Fractional Integro-Differential Equations for Electromagnetic Waves in Dielectric Media

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Abstract

We prove that the electromagnetic fields in dielectric media whose susceptibility follows a fractional power-law dependence in a wide frequency range can be described by differential equations with time derivatives of noninteger order. We obtain fractional integro-differential equations for electromagnetic waves in a dielectric. The electromagnetic fields in dielectrics demonstrate a fractional power-law relaxation. The fractional integro-differential equations for electromagnetic waves are common to a wide class of dielectric media regardless of the type of physical structure, the chemical composition, or the nature of the polarizing species (dipoles, electrons, or ions).

1 Introduction

Debye formulated his theory of dipole relaxation in dielectrics in 1912 [1]. A large number of dielectric relaxation measurements show that the classical Debye behavior is very rarely observed experimentally [2, 3, 4]. Dielectric measurements by Jonscher for a wide class of various substances confirm that different dielectric spectra are described by power laws [2, 3].

For the majority of materials, the dielectric susceptibility in a wide frequency range follows a fractional power-law called the universal response [2, 3]. This law is found both in dipolar media beyond their loss peak frequency and in media where the polarization arises from movements of either ionic or electronic hopping charge carriers. It was shown in [5] that the frequency dependence of the dielectric susceptibility $\tilde{\chi}(\omega) = \chi'(\omega) - i\chi''(\omega)$ follows a common universal pattern for virtually all kinds of media over many decades of frequency,

$$\chi'(\omega) \sim \omega^{n-1}, \quad \chi''(\omega) \sim \omega^{n-1}, \quad (\omega \gg \omega_p), \tag{1}$$

and

$$\chi'(0) - \chi'(\omega) \sim \omega^m, \quad \chi''(\omega) \sim \omega^m, \quad (\omega \ll \omega_p), \tag{2}$$

where $\chi'(0)$ is the static polarization, $0 < n, m < 1$, and ω_p is the loss peak frequency. We note that the ratio of the imaginary to the real component of the susceptibility is independent of frequency. The frequency dependence given by equation (1) implies that the imaginary and real components of the complex susceptibility at high frequencies satisfy the relation

$$\frac{\chi''(\omega)}{\chi'(\omega)} = \cot\left(\frac{\pi n}{2}\right), \quad (\omega \gg \omega_p). \tag{3}$$

Experimental behavior (2) leads to a similar frequency-independent rule for

the low-frequency polarization decrement,

$$\frac{\chi''(\omega)}{\chi'(0) - \chi'(\omega)} = \tan\left(\frac{\pi m}{2}\right), \quad (\omega \ll \omega_p). \tag{4}$$

The laws of universal response for dielectric media [2, 3] can be described using fractional calculus [6]. The theory of integrals and derivatives of non-integer order goes back to Leibniz, Liouville, Riemann, Grunwald, and Let-nikov [6]. Fractional analysis has found many applications in recent studies in mechanics and physics. The interest in fractional equations has been growing continuously during the last few years because of numerous applications. In a short time, the list of applications has becomes long (see, e.g., [7, 8, 9]). In Refs. [10, 11, 12, 13], fractional calculus has used to explain the nature of nonexponential relaxation, and equations were obtained containing operators of fractional integration and differentiation.

Here, we prove that a fractional power-law frequency dependence in a time domain gives integro-differential equations with derivatives and integrals of noninteger order. We obtain fractional equations that describe electromagnetic waves for a wide class of dielectric media. The power laws of Jonscher are represented by fractional integro-differential equations. The electromagnetic fields in the dielectric media demonstrate universal fractional damping. The suggested fractional equations are common (universal) to a wide class of materials regardless of the type of physical structure, the chemical composition, or the nature of the polarizing species.

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2 Fractional equations for universal laws

We consider Eqs. (1) and (3). For the region $\omega \gg \omega_p$, universal fractional power law (1) can be presented in the form

$$\tilde{\chi}(\omega) = \chi_\alpha (i\omega)^{-\alpha}, \quad (0 < \alpha < 1) \tag{5}$$

with some positive constant χ_α and $\alpha = 1 - n$. Here,

$$(i\omega)^\alpha = |\omega|^\alpha \exp\{i\,\alpha\,\pi\,sgn(\omega)/2\}.$$

It is easy to see that relation (3) is satisfied for (5).

