TYING UP BARIC ALGEBRAS

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Abstract. Given two baric algebras (A_1, ω_1) and (A_2, ω_2) we describe a way to define a new baric algebra structure over the vector space $A_1 \oplus A_2$, which we shall denote $(A_1 \bowtie A_2, \omega_1 \bowtie \omega_2)$. We present some easy properties of this construction and we show that in the commutative and unital case it preserves indecomposability. Algebras of the form $A_1 \bowtie A_2$ in the associative, coutable-dimensional, zero-characteristic case are classified.

1. Introduction

Baric algebras play an important role in the theory of genetic algebras. The use of algebraic formalism to study genetic inheritance was introduced by I.M.H. Etherington [3] in the first half of the last century and has revealed fruitful giving rise to many interesting classes of algebras such as train or Bernstein. For a brief survey of this subject we refer to [5] and for an introductory but deeper approach to [6].

In [1] the notion of decomposable baric algebras was introduced. In the same paper it was also presented a way to construct decomposable baric algebras starting from two baric algebras with an idempotent of weight one. Furthermore in [1] and in [2] the authors analized the indecomposability of some well-known examples of algebras arising in genetics. In this work we define a new way to construct a baric algebra starting from two given baric algebras. Our construction, although similar, is different than that in [1]. In particular, while the construction in [1] always gives rise to decomposable baric algebras, we will show that in the commutative unital case our construction preserves indecomposability.

We will also show that baric algebras obtained by our method always have a unique weight homomorphism. Thus, as a consequence, we show that every baric algebra can be embedded in a baric algebra with a unique weight homomorphism.

The paper is organized as follows. The second section presents the construction and the third one gives some properties following easily frome the definition. In the fourth section we study the uniqueness of the weight homomorphism. In the fifth section we study the ideals and focus on the case when our original

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algebras are commutative and unital, showing that in this case our construction preserves indecomposability. Finally we study the associative case and give a classification when we are in countable dimension and the base field is of characteristic zero.

2. The construction

Let (A_1,ω_1) and (A_2,ω_2) be two baric algebras; i.e, A_1 and A_2 are algebras over a field K and $\omega_i : A_i \longrightarrow K$ is a non-zero K-algebra homomorphism for $i=1,2$. Now, in the K-vector space $A_1 \oplus A_2$ we define a product

$$(a_1,a_2)(b_1,b_2) = (a_1b_1 + \omega_2(b_2)a_1, a_2b_2 + \omega_1(b_1)a_2) \tag{1}$$

which is easily seen to define a K-algebra structure on $A_1 \oplus A_2$.

Definition 2.1. Given (A_i,ω_i) with $i=1,2$ two baric algebras we define $A_1 \bowtie A_2$ to be the K-vector space $A_1 \oplus A_2$ with the algebra structure given by the product (1).

We can now define an application $\omega_1 \bowtie \omega_2 : A_1 \oplus A_2 \longrightarrow K$ given by the formula $\omega_1 \bowtie \omega_2(a_1,a_2) = \omega_1(a_1) + \omega_2(a_2)$. Trivially $\omega_1 \bowtie \omega_2$ is K-linear and, also, we have that

$$\begin{aligned} \omega_1 \bowtie \omega_2((a_1,a_2)(b_1,b_2)) &= \omega_1 \bowtie \omega_2(a_1b_1 + \omega_2(b_2)a_1, a_2b_2 + \omega_1(b_1)a_2) \\ &= \omega_1(a_1b_1) + \omega_2(b_2)\omega_1(a_1) + \omega_2(a_2b_2) + \omega_1(b_1)\omega_2(a_2) \\ &= (\omega_1(a_1) + \omega_2(a_2))(\omega_1(b_1) + \omega_2(b_2)) \\ &= \omega_1 \bowtie \omega_2(a_1,a_2)\omega_1 \bowtie \omega_2(b_1,b_2). \end{aligned}$$

Thus, $\omega_1 \bowtie \omega_2$ is a K-homomorphism and we have the following:

Proposition 2.1. *Let (A_i,ω_i) with $i=1,2$ be baric algebras. Then so is the pair $(A_1 \bowtie A_2, \omega_1 \bowtie \omega_2)$.*

Remark. By means of the inclusions $\iota_i : A_i \hookrightarrow A_1 \bowtie A_2$ $(i=1,2)$ given by $\iota_i(a_i) = (\delta_i^1 a_i, \delta_i^2 a_i)$ we can see each A_i as a subalgebra of $A_1 \bowtie A_2$ and we will identify A_i with $\iota_i(A_i)$. With this identification it is easy to see that $A_i \trianglelefteq_r A_1 \bowtie A_2$.

