \in \mathrm{Lip}_1(B, X)$. To check that $\overline{f}$ extends f, let $b \in A$, and let (a_i) be a sequence in A such that $\varrho(a_i, b) \to \bar{\varrho}(b)$. We have $\varrho(b, b) = d_z(f(b)) = \bar{\varrho}(b)$ and $x(b, b) = f(b)$, so $d(x(a_i, b), f(b)) \leq |\varrho(a_i, b) - \bar{\varrho}(b)|$ by (2.2). Since $x(a_i, b) \to \overline{f}(b)$, this gives $\overline{f}(b) = f(b)$. □

The l_∞ product of a non-empty family $\{(X_i, d_i, z_i)\}_{i \in I}$ of pointed metric spaces is defined as the set of all $x = (x_i)_{i \in I}$ with $x_i \in X_i$ and $\sup_{i \in I} d_i(x_i, z_i) < \infty$, endowed with the metric $(x, x') \mapsto \sup_{i \in I} d_i(x_i, x'_i)$. Here $I \neq \emptyset$ is an arbitrary index set; if I is finite or the diameters of the X_i are uniformly bounded, base points may be disregarded. It is easy to see that if each (X_i, d_i) is injective, then so is the l_∞ product. In case $(X_i, z_i) = (\mathbb{R}, 0)$ for all $i \in I$, the corresponding l_∞ product is the Banach space $l_\infty(I)$, which is thus an injective metric space. Similarly, $L_\infty(Y, \mu)$ is injective for every measure space (Y, μ).

Next we recall some well-known characterizations of injective metric spaces. A metric space X is called an *absolute* 1*-Lipschitz retract* if, whenever $i\colon X \to Y$ is an isometric embedding into another metric space Y, there exists a 1-Lipschitz retraction of Y onto $i(X)$. If X is injective and $i\colon X \to Y$ is an isometric embedding, then $i(X)$ is injective and thus the identity map on $i(X)$ extends to a 1-Lipschitz retraction $\pi\colon Y \to i(X)$. On the other hand, every metric space X embeds isometrically into $l_\infty(X)$ via the Kuratowski map $k_z\colon x \mapsto d_x - d_z$, for any base point $z \in X$. Hence, if $k_z(X)$ is a 1-Lipschitz retract in $l_\infty(X)$, then X is injective since $l_\infty(X)$ is. This shows:

2.2 Proposition. *A metric space X is injective if and only if it is an absolute* 1-*Lipschitz retract.*

At this point, we note that every injective metric space X is contractible. If $\pi\colon l_\infty(X) \to X' := k_z(X)$ is a 1-Lipschitz retraction onto the image of the Kuratowski embedding, then $h(x',t) := \pi(tx')$ defines a homotopy $h\colon X' \times [0,1] \to X'$ from the constant map with value $0 = k_z(z)$ to the identity map. A map γ as in (1.1) can be obtained in a similar way. An additional equivariance property will be shown in Theorem 3.8. Compare also [23, Theorem 1.1].

Another characterization of injective metric spaces relies on pointwise extensions of 1-Lipschitz maps. A metric space X is said to be *hyperconvex* if every family $((x_i, r_i))_{i\in I}$ in $X \times \mathbb{R}$ with the property that $r_i + r_j \ge d(x_i, x_j)$ for all pairs of indices $i, j \in I$ satisfies $X \cap \bigcap_{i\in I} B(x_i, r_i) \neq \emptyset$. (For $I = \emptyset$ this gives $X \neq \emptyset$, in accordance with the fact that injective metric spaces are non-empty.) This terminology was introduced by Aronszajn and Panitchpakdi in [3], who also observed Proposition 2.2 and the next result.

2.3 Proposition. *A metric space X is injective if and only if it is hyperconvex.*

For the proof, one notes first that if $f \in \mathrm{Lip}_1(A, X)$, $\emptyset \neq A \subset B$, and $b \in B \setminus A$, then $d(a,b) + d(a',b) \ge d(a,a') \ge d(f(a), f(a'))$ for all $a, a' \in A$. Hence, if X is hyperconvex, then $\bigcap_{a\in A} B(f(a), d(a,b))$ is non-empty, and one obtains an extension $f_b \in \mathrm{Lip}_1(A \cup \{b\}, X)$ of f by declaring $f_b(b)$ to be any point in this intersection. By iterating this process transfinitely, in general, one infers that X is injective. Conversely, if X is injective, a similar argument as for the existence of median points shows that X is hyperconvex. A useful direct consequence of this characterization is that the intersection of a family of closed balls in an injective metric space is injective, whenever the intersection is non-empty. Some key results on hyperconvex metric spaces were shown by Baillon [4]. Proposition 2.3 will be used in Section 6 for the proof of Theorem 1.4.

A concept very close to hyperconvexity is the *binary intersection property*, obtained by replacing the inequality $r_i + r_j \ge d(x_i, x_j)$ in the above definition by the condition $B(x_i, r_i) \cap B(x_j, r_j) \neq \emptyset$. The two concepts agree for geodesic metric spaces. Nachbin [31, Theorem 1] showed that a normed real vector space X has the binary intersection property if and only if X is *linearly injective*, that is, for every real normed space B, every linear subspace $A \subset B$, and every bounded linear operator $f\colon A \to X$ there exists a linear extension $\overline{f}\colon B \to X$ with norm $\|\overline{f}\| = \|f\|$. (The Hahn–Banach Theorem thus asserts that $\mathbb{R}$ is linearly injective.) Hence, a real normed space X is injective as a metric space if and only if X is injective in the linear category, and no ambiguity arises. By [31, Theorem 3], an n-dimensional normed space X is injective if and only if X is linearly isometric to l_∞^n or, in other words, balls in X are parallelotopes. The final classification result, usually attributed to Nachbin–Goodner–Kelley, asserts that a real normed space is injective if and only if it is isometrically isomorphic to the Banach space $C(K)$ of continuous real valued functions on some extremally disconnected compact Hausdorff space K, endowed with the supremum norm. See [25].

It is clear that linear subspaces of injective normed spaces need not be injective. A familiar example is the plane $H = \{x_1+x_2+x_3 = 0\}$ in l_∞^3, whose norm ball is hexagonal. One may also check directly that the triple of points $(1, 1, -2), (1, -2, 1), (-2, 1, 1)$ has no median point in H. We conclude this section by showing that certain subsets of l_∞^n (or $l_\infty(I)$) defined by linear inequalities involving at most two variables are injective. We shall use this to prove that the polyhedral cells of $\mathrm{E}(X)$ are themselves injective (compare Theorem 1.1).

2.4 Proposition. *Let $I \neq \emptyset$ be any index set. Suppose that Q is a non-empty subset of $l_\infty(I)$ given by an arbitrary system of inequalities of the form $\sigma x_i \le C$ or $\sigma x_i + \tau x_j \le C$ with $|\sigma|, |\tau| = 1$ and $C \in \mathbb{R}$. Then Q is injective.*

We use a similar explicit construction as for $\mathbb{R}$-trees. Some further results on injective polyhedral sets in l_∞^n can be found in [29, Section 1.8.2]. A good characterization of such sets seems to be missing.

Proof. Assume that $0 \in Q$, so that all constants on the right sides of the inequalities describing Q are nonnegative. For $i \in I$, denote by R_i the reflection of $l_\infty(I)$ that interchanges x_i with $-x_i$. Let B be a metric space and $\emptyset \neq A \subset B$. We show that there exists an extension operator $\phi\colon \mathrm{Lip}_1(A, l_\infty(I)) \to \mathrm{Lip}_1(B, l_\infty(I))$ such that

$$\phi(R_i \circ f) = R_i \circ \phi(f) \tag{2.3}$$

for every i, and such that the components of $\phi(f)$ satisfy

$$\phi(f)_i + \phi(f)_j \le C \tag{2.4}$$

whenever $f_i + f_j \le C$ for some pair of possibly equal indices i, j and some constant $C \ge 0$. This clearly gives the result.

First, for a real valued function $f \in \mathrm{Lip}_1(A, \mathbb{R})$, we combine the smallest and largest 1-Lipschitz extensions and define $\overline{f}\colon B \to \mathbb{R}$ by

$$\overline{f}(b) := \sup\Big\{0, \sup_{a\in A}(f(a) - d(a, b))\Big\} + \inf\Big\{0, \inf_{a'\in A}(f(a') + d(a', b))\Big\}.$$

Note that at most one of the two summands is nonzero since $f(a) - d(a, b) \le f(a') + d(a, a') - d(a, b) \le f(a') + d(a', b)$. It is not difficult to check that $\overline{f}$ is a 1-Lipschitz extension of f and that $\overline{R \circ f} = R \circ \overline{f}$ for the reflection $R\colon x \mapsto -x$ of $\mathbb{R}$. (The proof of Proposition 2.1 yields precisely this extension $\overline{f}$ in the case $(X, z) = (\mathbb{R}, 0)$.)

Now, for $f \in \mathrm{Lip}_1(A, l_\infty(I))$, define $\phi(f)$ such that $\phi(f)_i = \overline{f_i}$ for every i. Clearly $\phi(f) \in \mathrm{Lip}_1(B, l_\infty(I))$, and (2.3) holds. As for (2.4), suppose that $f_i+f_j \le C$ for some indices i, j and some constant $C \geq 0$. Let $b \in B$, and assume that $\phi(f)_i(b) \geq \phi(f)_j(b)$. If $\phi(f)_j(b) > 0$, then

$$\begin{aligned}\phi(f)_i(b) + \phi(f)_j(b) &= \sup_{a,a' \in A} (f_i(a) + f_j(a') - d(a,b) - d(a',b)) \\ &\leq \sup_{a,a' \in A} (f_i(a) + f_j(a') - d(a,a')) \\ &\leq \sup_{a \in A}(f_i(a) + f_j(a)) \leq C.\end{aligned}$$

If $\phi(f)_i(b) > 0 \geq \phi(f)_j(b)$, then

$$\begin{aligned}\phi(f)_i(b) + \phi(f)_j(b) &\leq \sup_{a \in A}(f_i(a) - d(a,b)) + \inf_{a' \in A}(f_j(a') + d(a',b)) \\ &\leq \sup_{a \in A}(f_i(a) + f_j(a)) \leq C.\end{aligned}$$

Finally, if $\phi(f)_i(b) \leq 0$, then $\phi(f)_i(b) + \phi(f)_j(b) \leq 0 \leq C$. □

3 Injective hulls

We now review Isbell's [23] construction $X \mapsto \mathrm{E}(X)$. Our proof of the injectivity of $\mathrm{E}(X)$ differs from Isbell's in that it does not appeal to Zorn's Lemma or the like. We employ an observation by Dress [12] which also gives further information.

Let X be a metric space. Denote by $\mathbb{R}^X$ the vector space of all real valued functions on X, and define

$$\Delta(X) := \{f \in \mathbb{R}^X : f(x) + f(y) \geq d(x,y) \text{ for all } x, y \in X\}$$

(compare [31, p. 35]). By the triangle inequality, the distance function d_z belongs to $\Delta(X)$ for each $z \in X$, and clearly all elements of $\Delta(X)$ are nonnegative. Isbell called a function $f \in \mathbb{R}^X$ *extremal* if it is a minimal element of the partially ordered set $(\Delta(X), \leq)$, where $g \leq f$ means $g(x) \leq f(x)$ for all $x \in X$ as usual. Thus

$$\mathrm{E}(X) := \{f \in \Delta(X) : \text{if } g \in \Delta(X) \text{ and } g \leq f, \text{ then } g = f\}$$

is the set of extremal functions on X. In case X is compact, $f \in \Delta(X)$ is extremal if and only if for every $x \in X$ there exists $y \in X$ such that $f(x) + f(y) = d(x,y)$. In general, $f \in \mathbb{R}^X$ is extremal if and only if

$$f(x) = \sup_{y \in X}(d(x,y) - f(y)) \tag{3.1}$$

for all $x \in X$. Each d_z is extremal. Applying (3.1) twice one obtains

$$f(x) - d(x,x') = \sup_{y \in X}(d(x,y) - d(x,x') - f(y)) \leq f(x')$$

for all $x, x' \in X$, so every $f \in \mathrm{E}(X)$ is 1-Lipschitz. Now consider the set

$$\Delta_1(X) := \Delta(X) \cap \mathrm{Lip}_1(X, \mathbb{R}),$$

equipped with the metric

$$(f, g) \mapsto \|f - g\|_\infty = \sup_{x \in X} |f(x) - g(x)|.$$

To see that the supremum is finite, note that a function $f \in \mathbb{R}^X$ belongs to $\Delta_1(X)$ if and only if $|f(x) - d(x, y)| \le f(y)$ for all $x, y \in X$ or, equivalently,

$$\|f - d_y\|_\infty = f(y) \tag{3.2}$$

for all $y \in X$. Hence, $\|f - g\|_\infty \le \inf(f + g)$. The set $\mathrm{E}(X)$ is contained in $\Delta_1(X)$ and is equipped with the induced metric. The map

$$\mathrm{e}\colon X \to \mathrm{E}(X), \quad \mathrm{e}(y) = d_y,$$

is a canonical isometric embedding of X into $\mathrm{E}(X)$, as $\|d_y - d_z\|_\infty = d(y, z)$. Equation (3.2) will be used frequently. It shows that a function $f \in \Delta_1(X)$ corresponds, after identification of X with $\mathrm{e}(X)$, to the restriction of a distance function to a point in $\Delta_1(X)$, namely f itself.