The polarization density $\mathbf{P}(t,r)$ can be written as

$$\mathbf{P}(t,r) = \mathcal{F}^{-1}\left(\tilde{\mathbf{P}}(\omega,r)\right) = \varepsilon_0 \mathcal{F}^{-1}\left(\tilde{\chi}(\omega)\tilde{\mathbf{E}}(\omega,r)\right), \tag{6}$$

where $\tilde{\mathbf{P}}(\omega,r)$ is the Fourier transform $\mathcal{F}$ of $\mathbf{P}(t,r)$. Substitution of (5) into (6) gives

$$\mathbf{P}(t,r) = \varepsilon_0 \chi_\alpha \mathcal{F}^{-1}\left((i\omega)^{-\alpha}\tilde{\mathbf{E}}(\omega,r)\right).$$

We note that the Fourier transform of the fractional Liouville integral [6, 14]

$$(I_+^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^t \frac{f(t')dt'}{(t-t')^{1-\alpha}}$$

is given by the relation (see Theorem 7.1 in [6] and Theorem 2.15 in [14]):

$$(\mathcal{F} I_+^\alpha f)(\omega) = \frac{1}{(i\omega)^\alpha}(\mathcal{F}f)(\omega),$$

where $0 < Re(\alpha) < 1$ and $f(t) \in L_1(\mathbb{R})$, or $1 \le p < 1/Re(\alpha)$ and $f(t) \in L_p(\mathbb{R})$.

Using the fractional Liouville integral and the fractional power law (5) for $\tilde{\chi}(\omega)$ in the frequency domain, we obtain

$$\mathbf{P}(t,r) = \varepsilon_0 \chi_\alpha (I_+^\alpha \mathbf{E})(t,r), \quad (0 < \alpha < 1). \tag{7}$$

This equation shows that the polarization density $\mathbf{P}(t, r)$ in the high-frequency region is proportional to the fractional Liouville integral of the electric field $\mathbf{E}(t, r)$.

We consider Eqs. (2) and (4). For the region $\omega \ll \omega_p$, universal fractional power law (2) can be presented as

$$\tilde{\chi}(\omega) = \tilde{\chi}(0) - \chi_\beta(i\omega)^\beta, \quad (0 < \beta < 1) \tag{8}$$

with some positive constants χ_β, $\tilde{\chi}(0)$, and $\beta = m$. It is easy to prove that equation (4) is satisfied.

We note that the Fourier transforms of the fractional Liouville derivative [6, 14]

$$(D_+^\beta f)(t) = \frac{\partial^k}{\partial t^k}(I_+^{k-\beta} f)(t) = \frac{1}{\Gamma(k - \beta)} \frac{\partial^k}{\partial t^k} \int_{-\infty}^t \frac{f(t')dt'}{(t - t')^{\beta - k + 1}},$$

where $k - 1 < \beta < k$, are given by the formula (see Theorem 7.1 in [6] and Theorem 2.15 in [14]):

$$(\mathcal{F}D_+^\beta f)(\omega) = (i\omega)^\beta(\mathcal{F}f)(\omega),$$

where $0 < Re(\beta) < 1$ and $f(t) \in L_1(\mathbb{R})$, or $1 \le p < 1/Re(\beta)$ and $f(t) \in L_p(\mathbb{R})$.

Using the definition of the fractional Liouville derivative and fractional power laws (8), we can represent polarization density (6) in the form

$$\mathbf{P}(t, r) = \varepsilon_0 \tilde{\chi}(0)\mathbf{E}(t, r) - \varepsilon_0 \chi_\beta(D_+^\beta \mathbf{E})(t, r), \quad (0 < \beta < 1). \tag{9}$$

This equation shows that the polarization density $\mathbf{P}(t, r)$ in the low-frequency region is determined by the fractional Liouville derivative of the electric field $\mathbf{E}(t, r)$.

Relations (7) and (9) can be considered universal laws. These equations with integro-differentiation of noninteger order allow obtaining fractional wave equations for electric and magnetic fields.

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3 Universal electromagnetic wave equation

Here, we obtain fractional equations for electromagnetic fields in dielectric media. Using the Maxwell equations, we obtain

$$\varepsilon_0 \frac{\partial^2 \mathbf{E}(t,r)}{\partial t^2} + \frac{\partial^2 \mathbf{P}(t,r)}{\partial t^2} + \frac{1}{\mu}\left(grad\,div\mathbf{E} - \nabla^2\mathbf{E}\right) + \frac{\partial \mathbf{j}(t,r)}{\partial t} = 0. \qquad (10)$$

For $\omega \gg \omega_p$, the polarization density $\mathbf{P}(t,r)$ is related with $\mathbf{E}(t,r)$ by equation (7). Substituting (7) in (10), we obtain the fractional differential equation for the electric field

$$\frac{1}{v^2}\frac{\partial^2 \mathbf{E}(t,r)}{\partial t^2} + \frac{\chi_\alpha}{v^2}(D_+^{2-\alpha}\mathbf{E})(t,r) + \left(grad\,div\mathbf{E} - \nabla^2\mathbf{E}\right) = -\mu\frac{\partial \mathbf{j}(t,r)}{\partial t}, \quad (11)$$

where $0 < \alpha < 1$, and $v^2 = 1/(\varepsilon_0\mu)$. We note that $div\mathbf{E} \neq 0$ for $\rho(t,r) = 0$.