Example 2.1. Let K be a field. Obviously (K, id_K) is a baric algebra, then $K \bowtie K$ is the vector space K^2 endowed with the product

$$(\alpha,\beta)(\alpha',\beta') = (\alpha' + \beta')(\alpha,\beta).$$

In this case we have

$$\mathrm{id}_K \bowtie \mathrm{id}_K(\alpha,\beta) = \alpha + \beta.$$

We will come back to this example later on.

3. Some easy properties

This section is devoted to present some properties arising easily from the previous construction.

We recall that two baric algebras (A, ω) and (B, φ) are said to be isomorphic if there exists a K-algebra isomorphism $f : A \longrightarrow B$ such that $\varphi \circ f = \omega$. The following propositions show some nice properties of this construction.

Proposition 3.1. *Let (A_i, ω_i) with $i = 1, 2, 3$ be baric algebras. Then we have the following isomorphisms:*

(i) $(A_1 \bowtie A_2, \omega_1 \bowtie \omega_2) \cong (A_2 \bowtie A_1, \omega_2 \bowtie \omega_1)$.
(ii) $((A_1 \bowtie A_2) \bowtie A_3, (\omega_1 \bowtie \omega_2) \bowtie \omega_3) \cong (A_1 \bowtie (A_2 \bowtie A_3), \omega_1 \bowtie (\omega_2 \bowtie \omega_3))$.

Proof. Define $f_1 : A_1 \bowtie A_2 \longrightarrow A_2 \bowtie A_1$ by $f(a_1, a_2) = (a_2, a_1)$, in the same way, define $f_2 : (A_1 \bowtie A_2) \bowtie A_3 \longrightarrow A_1 \bowtie (A_2 \bowtie A_3)$ by $f_2((a_1, a_2), a_3) = (a_1, (a_2, a_3))$. It is easy to see that both maps are weight-preserving K-isomorphisms. □

Proposition 3.2. *Let (A_1, ω_1), (A'_1, ω'_1) and (A_2, ω_2) be baric algebras and let us suppose that $(A_1, \omega_1) \cong (A'_1, \omega'_1)$, then $(A_1 \bowtie A_2, \omega_1 \bowtie \omega_2) \cong (A'_1 \bowtie A_2, \omega'_1 \bowtie \omega_2)$.*

Proof. We know by hypothesis that there exists an isomorphism $f : A_1 \longrightarrow A'_1$ such that $\omega'_1 \circ f = \omega_1$. We can define a map $\widetilde{f} : A_1 \bowtie A_2 \longrightarrow A'_1 \bowtie A_2$ in a natural way by $\widetilde{f}(a_1, a_2) = (f(a_1), a_2)$. This map is obviously a K-homomorphism and, moreover, $\omega'_1 \bowtie \omega_2(\widetilde{f}(a_1, a_2)) = \omega'_1(f(a_1)) + \omega_2(a_2) = \omega_1(a_1) + \omega_2(a_2) = \omega_1 \bowtie \omega_2(a_1, a_2)$ and this completes the proof. □

Given a K-algebra A and elements $x, y, z \in A$, the definitions of the commutator $[x, y] = xy - yx$ and of the associator $(x, y, z) = (xy)z - x(yz)$ are well known; A being commutative or associative if and only if $[x, y] = 0$ for all $x, y \in A$ or $(x, y, z) = 0$ for all $x, y, z \in A$ respectively.

