SINGULAR EQUIVALENCES INDUCED BY HOMOLOGICAL EPIMORPHISMS

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Abstract. We prove that a certain homological epimorphism between two algebras induces a triangle equivalence between their singularity categories. Applying the result to a construction of matrix algebras, we describe the singularity categories of some non-Gorenstein algebras.

1. Introduction

Let A be a finite dimensional algebra over a field k. Denote by A-mod the category of finitely generated left A-modules, and by $\mathbf{D}^b(A\text{-mod})$ the bounded derived category. Following [17], the *singularity category* $\mathbf{D}_{\rm sg}(A)$ of A is the Verdier quotient triangulated category of $\mathbf{D}^b(A\text{-mod})$ with respect to the full subcategory formed by perfect complexes; see also [4, 13, 11, 20, 2, 14] and [6].

The singularity category measures the homological singularity of an algebra: the algebra A has finite global dimension if and only if its singularity category $\mathbf{D}_{\rm sg}(A)$ is trivial. Meanwhile, the singularity category captures the stable homological features of an algebra ([4]).

A fundamental result of Buchweitz and Happel states that for a Gorenstein algebra A, the singularity category $\mathbf{D}_{\rm sg}(A)$ is triangle equivalent to the stable category of (maximal) Cohen-Macaulay A-modules ([4] and [11]). This result specializes to Rickard's result ([20]) on self-injective algebras. For non-Gorenstein algebras, not much is known about their singularity categories ([5, 7]).

The following concepts might be useful in the study of singularity categories. Two algebras A and B are said to be *singularly equivalent* provided that there is a triangle equivalence between $\mathbf{D}_{\rm sg}(A)$ and $\mathbf{D}_{\rm sg}(B)$. Such an equivalence is called a *singular equivalence*; compare [18]. In this case, if A is non-Gorenstein and B is Gorenstein, then Buchweitz-Happel's theorem applies to give a description of $\mathbf{D}_{\rm sg}(A)$ in terms of Cohen-Macaulay modules over B. We observe that a derived equivalence of two algebras, that is, a triangle equivalence between their bounded derived categories, induces naturally a singular equivalence. The converse is not true in general.

Let A be an algebra and let $J \subseteq A$ be a two-sided ideal. Following [19], we call J a *homological ideal* provided that the canonical map $A \to A/J$ is a homological epimorphism ([9]), meaning that the naturally induced functor $\mathbf{D}^b(A/J\text{-mod}) \to \mathbf{D}^b(A\text{-mod})$ is fully faithful.

The main observation we make is as follows.

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Theorem. *Let A be a finite dimensional k-algebra and let $J \subseteq A$ be a homological ideal which has finite projective dimension as an A-A-bimodule. Then there is a singular equivalence between A and A/J.*

The paper is structured as follows. In Section 2, we recall some ingredients and then prove Theorem. In Section 3, we apply Theorem to a construction of matrix algebras, and then describe the singularity categories of some non-Gorenstein algebras. In particular, we give two examples, which extend in different manners an example considered by Happel in [11].

2. Proof of Theorem

We will present the proof of Theorem in this section. Before that, we recall from [22] and [12] some known results on triangulated categories and derived categories.

Let $\mathcal{T}$ be a triangulated category. We will denote its translation functor by [1]. For a triangulated subcategory $\mathcal{N}$, we denote by $\mathcal{T}/\mathcal{N}$ the Verdier quotient triangulated category. The quotient functor $q\colon \mathcal{T} \to \mathcal{T}/\mathcal{N}$ has the property that $q(X) \simeq 0$ if and only if X is a direct summand of an object in $\mathcal{N}$. In particular, if $\mathcal{N}$ is a *thick* subcategory, that is, it is closed under direct summands, we have that $\mathrm{Ker}\ q = \mathcal{N}$. Here, for a triangle functor F, $\mathrm{Ker}\ F$ denotes the (thick) triangulated subcategory consisting of objects on which F vanishes.

The following result is well known.

