0-Schur 代数

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摘要：利用 Beilinson, Lusztig 和 MacPherson 关于量子 Schur 代数的一个几何构造以及 0-Hecke 代数的结构，我们给出了 0-Schur 代数的一个表现并确定了它们的表示型。

关键词：0-Schur 代数, 表示型, 表现

中图分类号： O153.3, O154.2

On 0-Schur algebras

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Abstract: Based on a geometric construction of quantum Schur algebras due to Beilinson, Lusztig and MacPherson and the structure of 0-Hecke algebras, we give a presentation for 0-Schur algebras and determine their representation type.

Key words: 0-Schur algebra, representation type, presentation

0 Introduction

Quantum Schur algebras (or q-Schur algebras) were studied independently by Jimbo [1] and Dipper and James [2]. This class of algebras plays a central role in linking the representations of quantum general linear groups, quantum enveloping algebras of type A and Hecke algebras of symmetric groups; see for example [3, 4, 5]. This provides a q-analogue of the classical theory relating representation theories of Schur algebras, general linear groups and symmetric groups; see a thorough treatment in [6]. The structure and representation theory of quantum Schur algebras have been widely studied in the literature; see [5, 7] and the references given there. Recently, the representation type of quantum Schur algebras was completely determined in [8], and a presentation for quantum Schur algebras was given in [9, 10].

It is known that the classical Schur algebras are the degeneration of quantum Schur algebras at $q = 1$. Analogously, by considering their degeneration at $q = 0$, we obtain the so-called 0-Schur algebras which have been studied by Donkin [5, §2.2] in terms of 0-Hecke algebras of symmetric groups, as well as by Krob–Thibon [11] in connection with noncommutative

基金项目： 国家自然科学基金项目 (10731070)，高等学校博士学科点专项科研基金资助项目 (2007030061)

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symmetric functions. Also, Su [12] has defined generic multiplication in certain subalgebras of 0-Schur algebras and related them with the degenerate Ringel–Hall algebras.

The present paper is devoted to the study of the structure and representation type of 0-Schur algebras. We first give a presentation for 0-Schur algebras based on a geometric construction of quantum Schur algebras due to Beilinson, Lusztig and MacPherson [13] and a presentation for the degenerate Ringel–Hall algebras of linear quivers given in [14, 15]. We then determine the representation type of 0-Schur algebras by using the structure and the representation theory of 0-Hecke algebras developed in [16, 17, 18, 19] and some techniques in the representation theory of algebras.

1 Quantum Schur algebras

In this section we recall the definition of quantum Schur algebras $S_{\boldsymbol{q}}(n,r)$ due to Dipper–James [2] and also review the geometric construction of $S_{\boldsymbol{q}}(n,r)$ given by Beilinson–Lusztig–MacPherson [13]. We then apply multiplication formulas in [13, Lem. 3.2] to obtain certain relations in $S_{\boldsymbol{q}}(n,r)$. Finally, we introduce the notion of Ringel–Hall algebras defined by Ringel [20].

Let $\mathfrak{S} = \mathfrak{S}_r$ denote the symmetric group on r letters with generating set $\{s_i = (i, i+1) \mid i \in I\}$, where $I = \{1, 2, \ldots, r-1\}$. Let $\mathscr{A} = \mathbb{Z}[\boldsymbol{q}]$ be the polynomial ring with indeterminate $\boldsymbol{q}$. By definition, the Hecke algebra $H_{\boldsymbol{q}}(r) = H_{\boldsymbol{q}}(\mathfrak{S})$ of $\mathfrak{S}$ is the $\mathscr{A}$-algebra with generators T_i, for $i \in I$, and relations

$$\begin{cases} T_i^2 = (\boldsymbol{q}-1)T_i + \boldsymbol{q}, & \text{for } i \in I; \\ T_iT_j = T_jT_i, & \text{for } i, j \in I \text{ with } |i-j| > 1; \\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, & \text{for } 1 \leqslant i < r-1. \end{cases}$$

If $w = s_{i_1} \cdots s_{i_t} = s_{j_1} \cdots s_{j_t}$ are two reduced expressions of $w \in \mathfrak{S}$, then $T_{i_1} \cdots T_{i_t} = T_{j_1} \cdots T_{j_t}$. Thus, the element $T_w := T_{i_1} \cdots T_{i_t}$ is well defined. It is well known that $H_{\boldsymbol{q}}(r)$ is a free $\mathscr{A}$-module with basis $\{T_w \mid w \in \mathfrak{S}\}$.

Fix a positive integer n and let $\Lambda(n,r)$ be the set of compositions of r into n parts. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda(n,r)$, define for $1 \leqslant i \leqslant n$,

$$R_i^\lambda = \{x \mid \lambda_1 + \cdots + \lambda_{i-1} + 1 \leqslant x \leqslant \lambda_1 + \cdots + \lambda_i\},$$

where $\lambda_0 = 0$. If $\lambda_i = 0$, put $R_i^\lambda := \emptyset$ by convention. In this way, we get a decomposition

$$\{1, 2, \ldots, r\} = R_1^\lambda \cup R_2^\lambda \cup \cdots \cup R_n^\lambda$$

of $\{1, 2, \ldots, r\}$ into a disjoint union of subsets. The subgroup

$$\mathfrak{S}_\lambda := \{w \in \mathfrak{S} \mid wR_i^\lambda = R_i^\lambda, 1 \leqslant i \leqslant n\}$$

is called a Young subgroup of $\mathfrak{S}$ defined by the composition λ. We then define

$$x_\lambda = \sum_{w\in\mathfrak{S}_\lambda} T_w \in H_{\boldsymbol{q}}(r)$$

which satisfies (see for example [7, Lem. 7.32])

$$x_\lambda T_i = \boldsymbol{q} x_\lambda \text{ for each } i \in I \text{ with } s_i \in \mathfrak{S}_\lambda. \tag{1.0.1}$$

Following Dipper and James [2], the endomorphism algebra

$$S_{\boldsymbol{q}}(n,r) := \mathrm{End}_{H_{\boldsymbol{q}}(r)}\Big(\bigoplus_{\lambda\in\Lambda(n,r)} x_\lambda H_{\boldsymbol{q}}(r)\Big)$$

is called the (integral) quantum Schur algebra of bidegree (n,r) over $\mathscr{A}$. For $\lambda,\mu\in\Lambda(n,r)$ and $w\in\mathfrak{S}$, define $\phi^w_{\lambda,\mu}\in S_{\boldsymbol{q}}(n,r)$ by

$$\phi^w_{\lambda,\mu} : \bigoplus_{\nu\in\Lambda(n,r)} x_\nu H_{\boldsymbol{q}}(r) \longrightarrow \bigoplus_{\nu\in\Lambda(n,r)} x_\nu H_{\boldsymbol{q}}(r), \quad x_\nu h \longmapsto \delta_{\mu,\nu} T_{\mathfrak{S}_\lambda w \mathfrak{S}_\mu} h,$$

where $T_{\mathfrak{S}_\lambda w\mathfrak{S}_\mu} = \sum_{x\in\mathfrak{S}_\lambda w\mathfrak{S}_\mu} T_x$.