To prove that $(\mathrm{e}, \mathrm{E}(X))$ is an injective hull of X we shall make use of the following basic fact.

3.1 Proposition. *For every metric space X there exists a map $p\colon \Delta(X) \to \mathrm{E}(X)$ such that*

(1) *$p(f) \le f$ for all $f \in \Delta(X)$, hence $p(f) = f$ for all $f \in \mathrm{E}(X)$;*

(2) *$\|p(f) - p(g)\|_\infty \le \|f - g\|_\infty$ for all $f, g \in \Delta(X)$.*

In (2) the right side is possibly infinite, but it is finite if $f, g \in \Delta_1(X)$, thus the restriction of p to $\Delta_1(X)$ is a 1-Lipschitz retraction onto $\mathrm{E}(X)$. The existence of such a map p could be shown by means of Zorn's Lemma, however Dress [12, Section (1.9)] (compare [9, Lemma 5.3]) also found the following construction, which is canonical in the sense that no choices need to be made.

Proof. For every $f \in \Delta(X)$, define $f^* \in \mathbb{R}^X$ such that

$$f^*(x) = \sup_{z \in X}(d(x, z) - f(z))$$

for all $x \in X$. Clearly $f^* \le f$, and equality holds if and only if $f \in \mathrm{E}(X)$, cf. (3.1). For every pair of points $x, y \in X$, the definition of f^* gives $f^*(x) + f(y) \ge d(x, y)$ and $f(x) + f^*(y) \ge d(x, y)$. It follows that the function

$$q(f) := \frac{1}{2}(f + f^*)$$

belongs to $\Delta(X)$, and $q(f) \le f$. For all $f, g \in \Delta(X)$ and $x \in X$,

$$g^*(x) = \sup_{z \in X} (d(x,z) - f(z) + f(z) - g(z)) \le f^*(x) + \|f - g\|_\infty,$$

hence $\|f^* - g^*\|_\infty \le \|f - g\|_\infty$ and thus

$$\|q(f) - q(g)\|_\infty \le \frac{1}{2}\|f - g\|_\infty + \frac{1}{2}\|f^* - g^*\|_\infty \le \|f - g\|_\infty.$$

Iterating the map q, one obtains for every $f \in \Delta(X)$ a sequence of functions $q(f) \ge q^2(f) \ge q^3(f) \ge \dots$ in $\Delta(X)$, then $p(f)$ is defined as the pointwise limit. Clearly $p(f) \in \Delta(X)$, and (1) and (2) hold. For all $n \ge 1$, $p(f) \le q^n(f)$ and hence $p(f)^* \ge q^n(f)^*$, so

$$0 \le p(f) - p(f)^* \le q^n(f) - q^n(f)^* = 2(q^n(f) - q^{n+1}(f)).$$

As $n \to \infty$, the last term converges pointwise to 0, thus $p(f)^* = p(f)$ and therefore $p(f) \in \mathrm{E}(X)$. □

Now, since $\mathrm{E}(X)$ is a 1-Lipschitz retract of $\Delta_1(X)$, to prove the injectivity of $\mathrm{E}(X)$ it remains to show that $\Delta_1(X)$ is injective. A simple component-wise extension procedure applies, like for $l_\infty(I)$.

3.2 Proposition. *For every metric space X the metric spaces $\Delta_1(X)$ and $\mathrm{E}(X)$ are injective.*

Proof. As just mentioned, in view of Proposition 3.1 it suffices to prove the result for $\Delta_1(X)$. We could embed $\Delta_1(X)$ into $l_\infty(X)$ via $f \mapsto f - h$ for some fixed $h \in \Delta_1(X)$ and then refer to Proposition 2.4, but the following argument is slightly more direct. Let B be a metric space, $\emptyset \ne A \subset B$, and let $F \colon A \to \Delta_1(X)$ be a 1-Lipschitz map, $F \colon a \mapsto f_a$. For $b \in B$, put

$$\bar{f}_b(x) := \inf_{a \in A} (f_a(x) + d(a,b))$$

for all $x \in X$. Clearly $\bar{f}_b$ is a nonnegative 1-Lipschitz function on X, as the infimum of a family of such. For $a, a' \in A$ and $y \in X$, we have $f_a(y) - f_{a'}(y) \le \|f_a - f_{a'}\|_\infty = \|F(a) - F(a')\|_\infty \le d(a,a')$ and so

$$\begin{aligned} \bar{f}_b(x) + \bar{f}_b(y) &\ge \inf_{a,a' \in A} (f_a(x) + f_{a'}(y) + d(a,a')) \\ &\ge \inf_{a \in A} (f_a(x) + f_a(y)) \\ &\ge d(x,y). \end{aligned}$$

This shows that $\bar{f}_b \in \Delta_1(X)$. For $b, b' \in B$ and $x \in X$,

$$\bar{f}_b(x) - d(b,b') = \inf_{a \in A}(f_a(x) + d(a,b) - d(b,b')) \le \bar{f}_{b'}(x),$$

hence $\|\bar{f}_b - \bar{f}_{b'}\|_\infty \le d(b,b')$. If $b \in A$, then $\bar{f}_b(x) \le f_b(x)$ and $f_b(x) \le f_a(x) + \|f_a - f_b\|_\infty \le f_a(x) + d(a,b)$ for all $x \in X$ and $a \in A$, so that $\bar{f}_b = f_b$. Thus $\overline{F} \colon b \mapsto \bar{f}_b$ is a 1-Lipschitz extension of F. □

If X is finite, so that the supremum norm gives a metric on $\Delta(X)$, the same argument also shows that $\Delta(X)$ is injective.

We now state Isbell's result about $\mathrm{E}(X)$. For brevity, isometric embeddings will just be called *embeddings*. An embedding i of X into some metric space Y is called *essential* if for every metric space Z and every 1-Lipschitz map $h \colon Y \to Z$ with the property that $h \circ i \colon X \to Z$ is an embedding, h is an embedding as well. If $i \colon X \to Y$ is essential and Y is injective, then (i, Y) is an *injective hull* of X; see [1, Section 9]. In the terminology of [12], an essential extension (i, Y) of X is called a *tight extension* (and $\Delta(X)$ and $\mathrm{E}(X)$ are denoted P_X and T_X, respectively).

3.3 Theorem. *For every metric space X, the following hold:*

(1) *If $L \colon \mathrm{E}(X) \to \mathrm{E}(X)$ is a 1-Lipschitz map that fixes $\mathrm{e}(X)$ pointwise, then L is the identity on $\mathrm{E}(X)$;*

(2) *$(\mathrm{e}, \mathrm{E}(X))$ is an injective hull of X;*

(3) *if (i, Y) is another injective hull of X, then there exists a unique isometry $I \colon \mathrm{E}(X) \to Y$ with the property that $I \circ \mathrm{e} = i$.*

Proof. For (1) we use (3.2). The map L takes $f \in \mathrm{E}(X)$ to some $g \in \mathrm{E}(X)$ such that

$$g(x) = \|g - d_x\|_\infty = \|L(f) - L(d_x)\|_\infty \le \|f - d_x\|_\infty = f(x)$$

for all $x \in X$, so $g = f$ by the minimality of f.

By Proposition 3.2, $\mathrm{E}(X)$ is injective, so for (2) it remains to show that e is essential. Suppose $h \colon \mathrm{E}(X) \to Z$ is 1-Lipschitz and $h \circ \mathrm{e} \colon X \to Z$ is an embedding. Since $\mathrm{E}(X)$ is injective, $\mathrm{e} \colon X \to \mathrm{E}(X)$ extends to a 1-Lipschitz map $\bar{\mathrm{e}} \colon Z \to \mathrm{E}(X)$, thus $\bar{\mathrm{e}} \circ h \circ \mathrm{e} = \mathrm{e}$. The map $L := \bar{\mathrm{e}} \circ h$ is 1-Lipschitz and fixes $\mathrm{e}(X)$ pointwise, so L is the identity on $\mathrm{E}(X)$ by (1). As both h and $\bar{\mathrm{e}}$ are 1-Lipschitz, h is in fact an embedding.

As for (3), if (i, Y) is another injective hull of X, then i extends to a 1-Lipschitz map $I \colon \mathrm{E}(X) \to Y$, so $I \circ \mathrm{e} = i$. Likewise, there is a 1-Lipschitz map $\bar{\mathrm{e}} \colon Y \to \mathrm{E}(X)$ with $\bar{\mathrm{e}} \circ i = \mathrm{e}$. Since i is essential, $\bar{\mathrm{e}}$ is an embedding; furthermore, $\bar{\mathrm{e}} \circ I \circ \mathrm{e} = \mathrm{e}$, thus $\bar{\mathrm{e}} \circ I = \mathrm{id}_{\mathrm{E}(X)}$ by (1). Hence $\bar{\mathrm{e}}$ is an isometry onto $\mathrm{E}(X)$, and I is its inverse. □

Injective hulls can be characterized in a number of different ways. We just state the following proposition, which is independent of the construction described above, except that the proof relies on the existence of *some* injective hull of X. For the details we refer again to the general discussion in [1, Section 9].

3.4 Proposition. *Let X and Y be metric spaces, and let $i\colon X \to Y$ be an embedding. Then the following are equivalent:*

(1) *(i, Y) is an injective hull of X, that is, i is essential and Y is injective;*

(2) *(i, Y) is a maximal essential extension of X, that is, i is essential and Y has no proper essential extension;*

(3) *(i, Y) is a minimal injective extension of X, that is, Y is injective and no proper subspace of Y containing $i(X)$ is injective;*

(4) *(i, Y) is a smallest injective extension of X, that is, Y is injective and whenever $j\colon X \to Z$ is an embedding into some injective metric space Z, there is an embedding $h\colon Y \to Z$ such that $h \circ i = j$.*

In fact, (3) is the definition of injective hulls adopted by Isbell in [23], and (2) corresponds to the notion of *tight span* introduced by Dress [12]. In the introduction we used property (4), a concrete instance of which is given in the next result (compare [12, Section (1.11)]).

3.5 Proposition. *Let X be a subspace of the metric space X'. Then:*

(1) *There exists an isometric embedding $h\colon \mathrm{E}(X) \to \mathrm{E}(X')$ such that $h(f)|_X = f$ for every $f \in \mathrm{E}(X)$.*

(2) *For every pair of functions $g \in \mathrm{E}(X)$ and $f' \in \mathrm{E}(X')$ there exists $g' \in \mathrm{E}(X')$ such that $g'|_X = g$ and $\|g' - f'\|_\infty = \|g - f'|_X\|_\infty$.*

Proof. For $f \in \mathrm{E}(X)$, let first $\overline{f}\colon X' \to \mathbb{R}$ be the 1-Lipschitz extension defined by

$$\overline{f}(y) := \inf_{x \in X}(f(x) + d(x, y)).$$

Clearly $\overline{f} \in \Delta(X')$. Now put $h(f) := p(\overline{f}) \in \mathrm{E}(X')$, where p is as in Proposition 3.1. We have $h(f)|_X = p(\overline{f})|_X \le \overline{f}|_X = f$; since $h(f)|_X \in \Delta(X)$, this gives $h(f)|_X = f$ by the minimality of f. For $f, g \in \mathrm{E}(X)$ and $y \in X'$,

$$\overline{f}(y) - \|f - g\|_\infty = \inf_{x \in X}(f(x) - \|f - g\|_\infty + d(x, y)) \le \overline{g}(y),$$

hence $\|h(f) - h(g)\|_\infty = \|p(\overline{f}) - p(\overline{g})\|_\infty \le \|\overline{f} - \overline{g}\|_\infty = \|f - g\|_\infty$. This yields (1).

As for (2), suppose that $\nu := \|g - f'|_X\|_\infty < \infty$. Define $\tilde{g}\colon X' \to \mathbb{R}$ such that $\tilde{g}|_X = g$ and $\tilde{g}(y) = f'(y) + \nu$ for all $y \in X' \setminus X$. Since $\tilde{g}(x) = g(x) \geq f'(x) - \nu$ for $x \in X$, it follows that $\tilde{g} \in \Delta(X')$. Now let $g' \in \mathrm{E}(X')$ be any extremal function with $g' \leq \tilde{g}$. Similarly as above, $g'|_X \leq \tilde{g}|_X = g$ and thus $g'|_X = g$ by the minimality of g. Furthermore, $g' \leq f' + \nu$ and hence

$$g'(y) \geq \sup_{y' \in X'} (d(y, y') - f'(y') - \nu) = f'(y) - \nu$$

for all $y \in X'$ by (3.1). This gives the result. □

A number of properties of $\mathrm{E}(X)$ are more or less obvious from the construction. If X is bounded, then $0 \leq f \leq \mathrm{diam}(X) := \sup_{x,y \in X} d(x, y)$ for all $f \in \mathrm{E}(X)$ by (3.1), thus

$$\mathrm{diam}(\mathrm{E}(X)) \leq \mathrm{diam}(X).$$

If X is compact, then so is $\mathrm{E}(X)$, as a consequence of the Arzelà-Ascoli Theorem. If X is finite, $\mathrm{E}(X)$ is a polyhedral subcomplex of the boundary of the polyhedral set $\Delta(X) \subset \mathbb{R}^X$. The faces of $\Delta(X)$ that belong to $\mathrm{E}(X)$ are exactly those whose affine hull $H \subset \mathbb{R}^X$ is determined by a system of equations of the form $f(x_i) + f(x_j) = d(x_i, x_j)$ involving each point $x_i \in X$ at least once. (Note that these are precisely the bounded faces of $\Delta(X)$; compare [13, Lemma 1].) It follows that $\mathrm{E}(X)$ has dimension at most $\frac{1}{2}|X|$. The possible combinatorial types of the injective hulls of metric spaces up to cardinality 5 are depicted in [12, Section (1.16)], and a classification for 6-point metrics is given in [36].

