APPENDIX: ADEQUATE SUBGROUPS

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Let l be a prime, and let Γ be a finite subgroup of $\mathrm{GL}_n(\overline{\mathbb{F}}_l) = \mathrm{GL}(V)$. With these assumptions we say that *Condition* (C) *holds* if for every irreducible Γ-submodule $W \subset \mathrm{ad}^0 V$ there exists an element $g \in \Gamma$ with an eigenvalue α such that $\mathrm{tr}\, e_{g,\alpha} W \neq 0$. Here, $e_{g,\alpha}$ denotes the projection to the generalised α-eigenspace of g. This condition arises in the definition of adequacy in section 2.

Let Γ^{ss} denote the subset of Γ consisting of the elements that are semisimple (i.e. of order prime to l).

Lemma 1. *Suppose that Γ acts irreducibly on V. The following are equivalent.*

(i) *Condition* (C).
(ii) *For every irreducible submodule $W \subset \mathrm{ad}^0 V$ there exists $g \in \Gamma^{\mathrm{ss}}$ and $\alpha \in \overline{\mathbb{F}}_l$ such that $\mathrm{tr}\, e_{g,\alpha} W \neq 0$.*
(iii) *The set Γ^{ss} spans $\mathrm{ad}\, V$ as an $\overline{\mathbb{F}}_l$-vector space.*

Proof. Note that for any $g \in \Gamma$, Γ contains both its semisimple and unipotent parts g_s and g_u, respectively. (They are powers of g, as we work over $\overline{\mathbb{F}}_l$.) Since $e_{g,\alpha} = e_{g_s,\alpha}$ for all $g \in \Gamma$, the first two conditions are equivalent.

To show that the last two conditions are equivalent, let $Z \subset \mathrm{ad}\, V$ be the span of the semisimple elements in Γ. Let U denote the annihilator of Z under the (non-degenerate, Γ-invariant) trace pairing:

$$U = \{w \in \mathrm{ad}\, V : \mathrm{tr}(gw) = 0 \quad \forall g \in \Gamma^{\mathrm{ss}}\} \tag{1}$$

$$= \{w \in \mathrm{ad}\, V : \mathrm{tr}(e_{g,\alpha} w) = 0 \quad \forall g \in \Gamma^{\mathrm{ss}},\ \alpha \in \overline{\mathbb{F}}_l\}, \tag{2}$$

where we used that $e_{g,\alpha}$ is a polynomial in g and that $g = \sum \alpha e_{g,\alpha}$ for g semisimple.

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Note that $U \subset \mathrm{ad}^0 V$ by taking $g = 1$ in (1). From (2) it thus follows that the second condition is equivalent to $U = 0$. Equivalently, $Z = \mathrm{ad}\, V$, which is the third condition. □

Lemma 2.

(i) *Suppose that* Γ *acts irreducibly on* V*. Condition* (C) *holds whenever* Γ *has order prime to* l*.*

(ii) *Suppose that* V*,* V' *are finite-dimensional vector spaces over* $\overline{\mathbb{F}}_l$ *and that* $\Gamma \subset \mathrm{GL}(V)$*,* $\Gamma' \subset \mathrm{GL}(V')$ *are finite subgroups that act irreducibly. If they both satisfy* (C)*, then the image of* $\Gamma \times \Gamma'$ *in* $\mathrm{GL}(V \otimes V')$ *also satisfies* (C)*.*

Proof. By Burnside's theorem, Γ spans $\mathrm{ad}\, V$. If Γ has order prime to l, then every element is semisimple, so the lemma above applies.

The second part of the proposition follows on noting that if g, h are semisimple elements then $g \otimes h$ is semisimple, and appealing to the third characterization of condition (C) in the lemma above. □

Next we establish some preliminary results to prepare for our main theorem.

Lemma 3. *Suppose that* T *is a torus over* $\mathbb{F}_l$*. Let* $X^* = X^*(T_{/\overline{\mathbb{F}}_l})$ *and* $X_* = X_*(T_{/\overline{\mathbb{F}}_l})$*. There is a natural action of Frobenius* Fr *as an automorphism of* X^* *and* X_**. Suppose that* $\Delta_* \subset X_*$ *is a finite subset that is stable under the action of* Fr *and spans* $X_* \otimes \mathbb{Q}$*.*

(i) *If* $\mu \in X^*$ *with* $|\langle \mu, \delta \rangle| < l - 1$ *for all* $\delta \in \Delta_*$ *then* $\mu(T(\mathbb{F}_l))$ *is trivial iff* $\mu = 0$*.*

(ii) *If* V *is a* $T_{/\overline{\mathbb{F}}_l}$*-module and all the weights* μ *of* $T_{/\overline{\mathbb{F}}_l}$ *on* V *satisfy* $|\langle \mu, \delta \rangle| < (l-1)/2$ *for all* $\delta \in \Delta_*$ *then the* $\overline{\mathbb{F}}_l$*-span of* $T(\mathbb{F}_l)$ *in* $\mathrm{ad}\, V$ *equals the* $\overline{\mathbb{F}}_l$*-span of* $T(\overline{\mathbb{F}}_l)$*.*

Proof. We can identify $\mathrm{Hom}(T(\mathbb{F}_l), \overline{\mathbb{F}}_l^\times)$ with $X^*/(l - \mathrm{Fr})X^*$. To prove the first part, suppose that $|\langle \mu, \delta \rangle| < l-1$ for $\delta \in \Delta_*$ and that $\mu(T(\mathbb{F}_l))$ is trivial, so $\mu = (l - \mathrm{Fr})\lambda$. Choose δ_1 in Δ_* with $|\langle \lambda, \delta_1 \rangle|$ maximal. If $\langle \lambda, \delta_1 \rangle \neq 0$ then

$$l - 1 > |\langle \mu, \delta_1 \rangle| \geq l|\langle \lambda, \delta_1 \rangle| - |\langle \lambda, \mathrm{Fr}^{-1} \delta_1 \rangle| \geq (l-1)|\langle \lambda, \delta_1 \rangle| \geq l - 1,$$

a contradiction. Therefore $\langle \lambda, \delta_1 \rangle = 0$, so $\lambda = 0$ and $\mu = 0$. In particular we see that if μ_1 and μ_2 are two elements of X^* with $|\langle \mu_i, \delta \rangle| < (l-1)/2$ for $\delta \in \Delta_*$ and $i = 1$, 2 then $\mu_1|_{T(\mathbb{F}_l)} = \mu_2|_{T(\mathbb{F}_l)}$ iff $\mu_1 = \mu_2$. The second part now follows since both subspaces of $\mathrm{ad}\, V$ equal the $\overline{\mathbb{F}}_l$-linear span of the $T_{/\overline{\mathbb{F}}_l}$-equivariant projectors onto the weight spaces of $T_{/\overline{\mathbb{F}}_l}$ in V. □

