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ON PROLONGATIONS OF CONTACT MANIFOLDS

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Abstract. In this note, we apply spectral sequences to both derive an obstruction to the existence of n-fold prolongations and to derive a topological classification. Prolongations were formerly used in an attempt to prove that every Engel structure on $M \times \mathbb{S}^1$ with characteristic line field tangent to the fibers is determined by both the contact structure induced on a cross section and the twisting of the Engel structure along the fibers. Our results show that the former results need some modification: to determine the Engel structure we have to additionaly fix a class in the first cohomology of M.

1. Introduction

The current note goals to present a discussion of prolongations of contact manifolds. For an introduction to the notions we point the reader to [7, §2.2] and [1, §1.2] (or to [8] for Engel structures). Given a 3-dimensional contact manifold (M, ξ), we define its prolongation $\mathbb{P}(\xi)$ as the $\mathbb{S}^1$-bundle over M we obtain from projectivizing the contact planes ξ (cf. §4 or see [7, §2.2] and [1, §1.2]). In [1], Adachi discusses prolongations of contact manifolds and introduces a notion of an *n-fold prolongation* which he defines as a *fiberwise n-fold covering* of $\mathbb{P}(\xi)$. This notion is then employed in an attempt to prove that Engel structures $\mathcal{D}$ on 4-manifolds $M \times \mathbb{S}^1$, where M is a closed, oriented 3-manifold and $\mathcal{D}$ is an Engel structure with characteristic foliation tangent to the $\mathbb{S}^1$-fibers, are determined by the contact structure induced on a cross section of $M \times \mathbb{S}^1$ and the *twisting* of $\mathcal{D}$ along the $\mathbb{S}^1$-fibers (see [1, Theorem 1(2)]). His introduction of the n-fold prolongation and the statement in his Theorem 1(2) in [1] suggest that n-fold prolongations of contact manifolds always exist and are unique. Additionally, the proof of Theorem 1(2) in [1] seems to essentially rest on a lifting argument which however does not work, generally (cf. §2).

By applying methods from spin geometry, more precisely from the characterization of spin structures, we see that there exists a topological obstruction to the existence of n-fold prolongations. This obstruction is non-zero, mostly, showing that n-fold prolongations do often not exist (see §5). On the other hand, we are able to show a general existence result of 2-fold and 4-fold prolongations (cf. Proposition 5.1). Along these lines we notice the following phenomenon: although there is a unique connected, n-fold covering of the circle $\mathbb{S}^1$, $\varphi_n \colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ say, there might be plenty of n-fold coverings of the $\mathbb{S}^1$-bundle $\mathbb{P}(\xi)$ that look like φ_n, fiberwise (see example in §2). A similar subtlety arises in the characterization of spin structures on vector bundles (cf. [5, Theorem 1.7]). The results of this article are based on the following observation.

Theorem 1.1. *Given an $\mathbb{S}^1$-bundle $P \to M$ a fiberwise n-fold covering of M exists if and only if the mod-n reduction $e_n(E)$ of the Euler class $e(E)$ is zero. In case of existence, the isomorphism classes of fiberwise n-fold coverings of M stay in one-to-one correspondence with elements of $H^1(M;\mathbb{Z}_n)$.*

In fact, the one-to-one correspondence results from a free and transitive $H^1(M;\mathbb{Z}_n)$-action on the set of fiberwise n-fold coverings. With this in place, we are able to show that every Engel structure $\mathcal{D}$ on $M \times \mathbb{S}^1$ whose characteristic foliation is tangent to the fibers is determined up to isotopy by a set of data associated to $\mathcal{D}$, which are specified in the following Theorem 1.2. This is a modified (corrected) version of [1, Theorem 1(2)].