3.6 Remark. As mentioned in Section 2, a normed real vector space is injective as a metric space if and only if it is linearly injective, and the only n-dimensional example is l_∞^n, up to isometric isomorphism. Cohen [11] showed that every real or complex normed space has an essentially unique injective hull in the respective linear category. Isbell [24] and Rao [35] then proved that for a real normed space X the linearly injective hull is isometric to $\mathrm{E}(X)$; an explicit description of the Banach space structure on $\mathrm{E}(X)$ can be found in [10].

We conclude this section with some results involving isometries of X. The isometry group of X will be denoted by $\mathrm{Isom}(X)$.

3.7 Proposition. *Let X be a metric space. Then:*

(1) *For every $L \in \mathrm{Isom}(X)$ there is a unique isometry $\bar{L}\colon \mathrm{E}(X) \to \mathrm{E}(X)$ with the property that $\bar{L} \circ \mathrm{e} = \mathrm{e} \circ L$. One has $\bar{L}(f) = f \circ L^{-1}$ for all $f \in \mathrm{E}(X)$, and $(L, f) \mapsto \bar{L}(f)$ is an action of $\mathrm{Isom}(X)$ on $\mathrm{E}(X)$.*

(2) *The linear isomorphism $f \mapsto f \circ L^{-1}$ of $\mathbb{R}^X$ maps $\Delta(X)$ onto itself, and the map $p\colon \Delta(X) \to \mathrm{E}(X)$ constructed in the proof of Proposition 3.1 has the additional property that $\bar{L}(p(f)) = p(f \circ L^{-1})$ for all $L \in \mathrm{Isom}(X)$ and $f \in \Delta(X)$.*

Proof. For every $L \in \mathrm{Isom}(X)$, $\mathrm{e} \circ L$ is essential and so $(\mathrm{e} \circ L, \mathrm{E}(X))$ is an injective hull of X. Hence, by part (3) of Theorem 3.3, there is a unique isometry $\bar{L} \colon \mathrm{E}(X) \to \mathrm{E}(X)$ such that $\bar{L} \circ \mathrm{e} = \mathrm{e} \circ L$. If $f \in \mathrm{E}(X)$ and $x \in X$, then

$$\begin{aligned}(\bar{L}(f))(x) &= \|\bar{L}(f) - d_x\|_\infty = \|f - \bar{L}^{-1}(d_x)\|_\infty = \|f - d_{L^{-1}(x)}\|_\infty \\ &= f(L^{-1}(x)).\end{aligned}$$

Obviously $(L, f) \mapsto \bar{L}(f) = f \circ L^{-1}$ is an action of $\mathrm{Isom}(X)$ on $\mathrm{E}(X)$.

It is straightforward to check that the linear isomorphism $f \mapsto f \circ L^{-1}$ of $\mathbb{R}^X$ maps $\Delta(X)$ onto $\Delta(X)$ and that it commutes with the operators defined in the proof of Proposition 3.1, thus $f^* \circ L^{-1} = (f \circ L^{-1})^*$, $q(f) \circ L^{-1} = q(f \circ L^{-1})$, and

$$\bar{L}(p(f)) = p(f) \circ L^{-1} = p(f \circ L^{-1})$$

(compare [14, pp. 83–84]). □

As an application of the above result we show that the weak convexity property of injective metric spaces stated in (1.1) holds in an equivariant form. By a *geodesic bicombing* γ on a metric space X we mean a map $\gamma \colon X \times X \times [0,1] \to X$ such that, for every pair $(x,y) \in X \times X$, $\gamma_{xy} := \gamma(x,y,\cdot)$ is a geodesic from x to y with constant speed, that is, $\gamma_{xy}(0) = x$, $\gamma_{xy}(1) = y$, and $d(\gamma_{xy}(s), \gamma_{xy}(t)) = (t-s)d(x,y)$ for $0 \le s \le t \le 1$.

3.8 Theorem. *Every injective metric space X admits a geodesic bicombing γ such that, for all $x, y, x', y' \in X$ and $t \in [0,1]$,*

(1) $d(\gamma_{xy}(t), \gamma_{x'y'}(t)) \le (1-t)d(x,x') + td(y,y')$;

(2) $\gamma_{xy}(t) = \gamma_{yx}(1-t)$;

(3) $L \circ \gamma_{xy} = \gamma_{L(x)L(y)}$ *for every isometry L of X.*

Proof. Since X is injective, the canonical map $\mathrm{e} \colon x \mapsto d_x$ is an isometry of X onto $\mathrm{E}(X)$. Let $p \colon \Delta(X) \to \mathrm{E}(X)$ be the map from the proof of Proposition 3.1. For all $x, y \in X$ and $t \in [0,1]$, we have $(1-t)d_x + td_y \in \Delta_1(X)$, and we set

$$\gamma_{xy}(t) := (\mathrm{e}^{-1} \circ p)((1-t)d_x + td_y).$$

Since $p|_{\Delta_1(X)}$ is 1-Lipschitz, it follows that

$$\begin{aligned}d(\gamma_{xy}(t), \gamma_{x'y'}(t)) &\le \|((1-t)d_x + td_y) - ((1-t)d_{x'} + td_{y'})\|_\infty \\ &\le (1-t)\|d_x - d_{x'}\|_\infty + t\|d_y - d_{y'}\|_\infty \\ &= (1-t)d(x,x') + td(y,y')\end{aligned}$$

for $x, y, x', y' \in X$ and $t \in [0,1]$. Similarly,

$$\begin{aligned} d(\gamma_{xy}(s), \gamma_{xy}(t)) &\le \|((1-s)d_x + sd_y) - ((1-t)d_x + td_y)\|_\infty \\ &= (t-s)\|d_x - d_y\|_\infty \\ &= (t-s)d(x,y) \end{aligned}$$

for $x, y \in X$ and $0 \le s \le t \le 1$, and it is easy to see that equality must hold since $\gamma_{xy}(0) = x$ and $\gamma_{xy}(1) = y$. Thus γ is a geodesic bicombing on X that satisfies (1) and (2).

Now let $L \in \mathrm{Isom}(X)$, and recall that $\bar{L}(p(f)) = p(f \circ L^{-1})$ for all $f \in \Delta(X)$, by Proposition 3.7. Since $d_v \circ L^{-1} = d_{L(v)}$ for all $v \in X$, we have

$$\bar{L}(p((1-t)d_x + td_y)) = p((1-t)d_{L(x)} + td_{L(y)})$$

for $x, y \in X$ and $t \in [0,1]$. As $\mathrm{e}^{-1} \circ \bar{L} = L \circ \mathrm{e}^{-1}$, this gives (3). □

4 Polyhedral structure

The main purpose of this section is to show that under suitable discreteness and finite dimensionality assumptions on the metric space X and its injective hull $\mathrm{E}(X)$, respectively, the latter has the structure of a polyhedral complex, like for finite X. Some results of this type are contained in [12, Section 6]. For simplicity, we shall focus on integer valued metrics.

At first, let X be an arbitrary metric space. For $f \in \mathbb{R}^X$, we denote by $A(f)$ the set of all unordered pairs $\{x, y\}$ of points in X with the property that

$$f(x) + f(y) = d(x,y).$$

We consider the undirected graph $(X, A(f))$ with vertex set X, edge set $A(f)$, and with loops $\{x, x\} \in A(f)$ marking the zeros of f. If $f \in \Delta(X)$ and X is finite (or compact), then f is extremal if and only if $(X, A(f))$ has no isolated vertices, that is, $\bigcup A(f) = X$. For an infinite X, this need no longer be true. We therefore introduce the subset

$$\mathrm{E}'(X) := \bigl\{ f \in \Delta(X) : \bigcup A(f) = X \bigr\}$$

of $\mathrm{E}(X)$, whose structure can be analyzed more directly, but which is not injective unless it coincides with $\mathrm{E}(X)$. (In [12], $\mathrm{E}'(X)$ is denoted T_X^0.) Proposition 4.4 below will show that $\mathrm{E}'(X)$ is dense in $\mathrm{E}(X)$ in case the metric of X is integer valued.

A set A of unordered pairs of points in X is called an *admissible* edge set if there exists a function $f \in \mathrm{E}'(X)$ with $A(f) = A$, and $\mathscr{A}(X)$ denotes the set of all such admissible sets. Let $A \in \mathscr{A}(X)$. Note that the graph (X, A) has no isolated vertices but need not be connected. We associate with A the affine subspace

$$\begin{aligned} H(A) &:= \bigl\{g \in \mathbb{R}^X : A \subset A(g)\bigr\} \\ &= \bigl\{g \in \mathbb{R}^X : g(x) + g(y) = d(x, y) \text{ for all } \{x, y\} \in A\bigr\} \end{aligned}$$

of $\mathbb{R}^X$, and we define the *rank* of A as the dimension of $H(A)$,

$$\mathrm{rk}(A) := \dim(H(A)) \in \{0, 1, 2, \dots\} \cup \{\infty\}.$$

An A-*path* in X of length $l \geq 0$ is an $(l+1)$-tuple $(v_0, \dots, v_l) \in X^{l+1}$ with $\{v_{i-1}, v_i\} \in A$ for $i = 1, \dots, l$. An A-*cycle* is an A-path $(v_0, \dots, v_l)$ with $v_l = v_0$. Note that (x, x) is an A-cycle of length 1 if $\{x, x\} \in A$. The A-*component* $[x]$ of a point $x \in X$ is the set

$$[x] := \{y \in X : \text{there exists an } A\text{-path from } x \text{ to } y\}.$$

Whenever $g, h \in H(A)$ and $\{v, v'\} \in A$, we have $g(v) + g(v') = d(v, v') = h(v) + h(v')$ and so $g(v') - h(v') = -(g(v) - h(v))$. It follows that

$$g(y) - h(y) = (-1)^l (g(x) - h(x)) \tag{4.1}$$

whenever there is an A-path of length l from x to y. As a consequence, if there exists an A-cycle of odd length in $[x]$, then $g|_{[x]} = h|_{[x]}$ for all $g, h \in H(A)$. We call $[x]$ an *odd A-component* in this case. In the opposite case, if $[x]$ contains no A-cycle of odd length, $[x]$ is called an *even A-component*. Then the set $\{g|_{[x]} : g \in H(A)\}$ forms a one-parameter family. In fact, every even A-component admits a unique partition

$$[x] = [x]_1 \cup [x]_{-1} \tag{4.2}$$

such that $x \in [x]_1$ and every edge $\{v, v'\} \in A$ with $\{v, v'\} \subset [x]$ connects $[x]_1$ and $[x]_{-1}$; that is, the subgraph of (X, A) induced by $[x]$ is bipartite. Then, by (4.1), $g(y) - h(y) = \sigma(g(x) - h(x))$ whenever $g, h \in H(A)$, $\sigma \in \{1, -1\}$, and $y \in [x]_\sigma$. It is now clear that $\mathrm{rk}(A)$ is exactly the number of even A-components of X. If $\mathrm{rk}(A) = 0$, $H(A)$ consists of a single function. This occurs in particular if $A = A(d_x)$ for some $x \in X$; then $\{x, y\} \in A$ for every $y \in X$, so X is A-connected, and (x, x) is an A-cycle of length 1.

4.1 Lemma. *Suppose that X is a metric space, $A \in \mathscr{A}(X)$, and $1 \leq n := \mathrm{rk}(A) < \infty$. Then the difference of any two elements of $H(A)$ is uniformly bounded on X, so the supremum norm gives a metric on $H(A)$, and there exists an affine isometry from $H(A)$ onto l_∞^n. In particular $H(A)$ is injective.*

Proof. Choose reference points $x_1, \dots, x_n \in X$ such that $[x_1], \dots, [x_n]$ are precisely the n even A-components of X. Let $I \colon H(A) \to l_\infty^n$ be the affine map defined by

$$I(g) := (g(x_1), \dots, g(x_n)).$$

It follows from (4.1) that $\|g - h\|_\infty = \max_{1 \le k \le n} |g(x_k) - h(x_k)| = \|I(g) - I(h)\|_\infty$ for all $g, h \in H(A)$. $\square$

For every $A \in \mathscr{A}(X)$ we consider the set

$$P(A) := \mathrm{E}'(X) \cap H(A) = \{g \in \mathrm{E}'(X) : A \subset A(g)\}.$$

First we note that

$$P(A) = \mathrm{E}(X) \cap H(A) = \Delta(X) \cap H(A). \tag{4.3}$$

To see this, let $f \in \mathrm{E}'(X)$ be such that $A(f) = A$, and let $g \in \Delta(X) \cap H(A)$. Every $x \in X$ is part of an edge $\{x, y\} \in A(f) = A$; then $\{x, y\} \in A(g)$ because $g \in H(A)$. Since $g \in \Delta(X)$, this shows that $g \in \mathrm{E}'(X)$. In view of the inclusions $\mathrm{E}'(X) \subset \mathrm{E}(X) \subset \Delta(X)$ we get (4.3). As $\Delta(X)$ is convex, so is $P(A)$. For every $f \in \mathrm{E}'(X)$ we have $f \in P(A(f))$, thus

$$\mathscr{P} := \{P(A)\}_{A \in \mathscr{A}(X)}$$

is a family of convex subsets of $\mathbb{R}^X$ whose union equals $\mathrm{E}'(X)$. Note that $P(A') \subset P(A)$ if and only if $A \subset A'$. The next result lists some basic properties of $\mathscr{A}(X)$ and $\mathscr{P}$.