Lemma 4. *Suppose that G is a connected simply connected semisimple algebraic group over $\overline{\mathbb{F}}_l$ and $\phi : G \to \mathrm{GL}(V)$ a finite-dimensional representation. Let $G \supset B \supset T$ denote a Borel and maximal torus, and suppose that $|\langle \mu_1 - \mu_2, \alpha^\vee \rangle| < l$ for all weights μ_1, μ_2 of T on V and all simple roots α. Then there exist connected simply connected semisimple algebraic subgroups I and J of G such that $G = I \times J$, $\phi(J) = 1$, and ϕ induces a central isogeny of I onto its image $\overline{I}$, which is a semisimple algebraic group.*

Proof. Let J denote the connected component of the kernel of ϕ with its reduced scheme structure. Then J is smooth ([Mil], Proposition I.5.18). By Theorem 8.1.5 of [Spr09] and its proof, J is semisimple and there is a second semisimple algebraic group $I \subset G$ which commutes with J and such that $I \times J \to G$ is a central isogeny. It follows from the simply-connectedness of G that it is an isomorphism of $I \times J$ onto G. In particular, I and J are simply connected. Note that $T = T_I \times T_J$ and that $B = B_I \times B_J$ where (B_I, T_I) (resp. (B_J, T_J)) is a Borel and maximal torus in I (resp. J). (This follows from the fact that any smooth connected soluble subgroup of (resp. torus in) G is conjugate to a subgroup of B (resp. T).) Moreover $U = U_I \times U_J$, where U denotes the unipotent radical of B. Let $\overline{I}$ denote the image of I under ϕ. Then $\overline{I}$ is again reduced and connected and hence also smooth. In fact it is semisimple. (See Proposition 14.10(1)(c) of [Bor91].) The map ϕ factors through an isogeny $I \to \overline{I} \subset \mathrm{GL}(V)$. Let $\overline{B}$, $\overline{T}$, $\overline{U}$ denote the images of B_I, T_I, U_I in $\overline{I}$. Then these are all reduced and hence smooth. Moreover $\overline{T}$ is a torus, $\overline{B}$ is connected and soluble, $\overline{U}$ is connected unipotent and $\overline{B} = \overline{TU}$. As $\dim \overline{I} = \dim I = \dim T_I + 2 \dim U_I = \dim \overline{T} + 2 \dim \overline{U}$ we see that $\overline{B}$ must be a Borel subgroup of $\overline{I}$ with unipotent radical $\overline{U}$ and that $\overline{T}$ is a maximal torus in $\overline{I}$. The isogeny $I \to \overline{I}$ induces an l-morphism from the root datum of $\overline{I}$ to the root datum of I. (See section 9.6.3 of [Spr09].) Then $I \to \overline{I}$ is a central isogeny, as otherwise T would have a weight occurring in $\mathrm{Lie}\,\overline{I} \subset \mathrm{ad}\,V$ of the form $l\mu$ with μ non-zero and this would contradict our assumption on the weights of T on V. $\square$

Suppose that we are given $\overline{\mathbb{F}}_l$-vector spaces W_i with $\dim W_i \leq l$ for $i = 1, \ldots, r$. Then the maps

$$\exp : X \mapsto 1 + X + \frac{X^2}{2!} + \cdots + \frac{X^{l-1}}{(l-1)!}$$
$$\log : 1 + u \mapsto u - \frac{u^2}{2} + \frac{u^3}{3} \pm \cdots - \frac{u^{l-1}}{l-1}$$

define inverse bijections between the set of nilpotent elements in $\prod \mathrm{End}(W_i)$ and the set of unipotent elements in $\prod \mathrm{GL}(W_i)$.

Lemma 5. *Suppose that $G \subset \prod \mathrm{GL}(W_i)$ is a connected reductive group over $\overline{\mathbb{F}}_l$ with $\dim W_i \leq l$ for all i. Let T be a maximal torus and U be the unipotent radical of a Borel subgroup of G that contains T. Suppose that $|\langle \mu_1 - \mu_2, \alpha^\vee \rangle| < l$ for all weights μ_1, μ_2 of T on $V = \bigoplus W_i$ and all simple roots α.*

(i) *The maps* exp *and* log *induce inverse isomorphisms of varieties between* $\mathrm{Lie}\, U \subset \mathrm{End}(V)$ *and* $U \subset \mathrm{GL}(V)$.
(ii) *For any positive root α we have* $\exp(\mathrm{Lie}\, U_\alpha) = U_\alpha$.
(iii) *The map* $\exp : \mathrm{Lie}\, U \to U$ *depends only on G and U, but not on V, W_i, or the representation* $G \hookrightarrow \mathrm{GL}(V)$.
(iv) *If θ is an automorphism of G that preserves T and U, then we have a commutative diagram:*

$$\begin{tikzcd}
\mathrm{Lie}\, U \arrow[r, "d\theta"] \arrow[d, "\exp"'] & \mathrm{Lie}\, U \arrow[d, "\exp"] \\
U \arrow[r, "\theta"] & U
\end{tikzcd}$$

Proof. By the Lie–Kolchin theorem we may suppose U is contained in the group $U' = \prod U'_i$, where U'_i denotes the unipotent radical of a Borel subgroup of $\mathrm{GL}(W_i)$. The maps exp and log provide mutually inverse isomorphisms of varieties between U' and $\mathrm{Lie}\, U'$. It remains to show that $\exp \mathrm{Lie}\, U = U$. Note that the product of any l elements of $\mathrm{Lie}\, U'$ is zero. Thus the Zassenhaus formula (see [Mag54], section IV) tells us that to check that $\exp \mathrm{Lie}\, U \subset U$ it suffices to check that for any root α we have $\exp(\mathrm{Lie}\, U_\alpha) \subset U$. Let $x_\alpha : \mathbb{G}_a \to U_\alpha$ be the root homomorphism corresponding to α and let $X_\alpha = dx_\alpha(1) \in \mathrm{Lie}\, U_\alpha$. Then formula II.1.19(6) of [Jan03] shows that for $a \in \overline{\mathbb{F}}_l$,

$$x_\alpha(a) = \sum_{n=0}^{l-1} a^n \frac{X_\alpha^n}{n!} = \exp(aX_\alpha) \tag{3}$$

in $\mathrm{GL}(V)$, on noting that for $n < l$ we have $X_{\alpha,n} = X_\alpha^n/n!$ while $X_{\alpha,n}$ acts trivially on V for $n \geq l$. (This latter assertion follows from formula II.1.19(5) of [Jan03] because V_λ and $V_{\lambda+n\alpha}$ cannot both be non-zero.) Now by the Baker–Campbell–Hausdorff formula (see section IV.8 in part I of [Ser92]) and the fact that the product of any l elements of $\mathrm{Lie}\, U'$ is zero we see that $\exp \mathrm{Lie}\, U$ is a subgroup of U. As U is connected and smooth and $\dim \mathrm{Lie}\, U \geq \dim U$ we deduce that $\exp \mathrm{Lie}\, U = U$. This proves the first two parts.