Lemma 3.1. *Let (A_i, ω_i) for $i = 1, 2$ be baric algebras. Let $x = (a_1, a_2)$ and $y = (b_1, b_2)$ be elements of $A_1 \bowtie A_2$. Then we have that:*

$$[x, y] = ([a_1, b_1] + \omega_2(b_2)a_1 - \omega_2(a_2)b_1, [a_2, b_2] + \omega_1(b_1)a_2 - \omega_1(a_1)b_2).$$

Recall that the commutative center of an algebra A is the set

$$\mathcal{K}(A) = \{a \in A \mid [a, b] = 0 \ \forall b \in A\}$$

In view of the previous lemma, we have the following corollary.

Corollary 3.1. *If (A_i, ω_i) with $i = 1, 2$ are baric algebras, then $\mathcal{K}(A_1 \bowtie A_2) = 0$*

As usual it is interesting to search for idempotents. In the case of baric algebras we look for idempotents of weight 1. In this direction we have the following easy result.

Proposition 3.3. *Let* (A_i, ω_i) *for* $i = 1, 2$ *be baric* K*-algebras and let* $e_i \in A_i$ *be idempotents such that* $\omega_i(e_i) = 1$*. Consider the set* $\mathfrak{I} = \{(\lambda e_1, \mu e_2) \mid \lambda + \mu = 1\}$*, then* $ef = e$ *for all* $e, f \in \mathfrak{I}$ *and, in particular,* $\mathfrak{I}$ *consists of idempotents of weight 1.*

Proof. $(\lambda_1 e_1, \mu_1 e_2)(\lambda_2 e_1, \mu_2 e_2) = (\lambda_1(\lambda_2 + \mu_2)e_1, \mu_1(\lambda_2 + \mu_2)e_2)$ □

4. Uniqueness of the weight homomorphism

In a baric algebra, the weight homomorphism is not uniquely determined in general (see [4] for example). Nevertheless, the following result shows that our construction behaves quite nicely in this sense.

Proposition 4.1. *Let* (A_i, ω_i) *for* $i = 1, 2$ *be baric algebras. Then* $\omega_1 \bowtie \omega_2$ *is uniquely determined.*

Proof. Let us suppose that $\varphi : A_1 \bowtie A_2 \longrightarrow K$ is a non-trivial homomorphism of K-algebras. Then for $i = 1, 2$ we can define $\varphi_i : A_i \longrightarrow K$ by $\varphi_1(a_1) = \varphi(a_1, 0)$ and $\varphi_2(a_2) = \varphi(0, a_2)$. It is easy to check that both φ_i are K-homomorphisms and $\varphi(a_1, a_2) = \varphi_1(a_1) + \varphi_2(a_2)$.

Now, as φ is a K-homomorphism, $\varphi(a_1, a_2)\varphi(b_1, b_2) = \varphi\big((a_1, a_2)(b_1, b_2)\big)$ and thus, by the preceding considerations:

$$(\varphi_1(a_1) + \varphi_2(a_2))(\varphi_1(b_1) + \varphi_2(b_2)) = \varphi(a_1 b_1 + \omega_2(b_2)a_1, a_2 b_2 + \omega_1(b_1)a_2)$$

From this it follows that:

$$\varphi_1(a_1)(\omega_2(b_2) - \varphi_2(b_2)) = \varphi_2(a_2)(\varphi_1(b_1) - \omega_1(b_1)),\ \forall a_i, b_i \in A_i$$

So choosing $a_1 \in \operatorname{Ker} \varphi_1$ and $a_2 \notin \operatorname{Ker} \varphi_2$ we have that $\varphi_1 = \omega_1$. Similarly we obtain $\varphi_2 = \omega_2$ and the proof is complete. □

As a consequence of this result, together with the fact that A_i is a subalgebra of $A_1 \bowtie A_2$ we have the following corollary.

Corollary 4.1. *Every baric algebra is a subalgebra of a baric algebra with a unique weight homomorphism.*

In [6] it is shown that if a baric algebra (A, ω) is such that $\operatorname{Ker} \omega$ is nil, then the weight homomorphism is uniquely determined. Clearly our construction provides a family of examples showing that the converse is false.

5. Ideals and indecomposability.

Let (A_i, ω_i) with $i = 1, 2$ be baric algebras, then each A_i can be seen as a subalgebra of $A_1 \bowtie A_2$. Now let $I \trianglelefteq_r A_1$ be a right ideal. We can identify I with $\iota_1(I)$ and it is easy to see that with this identification $I \trianglelefteq_r A_1 \bowtie A_2$ remains a right ideal.