We now recall the geometric construction of quantum Schur algebras given by Beilinson–Lusztig–MacPherson [13]. Let $\mathbb{F}$ be a field and let V be an $\mathbb{F}$-vector space of dimension r. Let $\mathfrak{F}=\mathfrak{F}(n,V)$ be the set of n-step flags

$$V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V.$$

The group $G=\mathrm{GL}(V)$ acts naturally on $\mathfrak{F}$. This induces a diagonal action of G on $\mathfrak{F}\times\mathfrak{F}$ defined by $g(\mathfrak{f},\mathfrak{f}')=(g\mathfrak{f},g\mathfrak{f}')$, where $g\in G$ and $\mathfrak{f},\mathfrak{f}'\in\mathfrak{F}$.

Let $\Xi(n,r)$ denote the set of matrices $A=(a_{i,j})\in\mathbb{N}^{n\times n}$ with $a_{i,j}$ nonnegative integers and $\sum_{1\leqslant i,j\leqslant n} a_{i,j}=r$. Then there is a bijection from $\mathfrak{F}\times\mathfrak{F}/G$ to $\Xi(n,r)$ sending the orbit of $(\mathfrak{f},\mathfrak{f}')$ to $A=(a_{i,j})$ with

$$a_{i,j} = \dim_{\mathbb{F}} \frac{V_i\cap V'_j}{V_{i-1}\cap V'_j + V_i\cap V'_{j-1}} \text{ for } 1\leqslant i,j\leqslant n, \tag{1.0.2}$$

where $\mathfrak{f}=(V_1\subseteq V_2\subseteq\cdots\subseteq V_n=V)$, $\mathfrak{f}'=(V'_1\subseteq V'_2\subseteq\cdots\subseteq V'_n=V)$ and $V_0=V'_0=0$ by convention.

For $A\in\Xi(n,r)$, we denote by $\mathcal{O}_A$ the orbit in $\mathfrak{F}\times\mathfrak{F}$ corresponding to A. For each matrix $A=(a_{i,j})\in\mathbb{N}^{n\times n}$, define

$$\mathrm{row}(A) = (\sum_{j=1}^n a_{1,j},\ldots,\sum_{j=1}^n a_{n,j})\in\mathbb{N}^n \text{ and } \mathrm{col}(A)=(\sum_{i=1}^n a_{i,1},\ldots,\sum_{i=1}^n a_{i,n})\in\mathbb{N}^n.$$

If $\mathbb{F}=\mathbb{F}_q$ is a finite field of q elements. For $A,B,C\in\Xi(n,r)$, fix a representative $(\mathfrak{f}',\mathfrak{f}'')\in\mathcal{O}_C$ and put

$$c_{A,B,C;q} = |\,\{\,\mathfrak{f}\in\mathfrak{F}\mid(\mathfrak{f}',\mathfrak{f})\in\mathcal{O}_A, (\mathfrak{f},\mathfrak{f}'')\in\mathcal{O}_B\,\}\,|.$$

Clearly, $c_{A,B,C;q}$ is independent of the choice of $(\mathfrak{f}', \mathfrak{f}'')$, and a necessary condition for $c_{A,B,C;q} \neq 0$ is that

$$\mathrm{row}(A) = \mathrm{row}(C),\ \mathrm{col}(A) = \mathrm{row}(B)\ \text{ and }\ \mathrm{col}(B) = \mathrm{col}(C). \tag{1.0.3}$$

Following [13, Prop. 1.2], for any given $A, B, C \in \Xi(n,r)$, there is a polynomial $g_{A,B,C}(\boldsymbol{q}) \in \mathscr{A} = \mathbb{Z}[\boldsymbol{q}]$ such that for all prime powers $q \neq 1$, the equality $g_{A,B,C}(q) = c_{A,B,C;q}$ holds.

Definition 1.1 ([13])**.** Let $S'_{\boldsymbol{q}}(n,r)$ be the free $\mathbb{Z}[\boldsymbol{q}]$-module with basis $\{\zeta_A \mid A \in \Xi(n,r)\}$ and with multiplication given by

$$\zeta_A\zeta_B = \sum_{C\in\Xi(n,r)} g_{A,B,C}(\boldsymbol{q})\zeta_C, \quad \text{for all } A, B \in \Xi(n,r).$$

Then $S'_{\boldsymbol{q}}(n,r)$ is an associative algebra with identity described as follows. For each $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda(n,r)$, let $\mathrm{diag}(\lambda)$ denote the diagonal matrix $\mathrm{diag}(\lambda_1, \dots, \lambda_n)$ and write $\zeta_\lambda = \zeta_{\mathrm{diag}(\lambda)}$. By definition, for each $A \in \Xi(n,r)$,

$$\zeta_\lambda\zeta_A = \begin{cases} \zeta_A, & \text{if } \lambda = \mathrm{row}(A); \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \zeta_A\zeta_\lambda = \begin{cases} \zeta_A, & \text{if } \lambda = \mathrm{col}(A); \\ 0, & \text{otherwise.} \end{cases} \tag{1.1.1}$$

Thus, $\sum_{\lambda\in\Lambda(n,r)} \zeta_{\mathrm{diag}(\lambda)}$ is the identity of $S'_{\boldsymbol{q}}(n,r)$.

For each $A \in \Xi(n,r)$, let $\lambda = \mathrm{row}(A)$ and $\mu = \mathrm{col}(A)$ and choose $w_A \in \mathfrak{S}$ such that for $1 \leqslant i, j \leqslant n$,

$$a_{i,j} = |R_i^\lambda \cap (w_A R_j^\mu)|.$$

By [21] (see [7] for the details), the correspondence

$$\zeta_A \longmapsto \phi^{d_A}_{\mathrm{row}(A),\mathrm{col}(A)} \ \ (A \in \Xi(n,r)) \tag{1.1.2}$$

induces an algebra isomorphism $S'_{\boldsymbol{q}}(n,r) \to S_{\boldsymbol{q}}(n,r)$, where d_A is the shortest element in the double coset $\mathfrak{S}_{\mathrm{row}(A)} w_A \mathfrak{S}_{\mathrm{col}(A)}$. In what follows, we will identify $S'_{\boldsymbol{q}}(n,r)$ with $S_{\boldsymbol{q}}(n,r)$ under the isomorphism above.

For each $d \geqslant 1$, we define in $\mathscr{A}$:

$$[\![d]\!]^! = [\![1]\!][\![2]\!]\cdots[\![d]\!] \ \text{ with } \ [\![s]\!] = \frac{\boldsymbol{q}^s - 1}{\boldsymbol{q} - 1},$$

and set $[\![0]\!]^! = 1$ by convention.

Let $\Xi(n)^\pm$ be the set of all matrices $A = (a_{i,j}) \in \mathbb{N}^{n\times n}$ such that $a_{i,i} = 0$ for all $1 \leqslant i \leqslant n$ and let $\Xi(n, \leqslant r)^\pm$ be the subset of matrices $A = (a_{i,j}) \in \Xi(n)^\pm$ satisfying $|A| = \sum_{1\leqslant i,j\leqslant n} a_{i,j} \leqslant r$.