4.2 Proposition. *Let X be a metric space. Then:*

(1) *If $f_0, f_1 \in \mathrm{E}'(X)$ and $\lambda \in (0,1)$, then $f := (1-\lambda) f_0 + \lambda f_1 \in \Delta(X)$ and $A(f) = A(f_0) \cap A(f_1)$, so $f \in \mathrm{E}'(X)$ if and only if $\bigcup (A(f_0) \cap A(f_1)) = X$.*

(2) *For $A_0, A_1 \in \mathscr{A}(X)$, the following are equivalent:*

  (i) *$P(A_0) \cup P(A_1) \subset P(A)$ for some $A \in \mathscr{A}(X)$;*

  (ii) *$\bigcup (A_0 \cap A_1) = X$;*

  (iii) *$A_0 \cap A_1 \in \mathscr{A}(X)$.*

  If conditions (i)–(iii) *hold, then $P(A_0) \cup P(A_1) \subset P(A_0 \cap A_1) \subset P(A)$.*

Proof. Let f be given as in (1). Since $f_0, f_1 \in \Delta(X)$ and $\lambda \in (0,1)$, we have

$$\begin{aligned} f(x) + f(y) &= (1-\lambda)(f_0(x) + f_0(y)) + \lambda (f_1(x) + f_1(y)) \\ &\ge (1-\lambda) d(x,y) + \lambda d(x,y) = d(x,y) \end{aligned}$$

for every pair of points $x, y \in X$, and equality holds if and only if $\{x, y\} \in A(f_0) \cap A(f_1)$.

Regarding (2), choose $f_0, f_1 \in \mathrm{E}'(X)$ such that $A(f_i) = A_i$, and put $f := \frac{1}{2}(f_0+f_1)$. If $P(A_0) \cup P(A_1) \subset P(A)$ for some $A \in \mathscr{A}(X)$, then $f \in P(A)$ by convexity and hence $f \in \mathrm{E}'(X)$, so $\bigcup(A_0 \cap A_1) = X$ by (1). If (ii) holds, then, again by (1), $f \in \mathrm{E}'(X)$ and so $A_0 \cap A_1 = A(f) \in \mathscr{A}(X)$. Finally, assuming (iii), we obtain $P(A_0) \cup P(A_1) \subset P(A_0 \cap A_1)$ and thus (i), and for every $A \in \mathscr{A}(X)$ with $P(A_0) \cup P(A_1) \subset P(A)$ we have $A \subset A_0 \cap A_1$ and hence $P(A_0 \cap A_1) \subset P(A)$. □

We now pass to integer valued metrics. Then the sets $P(A)$ with $\mathrm{rk}(A) = n < \infty$ turn out to be n-dimensional polytopes:

4.3 Theorem. *Suppose that X is a metric space with integer valued metric. Let $A \in \mathscr{A}(X)$, and assume that $1 \le n := \mathrm{rk}(A) < \infty$. Then:*

(1) *The set $P(A) \subset H(A) \subset \mathbb{R}^X$ is an injective n-dimensional polytope.*

(2) *The interior of $P(A)$ relative to $H(A)$ is the set $\{g \in \mathrm{E}'(X) : A(g) = A\}$.*

(3) *The faces of $P(A)$ are precisely the sets $P(A')$ with $A' \in \mathscr{A}(X)$ and $A \subset A'$.*

The proof will also give information on the possible isometry types of $P(A)$.

Proof. We fix reference points $x_1, \dots, x_n \in X$ representing the n even A-components. For each k we consider the partition $[x_k] = [x_k]_1 \cup [x_k]_{-1}$ as in (4.2). We also fix an element $f \in \mathrm{E}'(X)$ with $A(f) = A$. For $y \in [x_k]_\sigma$, $\sigma \in \{1, -1\}$, we have

$$f(y) \in \mathbb{Z} + \sigma f(x_k). \tag{4.4}$$

By contrast, if $y \in X_0 := X \setminus \bigcup_{k=1}^n [x_k]$, then there is an A-path from y to itself of odd length, so $f(y) \in \mathbb{Z} - f(y)$ and thus

$$f(y) \in \frac{1}{2}\mathbb{Z}. \tag{4.5}$$

Now let $I_f \colon H(A) \to l_\infty^n$ be the affine isometry defined by

$$I_f(g) := (g(x_1) - f(x_1), \dots, g(x_n) - f(x_n));$$

compare the proof of Lemma 4.1. To show that $I_f(P(A))$ is a polytope, we introduce constants as follows. First, for $1 \le k \le n$ and $\sigma \in \{1, -1\}$, put

$$C_{k\sigma} := \sup\left\{ \frac{d(x, y) - f(x) - f(y)}{2} : x, y \in [x_k]_\sigma \right\}.$$

For $x, y \in [x_k]_\sigma$ we have $\{x, y\} \notin A = A(f)$, hence $0 > d(x, y) - f(x) - f(y) \in \mathbb{Z} - 2\sigma f(x_k)$ by (4.4). Thus the supremum is attained, and $C_{k\sigma} < 0$. Next, if $X_0 \neq \emptyset$, then for $1 \leq k \leq n$ and $\sigma \in \{1, -1\}$, define

$$C_{k\sigma 0} := \sup\bigl\{d(x, y) - f(x) - f(y) : (x, y) \in [x_k]_\sigma \times X_0\bigr\}.$$

For every such pair (x, y), we have $0 > d(x, y) - f(x) - f(y) \in \frac{1}{2}\mathbb{Z} - \sigma f(x_k)$, hence $C_{k\sigma 0} < 0$. Set $\bar{C}_{k\sigma} := C_{k\sigma}$ if $X_0 = \emptyset$ and

$$\bar{C}_{k\sigma} := \max\bigl\{C_{k\sigma}, C_{k\sigma 0}\bigr\}$$

if $X_0 \neq \emptyset$. Finally, if $n \geq 2$, then for $1 \leq k < l \leq n$ and $\sigma, \tau \in \{1, -1\}$, define

$$C_{k\sigma l\tau} := \sup\bigl\{d(x, y) - f(x) - f(y) : (x, y) \in [x_k]_\sigma \times [x_l]_\tau\bigr\}.$$

For every such pair (x, y), we have $0 > d(x, y) - f(x) - f(y) \in \mathbb{Z} - \sigma f(x_k) - \tau f(x_l)$ and so $C_{k\sigma l\tau} < 0$. Now let Q denote the set of all $t = (t_1, \dots, t_n) \in l_\infty^n$ satisfying the system of $2n + 4\binom{n}{2} = 2n^2$ relations

$$\sigma t_k \geq \bar{C}_{k\sigma} \qquad (1 \leq k \leq n,\ \sigma \in \{1, -1\}), \tag{4.6}$$

$$\sigma t_k + \tau t_l \geq C_{k\sigma l\tau} \quad (1 \leq k < l \leq n,\ \sigma, \tau \in \{1, -1\}). \tag{4.7}$$

By the first $2n$ inequalities Q is bounded. Since all constants on the right side are strictly negative, Q is a polytope containing $I_f(f) = 0$ in its interior, so Q has dimension n. It follows readily from Proposition 2.4 that Q is itself injective.

We claim that $I_f(P(A)) = Q$. Let $g \in H(A)$ and $t := I_f(g)$. In view of (4.3), we need to check that $t \in Q$ if and only if $g(x) + g(y) \geq d(x, y)$ for all pairs $\{x, y\} \notin A$. First, consider pairs of points $x, y \in [x_k]$, for some k. If $\sigma \in \{1, -1\}$ and $x, y \in [x_k]_\sigma$, then

$$2\sigma t_k = 2\sigma(g(x_k) - f(x_k)) = g(x) - f(x) + g(y) - f(y)$$

by (4.1). Hence, we have $\sigma t_k \geq C_{k\sigma}$ if and only if the inequality $g(x) + g(y) \geq d(x, y)$ holds for all such pairs (x, y) in $[x_k]_\sigma$. If $x \in [x_k]_1$ and $y \in [x_k]_{-1}$, then $g(x) + g(y) = f(x) + f(y) > d(x, y)$ by (4.1) and since $\{x, y\} \notin A$ by assumption. Next, in case $X_0 \neq \emptyset$, consider pairs $(x, y) \in [x_k]_\sigma \times X_0$, for some k and σ. Then $g(y) = f(y)$ and so

$$\sigma t_k = \sigma(g(x_k) - f(x_k)) = g(x) - f(x) + g(y) - f(y),$$

hence $\sigma t_k \geq C_{k\sigma 0}$ if and only if $g(x) + g(y) \geq d(x, y)$ for all such (x, y). If $x, y \in X_0$ and $\{x, y\} \notin A$, then $g(x) + g(y) = f(x) + f(y) > d(x, y)$. Finally, in case $n \geq 2$, consider pairs $(x, y) \in [x_k]_\sigma \times [x_l]_\tau$ for some $k < l$ and σ, τ. Then

$$\sigma t_k + \tau t_l = \sigma(g(x_k) - f(x_k)) + \tau(g(x_l) - f(x_l)) = g(x) - f(x) + g(y) - f(y),$$

therefore $\sigma t_k + \tau t_l \geq C_{k\sigma l\tau}$ if and only if $g(x) + g(y) \geq d(x,y)$ for all such (x,y). This yields $I_f(P(A)) = Q$ and completes the proof of (1).

Since f was an arbitrary element of $\mathrm{E}'(X)$ with $A(f) = A$ and $I_f(f)$ is an inner point of Q, it follows that the set $\{g \in \mathrm{E}'(X) : A(g) = A\}$ is contained in the relative interior of $P(A)$. Furthermore, if $g \in P(A)$ is such that the inclusion $A \subset A(g)$ is strict, then $g(x) + g(y) = d(x,y)$ for some pair $\{x,y\} \notin A$ and we see from the above argument that equality holds in at least one of the $2n^2$ inequalities (4.6), (4.7); thus $I_f(g)$ is a boundary point of Q. This shows (2).

Now suppose that F is face of $P(A)$ of dimension $n-1$. Choose a point g in the relative interior of F. Since $g \in H(A)$, we have $H(A(g)) \subset H(A)$. For $t := I_f(g)$, exactly one of the $2n^2$ inequalities (4.6), (4.7) is an equality and the others are strict. Reviewing the above argument again we see that then the inclusion $A \subset A(g)$ is strict and exactly one of the following two cases occurs:

(i) there exist k and σ such that every edge in $A(g) \setminus A$ relates two (possibly equal) points of $[x_k]_\sigma$ or connects $[x_k]_\sigma$ with X_0;

(ii) there exist $k < l$ and σ, τ such that every edge in $A(g) \setminus A$ connects $[x_k]_\sigma$ with $[x_l]_\tau$.

In either case, X has $n-1$ even $A(g)$-components, thus $H(A(g))$ is an $(n-1)$-dimensional affine subspace of $H(A)$. For all $h \in H(A(g))$, $A \subset A(g) \subset A(h)$ and the first inclusion is strict, so $H(A(g))$ contains no inner points of $P(A)$ by (2). As $g \in H(A(g))$ is in the relative interior of F, we have $F = P(A) \cap H(A(g))$ and hence $F = P(A(g))$. Now it follows easily by downward induction on k that every face F of $P(A)$ of dimension $k \in \{0,\dots,n\}$ satisfies $F = P(A_F)$ for some $A_F \in \mathscr{A}(X)$ with $A \subset A_F$ and $\mathrm{rk}(A) = k$. Conversely, let $A' \in \mathscr{A}(X)$ with $A \subset A'$ be given. Then $P(A') \subset P(A)$, so the relative interior of $P(A')$ meets the relative interior of some face $F = P(A_F)$ of $P(A)$. Applying (2) to both $P(A')$ and $P(A_F)$ we obtain $A' = A_F$, thus $P(A') = P(A_F) = F$. This concludes the proof of (3). □

Next we show that $\mathrm{E}'(X)$ is dense in $\mathrm{E}(X)$, provided the metric of X is integer valued. A different criterion is given in [12, (5.17)].

4.4 Proposition. *Let X be a metric space with integer valued metric. Then for every $f \in \mathrm{E}(X)$ and every integer $m \geq 1$ there exists a function $f' \in \mathrm{E}'(X)$ with values in $\frac{1}{m}\mathbb{Z}$ such that $\|f - f'\|_\infty \leq \frac{1}{2m}$.*

Proof. Let $f \in \mathrm{E}(X)$, $m \geq 1$, and put $\varepsilon := \frac{1}{2m}$. Denote by $\mathscr{F}$ the set of all functions $g \in \Delta(X)$ with values in $2\varepsilon\mathbb{Z}$ and with $\|f - g\|_\infty \leq \varepsilon$. To see that $\mathscr{F}$ is non-empty, let $g_0 \in \mathbb{R}^X$ be the largest function less than or equal to $f + \varepsilon$ with values in $2\varepsilon\mathbb{Z}$. Then $g_0 > f - \varepsilon$, in particular $\|f - g_0\|_\infty \leq \varepsilon$. For $x, y \in X$, we have $g_0(x) + g_0(y) > f(x) + f(y) - 2\varepsilon \geq d(x,y) - 2\varepsilon$, and since both the first and the last term are in $2\varepsilon\mathbb{Z}$, this gives $g_0(x) + g_0(y) \geq d(x,y)$. So $g_0 \in \mathscr{F}$.