Theorem 1.2. *Suppose we are given an oriented Engel structure $\mathcal{D}$ on $M \times \mathbb{S}^1$ whose characteristic line field is tangent to the $\mathbb{S}^1$-fibers. Denote by ξ the contact structure on the base given by $\xi = \pi_*([\mathcal{D},\mathcal{D}])$ where π is the projection of $M \times \mathbb{S}^1$ onto M. Let $n \in \mathbb{N}$ denote the twisting number of $\mathcal{D}$ and α is a suitable class in $H^1(M;\mathbb{Z}_n)$ associated to $\mathcal{D}$. Then $\mathcal{D}$ is determined up to isotopy by the set of data (ξ, n, α).*

It is possible to give explicit descriptions of model Engel structures $\mathcal{D}_n^\alpha(\xi)$ for every set of data (ξ, n, α). In fact, using appropriate *twistings* of a model Engel structure $\mathcal{D}_n^0(\xi)$ on $M \times \mathbb{S}^1$ along a set of generators for $H_1(M;\mathbb{Z})$ it is possible to give models for $\mathcal{D}_n^\alpha(\xi)$. These descriptions are not necessary for our purposes, so we omit a general discussion. However, the considerations presented in §2 may be thought of as a model example to which we point the interested reader.

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2. An Introducing Example

In this section we discuss an introducing example of prolongations and fiberwise coverings. For a brief introduction of the relation between these two notions we point the reader to §4. What is done in the remainder of this article can be explicitly checked in terms of this example which we leave to the interested reader.

Let $M = T^3$ be the 3-dimensional torus with coordinates (x, y, z) and consider the trivial $\mathbb{S}^1$-bundle $P = T^3 \times \mathbb{S}^1$. Given some vector $\boldsymbol{\alpha} \in \mathbb{Z}^3 \cap [0, n-1]^3$ we define a fiberwise n-fold covering $\phi_{\boldsymbol{\alpha}} \colon T^3 \times \mathbb{S}^1 \to T^3 \times \mathbb{S}^1$ by setting

$$\phi_{\boldsymbol{\alpha}}(\boldsymbol{p}, \theta) = \big(\boldsymbol{p}, n\,\theta + \langle \boldsymbol{\alpha}, \boldsymbol{p}\rangle\,\big),$$

where we implicitly used the identification $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. When restricting to fibers of the $\mathbb{S}^1$-bundle $T^3 \times \mathbb{S}^1$ the map $\phi_{\boldsymbol{\alpha}}$ looks like the unique (connected) non-trivial n-fold covering of $\mathbb{S}^1$. Observe, that $\boldsymbol{\alpha}$ determines a cohomology class in $H^1(T^3;\mathbb{Z}_n)$ which we also denote by $\boldsymbol{\alpha}$: By the universal coefficient theorem we know that the group $H^1(T^3;\mathbb{Z}_n)$ is isomorphic to $\mathrm{Hom}(H_1(T^3;\mathbb{Z});\mathbb{Z}_n)$ whose every element – by fixing the *standard* generators of $H_1(T^3;\mathbb{Z})$ – is uniquely determined by a vector in $\mathbb{Z}^3 \cap [0, n-1]^3$. For $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$ with $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}'$ the coverings $\phi_{\boldsymbol{\alpha}}$ and $\phi_{\boldsymbol{\alpha}'}$ are not equivalent: Suppose they are equivalent, then there exists a covering isomorphism ψ such that $\phi_{\boldsymbol{\alpha}} \circ \psi = \phi_{\boldsymbol{\alpha}'}$. The morphism ψ_* induced on the fundamental group of $T^3 \times \mathbb{S}^1$ is given by the matrix

$$\psi_* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & 1 \end{pmatrix}$$

for suitable integers $a, b, c \in \mathbb{Z}$ and we have that $(\phi_{\boldsymbol{\alpha}})_* \psi_* = (\phi_{\boldsymbol{\alpha}'})_*$. This implies the equality $\boldsymbol{\alpha}' = \boldsymbol{\alpha} + n \cdot (a, b, c)$ which means that $\boldsymbol{\alpha}' = \boldsymbol{\alpha}$ considering the fact that they are both vectors in $\mathbb{Z}^3 \cap [0, n-1]^3$.