Given $A \in \Xi(n, \leqslant r)^\pm$ and $\mathbf{j} = (j_1, \dots, j_n) \in \mathbb{N}^n$, define

$$\zeta_{A;\mathbf{j}} := \sum_{\lambda\in\mathbb{N}^n,\ A+\lambda\in\Xi(n,r)} \boldsymbol{q}^{\lambda\cdot\mathbf{j}}\zeta_{A+\lambda} \in S_{\boldsymbol{q}}(n,r),$$

where $A+\lambda := A+\mathrm{diag}(\lambda)$ and $\lambda\cdot\mathbf{j} = \lambda_1 j_1+\cdots+\lambda_n j_n$. Also, for any $A=(a_{i,j})\in M_n(\mathbb{Z})$, define $\zeta_{A;\mathbf{j}}=0$ if $a_{i,j}<0$ for some $i\neq j$, or $|A|>r$. Let O denote the $n\times n$ zero matrix and $E_{i,j}$ denote the matrix with (i,j)-entry 1 and all other entries 0. Let $\mathbf{0}=(0,\ldots,0)\in\mathbb{N}^n, \varepsilon_i=(0,\ldots,0,1,0,\ldots,0)\in\mathbb{N}^n$ (with 1 in the ith position) for $1\leqslant i\leqslant n$.

By [13, Lem. 3.2] and a direct calculation, we obtain the following extended multiplication formulas in $S_{\boldsymbol{q}}(n,r)$.

Lemma 1.2. *For $1\leqslant h<n$, $\mathbf{j},\mathbf{j}'\in\mathbb{N}^n$, and $A=(a_{k,l})\in\Xi(n,\leqslant r)^{\pm}$, the following equalities hold in $S_{\boldsymbol{q}}(n,r)$,*

(1) $\zeta_{O;\mathbf{j}}\cdot\zeta_{A;\mathbf{j}'}=\boldsymbol{q}^{\mathrm{row}(A)\cdot\mathbf{j}}\zeta_{A;\mathbf{j}+\mathbf{j}'}$;

(2) $\zeta_{A;\mathbf{j}'}\cdot\zeta_{O;\mathbf{j}}=\boldsymbol{q}^{\mathrm{col}(A)\cdot\mathbf{j}}\zeta_{A;\mathbf{j}+\mathbf{j}'}$;

(3)
$$\begin{aligned}\zeta_{E_{h,h+1};\mathbf{0}}\cdot\zeta_{A;\mathbf{j}} =& \textstyle\sum_{i<h;a_{h+1,i}\geqslant 1}\boldsymbol{q}^{\sum_{j>i}a_{h,j}}[[a_{h,i}+1]]\zeta_{A+E_{h,i}-E_{h+1,i};\mathbf{j}+\varepsilon_h}\\ &+\sum_{i>h+1;a_{h+1,i}\geqslant 1}\boldsymbol{q}^{\sum_{j>i}a_{h,j}}\zeta_{A+E_{h,i}-E_{h+1,i};\mathbf{j}}\\ &+\frac{\boldsymbol{q}^{(\sum_{j>h}a_{h,j}-\mathbf{j}_h)}}{\boldsymbol{q}-1}(\zeta_{A-E_{h+1,h};\mathbf{j}+\varepsilon_h}-\zeta_{A-E_{h+1,h};\mathbf{j}})\\ &+\boldsymbol{q}^{(\sum_{j>h+1}a_{h,j}+\mathbf{j}_{(h+1)})}[[a_{h,h+1}+1]]\,\zeta_{A+E_{h,h+1};\mathbf{j}}\ ;\end{aligned}$$

(4)
$$\begin{aligned}\zeta_{E_{h+1,h};\mathbf{0}}\cdot\zeta_{A;\mathbf{j}} =& \textstyle\sum_{i<h;a_{h,i}\geqslant 1}\boldsymbol{q}^{\sum_{j<i}a_{h+1,j}}[[a_{h+1,i}+1]]\,\zeta_{A-E_{h,i}+E_{h+1,i};\mathbf{j}}\\ &+\sum_{i>h+1;a_{h,i}\geqslant 1}\boldsymbol{q}^{\sum_{j<i}a_{h+1,j}}[[a_{h+1,i}+1]]\,\zeta_{A-E_{h,i}+E_{h+1,i};\mathbf{j}+\varepsilon_{h+1}}\\ &+\frac{\boldsymbol{q}^{(\sum_{j<h+1}a_{h+1,j}-\mathbf{j}_{(h+1)})}}{\boldsymbol{q}-1}(\zeta_{A-E_{h,h+1};\mathbf{j}+\varepsilon_{h+1}}-\zeta_{A-E_{h,h+1};\mathbf{j}})\\ &+\boldsymbol{q}^{(\sum_{j<h}a_{h+1,j}+\mathbf{j}_h)}[[a_{h+1,h}+1]]\,\zeta_{A+E_{h+1,h};\mathbf{j}}\ .\end{aligned}$$

If $a_{h+1,h}=0$ (resp., $a_{h,h+1}=0$), then $A-E_{h+1,h}$ (resp., $A-E_{h,h+1}$) has a negative entry. Hence, the third term in formula (3) (resp., (4)) is zero in this case.

For each $1\leqslant i<n$, define the elements

$$\mathfrak{e}_i=\zeta_{E_{i,i+1};\mathbf{0}}\ \text{ and }\ \mathfrak{f}_i=\zeta_{E_{i+1,i};\mathbf{0}} \tag{1.2.1}$$

in $S_{\boldsymbol{q}}(n,r)$. From Lemma 1.2 we deduce the following result.

Proposition 1.3. *The following relations hold in $S_{\boldsymbol{q}}(n,r)$ for $\lambda,\mu\in\Lambda(n,r)$ and $1\leqslant i,j<n$:*

(S1) $\zeta_\lambda\zeta_\mu=\delta_{\lambda,\mu}\zeta_\lambda,\ 1=\sum_{\lambda\in\Lambda(n,r)}\zeta_\lambda$,

(S2) $\mathfrak{e}_i\zeta_\lambda=\zeta_{\lambda+\varepsilon_i-\varepsilon_{i+1}}\mathfrak{e}_i$, *if* $\lambda_{i+1}\geqslant 1$, $\mathfrak{e}_i\zeta_\lambda=0=\zeta_\lambda\mathfrak{f}_i$ *if* $\lambda_{i+1}=0$, (S3) $\mathfrak{f}_i\zeta_\lambda = \zeta_{\lambda-\varepsilon_i+\varepsilon_{i+1}}\mathfrak{f}_i$ *if* $\lambda_i \geqslant 1$, $\mathfrak{f}_i\zeta_\lambda = 0 = \zeta_\lambda\mathfrak{e}_i$ *if* $\lambda_i = 0$,

(S4) $(\boldsymbol{q}-1)(\mathfrak{e}_i\mathfrak{f}_j - \mathfrak{f}_j\mathfrak{e}_i) = \delta_{i,j}\sum_{\lambda\in\Lambda(n,r)}(\boldsymbol{q}^{\lambda_i} - \boldsymbol{q}^{\lambda_{i+1}})\zeta_\lambda$,