Lemma 2.1. *Let $F\colon \mathcal{T} \to \mathcal{T}'$ be a triangle functor which allows a fully faithful right adjoint G. Then F induces uniquely a triangle equivalence $\mathcal{T}/\mathrm{Ker}\ F \simeq \mathcal{T}'$.*

Proof. The existence of the induced functor follows from the universal property of the quotient functor. The result is a triangulated version of [8, Proposition I. 1.3]. For details, see [3, Propositions 1.5 and 1.6]. □

Let $F\colon \mathcal{T} \to \mathcal{T}'$ be a triangle functor. Assume that $\mathcal{N} \subseteq \mathcal{T}$ and $\mathcal{N}' \subseteq \mathcal{T}'$ are triangulated subcategories satisfying $F\mathcal{N} \subseteq \mathcal{N}'$. Then there is a uniquely induced triangle functor $\bar{F}\colon \mathcal{T}/\mathcal{N} \to \mathcal{T}'/\mathcal{N}'$.

Lemma 2.2. ([17, Lemma 1.2]) *Let $F\colon \mathcal{T} \to \mathcal{T}'$ be a triangle functor which has a right adjoint G. Assume that $\mathcal{N} \subseteq \mathcal{T}$ and $\mathcal{N}' \subseteq \mathcal{T}'$ are triangulated subcategories satisfying that $F\mathcal{N} \subseteq \mathcal{N}'$ and $G\mathcal{N}' \subseteq \mathcal{N}$. Then the induced functor $\bar{F}\colon \mathcal{T}/\mathcal{N} \to \mathcal{T}'/\mathcal{N}'$ has a right adjoint $\bar{G}$. Moreover, if G is fully faithful, so is $\bar{G}$.*

Proof. The unit and counit of (F, G) induce uniquely two natural transformations $\mathrm{Id}_{\mathcal{T}/\mathcal{N}} \to \bar{G}\bar{F}$ and $\bar{F}\bar{G} \to \mathrm{Id}_{\mathcal{T}'/\mathcal{N}'}$, which are the corresponding unit and counit of the adjoint pair $(\bar{F}, \bar{G})$; consult [16, Chapter IV, Section 1, Theorem 2(v)]. Note that the fully-faithfulness of G is equivalent to that the counit of (F, G) is an isomorphism. It follows that the counit of $(\bar{F}, \bar{G})$ is also an isomorphism, which is equivalent to the fully-faithfulness of $\bar{G}$; consult [16, Chapter IV, Section 3, Theorem 1]. □

Let k be a field and let A be a finite dimensional k-algebra. Recall that A-mod is the category of finite dimensional left A-modules. We write ${}_AA$ for the regular left A-module. Denote by $\mathbf{D}(A\text{-mod})$ (*resp.* $\mathbf{D}^b(A\text{-mod})$) the (*resp.* bounded) derived category of A-mod. We identify A-mod as the full subcategory of $\mathbf{D}^b(A\text{-mod})$ consisting of stalk complex concentrated at degree zero; see [12, Proposition I. 4.3].

A complex of A-modules is usually denoted by $X^\bullet = (X^n, d^n)_{n\in\mathbb{Z}}$, where X^n are A-modules and the differentials $d^n\colon X^n \to X^{n+1}$ are homomorphisms of modules satisfying $d^{n+1} \circ d^n = 0$. Recall that a complex in $\mathbf{D}^b(A\text{-mod})$ is *perfect* provided that it is isomorphic to a bounded complex consisting of projective modules. The full subcategory consisting of perfect complexes is denoted by $\mathrm{perf}(A)$. Recall from

[4, Lemma 1.2.1] that a complex $X^\bullet$ in $\mathbf{D}^b(A\text{-mod})$ is perfect if and only if there is a natural number n_0 such that for each A-module M, $\mathrm{Hom}_{\mathbf{D}^b(A\text{-mod})}(X^\bullet, M[n]) = 0$ for all $n \geq n_0$. It follows that $\mathrm{perf}(A)$ is a thick subcategory of $\mathbf{D}^b(A\text{-mod})$. Indeed, it is the smallest thick subcategory of $\mathbf{D}^b(A\text{-mod})$ containing ${}_AA$.