The third part follows inductively from equation (3) and the Zassenhaus formula: fix a total order $<$ on the set of positive roots such that if α, β, $\alpha+\beta$ are positive roots, then $\max(\alpha,\beta) < \alpha+\beta$. We induct on the positive root γ. Suppose that we know that exp depends only on G and U on the subspace $\bigoplus_{\alpha>\gamma} \operatorname{Lie} U_\alpha$. Then the same is true for $\exp(X+Y)$ for any $X \in \operatorname{Lie} U_\gamma$ and $Y \in \bigoplus_{\alpha>\gamma} \operatorname{Lie} U_\alpha$ by the Zassenhaus formula. (Note that $[\operatorname{Lie} U_\alpha, \operatorname{Lie} U_\beta] \subset \operatorname{Lie} U_{\alpha+\beta}$ whenever α, β are positive roots.) This completes the proof of the third part.

The last part follows from the third part, by considering the representation $G \xrightarrow{\theta} G \hookrightarrow \mathrm{GL}(V)$. □

Lemma 6. *Suppose that G is a connected simply connected semisimple algebraic group over $\overline{\mathbb{F}}_l$. Suppose that $l > 3$ and that G has no simple factor isomorphic to SL_n with $l|n$. Let $\mathfrak{g}$ denote the Lie algebra of G. Then $\mathfrak{g}$ contains no non-trivial abelian ideal, and the natural map $\mathrm{Aut}(G) \to \mathrm{Aut}(\mathfrak{g})$ is a bijection. Moreover, a connected normal subgroup of G is preserved by an automorphism $\theta \in \mathrm{Aut}(G)$ if and only if its Lie algebra is preserved by $d\theta \in \mathrm{Aut}(\mathfrak{g})$.*

Here, $\mathrm{Aut}(G)$ (resp., $\mathrm{Aut}(\mathfrak{g})$) denotes the abstract group of automorphisms of the algebraic group G (resp., its Lie algebra $\mathfrak{g}$). In the proof we use Chevalley groups in the sense of Steinberg's Yale notes [Ste68b].

Proof. The universal Chevalley group over $\overline{\mathbb{F}}_l$ constructed using the complex semisimple Lie algebra $\mathcal{L}$ of the same root system as G is an algebraic group isomorphic to G (see [Ste68b], §5). (In the notation of [Ste68b], we can let V be any representation whose weights span the weight lattice, so that $\mathcal{L}_{\mathbb{Z}} \subset \mathcal{L}$ is the $\mathbb{Z}$-lattice spanned by the fixed Chevalley basis H_i, X_α; see Cor. 2 on p. 18 of [Ste68b].) In particular, $\mathfrak{g} \cong \mathcal{L}_{\mathbb{Z}} \otimes \overline{\mathbb{F}}_l$ (by the remark on p. 64 of [Ste68b]). Write $G = \prod G_i$ as a product of almost simple simply connected algebraic groups and correspondingly $\mathfrak{g} = \bigoplus \mathfrak{g}_i$. Then $Z(\mathfrak{g}_i) = 0$ by our assumption on l and G (see Theorem 2.3 in [Hur82]) and hence all $\mathfrak{g}_i$ are simple ([Ste61], 2.6(5)). Moreover $\mathfrak{g}_i \cong \mathfrak{g}_j$ implies $G_i \cong G_j$ ([Ste61], 8.1). The G_i (resp., $\mathfrak{g}_i$) are uniquely characterised as the minimal non-trivial connected normal subgroups of G (resp., minimal non-trivial ideals of $\mathfrak{g}$), so they are permuted by automorphisms. Therefore if $\mathrm{Aut}(G_i) \to \mathrm{Aut}(\mathfrak{g}_i)$ is a bijection for all i, then so is $\mathrm{Aut}(G) \to \mathrm{Aut}(\mathfrak{g})$, and also the final claim of the proposition follows. (Note that any connected normal subgroup is a product of some of the G_i.) We can thus assume, without loss of generality, that G is almost simple.

Let G^{ad} denote the adjoint form of G. As G is the universal cover of G^{ad} and as $G^{\mathrm{ad}} = G/Z(G)$, we have $\mathrm{Aut}(G) = \mathrm{Aut}(G^{\mathrm{ad}})$. As $Z(\mathfrak{g}) = 0$

we see that the natural map $\mathfrak{g} \to \operatorname{Lie} G^{\mathrm{ad}}$ is an isomorphism. Thus it suffices to show that $\operatorname{Aut}(G) = \operatorname{Aut}(\mathfrak{g})$ whenever G is simple of *adjoint* type and $\mathfrak{g} = \operatorname{Lie} G$. Thus we write G for G^{ad} from now on.

As an algebraic group G is isomorphic to the adjoint Chevalley group over $\overline{\mathbb{F}}_l$ (again by [Ste68b], §5). (In the notation of [Ste68b], we take V to be the adjoint representation $\mathfrak{g}$.) Thus we can identify $G(\overline{\mathbb{F}}_l)$ with the subgroup of $\mathrm{GL}(\mathfrak{g})$ generated by the elements $x_\alpha(t) := \exp(\operatorname{ad}(tX_\alpha))$, where $t \in \overline{\mathbb{F}}_l$ and α is any root. As each $\operatorname{ad}(tX_\alpha)$ is a derivation of $\mathfrak{g}$, the group $G(\overline{\mathbb{F}}_l)$ is actually contained in $\operatorname{Aut}(\mathfrak{g})$. For any $\eta \in \operatorname{Aut}(\mathfrak{g})$, we have $\eta \circ \operatorname{ad} X \circ \eta^{-1} = \operatorname{ad}(\eta X)$ in $\mathrm{GL}(\mathfrak{g})$. It follows that the natural action of $G(\overline{\mathbb{F}}_l) \subset \mathrm{GL}(\mathfrak{g})$ on $\mathfrak{g}$ agrees with the adjoint action of $G(\overline{\mathbb{F}}_l)$ on $\mathfrak{g} \subset \operatorname{End}(\mathfrak{g})$.

The choice of Chevalley basis gives rise to a maximal torus T and a Borel B that contains it ([Ste68b], §5). From Theorem 9.6.2 in [Spr09] we deduce the following, using that G is adjoint. For each symmetry π of the Dynkin diagram $\mathcal{D}$ there is a unique $\pi' \in \operatorname{Aut}(G)$ that preserves (B, T) and that permutes the $x_{\alpha_i}(1) \in B$ according to π (where α_i are the simple roots). Moreover, $\operatorname{Aut}(G)$ is the semidirect product of G (acting by inner automorphisms) and $\operatorname{Aut}(\mathcal{D})$. Also, the elements of $\operatorname{Aut}(\mathcal{D})$ biject with the "graph automorphisms" of $\mathfrak{g}$ ([Ste61], §3).

The result now follows from ([Ste61], 4.2 and 4.5), as the group $\mathfrak{H}$ in [Ste61] is actually contained in $G(\overline{\mathbb{F}}_l)$ since $\overline{\mathbb{F}}_l$ is algebraically closed (see Lemma 19 on p. 27 of [Ste68b]). (Note that the uniqueness statement in ([Ste61], 4.2) is incorrect and seems to be a typo.) □

The following proposition may be of independent interest. The proof uses the classification of finite simple groups. Without it, the proof still goes through for l sufficiently large (depending on d and ineffective) by appealing to [LP] instead of [Gur99].