(S5) $\mathfrak{e}_i\mathfrak{e}_j = \mathfrak{e}_j\mathfrak{e}_i$, $\mathfrak{f}_i\mathfrak{f}_j = \mathfrak{f}_j\mathfrak{f}_i$ $(|i-j|>1)$,

(S6) $\mathfrak{e}_i^2\mathfrak{e}_{i+1} - (\boldsymbol{q}+1)\mathfrak{e}_i\mathfrak{e}_{i+1}\mathfrak{e}_i + \boldsymbol{q}\mathfrak{e}_{i+1}\mathfrak{e}_i^2 = 0$,

(S7) $\mathfrak{e}_i\mathfrak{e}_{i+1}^2 - (\boldsymbol{q}+1)\mathfrak{e}_{i+1}\mathfrak{e}_i\mathfrak{e}_{i+1} + \boldsymbol{q}\mathfrak{e}_{i+1}^2\mathfrak{e}_i = 0$,

(S8) $\boldsymbol{q}\mathfrak{f}_i^2\mathfrak{f}_{i+1} - (\boldsymbol{q}+1)\mathfrak{f}_i\mathfrak{f}_{i+1}\mathfrak{f}_i + \mathfrak{f}_{i+1}\mathfrak{f}_i^2 = 0$,

(S9) $\boldsymbol{q}\mathfrak{f}_i\mathfrak{f}_{i+1}^2 - (\boldsymbol{q}+1)\mathfrak{f}_{i+1}\mathfrak{f}_i\mathfrak{f}_{i+1} + \mathfrak{f}_{i+1}^2\mathfrak{f}_i = 0$.

Remarks 1.4. (1) The relations (S5)–(S9) are the so-called fundamental relations appeared in Ringel–Hall algebras; see [20]. Indeed, (S1)–(S9) are the generating relations for the quantum Schur algebra $S_{\boldsymbol{q}}(n,r)\otimes_{\mathbb{Z}[\boldsymbol{q}]}\mathbb{Q}(\boldsymbol{q})$; see [22].

(2) Let $\boldsymbol{v}$ be an indeterminate satisfying $\boldsymbol{v}^2 = \boldsymbol{q}$ and let $\mathbb{Q}(\boldsymbol{v})$ be the field of rational functions in $\boldsymbol{v}$. In [9, 10] (see also [7, Ch. 13]), a presentation for the quantum Schur algebra $\mathbf{S}_{\boldsymbol{v}}(n,r) := S_{\boldsymbol{q}}(n,r)\otimes\mathbb{Q}(\boldsymbol{v})$ is given. However, the generators given there satisfy the quantum Serre relations, while the $\mathfrak{e}_i, \mathfrak{f}_i$ defined in (1.2.1) satisfy the fundamental relations. But, in order to obtain the quantum Serre relations in Ringel–Hall algebras, we have to twist the multiplication; see [23].

In the following we introduced the Ringel–Hall algebra of the linear quiver

Q:

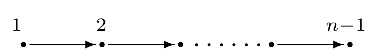

FIG.1. *Linear quiver with* $n-1$ *vertices*

Let $\mathbb{F}$ be a field. It is well known that for each $1\leqslant i<j\leqslant n$, there is a unique (up to isomorphism) indecomposable representation $M_{i,j}$ of Q over $\mathbb{F}$ whose dimension vector $\underline{\dim}M_{i,j}$ is $\alpha_i+\cdots+\alpha_{j-1}$, where $\alpha_1,\ldots,\alpha_{n-1}$ denote the standard basis of $\mathbb{Z}^{n-1}$. In particular, the $S_i := M_{i,i+1}$ $(1\leqslant i<n)$ are all simple representations of Q.

Let $\Xi(n)^+$ denote the set of all strictly upper triangular matrices in $\mathbb{N}^{n\times n}$. To each $A = (a_{i,j})\in\Xi(n)^+$ we can attach a representation of Q by setting

$$M(A) = M_{\mathbb{F}}(A) = \bigoplus_{i,j} a_{i,j}M_{i,j}.$$

By the Krull–Schmidt Theorem, the correspondence $A\mapsto M(A)$ induces a bijection from $\Xi(n)^+$ to the set of isoclasses of finite dimensional representations of Q over $\mathbb{F}$. Following [20], for $A,B,C\in\Xi(n)^+$, there exists $\varphi^A_{B,C}(\mathbf{q})\in\mathscr{A}=\mathbb{Z}[\mathbf{q}]$ (called the Hall polynomial) such that $\varphi^A_{B,C}(q) = F^{M_{\mathbb{F}}(A)}_{M_{\mathbb{F}}(B),M_{\mathbb{F}}(C)}$ for each finite field $\mathbb{F}$ with q elements, where $F^{M_{\mathbb{F}}(A)}_{M_{\mathbb{F}}(B),M_{\mathbb{F}}(C)}$ is the number of submodules X of $M_{\mathbb{F}}(A)$ such that $X \cong M_{\mathbb{F}}(C)$ and $M_{\mathbb{F}}(A)/X \cong M_{\mathbb{F}}(B)$. The (generic untwisted) Ringel–Hall algebra $\mathfrak{H}_{\boldsymbol{q}}(Q)$ of Q is by definition the free $\mathscr{A}$-module with basis $\{u_A \mid A \in \Xi(n)^+\}$ and with multiplication given by

$$u_A u_B = \sum_{C \in \Xi(n)^+} \varphi^C_{A,B}(\boldsymbol{q}) u_C \ \text{ for } A, B \in \Xi(n)^+.$$

We sometimes write $u_{[M(A)]}$ for u_A in order to make calculations in terms of representations of Q. In particular, we write $u_i = u_{[S_i]}$ for $1 \leqslant i < n$. It is known from [20] that the u_i satisfy the following fundamental relations ($1 \leqslant i, j < n$):

(H1) $u_i u_j = u_j u_i, \ \ (|i-j| > 1)$,

(H2) $u_i^2 u_{i+1} - (\boldsymbol{q}+1) u_i u_{i+1} u_i + \boldsymbol{q} u_{i+1} u_i^2 = 0$,

(H3) $\boldsymbol{q} u_{i+1}^2 u_i - (\boldsymbol{q}+1) u_{i+1} u_i u_{i+1} + u_i u_{i+1}^2 = 0$.

Let $\mathscr{R}$ be a commutative ring with identity and take an element $q \in \mathscr{R}$. By viewing $\mathscr{R}$ as an $\mathscr{A}$-module with the action of $\boldsymbol{q}$ the multiplication by q, we obtain $\mathscr{R}$-algebras

$$H_q(r)_{\mathscr{R}} := H_{\boldsymbol{q}}(r) \otimes_{\mathscr{A}} \mathscr{R} \ \text{ and } \ S_q(n,r)_{\mathscr{R}} = S_{\boldsymbol{q}}(n,r) \otimes_{\mathscr{A}} \mathscr{R}.$$

Moreover, by [7, Lem. 9.4], there is an $\mathscr{R}$-algebra isomorphism

$$S_q(n,r)_{\mathscr{R}} \cong \mathrm{End}_{H_q(r)_{\mathscr{R}}}\Big(\bigoplus_{\lambda \in \Lambda(n,r)} x_\lambda H_q(r)_{\mathscr{R}}\Big).$$

Similarly, we can consider the Ringel–Hall algebra of Q over $\mathscr{R}$

$$\mathfrak{H}_q(Q)_{\mathscr{R}} = \mathfrak{H}_{\boldsymbol{q}}(Q) \otimes_{\mathscr{A}} \mathscr{R}.$$

In the present paper we are mainly interested in the case where $\mathscr{R}$ is the ring $\mathbb{Z}$ of integers or a field $\mathbb{F}$ with $q = 0$.