Now let $I \trianglelefteq_r A_1 \bowtie A_2$ be a right ideal. Then we can define

$$I_1 = \{a_1 \in A_1 \mid \exists\, a_2 \in A_2 \text{ s.t. } (a_1, a_2) \in I\}$$

Again, it is easy to see that $I_1 \trianglelefteq_r A_1$ is also a right ideal. Note that if we define the projections $p_i : A_1 \bowtie A_2 \longrightarrow A_i$ in the obvious way, I_1 is just $p_1(I)$. In the same way we can define I_2.

In view of the previous considerations, it is natural to ask whether an ideal of A_i remains an ideal of $A_1 \bowtie A_2$.

Proposition 5.1. *Let $I \trianglelefteq A_i$ be an ideal. Then I is an ideal of $A_1 \bowtie A_2$ if and only if $I \subseteq Ker\ \omega_i$.*

Proof. If $I \trianglelefteq A_1$ (the case $I \trianglelefteq A_2$ is analogous), then clearly $I \trianglelefteq_r A_1 \bowtie A_2$. Now if $x \in I$ and $a_i \in A_i$ for $i = 1, 2$ we have that $(a_1, a_2)(x, 0) = (a_1 x, \omega_1(x) a_2) \in I$ if and only if $\omega_1(x) a_2 = 0$ for all $a_2 \in A_2$. Obviously this happens if and only if $\omega_1(x) = 0$ and the proof is complete. □

While, on the other hand, we have the following:

Proposition 5.2. *Let $I \trianglelefteq A_1 \bowtie A_2$ be an ideal such that $I_1 \neq A_1$. Then I_1 is an ideal of A_1 if and only if $I_2 \subseteq Ker\ \omega_2$.*

Proof. Given $I \trianglelefteq A_1 \bowtie A_2$ we already know that $I_1 \trianglelefteq_r A_1$ is a right ideal. Let us suppose that $I_2 \subseteq \text{Ker}\ \omega_2$, then if $a_1 \in I_1$ and $a \in A_1$, there exists $a_2 \in A_2$ such that $(a_1, a_2) \in I$; so we have that $(a, 0)(a_1, a_2) = (aa_1, 0) \in I$ and this implies that $aa_1 \in I_1$ as desired.

Conversely, suppose that there exists $a_2 \in I_2$ such that $\omega_2(a_2) \neq 0$. By definition, there exists $a_1 \in A_1$ such that $(a_1, a_2) \in I$; in particular $a_1 \in I_1$ so given any $a \in A_1$ we have that $aa_1 \in I_1$. Moreover, $(a, 0)(a_1, a_2) = (aa_1 + \omega_2(a_2)a, 0) \in I$ so $aa_1 + \omega_2(a_2)a \in I_1$. Then we have that $\omega_2(a_2)a \in I_1$ and that $a \in I_1$. This implies $A_1 = I_1$, a contradiction. □

Remark. If $I \trianglelefteq A_1 \bowtie A_2$ is an ideal, $I \neq A_1 \bowtie A_2$ does not imply $I_1 \neq A_1$. To see this it is enough to consider the ideal $I = \text{Ker}\ \omega_1 \bowtie \omega_2$, in this case we have that $I_1 = A_1$ although $I \neq A_1 \bowtie A_2$.

Proposition 5.3. *Let (A_i, ω_i) for $i = 1, 2$ be commutative baric algebras and let $I \trianglelefteq A_1 \bowtie A_2$ be an ideal such that $I \subseteq Ker\ \omega_1 \bowtie \omega_2$. Then $I_1 = A_1$ if and only if $I = Ker\ \omega_1 \bowtie \omega_2$.*

Proof. Let us suppose $I_1 = A_1$ and choose $a \in A_1$ such that $\omega_1(a) \neq 0$. Then there exists $b \in A_2$ such that $(a,b) \in I$, note that in particular $\omega_1(a) = -\omega_2(b) \neq 0$.