Let $\pi\colon A \to B$ be a homomorphism of algebras. The functor of restricting of scalars $\pi^*\colon B\text{-mod} \to A\text{-mod}$ is exact, and it extends to a triangle functor $\mathbf{D}^b(B\text{-mod}) \to \mathbf{D}^b(A\text{-mod})$, which will still be denoted by π^*. Following [9], we call the homomorphism π a *homological epimorphism* provided that $\pi^*\colon \mathbf{D}^b(B\text{-mod}) \to \mathbf{D}^b(A\text{-mod})$ is fully faithful. By [9, Theorem 4.1(1)] this is equivalent to that $\pi \otimes^{\mathbf{L}}_A B\colon B \simeq A \otimes^{\mathbf{L}}_A B \to B \otimes^{\mathbf{L}}_A B$ is an isomorphism in $\mathbf{D}(A^e\text{-mod})$. Here, $A^e = A \otimes_k A^{\mathrm{op}}$ is the enveloping algebra of A, and we identify A^e-mod as the category of A-A-bimodules.

Lemma 2.3. ([19, Proposition 2.2(a)]) *Let $J \subseteq A$ be an ideal and let $\pi\colon A \to A/J$ be the canonical projection. Then π is a homological epimorphism if and only if $J^2 = J$ and $\mathrm{Tor}_i^A(J, A/J) = 0$ for all $i \geq 1$.*

In the situation of the lemma, the ideal J is called a *homological ideal* in [19]. As a special case, we call an ideal J a *hereditary ideal* provided that $J^2 = J$ and J is a projective A-A-bimodule; compare [19, Lemma 3.4].

Proof. The natural exact sequence $0 \to J \to A \to A/J \to 0$ of A-A-bimodules induces a triangle $J \to A \to A/J \to J[1]$ in $\mathbf{D}^b(A^e\text{-mod})$. Applying the functor $- \otimes^{\mathbf{L}}_A A/J$, we get a triangle $J \otimes^{\mathbf{L}}_A A/J \to A/J \to A/J \otimes^{\mathbf{L}}_A A/J \to J \otimes^{\mathbf{L}}_A A/J[1]$. Then π is a homological epimorphism, or equivalently $\pi \otimes_A A/J$ is an isomorphism if and only if $J \otimes^{\mathbf{L}}_A A/J = 0$; see [10, Lemma I.1.7]. This is equivalent to that $\mathrm{Tor}_i^A(J, A/J) = 0$ for all $i \geq 0$. We note that $\mathrm{Tor}_0^A(J, A/J) \simeq J \otimes_A A/J \simeq J/J^2$. $\square$

Now we are in the position to prove Theorem. Recall that for an algebra A, its singularity category $\mathbf{D}_{\mathrm{sg}}(A) = \mathbf{D}^b(A\text{-mod})/\mathrm{perf}(A)$. Moreover, a complex $X^\bullet$ becomes zero in $\mathbf{D}_{\mathrm{sg}}(A)$ if and only if it is perfect. Here, we use the fact that $\mathrm{perf}(A) \subseteq \mathbf{D}^b(A\text{-mod})$ is a thick subcategory.

Proof of Theorem. Write $B = A/J$. Since J, as an A-A-bimodule, has finite projective dimension, so it has finite projective dimension both as a left and right A-module. Consider the natural exact sequence $0 \to J \to A \to B \to 0$. It follows that B, both as a left and right A-module, has finite projective dimension. Moreover, for a complex $X^\bullet$ in $\mathbf{D}^b(A\text{-mod})$, $J \otimes^{\mathbf{L}}_A X^\bullet$ is perfect. Indeed, take a bounded projective resolution $P^\bullet \to J$ as an A^e-module. Then $J \otimes^{\mathbf{L}}_A X^\bullet \simeq P^\bullet \otimes_A X^\bullet$. This is a perfect complex, since each left A-module $P^i \otimes_A X^j$ is projective.