Proposition 7. *Suppose that V is a finite-dimensional $\overline{\mathbb{F}}_l$-vector space and that $\Gamma \subset \mathrm{GL}(V)$ is a finite subgroup that acts semisimply on V. Let $\Gamma^0 \subset \Gamma$ be the subgroup generated by elements of l-power order. Then V is a semisimple Γ^0-module. Let $d \geq 1$ be the maximal dimension of an irreducible Γ^0-submodule of V. Suppose that $l \geq 2(d+1)$. Then there exists an algebraic group G over $\mathbb{F}_l$ and a semisimple representation $r : G_{/\overline{\mathbb{F}}_l} \to \mathrm{GL}(V)$ with the following properties:*

(i) *The connected component G^0 is semisimple, simply connected.*
(ii) *$G \cong G^0 \rtimes H$, where H is a finite group of order prime to l.*
(iii) *$r(G(\mathbb{F}_l)) = \Gamma$.*

Moreover, if $T \subset G^0$ is a maximal torus and if μ is a weight of $T_{/\overline{\mathbb{F}}_l}$ on V then $\sum |\langle \mu, \alpha^\vee \rangle| < 2d$, where α ranges over the roots of $G^0_{/\overline{\mathbb{F}}_l}$. Also, Γ does not have any composition factor of order l.

Proof. Write $V = \bigoplus_i W_i$ as a direct sum of irreducible Γ^0-modules. Since $\dim W_i \leq l$ for all i, we see that every element of l-power order in the image of $\Gamma^0 \to \mathrm{GL}(W_i)$ actually has order dividing l. Since $\Gamma^0 \hookrightarrow \prod \mathrm{GL}(W_i)$, we deduce that every element of Γ^0 of l-power order actually has order dividing l. Note that Γ/Γ^0 has order prime to l.

Step 1. We show that there exists a connected simply connected semisimple algebraic group G^0 over $\mathbb{F}_l$ and a finite central subgroup $Z_0 \subset G^0(\mathbb{F}_l)$ with $G^0(\mathbb{F}_l)/Z_0 \cong \Gamma^0$. Let Γ_i denote the image of Γ^0 in $\mathrm{GL}(W_i)$. Note that Γ_i has no non-trivial normal subgroup of l-power order (since Γ_i acts faithfully on W_i, and an l-group acting on a non-zero $\overline{\mathbb{F}}_l$-vector space has non-zero fixed points). So by Theorem B of [Gur99], Γ_i is a central product of quasisimple Chevalley groups. (Note that if $l = 11$ then $\dim W_i < 7$.) Now Γ^0 is a subgroup of $\prod \Gamma_i$ that surjects onto each factor, so $Z(\Gamma^0) = \Gamma^0 \cap \prod Z(\Gamma_i)$. Thus $\Gamma^0/Z(\Gamma^0)$ is a subgroup of $\prod \Gamma_i/Z(\Gamma_i)$, a product of simple Chevalley groups, that surjects onto each factor. By a theorem of Hall (Lemma 3.5 in [Kup]), $\Gamma^0/Z(\Gamma^0)$ is itself isomorphic to a direct product of simple Chevalley groups. It follows that $\Gamma^0 = [\Gamma^0, \Gamma^0]Z(\Gamma^0)$. Since Γ^0 is generated by elements of order l and $Z(\Gamma^0)$ is of order prime to l, it follows moreover that Γ^0 is perfect. Therefore Γ^0 is a perfect central extension of a product $\prod H_j$ of simple Chevalley groups H_j, so there exists a surjective homomorphism $\pi : \prod \widetilde{H}_j \to \Gamma^0$ with central kernel, where $\widetilde{H}_j$ is the universal perfect central extension of H_j.

As $l > 3$ (to rule out Suzuki and Ree groups) there exist connected simply connected algebraic groups G_j over $\mathbb{F}_l$ such that $H_j \cong G_j(\mathbb{F}_l)/Z(G_j(\mathbb{F}_l))$. (Note that G_j is the restriction of scalars of an absolutely almost simple algebraic group over a finite extension of $\mathbb{F}_l$.) Since $l > 3$ it is known that $\widetilde{H}_j \cong G_j(\mathbb{F}_l)$ (see section 6.1 in [GLS98], particularly table 6.1.3). So we can take $G^0 = \prod G_j$ and $Z_0 = \ker \pi$.

Since $\Gamma^0/Z(\Gamma^0)$ is a product of nonabelian simple groups and since $Z(\Gamma^0)$ and Γ/Γ^0 are of order prime to l, it follows that Γ does not have any composition factor of order l.

Let $G^0 \supset B \supset T$ denote a Borel and maximal torus defined over $\mathbb{F}_l$.

Step 2. We lift V to a $G^0_{/\overline{\mathbb{F}}_l}$-module and compare the actions of $T(\mathbb{F}_l)$ and $T(\overline{\mathbb{F}}_l)$ on V. Let U denote the unipotent radical of B and set $N = N_{G^0}(T)$. Let B^{op} denote the opposite Borel subgroup to B containing T and let U^{op} denote its unipotent radical. (See Theorem 14.1 of [Bor91].

By uniqueness we see it is defined over $\mathbb{F}_l$.) Let $X = X^*(T_{/\overline{\mathbb{F}}_l})$ with its subset Φ of roots and Φ^+ (resp. Δ) the set of positive (resp. simple) roots corresponding to B. Let $X^+ \subset X$ be the subset of dominant weights. There is a semisimple algebraic action of $G^0_{/\overline{\mathbb{F}}_l}$ on V, say $\phi : G^0_{/\overline{\mathbb{F}}_l} \to \mathrm{GL}(V)$, such that:

(i) the highest weight λ of a simple submodule is restricted (i.e. $0 \leq \langle \lambda, \alpha^\vee \rangle < l$ for all $\alpha \in \Delta$),
(ii) the action of $G^0(\mathbb{F}_l)$ is the one induced by the map $G^0(\mathbb{F}_l) \to \Gamma^0$,
(iii) the subspaces W_i are $G^0_{/\overline{\mathbb{F}}_l}$-stable.

(This follows from a result of Steinberg: see Theorem 2.11 in [Hum06]. Note that [Hum06] works with an algebraic group $\mathbf{G}$ that is simple, but the proof given does not depend on that assumption.) By Proposition 3 of [Ser94] we see that if λ in X^+ is a weight of $T_{/\overline{\mathbb{F}}_l}$ on V then $\sum_{\alpha \in \Phi^+} \langle \lambda, \alpha^\vee \rangle < d$; in particular, $\langle \lambda, \alpha^\vee \rangle < (l-1)/2$ for all $\alpha \in \Phi^+$. (Note that $\dim W_i \leq (l-1)/2$ and that the proof of that proposition does not require that $G^0_{/\overline{\mathbb{F}}_l}$ be almost simple.) If μ is a weight of $T_{/\overline{\mathbb{F}}_l}$ on V then we see that there is w in the Weyl group with $w\mu \in X^+$ and $0 \leq \langle w\mu, \alpha^\vee \rangle < (l-1)/2$ for all $\alpha \in \Phi^+$, and we deduce that $|\langle \mu, \alpha^\vee \rangle| < (l-1)/2$ for all $\alpha \in \Phi$. We also deduce that if μ is a weight of $T_{/\overline{\mathbb{F}}_l}$ on $\mathrm{ad}\, V$ then $|\langle \mu, \alpha^\vee \rangle| < l-1$ for all $\alpha \in \Delta$.