2 A presentation for 0-Schur algebras

In this section we show that by specializing at $\boldsymbol{q} = 0$, the relations (S1)–(S9) become the defining relations for the 0-Schur algebra. The proofs are modified from those in [10]; see also [7, Ch. 13]. Thus, some of them are omitted.

As defined in the previous section, $S_{\boldsymbol{q}}(n,r)$ is the quantum Schur algebra over $\mathscr{A} = \mathbb{Z}[\boldsymbol{q}]$ with basis $\{\zeta_A \mid A \in \Xi(n,r)\}$. Put

$$S_0(n,r) := S_0(n,r)_{\mathbb{Z}} = S_{\boldsymbol{q}}(n,r) \otimes_{\mathscr{A}} \mathbb{Z},$$

called the 0-Schur algebra over $\mathbb{Z}$. In other words, $S_0(n,r)$ is the free $\mathbb{Z}$-module with basis $\{\zeta_A = \zeta_A \otimes 1 \mid A \in \Xi(n,r)\}$, and the multiplication is defined by

$$\zeta_A\zeta_B = \sum_{C\in\Xi(n,r)} g_{A,B,C}(0)\zeta_C \text{ for all } A, B \in \Xi(n,r).$$

Given a polynomial $f(\boldsymbol{q})$ in $\mathscr{A}$ and an integer $a \in \mathbb{Z}$, we write $f(\boldsymbol{q})_a$ for $f(a)$. In particular, $([\![d]\!]^!)_0 = 1 = [\![d]\!]_0$ for each $d \geqslant 1$.

By letting $\boldsymbol{q} = 0$, we obtain the elements

$$\zeta_{A;\mathbf{j}} = \sum_{\substack{\lambda\in\mathbb{N}^n,\ \lambda\cdot\mathbf{j}=0\\ A+\lambda\in\Xi(n,r)}} \zeta_{A+\lambda} \text{ for } A \in \Xi(n, \leqslant r)^{\pm} \text{ and } \mathbf{j} \in \mathbb{N}^n.$$

in $S_0(n,r)$. In particular, we have

$$\mathfrak{e}_i = \sum_{\lambda\in\Lambda(n,r-1)} \zeta_{E_{i,i+1}+\lambda} \text{ and } \mathfrak{f}_i = \sum_{\lambda\in\Lambda(n,r-1)} \zeta_{E_{i+1,i}+\lambda}$$

in $S_0(n,r)$ for $1 \leqslant i < n$. Proposition 1.3 gives the following consequence.

Lemma 2.1. *The elements $\mathfrak{e}_i, \mathfrak{f}_i, \zeta_\lambda$ $(1 \leqslant i < n$ and $\lambda \in \Lambda(n,r))$ in $S_0(n,r)$ satisfy the following relations:*

(DS1) $\zeta_\lambda\zeta_\mu = \delta_{\lambda,\mu}\zeta_\lambda,\ 1 = \sum_{\lambda\in\Lambda(n,r)}\zeta_\lambda,$

(DS2) $\mathfrak{e}_i\zeta_\lambda = \zeta_{\lambda+\varepsilon_i-\varepsilon_{i+1}}\mathfrak{e}_i$ *if* $\lambda_{i+1} \geqslant 1$, $\mathfrak{e}_i\zeta_\lambda = 0 = \zeta_\lambda\mathfrak{f}_i$ *if* $\lambda_{i+1} = 0$,

(DS3) $\mathfrak{f}_i\zeta_\lambda = \zeta_{\lambda-\varepsilon_i+\varepsilon_{i+1}}\mathfrak{f}_i$ *if* $\lambda_i \geqslant 1$, $\mathfrak{f}_i\zeta_\lambda = 0 = \zeta_\lambda\mathfrak{e}_i$ *if* $\lambda_i = 0$,

(DS4) $\mathfrak{e}_i\mathfrak{f}_j - \mathfrak{f}_j\mathfrak{e}_i = \delta_{i,j}\big(\sum_{\lambda\in\Lambda(n,r),\lambda_i\neq 0,\lambda_{i+1}=0}\zeta_\lambda - \sum_{\lambda\in\Lambda(n,r),\lambda_i=0,\lambda_{i+1}\neq 0}\zeta_\lambda\big),$

(DS5) $\mathfrak{e}_i\mathfrak{e}_j = \mathfrak{e}_j\mathfrak{e}_i$, $\mathfrak{f}_i\mathfrak{f}_j = \mathfrak{f}_j\mathfrak{f}_i$ $(|i-j| > 1)$,

(DS6) $\mathfrak{e}_i^2\mathfrak{e}_{i+1} - \mathfrak{e}_i\mathfrak{e}_{i+1}\mathfrak{e}_i = 0,$

(DS7) $\mathfrak{e}_i\mathfrak{e}_{i+1}^2 - \mathfrak{e}_{i+1}\mathfrak{e}_i\mathfrak{e}_{i+1} = 0,$

(DS8) $\mathfrak{f}_{i+1}\mathfrak{f}_i^2 - \mathfrak{f}_i\mathfrak{f}_{i+1}\mathfrak{f}_i = 0,$

(DS9) $\mathfrak{f}_{i+1}^2\mathfrak{f}_i - \mathfrak{f}_{i+1}\mathfrak{f}_i\mathfrak{f}_{i+1} = 0.$

The main aim in this section is to show that $S_0(n,r)$ is generated by the elements $\mathfrak{e}_i, \mathfrak{f}_i, \zeta_\lambda$ with the defining relations (DS1)–(DS9).

First, we have the following lemma which can be proved by using the arguments completely analogous to those in [7, Th. 13.31].

Lemma 2.2. *The $\mathbb{Z}$-algebra $S_0(n,r)$ is generated by $\mathfrak{e}_i, \mathfrak{f}_i, \zeta_\lambda$ for $1 \leqslant i < n$ and $\lambda \in \Lambda(n,r)$.*

Now we define $U_0(n,r)$ to be the $\mathbb{Z}$-algebra generated by $\mathsf{x}_i, \mathsf{y}_i, \xi_\lambda$ for $1 \leqslant i < n$ and $\lambda \in \Lambda(n,r)$ subject to the relations (DS1′)–(DS9′) which are obtained from (DS1)–(DS9) by substituting the $\mathfrak{e}_i$, $\mathfrak{f}_i$ and ζ_λ for the x_i, y_i and ξ_λ, respectively. Therefore, there is a surjective algebra homomorphism

$$\rho : U_0(n,r) \longrightarrow S_0(n,r) \tag{2.2.1}$$

taking $\mathsf{x}_i \mapsto \mathfrak{e}_i$, $\mathsf{y}_i \mapsto \mathfrak{f}_i$ and $\xi_\lambda \mapsto \zeta_\lambda$. The rest of this section is to show that ρ is an isomorphism.