For $\omega \ll \omega_p$, the fields $\mathbf{P}(t,r)$ and $\mathbf{E}(t,r)$ are related by Eq. (9). In this case, equation (10) becomes

$$\frac{1}{v_\beta^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \frac{a_\beta}{v_\beta^2}(D_+^{2+\beta}\mathbf{E}) + \left(grad\,div\mathbf{E} - \nabla^2\mathbf{E}\right) = -\mu\frac{\partial \mathbf{j}}{\partial t}, \quad (0 < \beta < 1), \quad (12)$$

where

$$v_\beta^2 = \frac{1}{\varepsilon_0\mu\left[1 + \tilde{\chi}(0)\right]}, \quad a_\beta = \frac{\chi_\beta}{1 + \tilde{\chi}(0)}.$$

Equations (11) and (12) describe the time evolution of the electric field in dielectric media. These equations are fractional differential equations [14] with derivatives of the orders $2 - \alpha$ and $2 + \beta$.

Using the Maxwell equations, we obtain the equation for the magnetic field

$$\frac{\partial^2 \mathbf{B}(t,r)}{\partial t^2} = \frac{1}{\varepsilon_0\mu}\nabla^2\mathbf{B}(t,r) + \frac{1}{\varepsilon_0}\frac{\partial}{\partial t}curl\mathbf{P}(t,r) + \frac{1}{\varepsilon_0}curl\mathbf{j}(t,r). \qquad (13)$$

In experiments, the field $\mathbf{B}(t,r)$ can be presented as $\mathbf{B}(t,r) = 0$ for $t \leq 0$, and $\mathbf{B}(t,r) \neq 0$ for $t > 0$. For $\omega \gg \omega_p$, the polarization density $\mathbf{P}(t,r)$ is

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related to $\mathbf{E}(t,r)$ by equation (7), which leads to the fractional differential equation for magnetic field in the form

$$\frac{1}{v^2}\frac{\partial^2\mathbf{B}(t,r)}{\partial t^2} + \frac{\chi_\alpha}{v^2}\left(_0D_t^{2-\alpha}\mathbf{B}\right)(t,r) - \nabla^2\mathbf{B}(t,r) = \mu\,curl\,\mathbf{j}(t,r), \qquad (14)$$

where $0 < \alpha < 1$, $v^2 = 1/(\varepsilon_0\mu)$, and $_0D_t^{2-\alpha}$ is the Riemann-Liouville derivative [14] on $[0,\infty)$ such that

$$(_0D_t^{2-\alpha}f)(t) = \frac{1}{\Gamma(\alpha)}\frac{\partial^2}{\partial t^2}\int_0^t \frac{f(t')dt'}{(t-t')^{1-\alpha}}, \qquad (0 < \alpha < 1).$$

For $\omega \ll \omega_p$, we obtain

$$\frac{1}{v_\beta^2}\frac{\partial^2\mathbf{B}(t,r)}{\partial t^2} - \frac{a_\beta}{v_\beta^2}\left(_0D_t^{2+\beta}\mathbf{B}\right)(t,r) - \nabla^2\mathbf{B}(t,r) = \mu\,curl\,\mathbf{j}(t,r), \qquad (15)$$

where $0 < \beta < 1$, and

$$v_\beta^2 = \frac{1}{\varepsilon_0\mu\left[1 + \tilde{\chi}(0)\right]}, \qquad a_\beta = \frac{\chi_\beta}{1 + \tilde{\chi}(0)}.$$

Equations (14) and (15) are fractional differential equations that describe the magnetic field in dielectric media and demonstrate a power-law relaxation. They can be written in a general form. Such a general fractional differential equation for the magnetic field has the form

$$(_0D_t^\alpha\mathbf{B})(t,r) - \lambda_1\left(_0D_t^\beta\mathbf{B}\right)(t,r) - \lambda_2\nabla^2\mathbf{B}(t,r) = \mathbf{f}(t,r), \qquad (16)$$

where $1 \le \beta < \alpha < 3$. The curl of the current density of free charges is regarded as an external source: $\mathbf{f}(t,r) = \mu\lambda_2\,curl\,\mathbf{j}(t,r)$. Equation (16) yields Eq. (14) for $\alpha = 2$, $1 < \beta < 2$, and

$$\lambda_1 = -\chi_\alpha, \qquad \lambda_2 = v^2 = 1/(\varepsilon_0\mu).$$

Equation (15) can be written in form (16) for $2 < \alpha < 3$, $\beta = 2$, and

$$\lambda_1 = \frac{1}{a_\beta} = \frac{1 + \tilde{\chi}(0)}{\chi_\beta}, \qquad \lambda_2 = -\frac{v_\beta^2}{a_\beta} = \frac{-1}{\varepsilon_0\mu\chi_\beta}.$$

An exact solution of Eq. (16) can be written in terms of Wright functions [14] using Theorem 5.5 in [14]. We note that Wright functions can be represented as derivatives of the Mittag-Leffler function $E_{\alpha,\beta}[z]$ (see [14]). Solutions of equation (16) describe the fractional power-law damping of the magnetic field in dielectric media. An important property of the evolution described by the fractional differential equations is that the solutions have fractional power-law tails.