Step 3. The semisimple group $\overline{I} \subset \mathrm{GL}(V)$ and its simply connected cover $I \subset G^0_{/\overline{\mathbb{F}}_l}$. Since $|\langle \mu, \alpha^\vee \rangle| < l/2$ for all weights μ of $T_{/\overline{\mathbb{F}}_l}$ on V and all $\alpha \in \Delta$ we may apply Lemma 4 to $\phi : G^0_{/\overline{\mathbb{F}}_l} \to \mathrm{GL}(V)$. We obtain connected simply connected semisimple algebraic subgroups I, J of $G^0_{/\overline{\mathbb{F}}_l}$ such that $G^0_{/\overline{\mathbb{F}}_l} = I \times J$, $\phi(J) = 1$, and ϕ induces a central isogeny of I onto its image $\overline{I}$, which is a semisimple algebraic group. Note that $T_{/\overline{\mathbb{F}}_l} = T_I \times T_J$ and that $B_{/\overline{\mathbb{F}}_l} = B_I \times B_J$ where (B_I, T_I) (resp. (B_J, T_J)) is a Borel and maximal torus in I (resp. J). Moreover $U_{/\overline{\mathbb{F}}_l} = U_I \times U_J$. Let $\overline{B}$, $\overline{T}$, $\overline{U}$, $\overline{B}^{\mathrm{op}}$, $\overline{U}^{\mathrm{op}}$ denote the images of B_I, T_I, U_I, B_I^{op}, U_I^{op} in $\overline{I}$. Then $\overline{T}$ is a maximal torus of $\overline{I}$, and $\overline{B}$, $\overline{B}^{\mathrm{op}}$ are opposite Borel subgroups containing it. Also $\overline{U}$, $\overline{U}^{\mathrm{op}}$ are the unipotent radicals of $\overline{B}$, $\overline{B}^{\mathrm{op}}$. Since $I \to \overline{I}$ is a central isogeny, $U_I \to \overline{U}$ and $U_I^{\mathrm{op}} \to \overline{U}^{\mathrm{op}}$ are isomorphisms.

Step 4. The maps log *and* exp *provide inverse isomorphisms of varieties between $\overline{U} \subset \mathrm{GL}(V)$ and* $\mathrm{Lie}\, \overline{U} \subset \mathrm{ad}\, V$. This follows from Lemma 5 applied to $\overline{I} \subset \mathrm{GL}(V)$ since $\dim W_i \leq l$ for all i and $|\langle \mu, \alpha^\vee \rangle| < l/2$ for all weights μ of $T_{/\overline{\mathbb{F}}_l}$ on V and all $\alpha \in \Delta$. (Note that $T_I \to \overline{T}$

induces a bijection on coroots since $I \to \overline{I}$ is a central isogeny; thus $T \to \overline{T}$ induces a surjection on coroots.)

Step 5. The $\overline{\mathbb{F}}_l$*-span of* $\log U(\mathbb{F}_l)$ *is* $\mathrm{Lie}\,\overline{U}$*.* Since $d\phi : \mathrm{Lie}\,U \to \mathrm{Lie}\,\overline{U}$ is surjective, it suffices to show that there is an isomorphism $\log : U \to \mathrm{Lie}\,U$ defined over $\mathbb{F}_l$ such that $d\phi \circ \log = \log \circ \phi$. Pick an $\mathbb{F}_l$-structure on V. The map $G^0_{/\overline{\mathbb{F}}_l} \to \mathrm{GL}(V)$ can be defined over some $\mathbb{F}_{l^s}$ and so taking restrictions of scalars from $\mathbb{F}_{l^s}$ to $\mathbb{F}_l$ we get an $\mathbb{F}_l$-vector space V' and a map $\psi : G^0 \to \mathrm{GL}(V')$. The map $G^0_{/\overline{\mathbb{F}}_l} \to \mathrm{GL}(V)$ is obtained from ψ by extending scalars to $\overline{\mathbb{F}}_l$ and projecting to a direct summand V of $V' \otimes \overline{\mathbb{F}}_l$. The dimension of all irreducible factors of $V' \otimes \overline{\mathbb{F}}_l$ is at most l. Moreover for any weight λ of $T_{/\overline{\mathbb{F}}_l}$ on $V' \otimes \overline{\mathbb{F}}_l$ we have $|\langle \lambda, \alpha^\vee \rangle| < (l-1)/2$ for all $\alpha \in \Phi^+$.

By Lemma 4 we see that $\psi : G^0 \to \mathrm{GL}(V')$ is a central isogeny onto its image. (By construction we have $(\ker \psi)(\mathbb{F}_l) = Z_0$. Suppose that $\ker \psi$ is not finite. Then it has to contain one of the $\mathbb{F}_l$-almost simple factors of $G^0 = \prod G_j$. But $G_j(\mathbb{F}_l)$ is nonabelian.)

In particular, ψ induces an isomorphism $U \to \psi(U)$. Then Lemma 5 (applied to the image of $\psi_{/\overline{\mathbb{F}}_l}$) gives the desired map $\log : U \to \mathrm{Lie}\,U \subset \mathrm{ad}\,V'$.

Step 6: Some properties of $G^0(\mathbb{F}_l)$*.* The pair $(B(\mathbb{F}_l), N(\mathbb{F}_l))$ is a split BN pair in $G^0(\mathbb{F}_l)$ (see section 1.18 of [Car93]). Also $U(\mathbb{F}_l)$ is a Sylow l-subgroup of $G^0(\mathbb{F}_l)$ and $B(\mathbb{F}_l) = N_{G^0(\mathbb{F}_l)}(U(\mathbb{F}_l)) = N_{G^0(\mathbb{F}_l)}(B(\mathbb{F}_l))$ (see Proposition 2.5.1 of [Car93]).

Moreover $T(\mathbb{F}_l)$ is a Sylow l-complement in $B(\mathbb{F}_l)$. Note that $U^{\mathrm{op}}(\mathbb{F}_l)$ is $N(\mathbb{F}_l)$-conjugate to $U(\mathbb{F}_l)$. (The longest Weyl element w_0 is stable under Frobenius, hence represented by an element $n_0 \in N(\mathbb{F}_l)$. Then use that $U^{\mathrm{op}} = n_0 U n_0^{-1}$.) Moreover the second-last displayed equation on page 74 (section 2.9) of [Car93] shows that $U^{\mathrm{op}}(\mathbb{F}_l)$ is the unique $N(\mathbb{F}_l)$-conjugate of $U(\mathbb{F}_l)$ with trivial intersection with $U(\mathbb{F}_l)$.