Denote by $\pi\colon A \to B$ be the canonical projection. By the assumption, the functor $\pi^*\colon \mathbf{D}^b(B\text{-mod}) \to \mathbf{D}^b(A\text{-mod})$ is fully faithful. Since $\pi^*(B)$ is perfect, the functor π^* sends perfect complexes to perfect complexes. Then it induces a triangle functor $\bar{\pi^*}\colon \mathbf{D}_{\mathrm{sg}}(B) \to \mathbf{D}_{\mathrm{sg}}(A)$. We will show that $\bar{\pi^*}$ is an equivalence.

The functor $\pi^*\colon \mathbf{D}^b(B\text{-mod}) \to \mathbf{D}^b(A\text{-mod})$ has a left adjoint $F = B \otimes^{\mathbf{L}}_A -\colon \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}^b(B\text{-mod})$. Here we use the fact that the right A-module B has finite projective dimension. Since F sends perfect complexes to perfect complexes, we have the induced triangle functor $\bar{F}\colon \mathbf{D}_{\mathrm{sg}}(A) \to \mathbf{D}_{\mathrm{sg}}(B)$. By Lemma 2.2 we have the adjoint pair $(\bar{F}, \bar{\pi^*})$; moreover, the functor $\bar{\pi^*}$ is fully faithful. By Lemma 2.1 there is a triangle equivalence $\mathbf{D}_{\mathrm{sg}}(A)/\mathrm{Ker}\,\bar{F} \simeq \mathbf{D}_{\mathrm{sg}}(B)$.

It remains to show that the essential kernel $\mathrm{Ker}\,\bar{F}$ is trivial. For this, assume that a complex $X^\bullet$ lies in $\mathrm{Ker}\,\bar{F}$. This means that the complex $F(X^\bullet)$ in $\mathbf{D}^b(B\text{-mod})$ is perfect. Since π^* preserves perfect complexes, it follows that $\pi^* F(X^\bullet)$ is also perfect. The natural exact sequence $0 \to J \to A \to B \to 0$ induces a triangle

$J \otimes_A^{\mathbf{L}} X^\bullet \to X^\bullet \to \pi^* F(X^\bullet) \to J \otimes_A^{\mathbf{L}} X^\bullet[1]$ in $\mathbf{D}^b(A\text{-mod})$. Recall that $J \otimes_A^{\mathbf{L}} X^\bullet$ is perfect. It follows that $X^\bullet$ is perfect, since $\mathrm{perf}(A) \subseteq \mathbf{D}^b(A\text{-mod})$ is a triangulated subcategory. The proves that $X^\bullet$ is zero in $\mathbf{D}_{\rm sg}(A)$. □

The following special case of Theorem is of interest.

Corollary 2.4. *Let A be a finite dimensional algebra and $J \subseteq A$ a hereditary ideal. Then we have a triangle equivalence $\mathbf{D}_{\rm sg}(A) \simeq \mathbf{D}_{\rm sg}(A/J)$.*

Proof. It suffices to observe by Lemma 2.3 that J is a homological ideal. □

3. Examples

In this section, we will describe a construction of matrix algebras to illustrate Corollary 2.4. In particular, the singularity categories of sone non-Gorenstein algebras are studied.

The following construction is similar to [15, Section 4]. Let A be a finite dimensional algebra over a field k. Let ${}_AM$ and N_A be a left and right A-module, respectively. Then $M \otimes_k N$ becomes an A-A-bimodule. Consider an A-A-bimodule monomorphism $\phi\colon M \otimes_k N \to A$ such that $\mathrm{Im}\ \phi$ vanishes both on M and N. Here, we observe that $\mathrm{Im}\ \phi \subseteq A$ is an ideal. The matrix $\Gamma = \begin{pmatrix} A & M \\ N & k \end{pmatrix}$ becomes an associative algebra via the following multiplication

$$\begin{pmatrix} a & m \\ n & \lambda \end{pmatrix} \begin{pmatrix} a' & m' \\ n' & \lambda' \end{pmatrix} = \begin{pmatrix} aa' + \phi(m \otimes n) & am' + \lambda' m \\ na' + \lambda n' & \lambda\lambda' \end{pmatrix}.$$