Now let $g \in \mathscr{F}$ be arbitrary, and suppose that $x \in X \setminus \bigcup A(g)$. Then, for every $y \in X$, we have the strict inequality $g(x) > d(x,y) - g(y)$ in $2\varepsilon\mathbb{Z}$, so that

$$g(x) \geq \sup_{y \in X}(d(x,y) - g(y) + 2\varepsilon) \geq \sup_{y \in X}(d(x,y) - f(y) + \varepsilon) = f(x) + \varepsilon$$

by (3.1). Hence $g(x) = f(x) + \varepsilon$. Let the function g' be defined by

$$g'(x) := g(x) - 2\varepsilon = f(x) - \varepsilon$$

and $g'(y) := g(y)$ for all $y \in X \setminus \{x\}$. Note that $g(x) \geq \varepsilon$, thus in fact $g(x) \geq 2\varepsilon$ and $g'(x) \geq 0$. Since $g'(x) + g'(y) = g(x) + g(y) - 2\varepsilon \geq d(x,y)$ for all $y \in X \setminus \{x\}$, it follows that $g' \in \mathscr{F}$. This shows that every minimal element f' of $\mathscr{F}$ satisfies $\bigcup A(f') = X$, that is, $f' \in \mathrm{E}'(X)$. The existence of some minimal element is obvious if X is countable and a consequence of Zorn's Lemma in the general case. □

We now state the concluding result of this section. A metric space X with integer valued metric will be called *discretely path-connected* if for every pair of points $x, y \in X$ there exists a *discrete path* $\gamma\colon \{0,1,\dots,l\} \to X$ from x to y, that is, $\gamma(0) = x$, $\gamma(l) = y$, and $d(\gamma(k-1), \gamma(k)) = 1$ for $k = 1, \dots, l$.

4.5 Theorem. *Let X be a metric space with integer valued metric. Suppose that for every $f \in \mathrm{E}(X)$ there exist $\varepsilon, N > 0$ such that $\mathrm{rk}(A(g)) \leq N$ for all $g \in \mathrm{E}'(X)$ with $\|f - g\|_\infty < \varepsilon$. Then:*

(1) $\mathrm{E}'(X) = \mathrm{E}(X)$.

(2) *$\mathscr{P} = \{P(A)\}_{A \in \mathscr{A}(X)}$ is a polyhedral structure on $\mathrm{E}(X)$ with locally finite dimension, where $P(A')$ is a face of $P(A)$ if and only if $A \subset A'$.*

(3) *For every $n \geq 1$ and $D > 0$, $\mathscr{P}$ has only finitely many isometry types of n-cells with diameter at most D. If, in addition, X is discretely path-connected, then for every n there are only finitely many isometry types of n-cells.*

Proof. For (1), let $f \in \mathrm{E}(X)$. By Proposition 4.4 there exists a sequence (f_i) in $\mathrm{E}'(X)$ that converges to f, and by the assumption of the theorem there is no loss of generality in assuming that $\mathrm{rk}(A(f_i)) = n$ for all i and for some $n \geq 0$. It follows that for every i there exists a set $R_i \subset [0,1)$ with $|R_i| \leq 2n+2$ such that f_i takes values in $\mathbb{Z} + R_i$; see (4.4) and (4.5). Since $f_i \to f$, there also exists $R \subset [0,1)$ with $|R| \leq 2n+2$ such that $f(X) \subset \mathbb{Z} + R$. But then the supremum in (3.1) is attained for every $x \in X$, and so $f \in \mathrm{E}'(X)$. (Compare [12, (5.19)].)

The union of the family $\mathscr{P} = \{P(A)\}_{A \in \mathscr{A}(X)}$ equals $\mathrm{E}'(X) = \mathrm{E}(X)$. In view of Theorem 4.3, for (2) it remains to show that if $A_1, A_2 \in \mathscr{A}(X)$ and $C := P(A_1) \cap P(A_2) \neq \emptyset$, then $C \in \mathscr{P}$. For $i = 1, 2$, let $P(A_i')$ be the minimal face of $P(A_i)$ that contains C. By convexity, C has non-empty interior relative to its affine hull in $\mathbb{R}^X$, hence the relative interiors of $P(A_1')$ and $P(A_1')$ have a common point. It follows that $A_1' = A_2'$ and thus $P(A_1') = P(A_2') = C$.

As for (3), we first observe that if $f \in \mathrm{E}'(X)$ is a vertex of $\mathscr{P}$, then $\mathrm{rk}(A(f)) = 0$ and so $f(X) \subset \frac{1}{2}\mathbb{Z}$ by (4.5). In particular, all edges of $\mathscr{P}$ have length in $\frac{1}{2}\mathbb{Z}$. Now we show that if X is discretely path-connected, then all edges have length at most 2. Suppose that $A \in \mathscr{A}(X)$, $\mathrm{rk}(A) = 1$, and x_1 is a point in the only even A-component of X. Then clearly there exists a pair (x, y) with $d(x, y) = 1$ such that $x \in [x_1]_1$ and either $y \in [x_1]_{-1}$ or $y \in X \setminus [x_1]$. Let $g, h \in P(A)$. By (4.1),

$$\|g - h\|_\infty = |g(x) - h(x)|.$$

In case $y \in [x_1]_{-1}$, we have furthermore $g(x) - h(x) = -(g(y) - h(y))$; since g, h are 1-Lipschitz and $d(x, y) = 1$, it follows that

$$\begin{aligned} 2|g(x) - h(x)| &= |g(x) - h(x) - (g(y) - h(y))| \\ &\le |g(x) - g(y)| + |h(y) - h(x)| \le 2. \end{aligned}$$

In case $y \in X \setminus [x_1]$, we have $g(y) = h(y)$ and so

$$|g(x) - h(x)| \le |g(x) - g(y)| + |h(y) - h(x)| \le 2.$$

In either case, $\|g - h\|_\infty \le 2$. Hence the edge $P(A)$ has length at most 2. Finally, for $n \ge 2$, we see from (4.6), (4.7) that there are only finitely many isometry types of n-cells with diameter at most $D > 0$ and edge lengths in $\frac{1}{2}\mathbb{Z}$, and only finitely many isometry types of n-cells with edge lengths in $\{\frac{1}{2}, 1, \frac{3}{2}, 2\}$. □

The upper bound for the length of edges of $\mathscr{P}$ just derived is sharp:

4.6 Example. Let $X = \{x_1, x_2, y_1, y_2, y_3\}$ be the discretely geodesic metric space with $d(x_1, x_2) = 2$, $d(x_i, y_j) = 1$ for all i and j, and $d(y_j, y_k) = 2$ for $j < k$. The convex hull

$$P' := \mathrm{conv}\{d_{x_1}, d_{x_2}\} = \{(1 - \lambda)d_{x_1} + \lambda d_{x_2} : \lambda \in [0, 1]\}$$

of the distance functions to x_1 and x_2 is an edge of $\mathscr{P}$ of length 2. The injective hull $\mathrm{E}(X)$ consists of the three triangles $P_j := \mathrm{conv}\{d_{x_1}, d_{x_2}, d_{y_j}\}$ for $j = 1, 2, 3$ (the maximal cells of the partially ordered set $\mathscr{P}$), glued along P'. For instance, $A_1 := \{\{x_1, x_2\}, \{y_1, y_2\}, \{y_1, y_3\}\}$ and $A' := A_1 \cup \{\{y_2, y_3\}\}$ are the respective admissible edge sets with $P(A_1) = P_1$ and $P(A') = P'$.

The next example describes a discretely geodesic metric space whose injective hull contains 2-cells of minimal diameter $\frac{1}{2}$, isometric to $\mathrm{conv}\{(\frac{1}{2}, 0), (0, \frac{1}{2}), (0, 0)\}$ in l_∞^2.

4.7 Example. Let $X = \{x_1, x_2, x_3, y_1, y_2, y_3\}$, and suppose that all nonzero distances are 1 except that $d(x_i, x_j) = 2$ for $i \neq j$. The injective hull $\mathrm{E}(X)$ has the following maximal cells: 3 edges $\mathrm{conv}\{d_{x_i}, f_i\}$ of length $\frac{1}{2}$ for $i = 1, 2, 3$, where $f_i(x_i) = f_i(y_k) = \frac{1}{2}$ for all k and $f_i(x_j) = \frac{3}{2}$ for $j \neq i$, and 9 triangles of the type $\mathrm{conv}\{f_i, d_{y_k}, g\}$, where $g(x_i) = 1$ and $g(y_k) = \frac{1}{2}$ for $i, k = 1, 2, 3$. There are two odd $A(g)$-components, and the link of the central vertex g is a (complete bipartite) $K_{3,3}$ graph.

5 Cones

We now discuss geometric conditions that allow to verify the assumption on the rank in Theorem 4.5. Cones, as defined in (1.2), will be instrumental. We start with a basic fact.

5.1 Lemma. *Suppose that X is a metric space, $f \in \Delta_1(X)$, and $\{x, y\} \in A(f)$. Then $\{x, z\} \in A(f)$ and $f(z) = f(y) + d(y, z)$ for all $z \in C(x, y)$.*

Proof. For $f \in \Delta(X)$ and $z \in C(x, y)$, we have

$$f(x) + f(z) \geq d(x, z) = d(x, y) + d(y, z).$$

Furthermore, if $f \in \mathrm{Lip}_1(X, \mathbb{R})$ and $\{x, y\} \in A(f)$, then

$$f(x) + f(z) \leq f(x) + f(y) + d(y, z) = d(x, y) + d(y, z).$$

This gives the result. □

The next lemma, in particular criterion (4), will play a key role in the proof of Theorem 1.1. (For (2), compare [20, Thm. 3.12].)

5.2 Lemma. *Let X be a metric space, and suppose that $f \in \Delta(X)$, $x, y, \bar{x}, \bar{y} \in X$, and $\{x, y\}, \{\bar{x}, \bar{y}\} \in A(f)$. Then each of the following conditions implies that also $\{x, \bar{y}\}, \{\bar{x}, y\} \in A(f)$:*

(1) *$d(x, y) + d(\bar{x}, \bar{y}) \leq d(x, \bar{y}) + d(\bar{x}, y)$;*

(2) *$C(x, y) \cap C(\bar{x}, \bar{y}) \neq \emptyset$;*

(3) *$I(x, \bar{y}) \cap I(\bar{x}, y) \neq \emptyset$;*

(4) *there exists $v \in I(x, y) \cap I(\bar{x}, \bar{y})$ such that $C(x, v) = C(\bar{x}, v)$.*

Proof. Because $\{x, y\}, \{\bar{x}, \bar{y}\} \in A(f)$, (1) gives

$$f(x) + f(y) + f(\bar{x}) + f(\bar{y}) = d(x, y) + d(\bar{x}, \bar{y}) \leq d(x, \bar{y}) + d(\bar{x}, y).$$

It follows that each of the inequalities $f(x)+f(\bar{y}) \geq d(x,\bar{y})$ and $f(\bar{x})+f(y) \geq d(\bar{x},y)$ must in fact be an equality, that is, $\{x,\bar{y}\},\{\bar{x},y\} \in A(f)$. Now assume that (2) holds, and let $z \in C(x,y) \cap C(\bar{x},\bar{y})$. Then

$$d(x,y) + d(y,z) = d(x,z) \leq d(x,\bar{y}) + d(\bar{y},z)$$

and, likewise, $d(\bar{x},\bar{y}) + d(\bar{y},z) \leq d(\bar{x},y) + d(y,z)$. Adding these two inequalities one obtains (1). If $v \in I(x,\bar{y}) \cap I(\bar{x},y)$, then

$$d(x,y) + d(\bar{x},\bar{y}) \leq d(x,v) + d(v,y) + d(\bar{x},v) + d(v,\bar{y}) = d(x,\bar{y}) + d(\bar{x},y),$$

so (3) implies (1) as well. Finally, if (4) holds, then $\bar{y} \in C(\bar{x},v) = C(x,v)$ and $y \in C(x,v) = C(\bar{x},v)$, thus $v \in I(x,\bar{y}) \cap I(\bar{x},y)$. □

As a first simple application of these lemmas we note the following result.

5.3 Proposition. *Suppose that X is a metric space containing at most k pairwise disjoint cones, that is, $|I| \leq k$ for every disjoint family $(C(x_i,y_i))_{i\in I}$ of cones in X. Then* $\mathrm{rk}(A) \leq \frac{1}{2}k$ *for all $A \in \mathscr{A}(X)$.*

Proof. Let $A \in \mathscr{A}(X)$, and suppose that the two edges $\{x,y\},\{\bar{x},\bar{y}\} \in A$ belong to different even A-components of X. It follows from either Lemma 5.1 or Lemma 5.2 that the four cones $C(x,y), C(y,x), C(\bar{x},\bar{y}), C(\bar{y},\bar{x})$ are pairwise disjoint. For instance, if there was a point z in $C(x,y) \cap C(\bar{x},\bar{y})$, then $\{x,z\},\{\bar{x},z\} \in A$ by the first and $\{x,\bar{y}\},\{\bar{x},y\} \in A$ by the second lemma, so x and $\bar{x}$ would be connected by A-paths of length 2. This clearly gives the result. □

An example will be given below. First we record another useful fact related to cones.