Now take $(a_1,a_2) \in \text{Ker}\ \omega_1 \bowtie \omega_2$, i.e., $\omega_1(a_1)+\omega_2(a_2) = 0$. Being $I \trianglelefteq A_1 \bowtie A_2$ and due to the commutativity of each A_i we have:

$$(a,b)(a_1,0) - (a_1,0)(a,b) = (-\omega_2(b)a_1, \omega_1(a_1)b) \in I$$

$$(a,b)(0,a_2) - (0,a_2)(a,b) = (\omega_2(a_2)a, -\omega_1(a)a_2) \in I$$

and there are two possible cases:

Firstly, if $\omega_1(a_1) = -\omega_2(a_2) = 0$, then we have

$$(a_1,a_2) = -(\omega_2(b))^{-1}(-\omega_2(b)a_1,0) - (\omega_1(a))^{-1}(0,-\omega_1(a)a_2) \in I$$

and secondly, if $\omega_1(a_1) = -\omega_2(a_2) \neq 0$, then

$$(a_1,a_2) = (\omega_2(b))^{-1}\Big((\omega_2(a_2)a, -\omega_1(a)a_2) - (-\omega_2(b)a_1, \omega_1(a_1)b) + \omega_1(a_1)(a,b)\Big) \in I$$

Thus, in both cases $(a_1,a_2) \in I$ and the equality holds. The converse was discussed in the previous remark. □

Definition 5.1. Let (A,ω) be a baric algebra. We define the set $\mathcal{I}(A,\omega)$ to be:

$$\mathcal{I}(A,\omega) = \{I \trianglelefteq A \mid I \subseteq \text{Ker}\ \omega\}$$

Proposition 5.4. *Let (A_i,ω_i) for $i = 1,2$ be commutative unital baric algebras. Then the sets $\mathcal{I}(A_1,\omega_1) \times \mathcal{I}(A_2,\omega_2)$ and $\mathcal{I}(A_1 \bowtie A_2, \omega_1 \bowtie \omega_2) \setminus \{Ker\ \omega_1 \bowtie \omega_2\}$ are bijective.*

Proof. Let us define maps

$$\varphi : \mathcal{I}(A_1,\omega_1) \times \mathcal{I}(A_2,\omega_2) \longrightarrow \mathcal{I}(A_1 \bowtie A_2, \omega_1 \bowtie \omega_2) \setminus \{\text{Ker}\ \omega_1 \bowtie \omega_2\}$$

and

$$\psi : \mathcal{I}(A_1 \bowtie A_2, \omega_1 \bowtie \omega_2) \setminus \{\text{Ker}\ \omega_1 \bowtie \omega_2\} \longrightarrow \mathcal{I}(A_1,\omega_1) \times \mathcal{I}(A_2,\omega_2)$$

by $\varphi(I,J) = I \bowtie J = \{(a,b) \in A_1 \bowtie A_2 \mid a \in I,\ b \in J\}$ and $\psi(I) = (I_1,I_2)$.

Proposition 5.1 implies that φ is well-defined. In the same way Propositions 5.2 and 5.3 imply that ψ is well-defined. Thus, it is enough to see that φ and ψ are each other's inverse.

First, let $I_i \in \mathcal{I}(A_i,\omega_i)$. Then, obviously $(I_1 \bowtie I_2)_i = I_i$ and this shows that $\psi\varphi(I_1,I_2) = (I_1,I_2)$.

On the other hand, let $I \in \mathcal{I}(A_1 \bowtie A_2) \setminus \{\text{Ker}\ \omega_1 \bowtie \omega_2\}$. Clearly $I \subseteq I_1 \bowtie I_2$. Conversely, let $(a,b) \in I_1 \bowtie I_2$. By definition there exists $b' \in A_2$ such that $(a,b') \in I$. Since $w_1(a) = 0$ it follows that $w_2(b') = 0$ and, since I is an ideal we have that $(a,0) = (1,0)(a,b') \in I$. In the same way $(0,b) \in I$ and we have that $I \subseteq I_1 \bowtie I_2$; i.e., that $\varphi\psi(I) = I$ and the result follows. □

Example 5.1. Let K be any field. We construct $(K \bowtie K,\ \mathrm{id}_K \bowtie \mathrm{id}_K)$ like in Example 2.1. Then a direct application of the previous proposition gives us the simplicity of Ker $\mathrm{id}_K \bowtie \mathrm{id}_K$.