Proposition 3.1. *Keep the notation and assumption as above. Then there is a triangle equivalence $\mathbf{D}_{\rm sg}(\Gamma) \simeq \mathbf{D}_{\rm sg}(A/\mathrm{Im}\ \phi)$.*

Proof. Set $J = \Gamma e \Gamma$ with $e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Observe that $\Gamma/J = A/\mathrm{Im}\ \phi$. The ideal J is hereditary: $J^2 = J$ is clear, while the natural map $\Gamma e \otimes_k e\Gamma \to J$ is an isomorphism of Γ-Γ-bimodules and then J is a projective Γ-Γ-bimodule. The isomorphism uses that ϕ is mono. Then we apply Corollary 2.4. □

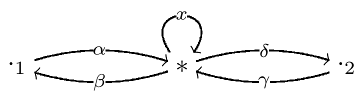
Remark 3.2. The above construction contains the one-point extension and coextension of algebras, where M or N is zero. Hence Proposition 3.1 contains the results in [7, Section 4].

We will illustrate Proposition 3.1 by three examples. Two of these examples extend an example considered by Happel in [11]. In particular, based on results in [7], we obtain descriptions of the singularity categories of some non-Gorenstein algebras.

Recall from [11] that an algebra A is *Gorenstein* provided that both as a left and right module, the regular module A has finite injective dimension. It follows from [4, Theorem 4.4] and [11, Theorem 4.6] that in the Gorenstein case, the singularity category $\mathbf{D}_{\rm sg}(A)$ is *Hom-finite*. This means that all Hom spaces in $\mathbf{D}_{\rm sg}(A)$ are finite dimensional over k.

For algebras given by quivers and relations, we refer to [1, Chapter III].

Example 3.3. Let Γ be the k-algebra given by the following quiver Q with relations $\{\delta x, \beta x, x\gamma, x\alpha, \beta\gamma, \delta\alpha, \beta\alpha, \delta\gamma, \alpha\beta - \gamma\delta\}$. We write the concatenation of paths from the left to the right.

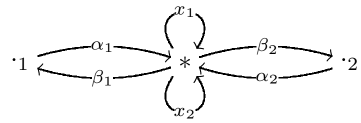


We have in Γ that $1 = e_1 + e_* + e_2$, where the e's are the primitive idempotents corresponding to the vertices. Set $\Gamma' = \Gamma/\Gamma e_1 \Gamma$. It is an algebra with radical square zero, whose quiver is obtained from Q by removing the vertex 1 and the adjacent arrows.

We identify Γ with $\begin{pmatrix} A & k\alpha \\ k\beta & k \end{pmatrix}$, where the k in the southeast corner is identified with $e_1\Gamma e_1$, and $A = (1-e_1)\Gamma(1-e_1)$. The corresponding Im ϕ equals $k\alpha\beta$, and we have $A/\text{Im } \phi = \Gamma'$; consult the proof of Proposition 3.1. Then Proposition 3.1 yields a triangle equivalence $\mathbf{D}_{\rm sg}(\Gamma) \simeq \mathbf{D}_{\rm sg}(\Gamma')$.

The triangulated category $\mathbf{D}_{\rm sg}(\Gamma')$ is completely described in [7]; also see [21]; in particular, it is not Hom-finite. More precisely, it is equivalent to the category of finitely generated projective modules on a von Neumann regular algebra. The algebra Γ', or rather its Koszul dual, is related to the noncommutative space of Penrose tiling via the work of Smith; see [21, Theorem 7.2 and Example]. We point out that the algebra Γ is non-Gorenstein, since $\mathbf{D}_{\rm sg}(\Gamma)$ is not Hom-finite.

Example 3.4. Let Γ be the k-algebra given by the following quiver Q with relations $\{x_1^2, x_2^2, x_1\alpha_1, x_2\alpha_1, \beta_2\alpha_1, \beta_2\alpha_1, x_1\alpha_2, x_2\alpha_2, \beta_1\alpha_2, \beta_2\alpha_2, \alpha_1\beta_1 - x_1x_2, \alpha_2\beta_2 - x_2x_1\}$.