4   Conclusion

We have prove that the electromagnetic fields and waves in a wide class of dielectric media must be described by fractional differential equations with derivatives of the order $2-\alpha$ and $2+\beta$, where $0 < \alpha < 1$ and $0 < \beta < 1$. The parameters $\alpha = 1-n$ and $\beta = m$ are defined by the exponents n and m in the experimentally measured frequency dependence of the dielectric susceptibility, called the universal response laws. An important property of the dynamics described by fractional differential equations for electromagnetic fields is that the solutions have fractional power-law tails. The suggested fractional integro-differential equations for the universal electromagnetic waves in dielectrics are common (universal) to a wide class of media regardless of the type of physical structure, the chemical composition, or the nature of the polarizing species (dipoles, electrons, or ions).

We note that the differential equations with derivatives of noninteger order proposed for describing the electromagnetic field in dielectric media can be solved numerically. For example, the Grunwald-Letnikov discretization scheme [6] is used for numerically model the electromagnetic field in dielectrics described by fractional differential equations. For small fraction-

ality of α (or β), an ε-expansion [15] in the small parameter $\varepsilon = \alpha$ (or $\varepsilon = 1 - \beta$) can be used. We note that a possible physical interpretation of fractional integrals and derivatives can be connected with memory effects or fractal properties of media (see, e.g., [16, 17]).

References

[1] P. Debye, "Some results of kinetic theory of isolators" Physikalische Zeitschrift **13** (1912) 97-100.

[2] A.K. Jonscher, *Universal Relaxation Law*, London: Chelsea Dielectrics Press, 1996.

[3] A.K. Jonscher, "Dielectric relaxation in solids" J. Physics D Appl. Phys. **32** (1999) R57-R70.

[4] T.V. Ramakrishnan, M.R. Lakshmi, (Eds.), *Non-Debye Relaxation in Condensed Matter*, Singapore: World Scientific, 1984.

[5] A.K. Jonscher, "Universal dielectric response" Nature **267** (1977) 673-679; "Low-frequency dispersion in carrier-dominated dielectrics" Philosophical Magazine B **38** (1978) 587-601.

[6] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Integrals and Derivatives of Fractional Order and Applications*, Minsk: Nauka i Tehnika, 1987 in Russian; and *Fractional Integrals and Derivatives Theory and Applications*, New York: Gordon and Breach, 1993.

[7] G.M. Zaslavsky, *Hamiltonian Chaos and Fractional Dynamics*, Oxsford: Oxford University Press, 2005.

[8] A. Carpinteri, F. Mainardi, *Fractals and Fractional Calculus in Continuum Mechanics*, New York: Springer, 1997.

[9] *Applications of Fractional Calculus in Physics*, Singapore: World Scientific, 2000.

[10] R.R. Nigmatullin, Ya.E. Ryabov, "Cole-Davidson dielectric relaxation as a self-similar relaxation process" Phys. Solid State **39** (1997) 87-90; Fizika Tverdogo Tela **39** (1997) 101-105 in Russian.

[11] V.V. Novikov, V.P. Privalko, "Temporal fractal model for the anomalous dielectric relaxation of inhomogeneous media with chaotic structure" Phys. Rev. E. **64** (2001) 031504.

[12] Y. Yilmaz, A. Gelir, F. Salehli, R.R. Nigmatullin, A. A. Arbuzov, "Dielectric study of neutral and charged hydrogels during the swelling process" J. Chem. Phys. **125** (2006) 234705.

[13] R.R. Nigmatullin, A.A. Arbuzov, F. Salehli, A. Giz, I. Bayrak, H. Catalgil-Giz "The first experimental confirmation of the fractional kinetics containing the complex-power-law exponents: Dielectric measurements of polymerization reactions" Physica B **388** (2007) 418-434.

[14] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Application of Fractional Differential Equations*, Amsterdam: Elsevier, 2006.

[15] V.E. Tarasov, G.M. Zaslavsky, "Dynamics with low-level fractionality" Physica A **368** (2006) 399-415.

[16] R.R. Nigmatullin, "Fractional integral and its physical interpretation" Theor. Math. Phys. **90** (1992) 242-251.

[17] A.A. Stanislavsky, "Probability interpretation of the integral of fractional order" Theor. Math. Phys. **138** (2004) 418-431.

See also V.E. Tarasov, *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media* (Springer, HEP, 2011) 504 pages.