Conversely, given a fiberwise n-fold covering $\phi\colon\ T^3 \times \mathbb{S}^1 \to T^3 \times \mathbb{S}^1$ we associate to it a cohomology class $\boldsymbol{\alpha}$ in $H^1(T^3;\mathbb{Z}_n)$ as follows: As outlined above we have to assign an element in $\mathbb{Z}_n$ to each of the *standard* generators of $H_1(T^3;\mathbb{Z})$. Take such a *standard* generator, c say, and lift it to an embedded circle $\tilde{c} \hookrightarrow T^3 \times \mathbb{S}^1$ which is unique up to multiplication (in the sense of homotopy groups) with a $\mathbb{S}^1$-fiber. Write p_2 for the projection of $T^3 \times \mathbb{S}^1$ to the $\mathbb{S}^1$-factor, then the composition of p_2 with ϕ and $\tilde{c}$ gives rise to a map $\mathbb{S}^1 \to \mathbb{S}^1$ whose degree is denoted by $k \in \mathbb{Z}$. We define

$$\boldsymbol{\alpha}(c) = [k],$$

where $[k] \in \mathbb{Z}_n$ denotes the mod-n reduction of k. Note that, since ϕ is a fiberwise n-fold covering, multiplication of $\tilde{c}$ with a $\mathbb{S}^1$-fiber increases the degree by n. Thus $\boldsymbol{\alpha}(c)$ is independent of the choice of the lift $\tilde{c}$ and, hence, well-defined.

2.1. **Non-equivalent n-fold Prolongations of (T^3, ξ).** Choose ξ to be the unique Stein fillable contact structure on T^3, i.e. ξ is defined by the kernel of the contact 1-form $\sin(2\pi\, z)\, dx + \cos(2\pi\, z)\, dy$. Hence, the contact planes are spanned by ∂_z and

$$V_{\boldsymbol{p}} = \cos(2\pi\, z)\, \partial_x + \sin(2\pi\, z)\, \partial_y$$

where $\boldsymbol{p} = (x, y, z)$. The prolongation $\mathbb{P}(\xi)$ can be naturally identified with the 4-dimensional torus $T^3 \times \mathbb{S}^1$ and the corresponding Engel structure $\mathcal{D}(\xi)$ is spanned by the tangent vectors ∂_θ and

$$\cos(\pi\, \theta)\, \partial_z + \sin(\pi\, \theta)\, V_{\boldsymbol{p}}.$$

Using the fiberwise n-fold covering $\phi_{\boldsymbol{\alpha}}$ defined above we can pull back the Engel structure $\mathcal{D}(\xi)$ to obtain a new Engel structure $\mathcal{D}^n_{\boldsymbol{\alpha}}(\xi)$. At the point $(\boldsymbol{p},\theta)$ the Engel plane $\mathcal{D}^n_{\boldsymbol{\alpha}}(\xi)_{(\boldsymbol{p},\theta)}$ is spanned by the tangent vectors ∂_θ and

$$\cos\big(\pi\big(n\,\theta+\langle\boldsymbol{\alpha},\boldsymbol{p}\rangle\,\big)\big)\,\partial_z+\sin\big(\pi\big(n\,\theta+\langle\boldsymbol{\alpha},\boldsymbol{p}\rangle\,\big)\big)\,V_{\boldsymbol{p}}.$$

The Engel manifolds $(T^3\times\mathbb{S}^1,\mathcal{D}^n_{\boldsymbol{\alpha}}(\xi))$ are non-equivalent pairwise if we consider isotopies through Engel structures with characteristic foliation tangent to the $\mathbb{S}^1$-fibers. These *special* isotopies for instance appear in the proof of Theorem 1.2. If considering general isotopies through Engel structures, then it is not clear which of them are isotopic and which are not.