We claim that there is a triangle equivalence $\mathbf{D}_{\rm sg}(\Gamma) \simeq \mathbf{D}_{\rm sg}(k\langle x_1, x_2\rangle/(x_1, x_2)^2)$. Here, $k\langle x_1, x_2\rangle$ is the free algebra with two variables.

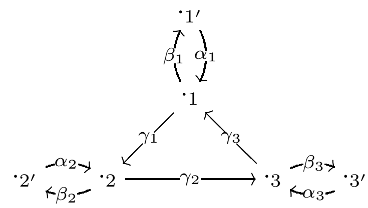
We point out that the triangulated category $\mathbf{D}_{\rm sg}(k\langle x_1, x_2\rangle/(x_1, x_2)^2)$ is described completely in [7, Example 3.11]. Similar as in the example above, this algebra Γ is non-Gorenstein.

To see the claim, we observe that the quiver Q has two loops and two 2-cycles. The proof is done by "removing the 2-cycles". We have a natural isomorphism $\Gamma = \begin{pmatrix} A & k\alpha_1 \\ k\beta_1 & k \end{pmatrix}$, where $k = e_1\Gamma e_1$ and $A = (1-e_1)\Gamma(1-e_1)$. We observe that Proposition 3.1 applies with the corresponding Im $\phi = k\alpha_1\beta_1$. Set $A/\text{Im } \phi = \Gamma'$. So $\mathbf{D}_{\rm sg}(\Gamma) \simeq \mathbf{D}_{\rm sg}(\Gamma')$. The quiver of Γ' is obtained from Q by removing the vertex 1 and the adjacent arrows, while its relations are obtained from the ones of Γ by replacing $\alpha_1\beta_1 - x_1x_2$ with x_1x_2. Similarly, $\Gamma' = \begin{pmatrix} A' & k\alpha_2 \\ k\beta_2 & k \end{pmatrix}$ with $k = e_2\Gamma' e_2$ and $A' = e_*\Gamma' e_*$. Then Proposition 3.1 applies and we get the equivalence $\mathbf{D}_{\rm sg}(\Gamma') \simeq \mathbf{D}_{\rm sg}(k\langle x_1, x_2\rangle/(x_1, x_2)^2)$.

This example generalizes directly to a quiver with n loops and n 2-cycles with similar relations. The corresponding statement for the case $n = 1$ is implicitly contained in [11, 2.3 and 4.8].

The last example is a Gorenstein algebra.

Example 3.5. Let $r \geq 2$. Consider the following quiver Q consisting of three 2-cycles and a central 3-cycle Z_3. We identify γ_3 with γ_0, and denote by p_i the path in the central cycle starting at vertex i of length 3.



Let Γ be the k-algebra given by the quiver Q with relations $\{\beta_i\alpha_i, \gamma_i\alpha_i, \beta_i\gamma_{i-1}, \beta_i\alpha_i - p_i^r \mid i = 1, 2, 3\}$. We point out that in Γ all paths in the central cycle of length strictly larger than $3r + 1$ vanish.

Set $A = kZ_3/(\gamma_1, \gamma_2, \gamma_3)^{3r}$, where kZ_3 is the path algebra of the central 3-cycle Z_3. The algebra A is self-injective and Nakayama ([1, p.111]). Denote by A-$\underline{\text{mod}}$ the stable category of A-modules; it is naturally a triangulated category; see [10, Theorem I.2.6].

We claim that there is a triangle equivalence $\mathbf{D}_{\text{sg}}(\Gamma) \simeq A$-$\underline{\text{mod}}$.