5.4 Proposition. *Let Y be a metric space, and let X be a non-empty subset. If for every pair of points $x,y \in Y$ there exists a point $z \in C(x,y) \cap X$, then* $\mathrm{E}(Y)$ *is isometric to* $\mathrm{E}(X)$ *via the restriction map $f \mapsto f|_X$.*

Proof. Let $f \in \mathrm{E}(Y)$, and let $x \in Y$. For every $\varepsilon > 0$ there exist $y \in Y$ and $z \in X$ such that $f(x)+f(y) \leq d(x,y)+\varepsilon$ and $d(x,y)+d(y,z) = d(x,z)$; since $f(z) \leq f(y)+d(y,z)$, this gives $f(x)+f(z) \leq d(x,z)+\varepsilon$. Hence

$$f(x) = \sup_{z\in X}(d(x,z) - f(z)). \tag{5.1}$$

For $x \in X$, this shows that $f|_X \in \mathrm{E}(X)$. Furthermore, for another function $g \in \mathrm{E}(Y)$, combining (5.1) with the inequality $d(x,z) \leq g(x)+g(z)$ we conclude that $f(x) - g(x) \leq \sup_{z\in X}(g(z) - f(z))$ for every $x \in Y$. So $\|f-g\|_\infty = \|f|_X - g|_X\|_\infty$, and by the second part of Proposition 3.5, the restriction operator $f \mapsto f|_X$ maps $\mathrm{E}(Y)$ onto $\mathrm{E}(X)$. □

We now illustrate Proposition 5.3, which turns out to be optimal in some instances.

5.5 Example. Consider the discretely geodesic metric space $X = \mathbb{Z}^n$ with the l_1 distance (the standard word metric of the group $\mathbb{Z}^n$). It is not difficult to see that X contains at most 2^n pairwise disjoint cones. By Theorem 4.5 and Proposition 5.3, $\mathrm{E}(X)$ is a polyhedral complex of dimension at most 2^{n-1}. For the subspace $W_n := \{0,1\}^n \subset X$ of diameter n, the constant function g on W_n with value $\frac{1}{2}n$ satisfies $g \in \mathrm{E}(W_n)$ and $\mathrm{rk}(A(g)) = 2^{n-1}$ (each pair of antipodal points is an $A(g)$-component of W_n); thus $\dim(\mathrm{E}(W_n)) = \dim(\mathrm{E}(X)) = 2^{n-1}$. Furthermore, it follows easily from Proposition 5.4 that $\mathrm{E}(X)$ is isometric to $\mathrm{E}(l_1^n)$. So $\mathrm{E}(X)$ is also a Banach space, isometric to $l_\infty^{2^{n-1}}$ (see Remark 3.6). Unless $n = 1, 2$, the dimension of $\mathrm{E}(X)$ is strictly larger than n and hence the canonical action of $\mathbb{Z}^n$ on $\mathrm{E}(X)$ is not cocompact. This can be remedied by taking the l_∞ distance on $\mathbb{Z}^n$ instead (which is again a word metric); then clearly the injective hull is isometric to l_∞^n.

Given a metric space X and a point $v \in X$, we denote by $\mathscr{C}(v)$ the set of all cones $C(x, v)$ for $x \in X$. The following result shows that if $\mathscr{C}(v)$ happens to be finite, then one obtains some control on the complexity of $\mathrm{E}'(X)$ near d_v. Note that here X is not assumed to be discrete.

5.6 Proposition. *Suppose that X is a metric space and $v \in X$ is a point with $|\mathscr{C}(v)| < \infty$. Consider the set $A_v := A(d_v) \in \mathscr{A}(X)$. Then:*

(1) *Every admissible set $A \in \mathscr{A}(X)$ with $A \subset A_v$ satisfies $\mathrm{rk}(A) \le \frac{1}{2}|\mathscr{C}(v)|$.*

(2) *There are at most $2^{|\mathscr{C}(v)|-1} - 1$ sets $A \in \mathscr{A}(X)$ with $A \subset A_v$ and $\mathrm{rk}(A) = 1$.*

Note that $\{x, y\} \in A_v = A(d_v)$ if and only if $d_v(x) + d_v(y) = d(x, y)$, that is, $v \in I(x, y)$.

Proof. Let $A \subset A_v$ be admissible. There exists a partition

$$X = X_0 \cup \bigcup_{j \in J} (X_{j,1} \cup X_{j,-1}), \tag{5.2}$$

where X_0 is the union of all odd A-components of X, $\{X_j\}_{j \in J}$ is the family of all even A-components, and the partition $X_j = X_{j,1} \cup X_{j,-1}$ is such that no edge in A relates points in the same subset. Let $x, \bar{x} \in X$. There exist $y, \bar{y}$ such that $\{x, y\}, \{\bar{x}, \bar{y}\} \in A \subset A_v$ and thus $v \in I(x, y) \cap I(\bar{x}, \bar{y})$. In case $C(x, v) = C(\bar{x}, v)$ it follows from Lemma 5.2 that $\{x, \bar{y}\}, \{\bar{x}, y\} \in A$, in particular x and $\bar{x}$ are connected by an A-path of length 2. Hence, if x and $\bar{x}$ lie in different sets of the above partition, then $C(x, v) \neq C(\bar{x}, v)$. Thus the number of even A-components is in fact finite and less than or equal to $\frac{1}{2}|\mathscr{C}(v)|$.

For the proof of (2), let $\mathscr{A}_v'$ denote the set of all $A \in \mathscr{A}(X)$ with $A \subset A_v$ and $\mathrm{rk}(A) = 1$. We show that there is an injective map S from $\mathscr{A}_v'$ into the set of all non-empty subsets of $\mathscr{C}(v)$ that do not contain $C(v,v) = X$. For each $A \in \mathscr{A}_v'$ there is a unique partition $X = X_0 \cup X_1 \cup X_{-1}$ such that every $g \in \mathrm{E}'(X)$ with $A(g) = A$ satisfies

$$g(x) = d_v(x) + \sigma \|g - d_v\|_\infty \tag{5.3}$$

for $\sigma \in \{0, 1, -1\}$ and $x \in X_\sigma$. Note that every such g is strictly positive since $\mathrm{rk}(A) > 0$, therefore $v \in X_1$ and thus $C(x,v) \neq X = C(v,v)$ for all $x \in X_{-1}$. The desired map S is defined by

$$S(A) := \{C(x,v) : x \in X_{-1}\}.$$

To show that S is injective, suppose that $S(A) = S(A')$ for some $A, A' \in \mathscr{A}_v'$, and let $X_0 \cup X_1 \cup X_{-1}$ and $X_0' \cup X_1' \cup X_{-1}'$ be the respective partitions of X. Now note, first, that

$$X_{-1} = \{x \in X : C(x,v) \in S(A)\}.$$

This holds since $C(\bar{x}, v) \neq C(x,v)$ for $\bar{x} \in X_0 \cup X_1$ and $x \in X_{-1}$, by the same argument as in the proof of (1). Similarly, $X_{-1}' = \{x \in X : C(x,v) \in S(A')\}$ and so $X_{-1} = X_{-1}'$. Second,

$$X_1 = \{y \in X : \text{there exists } x \in X_{-1} \text{ with } \{x,y\} \in A_v\}.$$

The inclusion $\subset$ is clear since $A \subset A_v$. For the other, if $x \in X_{-1}$ and $\{x,y\} \in A_v$, then every $g \in \mathrm{E}'(X)$ with $A(g) = A$ satisfies $g(y) \geq d(x,y) - g(x) = d_v(x) + d_v(y) - g(x) = d_v(y) + \|g - d_v\|_\infty$, so $y \in X_1$. Together with the corresponding characterization of X_1' and the fact that $X_{-1} = X_{-1}'$, this shows that $X_1 = X_1'$ and $X_0 = X_0'$ as well. Finally, using (5.3) again, we conclude that $A = A'$. □

The bound in the first part of Proposition 5.6 is sharp:

5.7 Example. The cyclic group of order $2n$, with the usual word metric of diameter n, satisfies $|\mathscr{C}(v)| = 2n$ for every element v. The constant function f with value $\frac{1}{2}n$ has $\mathrm{rk}(A(f)) = n$, and $A(f) \subset A(d_v)$ for all v. In fact, the injective hull is a combinatorial n-cube, as is shown in [20, Section 9].

We now turn to discretely geodesic metric spaces X with β-stable intervals, as defined in (1.3). The following observation goes back to Cannon [7, Section 7]. For $x, v \in X$, define $F_{xv} \colon B(v, \beta) \to \mathbb{Z}$ by $F_{xv}(u) := d_x(u) - d_x(v)$.

5.8 Lemma. *Let X be a discretely geodesic metric space with β-stable intervals, and let $x, x', v \in X$. If $F_{xv} \leq F_{x'v}$, then $C(x,v) \subset C(x',v)$. Hence, $F_{xv} = F_{x'v}$ implies that $C(x,v) = C(x',v)$.*

In particular, if for a fixed vertex v the closed ball $B(v,\beta)$ is finite, then there are only finitely many distinct such functions F_{xv} as x ranges over X and so $|\mathscr{C}(v)| < \infty$.

Proof. Suppose that $F_{xv} \le F_{x'v}$. We show by induction on $l \ge 0$ that every $y \in C(x,v)$ with $d(v,y) = l$ is an element of $C(x',v)$. The case $l = 0$ is trivial, so let $y \in C(x,v)$ with $d(v,y) = l \ge 1$. Choose a point $y' \in I(v,y)$ such that $d(v,y) = l-1$; note that $y' \in C(x,v)$. By the induction hypothesis, $y' \in C(x',v)$ and thus $v \in I(x',y')$. Since $d(y',y) = 1$ and X has β-stable intervals, there exists a point $u \in I(x',y) \cap B(v,\beta)$. We have

$$d_{x'}(u) - d_{x'}(v) = F_{x'v}(u) \ge F_{xv}(u) = d_x(u) - d_x(v).$$

Adding the term $d(u,y) - d(v,y)$ on either side and using the identities $d_{x'}(u) + d(u,y) = d_{x'}(y)$ and $d_x(v) + d(v,y) = d_x(y)$ we obtain

$$d_{x'}(y) - d_{x'}(v) - d(v,y) \ge d_x(u) + d(u,y) - d_x(y) \ge 0.$$

Thus $d_{x'}(y) = d_{x'}(v) + d(v,y)$ and so $y \in C(x',v)$. □

Lemma 5.8 shows that if a finitely generated group $\Gamma_S = (\Gamma, d_S)$ with the word metric has β-stable intervals, then $|\mathscr{C}(v)|$ is finite for every $v \in \Gamma_S$, and this number is of course independent of v. The cones $C(x,1)$ based at the identity element of Γ will be called *cone types*. For groups with finitely many cone types the language of all geodesic words is regular and the growth series is a rational function (see [7, 18]).

5.9 Remark. Neumann–Shapiro [32] introduced a similar criterion, the *falsification by fellow traveller* (FFT) property, which is easily seen to imply uniform stability of intervals. In particular it follows from Proposition 4.4 and Theorem 4.3 in their paper that all finitely generated abelian groups have β-stable intervals and that finitely generated virtually abelian groups as well as geometrically finite hyperbolic groups have β-stable intervals for *some* word metrics. The FFT property has been verified for further classes of groups, with respect to suitably chosen finite generating sets, in [33, 34, 22].

We also remark that the uniform stability of intervals is not a necessary condition for a finitely generated group Γ_S to have finitely many cone types:

5.10 Example. The finitely presented group $\Gamma = \langle a, t \,|\, t^2 = 1,\, atat = tata \rangle$ with generating set $S = \{a, t\}$ has finitely many cone types, but intervals are not uniformly stable. This example is discussed in [17].

The next result states another consequence of the stability assumption. For a pair of points x, y in a metric space X,

$$(x \,|\, y)_z := \frac{1}{2}\big(d_z(x) + d_z(y) - d(x,y)\big) \tag{5.4}$$

denotes their Gromov product with respect to $z \in X$. Note that $0 \le (x \,|\, y)_z \le \min\{d_z(x), d_z(y)\}$ and $(x \,|\, y)_z + (x \,|\, z)_y = d(y,z)$.

5.11 Lemma. *Let X be a discretely geodesic metric space with β-stable intervals. Whenever $x, y, z \in X$, there exists a point $v \in I(x,y)$ with $d_z(v) \le \beta \cdot 2(x \,|\, y)_z$.*

Proof. We proceed by induction on the integer $2(x \,|\, y)_z$. If $(x \,|\, y)_z = 0$, then $z \in I(x,y)$ and so we can take $v = z$. Now suppose that $(x \,|\, y)_z > 0$. Choose a discrete geodesic $\gamma \colon \{0,1,\dots,d_z(y)\} \to X$ from z to y, and let k be the largest parameter value such that $z \in I(x,\gamma(k))$. Note that $k < d_z(y)$ because $(x \,|\, y)_z > 0$. Let $y' := \gamma(k+1)$. Since X has β-stable intervals, there exists a point $z' \in I(x,y')$ with $d_z(z') \le \beta$. We have

$$d_{z'}(x) + d_{z'}(y') = d(x,y') \le d_z(x) + d_z(y') - 1$$

by the choice of k. Adding the term $d(y',y) - d(x,y)$ on either side we obtain $2(x \,|\, y)_{z'} \le 2(x \,|\, y)_z - 1$. Hence, by the induction hypothesis, there exists $v \in I(x,y)$ such that $d_{z'}(v) \le \beta \cdot 2(x \,|\, y)_{z'}$. So

$$d_z(v) \le d_z(z') + d_{z'}(v) \le \beta\bigl(1 + 2(x \,|\, y)_{z'}\bigr) \le \beta \cdot 2(x \,|\, y)_z,$$

as desired. □

We conclude this section with a partial generalization of Proposition 5.6. The above lemma will be used in combination with the following simple fact: if $f \in \Delta(X)$ and $\{x,y\} \in A(f)$, that is, $f(x) + f(y) = d(x,y)$, then

$$(x \,|\, y)_z \le \frac{1}{2}\bigl((f(x) + f(z)) + (f(y) + f(z)) - d(x,y)\bigr) = f(z) \tag{5.5}$$

for every $z \in X$. For a subset B of a metric space X we denote by $\mathscr{C}(B)$ the set of all pointed cones $(v, C(x,v))$ with $v \in B$ and $x \in X$.