Step 7. We have $N(\mathbb{F}_l) = N_{G^0(\mathbb{F}_l)}(T(\mathbb{F}_l))$ *so that* $N_{G^0(\mathbb{F}_l)}(T(\mathbb{F}_l)) \cap N_{G^0(\mathbb{F}_l)}(B(\mathbb{F}_l)) = T(\mathbb{F}_l)$ *and* $Z_0 \subset Z(G^0(\mathbb{F}_l)) \subset T(\mathbb{F}_l)$*.*

Suppose that g is in $N_{G^0(\mathbb{F}_l)}(T(\mathbb{F}_l))$. One can write g uniquely as unu' where $u \in U(\mathbb{F}_l), n \in N(\mathbb{F}_l)$ maps to w_n in the Weyl group and $u' \in U_{w_n}$ in the notation of Theorem 2.5.14 of [Car93]. Then for any h in $T(\mathbb{F}_l)$ we can find h' and h'' in $T(\mathbb{F}_l)$ such that

$$hunu' = unu'h' \quad \text{and} \quad h''unu' = unu'h,$$

i.e.,

$$(huh^{-1})(hn)u' = u(nh')(h'^{-1}u'h')$$

and

$$(h''uh''^{-1})(h''n)u' = u(nh)(h^{-1}u'h).$$

As $T(\mathbb{F}_l)$ normalizes $U(\mathbb{F}_l)$ and U_{w_n} and as $w_{nh} = w_n = w_{hn}$ the uniqueness assertion of Theorem 2.5.14 of [Car93] tells us that $huh^{-1} = u$ and $u' = h^{-1}u'h$. Thus $u \in Z_{U(\mathbb{F}_l)}(T(\mathbb{F}_l))$ and $u' \in Z_{U_{w_n}}(T(\mathbb{F}_l)) \subset Z_{U(\mathbb{F}_l)}(T(\mathbb{F}_l))$. So it suffices to prove that $Z_{U(\overline{\mathbb{F}}_l)}(T(\mathbb{F}_l)) = 1$. By Proposition 8.2.1 in [Spr09]

it suffices to show that $Z_{U_\alpha(\overline{\mathbb{F}}_l)}(T(\mathbb{F}_l)) = 1$ for all $\alpha \in \Phi^+$. By Proposition 8.1.1(i) in [Spr09]

it suffices that α is non-trivial on $T(\mathbb{F}_l)$ for all $\alpha \in \Phi^+$. As $l \geq 5$, this follows from Lemma 3(i) (applied with Δ_* the set of simple coroots).

Step 8. We find a subgroup H of order prime to l such that $\Gamma = \Gamma^0 H$. Let H denote the subgroup of $h \in \Gamma$ which normalize both the image of $B(\mathbb{F}_l)$ and the image of $T(\mathbb{F}_l)$ in Γ^0. Then by the previous paragraph we see that $H \cap \Gamma^0$ is $T(\mathbb{F}_l)/Z_0$. Thus H has order prime to l.

Moreover if $\gamma \in \Gamma$ we see that $\gamma(B(\mathbb{F}_l)/Z_0)\gamma^{-1}$ is the normalizer of a Sylow l-subgroup of $G^0(\mathbb{F}_l)/Z_0$ and hence $G^0(\mathbb{F}_l)$-conjugate to $B(\mathbb{F}_l)/Z_0$, say $\gamma(B(\mathbb{F}_l)/Z_0)\gamma^{-1} = k(B(\mathbb{F}_l)/Z_0)k^{-1}$ with $k \in G^0(\mathbb{F}_l)$. Then $k^{-1}\gamma(T(\mathbb{F}_l)/Z_0)\gamma^{-1}k$ is a Sylow l-complement in $B(\mathbb{F}_l)/Z_0$ and hence (by Hall's theorem) $B(\mathbb{F}_l)/Z_0$-conjugate to $T(\mathbb{F}_l)/Z_0$, say

$$k^{-1}\gamma(T(\mathbb{F}_l)/Z_0)\gamma^{-1}k = k'(T(\mathbb{F}_l)/Z_0)k'^{-1}$$

for some $k' \in B(\mathbb{F}_l)$. Then $(kk')^{-1}\gamma$ lies in H and we deduce that Γ is generated by H and $G^0(\mathbb{F}_l)/Z_0 = \Gamma^0$.

Step 9. Lifting the conjugation action of H on Γ^0 to G^0. We first show that $G^0_{/\overline{\mathbb{F}}_l}$ has no simple factor SL_n with $l|n$ by showing that any such factor would act trivially on $V = \bigoplus W_i$, contradicting that $G^0(\mathbb{F}_l)/Z_0$ acts faithfully. So suppose that $\mathrm{SL}_{n/\overline{\mathbb{F}}_l}$ has an irreducible module of dimension less than $l-1$. Then by Proposition 3 in [Ser94] its highest weight λ would satisfy $\sum\langle\lambda, \alpha^\vee\rangle < l-1$, where α runs through the set of positive roots. A calculation shows that the left-hand side is at least $n-1$ if λ is non-zero. So if $n \geq l$, then $\lambda = 0$.

Next we claim that $d\phi : (\mathrm{Lie}\, G^0)(\overline{\mathbb{F}}_l) \to \mathrm{ad}\, V$ is injective on the subspace $(\mathrm{Lie}\, G^0)(\mathbb{F}_l)$. Note first that it is injective on $(\mathrm{Lie}\, U)(\mathbb{F}_l)$ as ϕ is injective on $U(\mathbb{F}_l)$. (Consider the isomorphism $\log : U(\mathbb{F}_l) \to (\mathrm{Lie}\, U)(\mathbb{F}_l)$ constructed in Step 5.) Similarly $d\phi$ is injective on $(\mathrm{Lie}\, U^{\mathrm{op}})(\mathbb{F}_l)$. Since ϕ maps U to $\overline{U}$, T to $\overline{T}$, U^{op} to $\overline{U}^{\mathrm{op}}$, and since $\mathrm{Lie}\, G^0 = \mathrm{Lie}\, U \oplus \mathrm{Lie}\, T \oplus \mathrm{Lie}\, U^{\mathrm{op}}$, $\mathrm{Lie}\, \overline{I} = \mathrm{Lie}\, \overline{U} \oplus \mathrm{Lie}\, \overline{T} \oplus \mathrm{Lie}\, \overline{U}^{\mathrm{op}}$ it follows that the kernel of $d\phi$ on $(\mathrm{Lie}\, G^0)(\mathbb{F}_l)$ is contained in $(\mathrm{Lie}\, T)(\mathbb{F}_l)$. But $(\mathrm{Lie}\, G^0)(\overline{\mathbb{F}}_l)$ contains no non-trivial abelian ideal by Lemma 6. This proves the claim.