3. Characterization of Fiberwise n-fold Coverings

Let us denote by $\varphi_n\colon\ \mathbb{S}^1\to\mathbb{S}^1$ the connected n-fold covering of the unit circle $\mathbb{S}^1$, where $n\in\mathbb{N}$ is some positive integer. Suppose we are given a $\mathbb{S}^1$-bundle $P\to M$ over a closed, oriented manifold M. We define a **fiberwise n-fold covering of P** as a pair (Q,ϕ) where Q is a $\mathbb{S}^1$-bundle over M and ϕ is a smooth map $Q\to P$ such that its restriction $\phi|_{Q_x}$ for every $x\in M$ is a map $Q_x\to P_x$ which corresponds to the n-fold covering map $\varphi_n\colon\ \mathbb{S}^1\to\mathbb{S}^1$. Before we move our focus to the characterization of fiberwise n-fold coverings we show that their existence is tied to the following condition on the Euler classes of the bundles.

Lemma 3.1. *Let $\pi_Q\colon\ Q\to M$ and $\pi_P\colon\ P\to M$ be two principal $\mathbb{S}^1$-bundles such that there is a bundle map $\phi\colon\ Q\to P$ whose restriction to the fibers corresponds to the map φ_n, then $e(P)=n\cdot e(Q)$.*

Proof. Given vector fields R_Q and R_P, whose flows induce the $\mathbb{S}^1$-actions on the circle bundles Q and P. Let further α_P be a connection 1-form on P. Hence, we have that $\mathcal{L}_{R_P}\alpha_P=0$ and $\alpha_P(R_P)=1$ (cf. [3, Definition 7.2.3]). Further, denote by ω_P the curvature form on M. Then, $(\pi_P)^*\omega_P=d\alpha_P$ and according to [3, p. 340] we have

$$e(P)=-\Big[\frac{\omega_P}{2\pi}\Big].$$

Defining α_Q as $1/n\cdot\phi^*(\alpha_P)$ it is easy to verify that it is a connection 1-form on Q. Denote by ω_Q the associated curvature form. Then we have

$$(\pi_Q)^*\omega_Q=d\alpha_Q=1/n\cdot\phi^*(\pi_P)^*\omega_P=(\pi_Q)^*(1/n\cdot\omega_P).$$

By the injectivity of π_Q^* (cf. Proposition 3.2) we see that $\omega_Q=1/n\cdot\omega_P$ which implies

$$e(Q)=-\Big[\frac{\omega_P}{2\pi}\Big]=-\Big[\frac{\omega_Q}{2\pi n}\Big]=1/n\cdot e(P).$$

□

From the relationship of the Euler classes presented in Lemma 3.1 we see that the mod-n reduction of the Euler class is an obstruction to the existence of fiberwise n-fold coverings, i.e. the existence of fiberwise n-fold coverings imply the vanishing of the mod-n reduction of the Euler class. Even more, in Theorem 1.1 we will see that the vanishing of the mod-n reduction is equivalent to the existence of fiberwise n-fold coverings. As in the characterization of spin structures used in [5, §1] (see especially Sequences (1.2) and (1.4) of [5]), we find it opportune to work with Čech cohomology.

Proposition 3.2. *A $\mathbb{S}^1$-bundle $\mathbb{S}^1 \longrightarrow P \longrightarrow M$ induces the following long exact sequence*

$$0 \longrightarrow H^1(M;\mathbb{Z}_n) \xrightarrow{\pi^*} H^1(P;\mathbb{Z}_n) \xrightarrow{\iota^*} H^1(\mathbb{S}^1;\mathbb{Z}_n) \xrightarrow{d} H^2(M;\mathbb{Z}_n)$$

where the map d sends the generator of $H^1(\mathbb{S}^1;\mathbb{Z}_n)$ to the mod n reduction of the Euler class $e(P)$.