5.12 Proposition. *Let X be a discretely geodesic metric space with β-stable intervals, and assume that all bounded subsets of X are finite. Fix $z \in X$ and $\alpha > 0$, and let B be the closed ball $B(z, 2\alpha\beta)$. Then $|\mathscr{C}(B)| < \infty$, and*

(1) *every $f \in \mathrm{E}'(X)$ with $f(z) \le \alpha$ satisfies $\mathrm{rk}(A(f)) \le \frac{1}{2}|\mathscr{C}(B)|$;*

(2) *for every $f \in \mathrm{E}'(X)$ with $f(z) \le \alpha$ and $\mathrm{rk}(A(f)) = 0$ there are no more than $2^{|\mathscr{C}(B)|}$ sets $A \in \mathscr{A}(X)$ such that $A \subset A(f)$ and $\mathrm{rk}(A) = 1$.*

Proof. By the assumptions on X and Lemma 5.8, $\mathscr{C}(B)$ is finite. Now let $f \in \mathrm{E}'(X)$, and suppose that $f(z) \le \alpha$. For every ordered pair (x,y) with $\{x,y\} \in A(f)$, we choose a point $v_{xy} \in I(x,y) \cap B$ by means of Lemma 5.11 and (5.5), then we put

$$\widehat{C}(x,y) := \bigl(v_{xy}, C(x,v_{xy})\bigr) \in \mathscr{C}(B).$$

Consider the partition of X induced by $A(f)$, as in (5.2). Let $x, \bar{x} \in X$, and choose $y, \bar{y}$ such that $\{x, y\}, \{\bar{x}, \bar{y}\} \in A(f)$. If $\widehat{C}(x, y) = \widehat{C}(\bar{x}, \bar{y})$, Lemma 5.2 shows that x and $\bar{x}$ are connected by an $A(f)$-path of length 2. Hence, if x and $\bar{x}$ lie in different sets of the partition, then $\widehat{C}(x, y) \neq \widehat{C}(\bar{x}, \bar{y})$. It follows that $\mathrm{rk}(A(f)) \le \frac{1}{2}|\mathscr{C}(B)|$.

Now suppose in addition that $\mathrm{rk}(A(f)) = 0$, and define $\widehat{C}(x, y) \in \mathscr{C}(B)$ as above, for every pair (x, y) with $\{x, y\} \in A(f)$. Let $\mathscr{A}'_f$ denote the set of all $A \in \mathscr{A}(X)$ such that $A \subset A(f)$ and $\mathrm{rk}(A) = 1$. We show that there is an injective map S from $\mathscr{A}'_f$ into the set of all (non-empty) subsets of $\mathscr{C}(B)$. For every $A \in \mathscr{A}'_f$ there is a unique partition $X = X_0 \cup X_1 \cup X_{-1}$ such that every $g \in \mathrm{E}'(X)$ with $A(g) = A$ satisfies

$$g(x) = f(x) + \sigma\|g - f\|_\infty$$

for $\sigma \in \{0, 1, -1\}$ and $x \in X_\sigma$. Define

$$S(A) := \bigl\{\widehat{C}(x, y) : (x, y) \in X_{-1} \times X_1,\ \{x, y\} \in A\bigr\};$$

since $A \subset A(f)$, $\widehat{C}(x, y)$ is defined for all (x, y) with $\{x, y\} \in A$. We claim that

$$X_{-1} = \bigl\{x \in X : \widehat{C}(x, y) \in S(A) \text{ for all } y \in X \text{ with } \{x, y\} \in A(f)\bigr\}.$$

Let $x \in X_{-1}$. If $y \in X$ is such that $\{x, y\} \in A(f)$, then every $g \in \mathrm{E}'(X)$ with $A(g) = A$ satisfies $g(y) \ge d(x, y) - g(x) = f(x) + f(y) - g(x) = f(y) + \|g - f\|_\infty$, so $y \in X_1$ and $\{x, y\} \in A(g) = A$; thus $\widehat{C}(x, y) \in S(A)$. Conversely, suppose that $\bar{x} \in X$, and $\widehat{C}(\bar{x}, \bar{y}) \in S(A)$ for all $\bar{y} \in X$ with $\{\bar{x}, \bar{y}\} \in A(f)$. Among all such points $\bar{y}$, fix one with $\{\bar{x}, \bar{y}\} \in A$. Since $\widehat{C}(\bar{x}, \bar{y}) \in S(A)$, there is a pair $(x, y) \in X_{-1} \times X_1$ such that $\{x, y\} \in A$ and $\widehat{C}(x, y) = \widehat{C}(\bar{x}, \bar{y})$. By Lemma 5.2, x and $\bar{x}$ are connected by an A-path of length 2, thus $\bar{x} \in X_{-1}$. Hence, X_{-1} is characterized in terms of $S(A)$ as claimed. Now one can proceed as in the proof of Proposition 5.6, with f in place of d_v, to show that S is injective. This gives (2). □

6 Proofs of the main results

We now prove the theorems stated in the introduction and discuss some examples.

Proof of Theorem 1.1. Let X be a discretely geodesic metric space with β-stable intervals, and suppose that all bounded subsets of X are finite. Given $f \in \mathrm{E}(X)$, there exists a point $z \in X$ where f attains its minimum. Fix any $\varepsilon > 0$. By the first part of Proposition 5.12 there exists a number N such that $\mathrm{rk}(A(g)) \le N$ for all $g \in \mathrm{E}'(X)$ with $g(z) \le f(z) + \varepsilon$, in particular for all $g \in \mathrm{E}'(X)$ with $\|f - g\| < \varepsilon$. Now Theorem 4.5 shows that $\mathrm{E}'(X) = \mathrm{E}(X)$ and that $\mathscr{P} = \{P(A)\}_{A \in \mathscr{A}(X)}$ is a polyhedral structure on $\mathrm{E}(X)$ with locally finite dimension and with only finitely many isometry types of n-cells for every n. By Theorem 4.3 every n-cell $P(A)$ is isometric to an injective polytope in l_∞^n. To show that $\mathscr{P}$ is in fact locally finite, let $f \in \mathrm{E}(X) = \mathrm{E}'(X)$

be a vertex of $\mathscr{P}$; that is, $\mathrm{rk}(A(f)) = 0$. Let again z be a point where f attains the minimum, and put $\alpha := f(z)$. By the second part of Proposition 5.12 there is a number M such that there are at most M admissible sets $A \subset A(f)$ with $\mathrm{rk}(A) = 1$; in other words, there are at most M edges in $\mathscr{P}$ issuing from the vertex f. Thus $\mathscr{P}$ is locally finite, and $\mathrm{E}(X)$ is locally compact. Consequently, as a complete geodesic metric space, $\mathrm{E}(X)$ is proper. □

The following simple example shows that, with the assumptions of Theorem 1.1, the injective hull may be infinite dimensional.

6.1 Example. For every integer $n \geq 1$, $W_n := \{0,1\}^n$ with the l_1 distance (the vertex set of the n-cube graph) has 1-stable intervals, and the injective hull $\mathrm{E}(W_n)$ has dimension 2^{n-1} (compare Example 5.5; see also [20, Section 5] for more precise information in the case $n = 3$). Now let X be the space obtained from the disjoint union $\bigcup_{n=1}^{\infty} W_n$ by identifying $(1,1,\dots,1) \in W_n$ with $(0,0,\dots,0) \in W_{n+1}$ for $n = 1,2,\dots$, equipped with the obvious discretely geodesic metric, so that X contains an isometric copy of each W_n. Clearly X has 1-stable intervals, bounded subsets of X are finite, and $\mathrm{E}(X)$ is infinite dimensional.

We pass to δ-hyperbolic metric spaces, as defined in (1.4).

Proof of Theorem 1.2. To show that $\mathrm{E}(X)$ is δ-hyperbolic, let $e,f,g,h \in \mathrm{E}(X)$, and let $\varepsilon > 0$. There exist $w,x \in X$ such that either $\|e-f\|_\infty \leq e(x) - f(x) + \varepsilon$ and $e(x) \leq d(w,x) - e(w) + \varepsilon$, or $\|e-f\|_\infty \leq f(w) - e(w) + \varepsilon$ and $f(w) \leq d(w,x) - f(x) + \varepsilon$. Thus

$$\|e-f\|_\infty \leq d(w,x) - e(w) - f(x) + 2\varepsilon.$$

Likewise, $\|g-h\|_\infty \leq d(y,z) - g(y) - h(z) + 2\varepsilon$ for some $y,z \in X$. Put $\Sigma := e(w) + f(x) + g(y) + h(z)$. Using the δ-hyperbolicity of X we obtain

$$\begin{aligned}\|e-f\|_\infty + \|g-h\|_\infty &\leq d(w,x) + d(y,z) - \Sigma + 4\varepsilon \\ &\leq \max\{d(w,y) + d(x,z), d(x,y) + d(w,z)\} - \Sigma + \delta + 4\varepsilon.\end{aligned}$$

Now $d(w,y) + d(x,z) - \Sigma \leq e(y) + f(z) - g(y) - h(z) \leq \|e-g\|_\infty + \|f-h\|_\infty$ and $d(x,y) + d(w,z) - \Sigma \leq -e(w) - f(x) + g(x) + h(w) \leq \|f-g\|_\infty + \|e-h\|_\infty$. Since $\varepsilon > 0$ was arbitrary, this gives the desired inequality for e,f,g,h.

Suppose, in addition, that X is geodesic or discretely geodesic. Put $\nu := 0$ in the former and $\nu := \frac{1}{2}$ in the latter case. Let $f \in \mathrm{E}(X)$. For $\varepsilon > 0$, choose $x,y \in X$ such that $f(x) + f(y) \leq d(x,y) + \varepsilon$. Since $f(x) + f(y) \geq d(x,y)$, there is a point $v \in I(x,y)$ such that $d(v,x) \leq f(x) + \nu$ and $d(v,y) \leq f(y) + \nu$. Using the δ-hyperbolicity of the quadruple $\{f, d_v, d_x, d_y\} \subset \mathrm{E}(X)$, together with (3.2), we get

$$\begin{aligned}f(v) + d(x,y) &\leq \max\{f(x) + d(v,y), f(y) + d(v,x)\} + \delta \\ &\leq f(x) + f(y) + \delta + \nu \\ &\leq d(x,y) + \varepsilon + \delta + \nu,\end{aligned}$$

thus $f(v) \le \varepsilon+\delta+\nu$. Hence, for every $\varepsilon > 0$ there exists $v \in X$ such that $\|f-d_v\|_\infty = f(v) \le \varepsilon+\delta+\nu$. □

Next, let again Γ be a group with a finite generating set S, and let d_S denote the word metric on Γ with respect to the alphabet $S \cup S^{-1}$. We write Γ_S for the metric space (Γ, d_S). By Proposition 3.7, the isometric action $(x,y) \mapsto L_x(y) := xy$ of Γ on Γ_S induces an isometric action $(x,f) \mapsto \bar{L}_x(f) = f \circ L_x^{-1}$ of Γ on the injective hull $\mathrm{E}(\Gamma_S)$.

Proof of Theorem 1.3. First we show that for every bounded set $B \subset \mathrm{E}(\Gamma_S)$ there are only finitely many $x \in \Gamma$ such that $\bar{L}_x(B) \cap B \ne \emptyset$. Let $R > 0$ be such that $\|f-d_1\|_\infty \le R$ for all $f \in B$, where 1 is the identity element of Γ. We have $\bar{L}_x(d_1) = d_x$. Hence, if $f \in \bar{L}_x(B) \cap B$, then also $\|f-d_x\|_\infty \le R$ and so $d_S(1,x) = \|d_1 - d_x\|_\infty \le 2R$. This gives the result.

If Γ_S has β-stable intervals, we already know that $\mathrm{E}'(\Gamma_S) = \mathrm{E}(\Gamma_S)$ and that $\mathscr{P} = \{P(A)\}_{A \in \mathscr{A}(\Gamma_S)}$ is a locally finite polyhedral structure on $\mathrm{E}(\Gamma_S)$, which is thus a proper metric space. By what we have just shown, the action of Γ on $\mathrm{E}(\Gamma_S)$ is proper. For $x \in \Gamma$ and $f, g \in \mathrm{E}(\Gamma_S)$, it follows from the left-invariance of d_S that $A(f) = A(g)$ if and only if $A(f \circ L_x^{-1}) = A(g \circ L_x^{-1})$, which is in turn equivalent to $A(\bar{L}_x(f)) = A(\bar{L}_x(g))$. So $\bar{L}_x$ maps cells onto cells.