In [1] the notion of decomposable baric algebra was introduced. Namely, a baric algebra (A, ω) with an idempotent of weight 1 is decomposable if there are non-trivial ideals N_1 and N_2 of A, both contained in Ker ω and such that Ker $\omega = N_1 \oplus N_2$. Otherwise (A, ω) is indecomposable.

The following result shows that our construction works nicely with respect to indecomposability in the commutative case.

Proposition 5.5. *Let (A_i, ω_i) be commutative unital indecomposable baric algebras for $i = 1, 2$. Then $(A_1 \bowtie A_2, \omega_1 \bowtie \omega_2)$ is also indecomposable.*

Proof. Assume that $(A_1 \bowtie A_2, \omega_1 \bowtie \omega_2)$ is decomposable. Then there exist ideals S, T such that Ker $\omega_1 \bowtie \omega_2 = S \oplus T$. Since both S, T are non-trivial we can apply Proposition 5.4 to get that $S = (S_1, S_2)$ and $T = (T_1, T_2)$ with $S_i, T_i \trianglelefteq A_i$.

Clearly $S_i + T_i \subseteq$ Ker ω_i. Now, if $x \in S_i \cap T_i$ it follows that $(x, 0) \in S \cap T = 0$ so S_i and T_i have direct sum. Moreover, since Ker $\omega_i \trianglelefteq A_1 \bowtie A_2$, Ker $\omega_1 \cap T =$ Ker $\omega_2 \cap S = 0$ it follows that Ker $\omega_i = S_i \oplus T_i$.

If $S_1 = 0$ then it must be $S_2 \neq 0$. Moreover, Ker $\omega_1 \subseteq T_1$ and if it was $T_2 = 0$ it follows that Ker $\omega_2 \subseteq S_2$ and Ker $\omega_1 \oplus$ Ker $\omega_2 =$ Ker $\omega_1 \bowtie \omega_2$ which is false by Proposition 5.4 again. Consequently we have proved that if $S_1 = 0$, then $S_2, T_2 \neq 0$ and (A_2, ω_2) is decomposable.

In the same way it follows that $T_1 = 0$ implies that (A_2, ω_2) is decomposable.

If both S_1 and T_1 are non-zero, then (A_1, ω_1) is decomposable and the result follows. □

6. Associativity

We will start this section with the following lemma:

Lemma 6.1. *Let (A_i, ω_i) for $i = 1, 2$ be baric algebras. Let $x = (a_1, a_2)$, $y = (b_1, b_2)$ and $z = (c_1, c_2)$ be elements of $A_1 \bowtie A_2$. Then we have that:*

$$(x, y, z) = ((a_1, b_1, c_1) + \omega_2(b_2)(a_1c_1 - \omega_1(c_1)a_1), (a_2, b_2, c_2) + \omega_1(b_1)(a_2c_2 - \omega_2(c_2)a_2)).$$

We can use this to prove the following characterization:

Proposition 6.1. *Let (A_i, ω_i) with $i = 1, 2$ be baric algebras. Then the algebra $A_1 \bowtie A_2$ is associative if and only if $(a_1, a_2)(b_1, b_2) = (\omega_1 \bowtie \omega_2(b_1, b_2))(a_1, a_2)$ for all $(a_1, a_2), (b_1, b_2) \in A_1 \bowtie A_2$.*

Proof. Put $x = (a_1, a_2)$, $y = (b_1, b_2)$ and $z = (c_1, c_2)$. Let us suppose that $A_1 \bowtie A_2$ is associative. Then each A_i is also associative because they are subalgebras of $A_1 \bowtie A_2$. So, by Lemma 6.1:

$$0 = (x, y, z) = (\omega_2(b_2)(a_1c_1 - \omega_1(c_1)a_1), \omega_1(b_1)(a_2c_2 - \omega_2(c_2)a_2)).$$

and choosing $b_i \notin \text{Ker } \omega_i$ we have that $a_ic_i = \omega_i(c_i)a_i$ for all $a_i, c_i \in A_i$. Thus we have that $(a_1, a_2)(b_1, b_2)(= \omega_1 \bowtie \omega_2(b_1, b_2))(a_1, a_2)$ and the proof is complete as the converse is just an easy computation. □