To give a bit of explanation, observe, that an n-fold covering of the space P corresponds to an element in $H^1(P;\mathbb{Z}_n)$ (cf. [5, Appendix A]). A fiberwise n-fold covering (Q,ϕ) is an ordinary n-fold covering and, thus, we may think of the pair $[(Q,\phi)]$ (or simply $[Q]$) as an element in the first cohomology of P. The statement that it is φ_n fiberwise is equivalent to saying that the pullback bundle $\iota^*(Q,\phi)$ is isomorphic to φ_n. In terms of the exact sequence presented in Proposition 3.2, this amounts to saying that $\iota^*[Q] = [\varphi_n]$ where $[\varphi_n]$ is a generator of $H^1(\mathbb{S}^1;\mathbb{Z}_n)$ (cf. [5, §1] and [5, Appendix A]).

Proof of Theorem 1.1. By Proposition 3.2, the following sequence is exact

$$0 \longrightarrow H^1(M;\mathbb{Z}_n) \xrightarrow{\pi^*} H^1(P;\mathbb{Z}_n) \xrightarrow{\iota^*} H^1(\mathbb{S}^1;\mathbb{Z}_n) \xrightarrow{d} H^2(M;\mathbb{Z}_n).$$

In the following, the mod-n-reduction of the Euler class $e(P)$ will be denoted by e_n. Suppose that a fiberwise n-fold covering Q exists. Then, $[Q]$ is an element of $H^1(P;\mathbb{Z}_n)$ such that $\iota^*[Q] = [\varphi_n]$, because $Q \longrightarrow P$ is the n-fold covering φ_n, fiberwise. Thus, by exactness of the sequence,

$$0 = d(\iota^*[Q]) = d[\varphi_n] = e_n.$$

Conversely, assuming that $e_n = 0$ we have that $d[\varphi_n] = 0$. By exactness, this implies the existence of an element $q \in H^1(P;\mathbb{Z}_n)$ which is mapped to $[\varphi_n]$ under ι^*. But, q corresponds to a fiberwise n-fold covering of P. The isomorphism classes of fiberwise coverings correspond to the set $(\iota^*)^{-1}([\varphi_n])$ on which π^* – by the sequence above – induces a free and transitive $H^1(M;\mathbb{Z}_n)$-action. Hence, we obtain a one-to-one correspondence between $H^1(M;\mathbb{Z}_n)$ and the isomorphism classes of fiberwise coverings. □

It remains to prove Proposition 3.2. We just sketch the proof, since it is analogous to the proofs of the exact sequences used in the characterization of spin structures (see [5, §1]).

Sketch of Proof of Proposition 3.2. We look at the Leray-Serre spectral sequence with E_2-page given by $E_2^{p,q} = H^p(M; H^q(\mathbb{S}^1; \mathbb{Z}_n))$ (see [6]). By applying the fact that $H^q(\mathbb{S}^1; \mathbb{Z}_n)$ is non-zero for $q = 0, 1$ only, we see that $E_\infty^{1,0} = E_2^{1,0} = H^1(M; \mathbb{Z}_n)$ and that $E_\infty^{0,1} = E_3^{0,1}$. Thus, we obtain the following exact sequence

$$0 \longrightarrow H^1(M;\mathbb{Z}_n) \longrightarrow H^1(P;\mathbb{Z}_n) \longrightarrow E_2^{0,1} \xrightarrow{d_2^{0,1}} H^2(M;\mathbb{Z}_n).$$

Continuing with a careful analysis of the E_1-page, we see that $E_2^{0,1}$ equals $H^1(\mathbb{S}^1; \mathbb{Z}_n)$ and we obtain the exact sequence as proposed. With a discussion of $d_2^{0,1}$ as done similarly for the spin case (see [4] and cf. [5, §1]) it is possible to prove that $d_2^{0,1}$ sends the generator of $H^1(\mathbb{S}^1; \mathbb{Z}_n)$ to the mod-n reduction of the Euler class $e(P)$. □