Now suppose that Γ_S is δ-hyperbolic. It follows from Theorem 1.2 that for every $f \in \mathrm{E}(\Gamma_S)$ there is a point $z \in \Gamma_S$ with $\|f - d_z\| \le \delta + \frac{1}{2}$. Hence the closed ball in $\mathrm{E}(\Gamma_S)$ with center d_1 and radius $\delta + \frac{1}{2}$ is a compact set whose Γ-orbit covers $\mathrm{E}(\Gamma_S)$. As in the second half of the proof of Theorem 1.2 we see that whenever $f \in \mathrm{E}(\Gamma_S)$ and $\{x,y\} \in A(f)$, there exists a point $v_{xy} \in I(x,y)$ with $f(v_{xy}) \le \delta + \frac{1}{2}$. If $\{x',y'\}$ is another element of $A(f)$, then $d(v_{xy}, v_{x'y'}) \le f(v_{xy}) + f(v_{x'y'}) \le 2\delta+1$. The argument of the first part of Proposition 5.12 then shows that

$$\mathrm{rk}(A(f)) \le \frac{1}{2} \cdot \max\{|B| : B \subset \Gamma_S,\, \mathrm{diam}(B) \le 2\delta+1\} \cdot |\mathscr{C}(1)| \tag{6.1}$$

for all $f \in \mathrm{E}(\Gamma_S)$, where $|\mathscr{C}(1)|$ is the number of cone types of (Γ, S). So the dimension of $\mathrm{E}(\Gamma_S)$ is bounded by the right side of (6.1) too. □

In order for the injective hull $\mathrm{E}(\Gamma_S)$ to lie within finite distance of $\mathrm{e}(\Gamma_S)$, Γ_S need not be word hyperbolic, as is shown by $\mathbb{Z}^2$ (Example 5.5). A necessary condition is given next.

6.2 Remark. Let $\Gamma_S = (\Gamma, d_S)$ be a finitely generated group with the word metric, and suppose that there is a constant D such that for every $f \in \mathrm{E}(\Gamma_S)$ there is an element $z \in \Gamma_S$ with $\|f - d_z\|_\infty = f(z) \le D$. Note that Γ acts coboundedly on $\mathrm{E}(\Gamma_S)$. It follows from Theorem 3.8 that there is map $\sigma\colon \Gamma_S \times \Gamma_S \times [0,1] \to \Gamma_S$ with the following properties: for every pair $(x,y) \in \Gamma_S \times \Gamma_S$, the map $\sigma_{xy} := \sigma(x,y,\cdot)$

satisfies $\sigma_{xy}(0) = x$, $\sigma_{xy}(1) = y$, and $|d_S(\sigma_{xy}(s), \sigma_{xy}(t)) - (t-s)d_S(x,y)| \le 2D$ for $0 \le s \le t \le 1$; furthermore,

$$d_S(\sigma_{xy}(t), \sigma_{x'y'}(t)) \le (1-t)d(x,x') + td(y,y') + 2D$$

and $z \cdot \sigma_{xy}(t) = \sigma_{zx,zy}(t)$ for $x, y, x', y' \in \Gamma_S$, $t \in [0,1]$, and $z \in \Gamma$. In particular, Γ_S is semihyperbolic in the sense of Alonso–Bridson [2]. On the other hand, $\mathbb{Z}^n$ for $n \ge 3$ is an example of a semihyperbolic group that does not act coboundedly on its injective hull.

For a finitely generated group Γ_S with β-stable intervals that is not word hyperbolic, Theorem 1.3 leaves open the possibility of $\mathrm{E}(\Gamma_S)$ being infinite dimensional. An example of this type is missing at present. However, there are simple instances of finitely presented groups (without uniformly stable intervals) whose injective hull fails to be finite dimensional or locally compact. We also note that groups with β-stable intervals are easily seen to be almost convex in the sense of Cannon [8] and hence finitely presented.

6.3 Example. Let Γ_S be the Baumslag–Solitar group $\langle x, y \,|\, yx = x^2y \rangle$ with generating set $S = \{x, y\}$. Fix an integer $n \ge 1$. The word $w_n := u_n x u_n^{-1} x^{-1}$, where $u_n := y^n x^2 y^{-n}$, represents the identity. Let $\gamma\colon \{0, 1, \dots, l\} \to \Gamma_S$ be the corresponding discrete loop of length $l := 4n + 6$. This is similar to the loop depicted in [18, Figure 7.8] (where u_n is chosen to be $y^n x y^{-n}$). By inspecting this picture, one sees that for $k = 0, \dots, n$, the two points $\gamma(k) = y^k$ and $\gamma(\frac{1}{2}l + k) = u^n x y^k = x^{2^{n+1}+1} y^k$ are at distance $\frac{1}{2}l$ from each other. It follows that the constant function $f = \frac{1}{4}l$ on $Y := \bigcup_{k=0}^{n} \{\gamma(k), \gamma(\frac{1}{2}l + k)\}$ is an element of $\mathrm{E}(Y)$ with $\mathrm{rk}(A(f)) = n + 1$. As $n \ge 1$ was arbitrary, $\mathrm{E}(\Gamma_S)$ must be infinite dimensional.

The following example shows that for a finitely presented group Γ_S with infinitely many cone types, $\mathrm{E}(\Gamma_S)$ need not be locally finite near points in $\mathrm{e}(\Gamma_S)$. This contrasts with the second assertion of Proposition 5.6.

6.4 Example. Consider the group $\Gamma = \langle a, b, t \,|\, ab = ba,\, t^2 = 1,\, tab = abt \rangle$ with generating set $S = \{a, b, t\}$. For every integer $m \ge 1$, put $x_m := a^{-m}t$ and $y_m := b^m t$. Note that $d_S(1, x_m) = d_S(1, y_m) = m + 1$,

$$d_S(x_m, y_m) = d_S(1, ta^m b^m t) = d_S(1, t(ab)^m t) = 2m$$

for all $m \ge 1$, and $d_S(x_m, y_n) = m + n + 2$ if $n \ne m$. Hence, $C(x_m, 1)$ contains $\{y_n : n \ne m\}$ but not y_m, so Γ_S has infinitely many cone types. Now let $f_m \in \mathrm{E}(\Gamma_S)$ be a median point of the triple of distance functions d_1, d_{x_m}, d_{y_m}. Then $\|f_m - d_1\|_\infty = 1$ and $\|f_m - d_{x_m}\|_\infty = \|f_m - d_{y_m}\|_\infty = m$, and it follows from the triangle inequality that $\|f_n - f_m\|_\infty = 2$ whenever $n \ne m$. Hence, there is an isometrically embedded simplicial tree with infinite valence at the vertex d_1, which therefore has no compact neighborhood in $\mathrm{E}(\Gamma_S)$. Nevertheless, I suspect $\mathrm{E}(\Gamma_S)$ to be a polyhedral complex of finite dimension (equal to 3).

We proceed to Theorem 1.4 and Theorem 1.5. Given a metric space X and a group Λ of isometries of X, we write $\Lambda x := \{L(x) : L \in \Lambda\}$ for the orbit of x and $\Lambda \backslash X$ for the set of orbits; furthermore $\mathrm{Fix}(\Lambda) := \{x \in X : \Lambda x = \{x\}\}$ denotes the fixed point set.

Proof of Theorem 1.4. First we show that if X is a metric space and Λ is a subgroup of the isometry group of X with bounded orbits, there exists an extremal function $f \in \mathrm{E}(X)$ that is constant on each orbit. For $\Lambda x, \Lambda y \in \Lambda \backslash X$, define

$$D(\Lambda x, \Lambda y) := \sup\{d(x', y') : x' \in \Lambda x,\ y' \in \Lambda y\}.$$

Note that this is finite since the orbits are bounded, and D has all the properties of a metric except that $D(\Lambda x, \Lambda x) = \mathrm{diam}(\Lambda x) > 0$ if $x \notin \mathrm{Fix}(\Lambda)$. Denote by $\Delta(\Lambda \backslash X, D)$ the set of all functions $G\colon \Lambda \backslash X \to \mathbb{R}$ such that

$$G(\Lambda x) + G(\Lambda y) \geq D(\Lambda x, \Lambda y)$$

for all $\Lambda x, \Lambda y \in \Lambda \backslash X$. For $z \in X$, the function defined by $G_z(\Lambda x) := D(\Lambda x, \Lambda z)$ belongs to $\Delta(\Lambda \backslash X, D)$, due to the triangle inequality for D. By Zorn's Lemma, the partially ordered set $(\Delta(\Lambda \backslash X, D), \leq)$ has a minimal element F. Consider the respective function $f\colon X \to \mathbb{R}$, $f(x) := F(\Lambda x)$. For all $x, y \in X$,

$$f(x) + f(y) = F(\Lambda x) + F(\Lambda y) \geq D(\Lambda x, \Lambda y) \geq d(x, y),$$

so $f \in \Delta(X)$. Furthermore, by the minimality of F, for every $x \in X$ and $\varepsilon > 0$ there is a point $y \in X$ such that $F(\Lambda x) + F(\Lambda y) \leq D(\Lambda x, \Lambda y) + \varepsilon$ and $D(\Lambda x, \Lambda y) \leq d(x, y) + \varepsilon$, hence $f(x) + f(y) \leq d(x, y) + 2\varepsilon$. This shows that in fact $f \in \mathrm{E}(X)$.

Now suppose that X is injective. Then the only extremal functions on X are distance functions, so by the above result there exists a point $z \in X$ such that d_z is constant on each orbit of Λ. Thus $\Lambda z = \{z\}$ and so $z \in \mathrm{Fix}(\Lambda)$. We prove that $\mathrm{Fix}(\Lambda)$ is hyperconvex (recall Proposition 2.3). Since $\mathrm{Fix}(\Lambda) \neq \emptyset$, it suffices to show that if $((x_i, r_i))_{i \in I}$ is a non-empty family in $X \times \mathbb{R}$ such that $x_i \in \mathrm{Fix}(\Lambda)$ and $r_i + r_j \geq d(x_i, x_j)$ for all pairs of indices $i, j \in I$, then $Y := \bigcap_{i \in I} B(x_i, r_i)$ has non-empty intersection with $\mathrm{Fix}(\Lambda)$. Note that Y is bounded and hyperconvex, in particular $Y \neq \emptyset$. For all $i \in I$, $L \in \Lambda$, and $y \in Y$, we have

$$d(x_i, L(y)) = d(L(x_i), L(y)) = d(x_i, y) \leq r_i,$$

thus $L(Y) \subset Y$. In other words, for every $L \in \Lambda$, the restriction $L|_Y$ is an isometric embedding of Y into itself. In fact, since also $L^{-1}(Y) \subset Y$, $L|_Y$ is an isometry of Y. Since Y is bounded and injective, the group $\{L|_Y : L \in \Lambda\}$ must have a fixed point, as we already know, so $Y \cap \mathrm{Fix}(\Lambda) \neq \emptyset$. $\square$

Recall that the barycenter b of a finite family $(v_i)_{i=1}^k$ of points in a vector space Z is defined as $b := \frac{1}{k}\sum_{i=1}^k v_i$ or, equivalently, as the unique point b such that $\sum_{i=1}^k (v_i - b) = 0$. If $L\colon Z \to Z$ is an affine map, the barycenter of $(L(v_i))_{i=1}^k$ equals $L(b)$. If, in addition, $L(v_i) = v_{\sigma(i)}$ for some permutation σ of $\{1, \dots, k\}$, then $L(b) = b$. Now let again $\Gamma_S = (\Gamma, d_S)$ be a finitely generated group with β-stable intervals. For every cell $P(A)$ of $\mathscr{P} = \{P(A)\}_{A \in \mathscr{A}(\Gamma_S)}$ we define the barycenter $b(A)$ as the barycenter in $\mathbb{R}^\Gamma$ of the vertex set of $P(A)$; $b(A)$ is a point in the interior of $P(A)$ relative to the affine hull $H(A)$ of $P(A)$. Every isometry L of $P(A)$ is the restriction of an affine transformation of $H(A)$ that permutes the vertices of $P(A)$, so $L(b(A)) = b(A)$. We denote by $\mathscr{P}^1$ the first barycentric subdivision of $\mathscr{P}$, that is, the collection of all simplices $\mathrm{conv}\{b(A_0), b(A_1), \dots, b(A_j)\}$ corresponding to strictly ascending sequences $P(A_0) \subset P(A_1) \subset \dots \subset P(A_j)$ of cells in $\mathscr{P}$. We write $\mathrm{E}(\Gamma_S)^1$ for the metric space $\mathrm{E}(\Gamma_S)$ equipped with the simplicial structure $\mathscr{P}^1$. The group Γ still acts by cellular — now simplicial — isometries on $\mathrm{E}(\Gamma_S)^1$ via $x \mapsto \bar{L}_x$. Furthermore, if $\bar{L}_x$ maps a simplex in $\mathscr{P}^1$ to itself, then $\bar{L}_x$ fixes the simplex pointwise. In particular, $\mathrm{E}(\Gamma_S)^1$ is a Γ-CW-complex. We recall that a Γ-CW-complex W is a *model for the classifying space* $\underline{\mathrm{E}}\Gamma$ *for proper* Γ*-actions* if every cell stabilizer is finite, and for each finite subgroup Λ of Γ, the fixed point subcomplex $\mathrm{Fix}(\Lambda) \subset W$ is contractible (see [27, Sect. 1]).

Proof of Theorem 1.5. As discussed above, $\mathrm{E}(\Gamma_S)^1$ is a Γ-CW-complex. Since Γ acts properly, all cell stabilizers are finite. For every finite subgroup Λ of Γ, $\mathrm{Fix}(\Lambda)$ is contractible by Theorem 1.4. □

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