Note that H acts by conjugation on $\mathrm{GL}(V)$ and $\mathrm{ad}\, V$, in particular it preserves the Lie algebra structure of $\mathrm{ad}\, V$. By definition H stabilises the image of $U(\mathbb{F}_l)$ in $\mathrm{GL}(V)$ and hence by Step 5 it also stabilises $\log U(\mathbb{F}_l) = d\phi((\mathrm{Lie}\, U)(\mathbb{F}_l))$. Because $U^{\mathrm{op}}(\mathbb{F}_l)$ is the unique $N_{G^0(\mathbb{F}_l)}(T(\mathbb{F}_l))$-conjugate of $U(\mathbb{F}_l)$ that has trivial intersection with $U(\mathbb{F}_l)$, it is also stabilised by H. The previous argument then shows that H stabilises $d\phi((\mathrm{Lie}\, U^{\mathrm{op}})(\mathbb{F}_l))$. Since $[\mathrm{Lie}\, U, \mathrm{Lie}\, U^{\mathrm{op}}] = \mathrm{Lie}\, G^0$ (as we may check over $\overline{\mathbb{F}}_l$), it follows that H stabilises the image of $(\mathrm{Lie}\, G^0)(\mathbb{F}_l)$ in $\mathrm{ad}\, V$. By extending scalars, we get a natural action of H on $(\mathrm{Lie}\, G^0)(\overline{\mathbb{F}}_l)$. This action lifts uniquely to an action on $G^0_{/\overline{\mathbb{F}}_l}$ by Lemma 6.

We claim that with respect to the H-action on $G^0_{/\overline{\mathbb{F}}_l}$ just constructed, $\phi : G^0_{/\overline{\mathbb{F}}_l} \to \mathrm{GL}(V)$ is H-equivariant. We first show that the conjugation action of H on $\mathrm{GL}(V)$ stabilises $\overline{I}$. If $h \in H$ then h sends $U(\mathbb{F}_l)$ to itself and hence $\log U(\mathbb{F}_l)$ to itself and hence $\mathrm{Lie}\, \overline{U}$ to itself and hence $\overline{U}$ to itself. Similarly h stabilises $\overline{U}^{\mathrm{op}}$. As the root subgroups generate $\overline{I}$ (by Theorem 8.1.5 in [Spr09]), we see that h indeed stabilises $\overline{I}$. This action of H on $\overline{I}$ lifts uniquely to an action on the simply connected cover I of $\overline{I}$. (For existence use Theorem 9.6.5 of [Spr09] and the conjugation action of T_I. For uniqueness use the semisimplicity of I.) On the other hand, Lemma 6 shows that the H-action on $G^0_{/\overline{\mathbb{F}}_l}$ respects the decomposition $G^0_{/\overline{\mathbb{F}}_l} = I \times J$. Since J is killed by ϕ it suffices to show that the two H-actions on I (one coming from $\overline{I}$ and one from $G^0_{/\overline{\mathbb{F}}_l}$) agree. By Lemma 6 we can check this on the Lie algebra. The same lemma shows that $d\phi : \mathrm{Lie}\, I \to \mathrm{Lie}\, \overline{I}$ is an isomorphism, since $\mathrm{Lie}\, I$ contains no non-trivial abelian ideal. By construction both H-actions on $\mathrm{Lie}\, I$ are compatible with the H-action on $\mathrm{Lie}\, \overline{I}$, so the two H-actions on I indeed agree. Therefore ϕ is H-equivariant. A fortiori, it extends to a homomorphism $G^0_{/\overline{\mathbb{F}}_l} \rtimes H \to \mathrm{GL}(V)$.

Finally we show that the H-action on $G^0_{/\overline{\mathbb{F}}_l}$ descends to G^0. Suppose that $h \in H$ and $\sigma \in \mathrm{Gal}(\overline{\mathbb{F}}_l/\mathbb{F}_l)$. The automorphism $\sigma h \sigma^{-1} h^{-1}$ is trivial on $(\mathrm{Lie}\, G^0)(\mathbb{F}_l)$, hence trivial on $(\mathrm{Lie}\, G^0)(\overline{\mathbb{F}}_l)$, hence trivial on $G^0_{/\overline{\mathbb{F}}_l}$ by Lemma 6. Therefore the H-action indeed descends to G^0.

By construction, the image of $G^0(\mathbb{F}_l) \rtimes H$ is Γ. Let $G = G^0 \rtimes H$ and $r : G_{/\overline{\mathbb{F}}_l} \to \mathrm{GL}(V)$ the homomorphism we just obtained. It remains to show that r is semisimple. But this follows from Lemma 5(b) in [Ser94] since the restriction of r to $G^0_{/\overline{\mathbb{F}}_l}$ is semisimple and $(G : G^0)$ is prime to l. □

We remark that for the purpose of proving Theorem 9 we do not need an H-action on G^0, we only need an H-action on $G^0_{/\overline{\mathbb{F}}_l}$ that is compatible with the H-action on $\mathrm{GL}(V)$. Since $G^0_{/\overline{\mathbb{F}}_l} = I \times J$, we can lift the H-action on $\overline{I}$ to I as above and let H act arbitrarily on J; for this it is not necessary to appeal to Lemma 6.

Lemma 8. *Suppose that G is a linear algebraic group over $\overline{\mathbb{F}}_l$ such that the connected component G^0 is semi-simple and simply connected and such that l does not divide $(G : G^0)$. Let $G^0 \supset B \supset T$ denote a Borel subgroup and a maximal torus and let $\mathcal{T}$ denote the normalizer of the pair (B, T) in G. Then the $G^0(\overline{\mathbb{F}}_l)$-conjugates of $\mathcal{T}(\overline{\mathbb{F}}_l)$ equal the semisimple elements of $G(\overline{\mathbb{F}}_l)$ and they are Zariski dense in G. In particular, if V is an irreducible representation of G then the $G^0(\overline{\mathbb{F}}_l)$-conjugates of $\mathcal{T}(\overline{\mathbb{F}}_l)$ span $\operatorname{ad} V$ over $\overline{\mathbb{F}}_l$.*

Proof. By Theorem 7.5 in [Ste68a] every semisimple element of $G(\overline{\mathbb{F}}_l)$ is $G^0(\overline{\mathbb{F}}_l)$-conjugate to an element of $\mathcal{T}(\overline{\mathbb{F}}_l)$. The converse is clear as $\mathcal{T} \cap G^0 = T$, an element $g \in G(\overline{\mathbb{F}}_l)$ is semisimple iff g is of order prime to l, and l does not divide $(G : G^0)$. Next we have $G = G^0\mathcal{T}$ since Borel subgroups in G^0 are conjugate and maximal tori in B are conjugate. Consider a fixed coset $G^0 h$ with $h \in \mathcal{T}(\overline{\mathbb{F}}_l)$. By Lemma 4 of [Spr06] the elements $g(th)g^{-1} = [gt(hgh^{-1})^{-1}]h$ of $G^0 h$, where t runs over $T(\overline{\mathbb{F}}_l)$ and g runs over $G^0(\overline{\mathbb{F}}_l)$, are Zariski dense in $G^0 h$. (Lemma 4 of [Spr06] does not immediately apply to h as h is not a diagram automorphism. However for some $s \in T(\overline{\mathbb{F}}_l)$ the automorphism $g \mapsto shgh^{-1}s^{-1}$ is a diagram automorphism and hence the elements $gt(hgh^{-1})^{-1} = gts^{-1}(shgh^{-1}s^{-1})^{-1}s$ as t runs over $T(\overline{\mathbb{F}}_l)$ and g runs over $G^0(\overline{\mathbb{F}}_l)$ are Zariski dense in G^0.) Thus the $G^0(\overline{\mathbb{F}}_l)$-conjugates of $\mathcal{T}(\overline{\mathbb{F}}_l)$ are Zariski dense in $G(\overline{\mathbb{F}}_l)$. For the last claim note that if $\operatorname{tr}(gw) = 0$ for some $w \in \operatorname{ad} V$ and some Zariski dense subset of $g \in G(\overline{\mathbb{F}}_l)$, then $w = 0$. $\square$

The proof of our main theorem relies on Proposition 7 and thus on the classification of finite simple groups. (It still holds without it for l sufficiently large, depending on d and ineffective, due to the results of Larsen and Pink [LP].)