4. Engel Structures with trivial Characteristic Line Field

An **Engel structure** is a maximally non-integrable 2-plane distribution $\mathcal{D}$ on a 4-dimensional manifold Q, i.e. $\mathcal{D}$ is defined as a 2-plane bundle for which $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$ is of rank 3 and $[\mathcal{E}, \mathcal{E}]$ of rank 4. Inside the Engel structure $\mathcal{D}$ there is a line field $\mathcal{L}$ given by the condition that $[\mathcal{L}, \mathcal{E}] \subset \mathcal{E}$. This line field is called the **characteristic line field** and its induced foliation the **characteristic foliation** of $\mathcal{D}$. Engel structures arise in a natural way as prolongations of contact 3-manifolds. That is, given a contact 3-manifold (M, ξ) one can consider the bundle $\mathbb{P}\xi$ whose fibers are the projectivizations of the contact planes, i.e. for every $p \in M$ a point $q \in (\mathbb{P}\xi)_p$ accords a line $l \subset \xi_p$ in the contact plane. Note that by construction this 4-manifold carries the structure of a $\mathbb{S}^1$-bundle $\rho : \mathbb{P}\xi \to M$ over M. Furthermore, we obtain a natural plane distribution $\mathcal{D}\xi \subset T\,\mathbb{P}\xi$ given by

$$(\mathcal{D}\xi)_q = T_q\rho^{-1}(l).$$

This distribution defines an Engel structure whose characteristic line field is tangent to the $\mathbb{S}^1$-fibers of the bundle.

Now, assume we are given an oriented $\mathbb{S}^1$-bundle $\pi : Q \to M$ over some 3-manifold M carrying an Engel structure $\mathcal{D} \subset TQ$ with characteristic line field $\mathcal{L}$ tangent to the fibers. Since the induced distribution $[\mathcal{D}, \mathcal{D}] \subset TQ$ is preserved by any flow tangent to $\mathcal{L}$ we obtain a well defined contact structure $\xi = \pi_*([\mathcal{D}, \mathcal{D}])$ on M. Furthermore, one obtains the **development map**

$$\phi_{\mathcal{D}}\colon\ (Q, \mathcal{D}) \longrightarrow (\mathbb{P}\xi, \mathcal{D}\xi)$$

by assigning to a point $q \in Q$ the element $\phi_{\mathcal{D}}(q)$ in $(\mathbb{P}(\xi))_{\pi(q)}$ which corresponds to the 1-dimensional subspace $T_q\pi(\mathcal{D}_q)$ of the contact plane $\xi_{\pi(q)}$. Note, that $\phi_{\mathcal{D}}(\mathcal{D}) = \mathcal{D}\xi$

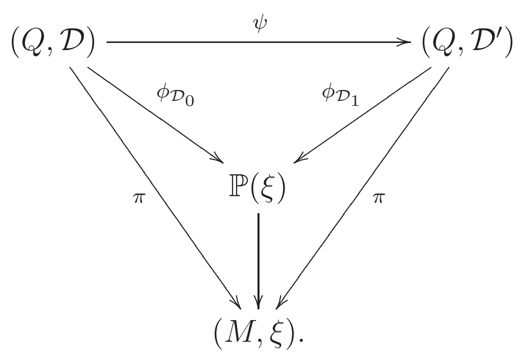
and that $\phi_{\mathcal{D}}$ defines a fiberwise n-fold covering of $\mathbb{P}(\xi)$, where $n \in \mathbb{N}$ denotes the degree of the development map restricted to a fiber. In contrast to the definition of Adachi, we will refer to such $(Q, \mathcal{D})$ as an **n-fold prolongation of** (M, ξ).

Conversely, given a fiberwise n-fold covering $\phi\colon Q \to \mathbb{P}(\xi)$, we obtain an Engel structure $\mathcal{D} = (T\phi)^{-1}\big(\mathcal{D}(\xi)\big)$ on Q whose associated development map $\phi_{\mathcal{D}}$ equals ϕ. Thus, according to the classification of fiberwise n-fold coverings presented in Theorem 1.1 we have proved the following statement.