Let $\mathfrak{H}_0(Q) = \mathfrak{H}_{\boldsymbol{q}}(Q) \otimes_{\mathscr{A}} \mathbb{Z}$ be the degenerate Ringel–Hall algebra of the linear quiver Q given in §2. The following result is taken from [14] and [15, Remarks 4.9(a)].

Lemma 2.3. *As a $\mathbb{Z}$-algebra, $\mathfrak{H}_0(Q)$ is generated by $u_i = u_i \otimes 1$ $(1 \leqslant i < n)$ subject to the relations:*

(DH1) $u_i u_j = u_j u_i, \ \ (|i-j| > 1)$,

(DH2) $u_i^2 u_{i+1} - u_i u_{i+1} u_i = 0$,

(DH3) $u_i u_{i+1}^2 - u_{i+1} u_i u_{i+1} = 0$.

By Lemma 2.1 and the lemma above, there are an algebra homomorphism

$$\phi : \mathfrak{H}_0(Q) \longrightarrow U_0(n,r), \ u_i \longmapsto \mathsf{x}_i \ \ (1 \leqslant i < n)$$

and an algebra anti-homomorphism

$$\psi : \mathfrak{H}_0(Q) \longrightarrow U_0(n,r), \ u_i \longmapsto \mathsf{y}_i \ \ (1 \leqslant i < n).$$

We set

$$U_0(n,r)^+ = \operatorname{Im}\phi \ \text{ and } \ U_0(n,r)^- = \operatorname{Im}\psi,$$

that is, $U_0(n,r)^+$ (resp., $U_0(n,r)^-$) is the $\mathbb{Z}$-subalgebra of $U_0(n,r)$ generated by the x_i (resp., y_i). Furthermore, let $U_0(n,r)^0$ be the $\mathbb{Z}$-subalgebra of $U_0(n,r)$ generated by the ξ_λ which is clearly $\mathbb{Z}$-free with basis $\{\xi_\lambda \mid \lambda \in \Lambda(n,r)\}$. From the relations (DS1′)–(DS9′) we easily deduce that

$$U_0(n,r) = U_0(n,r)^+ \cdot U_0(n,r)^0 \cdot U_0(n,r)^-. \tag{2.3.1}$$

We now fix a field $\mathbb{F}$. For $A, B \in \Xi(n)^+$, define $B \leqslant_{\rm dg} A$ if and only if $\underline{\dim} M_{\mathbb{F}}(B) = \underline{\dim} M_{\mathbb{F}}(A)$ and for each $C \in \Xi(n)^+$,

$$\dim \operatorname{Hom}_{\mathbb{F}Q}(M_{\mathbb{F}}(C), M_{\mathbb{F}}(B)) \geqslant \dim \operatorname{Hom}_{\mathbb{F}Q}(M_{\mathbb{F}}(C), M_{\mathbb{F}}(A)) \text{ for all } C \in \Xi(n)^+.$$

This is the so-called degeneration order on $\Xi(n)^+$ which is a partial order independent of the field $\mathbb{F}$; see [7, §1.6]. We write $B <_{\rm dg} A$ if $B \leqslant_{\rm dg} A$ and $B \neq A$.

For each pair $1 \leqslant i < j \leqslant n$ and an integer $a \geqslant 1$, define a monomial

$$u_{i,j}^a = u_i^a u_{i+1}^a \cdots u_{j-1}^a = ([\![a]\!]^!)^{j-i}\big(u_{aE_{i,j}} + \sum_{X <_{\rm dg} aE_{i,j}} u_X\big)$$

in $\mathfrak{H}_{\boldsymbol{q}}(Q)$. For $A=(a_{i,j})\in\Xi(n)^+$, define a monomial

$$\mathfrak{u}^A=u_{n-1,n}^{a_{n-1,n}}u_{n-2,n}^{a_{n-2,n}}\cdots u_{1,n}^{a_{1,n}}u_{n-2,n-1}^{a_{n-2,n-1}}u_{n-3,n-1}^{a_{n-3,n-1}}\cdots u_{1,n-1}^{a_{1,n-1}}\cdots u_{2,3}^{a_{2,3}}u_{1,3}^{a_{1,3}}u_{1,2}^{a_{1,2}} \tag{2.3.2}$$

in $\mathfrak{H}_{\boldsymbol{q}}(Q)$. By [23] and [24, §6], we have

$$\mathfrak{u}^A=\prod_{1\leqslant i<j\leqslant n}([\![a_{i,j}]\!]^!)^{j-i}\big(u_A+\sum_{B<_{\rm dg}A}f_{A,B}(\boldsymbol{q})u_B\big),$$

where $f_{A,B}(\boldsymbol{q})\in\mathscr{A}$. We denote by $\mathfrak{u}_0^A$ the monomial in (2.3.2) viewing as an element in $\mathfrak{H}_0(Q)$. Thus,

$$\mathfrak{u}_0^A=u_A+\sum_{B<_{\rm dg}A}f_{A,B}(0)u_B. \tag{2.3.3}$$

Since $\{u_A\mid A\in\Xi(n)^+\}$ is a $\mathbb{Z}$-basis of $\mathfrak{H}_0(Q)$, it follows that $\{\mathfrak{u}_0^A\mid A\in\Xi(n)^+\}$ is also a $\mathbb{Z}$-basis of $\mathfrak{H}_0(Q)$.

For $A\in\Xi(n)^+$, define

$$\mathsf{x}^A=\phi(\mathfrak{u}_0^A)\in U_0(n,r)^+.$$

Dually, let $\Xi(n)^-$ be the set of all strictly lower triangular matrices in $\mathbb{N}^{n\times n}$. For $A\in\Xi(n)^-$, define

$$\mathsf{y}^A=\psi(\mathfrak{u}_0^{A^t})\in U_0(n,r)^-,$$

where A^t denotes the transpose of A.

For $A\in\mathbb{N}^{n\times n}$, define $\sigma(A)=(\sigma_1(A),\ldots,\sigma_n(A))\in\mathbb{N}^n$ by setting for $1\leqslant i\leqslant n$,

$$\sigma_i(A)=a_{i,i}+\sum_{1\leq j<i}(a_{i,j}+a_{j,i}).$$

For $\lambda=(\lambda_i),\mu=(\mu_i)\in\mathbb{N}^n$, write $\lambda\leqslant\mu$ if $\lambda_i\leqslant\mu_i$ for all $1\leqslant i\leqslant n$. Applying an argument similar to that in the proof of [7, Prop. 13.41], we obtain the following result.