Theorem 9. *Suppose that V is a finite-dimensional $\overline{\mathbb{F}}_l$-vector space and that $\Gamma \subset \mathrm{GL}(V)$ is a finite subgroup that acts irreducibly on V. Let $\Gamma^0 \subset \Gamma$ be the subgroup generated by elements of l-power order. Then V is a semisimple Γ^0-module. Let $d \geq 1$ be the maximal dimension of an irreducible Γ^0-submodule of V. Suppose that $l \geq 2(d+1)$. Then:*

(i) $H^0(\Gamma, \operatorname{ad}^0 V) = H^1(\Gamma, \operatorname{ad}^0 V) = H^1(\Gamma, \overline{\mathbb{F}}_l) = 0$.

(ii) *The set Γ^{ss} spans $\operatorname{ad} V$ as an $\overline{\mathbb{F}}_l$-vector space.*

In particular, for any finite subfield k of $\overline{\mathbb{F}}_l$ containing the eigenvalues of all elements of Γ and such that $\Gamma \subset \mathrm{GL}_n(k)$, Γ is adequate.

Proof. Write $V = \bigoplus_i W_i$ as a direct sum of irreducible Γ^0-modules. Note that Γ/Γ^0 has order prime to l.

We claim that $\dim V$ is prime to l. Let U be an irreducible constituent of V as a Γ^0-module and let V' be the U-isotypic direct summand of V. Since Γ acts transitively on the set of isotypic components and as $(\Gamma : \Gamma^0)$ is prime to l, it suffices to show that $\dim V'$ is prime to l. Let $\Gamma' \supset \Gamma^0$ be the stabiliser of V'. Then V' is an irreducible Γ'-module. By Theorem 51.7 in [CR62], U extends to a projective representation of Γ' and there is an irreducible projective representation U' of Γ'/Γ^0 such that $V' \cong U \otimes U'$ (as projective Γ'-representation). The claim follows as $\dim U < l$ and Γ'/Γ^0 is of order prime to l.

By Proposition 7 there exists an algebraic group $G = G^0 \rtimes H$ over $\mathbb{F}_l$ and a semisimple representation $r : G_{/\overline{\mathbb{F}}_l} \to \mathrm{GL}(V)$, where G^0 is connected simply connected semisimple, H is a finite group of order prime to l, and $r(G(\mathbb{F}_l)) = \Gamma$. Moreover Γ has no composition factor of order l, which implies that no quotient of Γ^0 contains a non-trivial normal l-subgroup.

We have

$$H^1(\Gamma, \mathrm{ad}\, V) = \bigoplus_{i,j} H^1(\Gamma^0, \mathrm{Hom}(W_i, W_j))^\Gamma$$

and

$$H^1(\Gamma^0, \mathrm{Hom}(W_i, W_j)) = \mathrm{Ext}^1_{\Gamma^0}(W_i, W_j),$$

which vanishes by [Gur99], Theorem A, since $\dim W_i + \dim W_j \le l - 2$. (We apply that theorem to the quotient of Γ^0 that acts faithfully. Note that we saw above that this quotient does not have a non-trivial normal l-subgroup.) Similarly, $2 \le l - 2$ implies that $H^1(\Gamma, \overline{\mathbb{F}}_l) = 0$. Since $\dim V$ is prime to l it follows that $H^0(\Gamma, \mathrm{ad}^0 V) = 0$ and that $\mathrm{ad}^0 V$ is a direct summand of $\mathrm{ad}\, V$, so $H^1(\Gamma, \mathrm{ad}^0 V) = 0$. This proves the first part above.

Let $G^0 \supset B \supset T$ denote a Borel and maximal torus defined over $\mathbb{F}_l$. Proposition 7 also shows that $|\langle \mu, \alpha^\vee \rangle| < (l-1)/2$ for all weights μ of $T_{/\overline{\mathbb{F}}_l}$ on V and all $\alpha \in \Delta$. In particular, all dominant weights of $T_{/\overline{\mathbb{F}}_l}$ on V and $\mathrm{ad}\, V$ are restricted. Note that if W is a semisimple $G^0_{/\overline{\mathbb{F}}_l}$-module such that all dominant weights of $T_{/\overline{\mathbb{F}}_l}$ on W are restricted, then every $G^0(\mathbb{F}_l)$-submodule of W is also a $G^0_{/\overline{\mathbb{F}}_l}$-submodule. We apply this first to V (which is semisimple as $G^0_{/\overline{\mathbb{F}}_l}$-module, since r is semisimple), so the W_i are $G^0_{/\overline{\mathbb{F}}_l}$-submodules. By Proposition 8 of [Ser94] we see that $\mathrm{ad}\, V = \bigoplus_{i,j} \mathrm{Hom}(W_i, W_j)$ is a semisimple $G^0_{/\overline{\mathbb{F}}_l}$-module. (Note that $\dim W_i + \dim W_j < l+2$.) Thus every $G^0(\mathbb{F}_l)$-submodule of $\operatorname{ad} V$ is also a $G^0_{/\overline{\mathbb{F}}_l}$-submodule.

By Lemma 3 (applied with Δ_* the set of simple coroots), the $\overline{\mathbb{F}}_l$-linear span of the image of $T(\mathbb{F}_l)$ in $\operatorname{ad} V$ equals the $\overline{\mathbb{F}}_l$-linear span of the image of $T(\overline{\mathbb{F}}_l)$. Thus the $G^0(\mathbb{F}_l)$-submodule of $\operatorname{ad} V$ generated by the $\overline{\mathbb{F}}_l$-linear span of $r(H)$ equals the $G^0(\overline{\mathbb{F}}_l)$-submodule generated by $r(T(\overline{\mathbb{F}}_l)H)$. By Lemma 8 (noting that $\mathcal{T}(\overline{\mathbb{F}}_l) = T(\overline{\mathbb{F}}_l)H$) it follows that $r(H)$ spans $\operatorname{ad} V$. As $r(H) \subset \Gamma^{\mathrm{ss}}$, this completes the proof. $\square$

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