Corollary 4.1. *A contact manifold (M, ξ) admits an n-fold prolongation if and only if the mod-n reduction $e_n(\mathbb{P}(\xi))$ of the Euler class $e(\mathbb{P}(\xi))$ vanishes. The isomorphism classes of n-fold prolongations stay in one-to-one correspondence with elements in $H^1(M; \mathbb{Z}_n)$.* □

Now we have everything ready to prove a modified version of [1, Theorem 1(2)].

Proof of Theorem 1.2. Let $\mathcal{D}_0$ and $\mathcal{D}_1$ be two Engel structures on Q inducing the same set of data (ξ, n, α). According to our classification of fiberwise n-fold coverings of $\mathbb{P}(\xi)$ the n-fold prolongations $(Q, \mathcal{D}_0)$ and $(Q, \mathcal{D}_1)$ are both equivalent, i.e. there is an isomorphism ψ which makes the following diagram commutative



Observe, that ψ is also an isomorphism of $\mathbb{S}^1$-bundles over the identity of M. Hence, assuming that Q is the trivial $\mathbb{S}^1$-bundle $M \times \mathbb{S}^1$, the bundle map ψ is isotopic to the identity, showing that $\mathcal{D}_0 \simeq \mathcal{D}_1$. □

5. On Euler Classes of Prolongations

In §3 we derived a characterization of fiberwise n-fold coverings and we have seen that the mod-n reduction of the Euler class determines their existence (see Theorem 1.1). In §4 we have seen that every n-fold prolongation $(Q, \mathcal{D})$ of a contact manifold (M, ξ) naturally carries the structure of a fiberwise n-fold covering of the prolongation $\mathbb{P}(\xi)$ via the development map $\phi_{\mathcal{D}}$ (see Corollary 4.1). Thus, the existence of n-fold prolongations of contact manifolds is connected to the vanishing of the mod-n reduction of the Euler class $e(\mathbb{P}(\xi))$. Since this represents a special situation it is natural to ask whether or not prolongations exist, always.

Proposition 5.1. *Every 3-dimensional contact manifold (M,ξ) admits both 2-fold prolongations and 4-fold prolongations.*

Proof. Suppose we are given an oriented contact manifold (M,ξ). Choose a Riemannian metric on M and a trivialization of the tangent bundle TM. Then the 2-plane field ξ can be described in terms of its corresponding Gauß map $f_\xi\colon M\to S^2$, assigning to each $x\in M$ the positive normal vector to $\xi(x)$. Consider the tangent bundle TS^2 over the 2-sphere. In fact, $f_\xi^*TS^2=\xi$. Combining naturality of the Euler class under pullback and the fact that $e(TS^2)=2\cdot u_0$, where u_0 is the positive generator of $H^2(\mathbb{S}^2;\mathbb{Z})$, we conclude that

$$e(\xi)=2\cdot f_\xi^*u_0.$$

Naturally, the unit-sphere bundle ξ_1 of ξ is a fiberwise 2-fold covering of the prolongation $\mathbb{P}(\xi)$ such that Lemma 3.1 implies $e(\mathbb{P}(\xi))=2\cdot e(\xi_1)$. Combining this with the equality above and the fact that $e(\xi)=e(\xi_1)$ we have

$$e(\mathbb{P}(\xi))=4\cdot f_\xi^*u_0.$$

And, hence, both the mod-4 reduction $e_4(\xi)$ and the mod-2 reduction $e_2(\xi)$ of the Euler class $e(\mathbb{P}(\xi))$ vanishes. By Theorem 1.1 the result follows. □

As a consequence of the last proof we, additionally, see that n-fold prolongations of contact manifolds (M,ξ) do often not exist: Recall that the class $f_\xi^*u_0$ classifies the homotopy class of ξ over the 2-skeleton of M (cf. [3, §4.2]). Since in every homotopy class of 2-plane fields there is an overtwisted contact structure it is easy to find examples for which $e(\mathbb{P}(\xi))=4\cdot f_\xi^*u_0$ is non-zero when reduced modulo n.

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