Proposition 2.4. *Given $A\in\Xi(n)^+$, $B\in\Xi(n)^-$ and $\lambda\in\Lambda(n,r)$, the following statements hold in the algebra $U_0(n,r)$.*

(1) *If $\lambda\geqslant\sigma(A)$, then $\mathsf{x}^A\xi_\lambda=\xi_{\lambda'}\mathsf{x}^A$, where $\lambda'=\lambda-\mathrm{col}(A)+\mathrm{row}(A)$,*

(2) *If $\lambda_i<\sigma_i(A)$ for some i, then $\mathsf{x}^A\xi_\lambda=0$,*

(3) *If $\lambda\geqslant\sigma(B)$, then $\xi_\lambda\mathsf{y}^B=\mathsf{y}^B\xi_{\lambda''}$, where $\lambda''=\lambda+\mathrm{col}(B)-\mathrm{row}(B)$,*

(4) *If $\lambda_i<\sigma_i(B)$ for some i, then $\xi_\lambda\mathsf{y}^B=0$.*

Corollary 2.5. *The algebra $U_0(n,r)^+$ (resp., $U_0(n,r)^-$) is spanned by the set*

$$\big\{\mathsf{x}^A\mid A\in\Xi(n)^+,\,|A|\leqslant r\big\}\quad\big(\textit{resp., }\big\{\mathsf{y}^A\mid A\in\Xi(n)^-,\,|A|\leqslant r\big\}\big).$$

Proof. Since $\{\mathfrak{u}_0^A \mid A \in \Xi(n)^+\}$ is a $\mathbb{Z}$-basis of $\mathfrak{H}_0(Q)$, it follows that $U_0(n,r)^+$ is spanned by $\mathsf{x}^A = \phi(\mathfrak{u}_0^A)$ for all $A \in \Xi(n)^+$. If $|A| = \sum_i \sigma_i(A) > r$, then applying Proposition 2.4(2) gives $\mathsf{x}^A = \sum_{\lambda\in\Lambda(n,r)} \mathsf{x}^A\xi_\lambda = 0$. This proves the assertion for $U_0(n,r)^+$.

The assertion for $U_0(n,r)^-$ can be proved similarly. □

For each matrix $A = (a_{i,j}) \in \mathbb{N}^{n\times n}$, let A^+ (resp., A^-) be the strictly upper (resp., lower) triangular part of A, i.e., $A^+ \in \Xi(n)^+$ and $A^- \in \Xi(n)^-$ with

$$A = A^+ + \mathrm{diag}(a_{1,1}, \dots, a_{n,n}) + A^-.$$

For any $A \in \Xi(n)^\pm$ and $\lambda \in \Lambda(n,r)$, set

$$\mathfrak{m}^{(A,\lambda)} := \mathsf{x}^{A^+}\xi_\lambda \mathsf{y}^{A^-}. \tag{2.5.1}$$

By (2.3.1) and Corollary 2.5, $U_0(n,r)$ is spanned by all such $\mathfrak{m}^{(A,\lambda)}$ with $\lambda \in \Lambda(n,r)$, $A \in \Xi(n)^\pm$ satisfying $|A^+| \leqslant r$ and $|A^-| \leqslant r$.

Lemma 2.6. *For all $s \geqslant 1$ and $1 \leqslant i < n$, the following equalities hold in $U_0(n,r)$:*

(1) $\mathsf{x}_i\mathsf{y}_i^s - \mathsf{y}_i^s\mathsf{x}_i = \mathsf{y}_i^{s-1}\big(\Theta - \sum\limits_{1\leqslant t\leqslant s-1}\ \sum\limits_{\substack{\lambda\in\Lambda(n,r),\\ \lambda_i=t,\lambda_{i+1}<r-t+1}} \xi_\lambda\big)$;

(2) $\mathsf{y}_i\mathsf{x}_i^s - \mathsf{x}_i^s\mathsf{y}_i = -\mathsf{x}_i^{s-1}\big(\Theta - \sum\limits_{1\leqslant t\leqslant s-1}\ \sum\limits_{\substack{\lambda\in\Lambda(n,r),\\ \lambda_i=t,\lambda_{i+1}<r-t+1}} \xi_\lambda\big)$,

where $\Theta = \sum\limits_{\lambda\in\Lambda(n,r),\lambda_i\neq0,\lambda_{i+1}=0} \xi_\lambda - \sum\limits_{\lambda\in\Lambda(n,r),\lambda_i=0,\lambda_{i+1}\neq0} \xi_\lambda.$

Proof. We prove the first equality by induction on s. The second one is proved similarly.

By definition, the equality holds for $s = 1$. Now suppose $s > 1$. Then we have

$$\begin{aligned}
\mathsf{x}_i\mathsf{y}_i^s - \mathsf{y}_i^s\mathsf{x}_i &= [\mathsf{x}_i, \mathsf{y}_i^{s-1}]\mathsf{y}_i + \mathsf{y}_i^{s-1}[\mathsf{x}_i, \mathsf{y}_i] \\
&= \mathsf{y}_i^{s-2}\big(\Theta - \sum_{1\leqslant t\leqslant s-2}\ \sum_{\substack{\lambda\in\Lambda(n,r),\\ \lambda_i=t,\lambda_{i+1}<r-t+1}} \xi_\lambda\big)\mathsf{y}_i + \mathsf{y}_i^{s-1}\Theta \ \text{(By induction hypothesis)} \\
&= \mathsf{y}_i^{s-1}\big(\Theta - \sum_{1\leqslant t\leqslant s-1}\ \sum_{\substack{\lambda\in\Lambda(n,r),\\ \lambda_i=t,\lambda_{i+1}<r-t+1}} \xi_\lambda\big).
\end{aligned}$$

This proves the first equality. □

For a monomial $\mathfrak{m} \in \mathfrak{H}_0(Q)$ (resp., $\mathfrak{m} \in U_0(n,r)$) in the u_i (resp., x_i and y_i), let $\deg(\mathfrak{m})$ be the number of the u_i (resp., x_i and y_i) occurring in $\mathfrak{m}$. In other words, if $\mathfrak{m}$ is regarded as a word, $\deg(\mathfrak{m})$ is the length of the word. If we define for each $A = (a_{i,j}) \in \mathbb{N}^{n\times n}$,

$$\deg(A) = \sum_{i,j} |i-j| a_{i,j},$$

then $\deg(A) = \deg \mathsf{x}^{A^+} + \deg \mathsf{y}^{A^-}$.

Lemma 2.7. *Let* $\mathfrak{m} \in U_0(n,r)^+$ *be a monomial in the* x_i*. Then* $\mathfrak{m}$ *is a* $\mathbb{Z}$*-linear combination of* x^A*,* $A \in \Xi(n)^+$ *(hence, a* $\mathbb{Z}$*-linear combination of* $\mathsf{x}^A\xi_\lambda$*,* $A \in \Xi(n)^+, \lambda \in \Lambda(n,r)$*) with* $\deg(A) = \deg(\mathfrak{m})$*. A similar result holds for monomials in the* y_i*.*

Proof. Let $\widetilde{\mathfrak{m}} \in \mathfrak{H}_0(Q)$ be a monomial in the u_i such that $\phi(\widetilde{\mathfrak{m}}) = \mathfrak{m}$. Then $\deg(\widetilde{\mathfrak{m}}) = \deg(\mathfrak{m})$. Since $\mathfrak{H}_0(Q)$ is $\mathbb{N}^{n-1}$-graded and $\{\mathfrak{u}_0^A \mid A \in \Xi(n)^+\}$ is a $\mathbb{Z}$-basis of $\mathfrak{H}_0(Q)$, $\widetilde{\mathfrak{m}}$ is a $\mathbb{Z}$-linear combination of $\mathfrak{u}_0^A$ with $\deg(A) = \deg(\widetilde{\mathfrak{m}})$. Hence, $\mathfrak{m}$ is a $\mathbb{Z}$-linear combination of $\mathsf{x}^A = \phi(\mathfrak{u}_0^A)$ with $\deg(A) = \deg(\mathfrak{m})$.