For the claim, we observe an isomorphism $A = \Gamma/\Gamma(e_{1'} + e_{2'} + e_{3'})\Gamma$. We argue as in Example 3.4 by removing the three 2-cycles and applying Proposition 3.1 repeatedly. Then we get a triangle equivalence $\mathbf{D}_{\text{sg}}(\Gamma) \simeq \mathbf{D}_{\text{sg}}(A)$. Finally, by [20, Theorem 2.1] we have a triangle equivalence $\mathbf{D}_{\text{sg}}(A) \simeq A$-$\underline{\text{mod}}$. Then we are done.

We observe that the algebra Γ is Gorenstein with self injective dimension two. Moreover, this example generalizes directly to a quiver with n 2-cycles and a central n-cycle with similar relations. The case where $n = 1$ and $r = 2$ coincides with the example considered in [11, 2.3 and 4.8].

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References

[1] M. Auslander, I. Reiten, and S.O. Smalø, Representation Theory of Artin Algebras, Cambridge Studies in Adv. Math. **36**, Cambridge Univ. Press, Cambridge, 1995.

[2] A. Beligiannis, *The homological theory of contravariantly finite subcategories: Auslander-Buchweitz contexts, Gorenstein categories and (co-)stabilization,* Comm. Algebra **28**(10) (2000), 4547–4596.

[3] A.I. Bondal and M.M. Kapranov, *Representable functors, Serre functors, and reconstructions,* Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), 1183–1205.

[4] R.O. Buchweitz, Maximal Cohen-Macaulay Modules and Tate Cohomology over Gorenstein Rings, Unpublished Manuscript, 1987.

[5] X.W. Chen, *Singularity categories, Schur functors and triangular matrix rings*, Algebr. Represent. Theor. **12** (2009), 181–191.

[6] X.W. Chen, *Relative singularity categories and Gorenstein-projective modules*, Math. Nathr. **284** no.2-3 (2011), 199–212.

[7] X.W. Chen, *The singularity category of an algebra with radical square zero*, arXiv:1104.4006v1.

[8] P. Gabriel and M. Zisman, Calculus of Fractions and Homotopy Theory, Springer-Verlag New York, Inc., New York, 1967.

[9] W. Geigle and H. Lenzing, *Perpendicular categories with applications to representations and sheaves*, J. Algebra **144** (1991), no. 2, 273–343.

[10] D. Happel, Triangulated Categories in the Representation Theory of Finite Dimensional Algebras, London Math. Soc., Lecture Notes Ser. **119**, Cambridge Univ. Press, Cambridge, 1988.

[11] D. Happel, *On Gorenstein algebras*, In: Progress in Math. **95**, Birkhäuser Verlag, Basel, 1991, 389–404.

[12] R. Hartshorne, Duality and Residue, Lecture Notes in Math. **20**, Springer, Berlin, 1966.

[13] B. Keller and D. Vossieck, *Sous les catégories dérivées,* C.R. Acad. Sci. Paris, t. **305** Série I (1987) 225–228.

[14] H. Krause, *The stable derived category of a noetherian scheme,* Compositio Math. **141** (2005), 1128–1162.

[15] S. Koenig and H. Nagase, *Hochschild cohomology and stratifying ideals*, J. Pure Appl. Algebra **213** (2009), 886–891.

[16] S. Mac Lane, Categories for the Working Mathematicians, Grad. Text in Math. **5**, Springer, 1998.

[17] D. Orlov, *Triangulated categories of singularities and D-branes in Landau-Ginzburg models*, Trudy Steklov Math. Institute **204** (2004), 240–262.

[18] D. Orlov, *Triangulated categories of singularities and equivalences between Landau-Ginzburg models*, Matem. Sbornik **197** (2006), 1827–1840.

[19] J.A. de la Pena and C.C. Xi, *Hochschild cohomology of algebras with homological ideals*, Tsukuba J. Math. **30** (1) (2006), 61–79.

[20] J. Rickard, *Derived categories and stable equivalence,* J. Pure Appl. Algebra **61** (1989), 303–317.

[21] S.P. Smith, *Equivalence of categories involving graded modules over path algebras of quivers*, arXiv:1107.3511.

[22] J.L. Verdier, *Categories deriveées*, in SGA 4 1/2, Lecture Notes in Math. **569**, Springer, Berlin, 1977.

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