A K-algebra A is called left (resp. right) alternative if $(x, x, y) = 0$ for all $x, y \in A$ (resp. $(x, y, y) = 0$ for all $x, y \in A$). We say that A is alternative if it is both left and right alternative. Of course an associative algebra is left and right alternative. As an easy consequence of Lemma 6.1 we have:

Proposition 6.2. *Let (A_i, ω_i) with $i = 1, 2$ be associative baric algebras. Then the following are equivalent:*

(i) *$A_1 \bowtie A_2$ is associative.*
(ii) *$A_1 \bowtie A_2$ is left alternative.*
(iii) *$A_1 \bowtie A_2$ is right alternative.*

Example 6.1. Let K be any field. Thanks to Proposition 3.1(ii) and recalling Example 2.1 , we can unambiguously define the baric algebra $(K^{\bowtie n}, \text{id}_K^{\bowtie n})$, where $K^{\bowtie n}$ stands for $K \bowtie \ldots \bowtie K$ and $\text{id}_K^{\bowtie n} = \text{id}_K \bowtie \ldots \bowtie \text{id}_K$ is defined by the formula $\text{id}_K^{\bowtie n}(\alpha_1, \ldots, \alpha_n) = \alpha_1 + \cdots + \alpha_n$. Then, due to Proposition 6.1, $K^{\bowtie n}$ is associative.

The remaining of this section will be devoted to show that, under certain assumptions, the previous example is the only situation in which our construction is associative.

Let (A, ω) be a baric algebra over a field K and let us choose $\{e_i \mid i \in I\}$ any K-basis for A. Put $\epsilon_i = \omega(e_i)$ for all $i \in I$ and observe that we can suppose, without loss of generality, that $\epsilon_i \in \{0, 1\}$ for all $i \in I$. Moreover we have:

Lemma 6.2. *Let K be a field with char $K = 0$ and let (A, ω) be a baric K-algebra of countable dimension. Then A admits a basis such that every element in the basis is of weight 1.*

Proof. Let $\{e_i \mid i \in I\}$ with $|I| \leq \aleph_0$ be a K-basis of A. We can suppose that $I \subseteq \mathbb{N}$ and that $\epsilon_1 = 1$. Now for each $n \in I$ we define $e'_n = \dfrac{1}{\sum_{j \leq n} \epsilon_j} \displaystyle\sum_{j \leq n} e_j$. Then $\{e'_i \mid i \in I\}$ is the desired basis. □

Proposition 6.3. *Let K be a field with char K=0 and let (A, ω) be a countable-dimensional baric K-algebra such that $xy = \omega(y)x$ for all $x, y \in A$. Then, if $\nu = \dim_K A$, we have $(A, \omega) \cong (K^{\bowtie \nu}, id_K^{\bowtie \nu})$ as baric algebras.*

Proof. We consider the K-basis of A $\{e_i \mid i \in I\}$ with $\nu = |I| \leq \aleph_0$ and $\omega(e_i) = 1$ for all $i \in I$ given by the previous lemma. We define $(A_i, \omega_i) = (Ke_i, \omega|_{Ke_i})$. Obviously $(A, \omega) = (\bowtie_{i=1}^{\nu} A_i, \bowtie_{i=1}^{\nu} \omega_i)$ and the proof is complete as $(Ke_i, \omega|_{Ke_i}) \cong (K, \mathrm{id}_K)$ trivially. □

Finally, as a consequence of this proposition we obtain the following:

Corollary 6.1. *Let (A_i, ω_i) for $i = 1, 2$ be countable-dimensional baric K-algebras with char $K = 0$. Then $A_1 \bowtie A_2$ is associative if and only if $(A_i, \omega_i) \cong (K^{\bowtie\nu_i}, id_K^{\bowtie\nu_i})$ with $\nu_i = dim_K\ A_i$. In particular, $(A_1 \bowtie A_2, \omega_1 \bowtie \omega_2) \cong (K^{\bowtie\nu}, id_K^{\bowtie\nu})$ with $\nu = \nu_1 + \nu_2$.*

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