The assertion for monomials in the y_i can be proved analogously. □

The following theorem is an analogue to [7, Th. 13.44] which has been proved in [9] (see also [10]). We provide a proof for completeness.

Theorem 2.8. *The algebra homomorphism* $\rho : U_0(n,r) \to S_0(n,r)$ *given in* (2.2.1) *is an isomorphism.*

Proof. Put

$$\mathcal{M} = \{\mathfrak{m}^{(A)} = \mathsf{x}^{A^+}\xi_{\sigma(A)}\mathsf{y}^{A^-} \mid A \in \Xi(n,r)\}.$$

We aim to prove that $\mathcal{M}$ is a $\mathbb{Z}$-basis for $U_0(n,r)$. Since $|\mathcal{M}| \leqslant |\Xi(n,r)|$, which is the rank of $S_0(n,r)$, it suffices to show that $\mathcal{M}$ spans $U_0(n,r)$. Let $B \in \Xi(n)^\pm$ with $|B^+| \leqslant r$ and $|B^-| \leqslant r$, and let $\lambda \in \Lambda(n,r)$. If $\lambda \geqslant \sigma(B)$, there is a unique $A = B + \mathrm{diag}(\lambda - \sigma(B)) \in \Xi(n,r)$ such that $\mathfrak{m}^{(B,\lambda)} = \mathfrak{m}^{(A)}$, which belongs to $\mathcal{M}$. It remains to prove that if $\lambda_i < \sigma_i(B)$ for some i, then $\mathfrak{m}^{(B,\lambda)}$ lies in the span of $\mathcal{M}$.

We proceed by induction on $\deg(B)$. If $\deg(B) = 1$, then $B = E_{i-1,i}$ or $E_{i,i-1}$, and so $\lambda_i = 0$ and $\mathfrak{m}^{(B,\lambda)} = \mathsf{x}_{i-1}\xi_\lambda$ or $\xi_\lambda\mathsf{y}_{i-1}$, which is zero by the definition. Assume now $\deg(B) > 1$ and let i be minimal with $\lambda_i < \sigma_i(B) = \sigma_i(B^+) + \sigma_i(B^-)$. Let B_i be the top left $i \times i$ submatrix of B, write $\mathsf{x}^{B^+} = \mathfrak{m}\mathsf{x}^{B_i^+}$ and $\mathsf{y}^{B^-} = \mathsf{y}^{B_i^-}\mathfrak{m}'$ for some monomials $\mathfrak{m}, \mathfrak{m}'$. Then $\mathfrak{m}^{(B,\lambda)} = \mathfrak{m}\mathsf{x}^{B_i^+}\xi_\lambda\mathsf{y}^{B_i^-}\mathfrak{m}'$. By Proposition 2.4(2), we can assume $\lambda_i \geq \sigma_i(B^+)$, otherwise $\mathfrak{m}^{(B,\lambda)} = 0$ which is obviously in $\mathcal{M}$. Now Proposition 2.4(1) implies that

$$\mathfrak{m}^{(B,\lambda)} = \mathfrak{m}(\mathsf{x}^{B_i^+}\xi_\lambda)\mathsf{y}^{B_i^-}\mathfrak{m}' = \mathfrak{m}\xi_{\lambda'}\mathsf{x}^{B_i^+}\mathsf{y}^{B_i^-}\mathfrak{m}',$$

where $\lambda' = \lambda - \mathrm{col}(B_i^+) + \mathrm{row}(B_i^+)$. Then $\lambda'_i = \lambda_i - (b_{1,i} + \cdots + b_{i-1,i}) = \lambda_i - \sigma_i(B_i^+) \geq 0$.

By repeatedly applying the commutator formula given in Lemma 2.6, we can write

$$\mathsf{x}^{B_i^+}\mathsf{y}^{B_i^-} = \mathsf{y}^{B_i^-}\mathsf{x}^{B_i^+} + f,$$

where f is a linear combination of monomials $\widehat{\mathfrak{m}}\xi_\lambda\widehat{\mathfrak{m}}'$ with $\lambda \in \Lambda(n,r)$ and $\deg(\widehat{\mathfrak{m}}\widehat{\mathfrak{m}}') < \deg(B_i)$. Hence,

$$\mathfrak{m}^{(B,\lambda)} = \mathfrak{m}\xi_{\lambda'}\mathsf{x}^{B_i^+}\mathsf{y}^{B_i^-}\mathfrak{m}' = \mathfrak{m}\xi_{\lambda'}\mathsf{y}^{B_i^-}\mathsf{x}^{B_i^+}\mathfrak{m}' + \mathfrak{m}\xi_{\lambda'}f\mathfrak{m}'.$$

Since $\lambda'_i = \lambda_i - \sigma_i(B_i^+) < \sigma_i(B_i^-)$, we have $\mathfrak{m}\xi_{\lambda'}\mathsf{y}^{B_i^-}\mathsf{x}^{B_i^+}\mathfrak{m}' = 0$ by Proposition 2.4(4). Furthermore, $\mathfrak{m}\xi_{\lambda'}f\mathfrak{m}'$ is a $\mathbb{Z}$-linear combination of $\mathfrak{m}^{(B',\mu)}$ with $\deg(B') < \deg(B)$. By the induction hypothesis, each $\mathfrak{m}^{(B',\mu)}$ lies in the span of $\mathcal{M}$. Then $\mathfrak{m}\xi_{\lambda'}f\mathfrak{m}'$ is in the span of $\mathcal{M}$, so is $\mathfrak{m}^{(B,\lambda)}$. The proof is completed. □

From the proof of the above theorem, we obtain the following result.

Corollary 2.9. *The algebra $S_0(n,r)$ is generated by the elements $\mathfrak{e}_i, \mathfrak{f}_i, \zeta_\lambda$ with* (DS1)–(DS9) *as the generating relations. Moreover, the set*

$$\{\mathfrak{e}^{A^+}\zeta_{\sigma(A)}\mathfrak{f}^{A^-} \mid A \in \Xi(n,r)\}$$

is a $\mathbb{Z}$-basis for $S_0(n,r)$, where $\mathfrak{e}^{A^+} = \rho(\mathsf{x}^{A^+})$ and $\mathfrak{f}^{A^-} = \rho(\mathsf{y}^{A^-})$.

3 Representation type of $S_0(n,r)$

This section is devoted to determining the representation type of $S_0(n,r)$. This is based on the representation theory of 0-Hecke algebras developed in [16, 17, 18, 19]. Throughout this section, we assume that $S_0(n,r) = S_0(n,r)_{\mathbb{F}}$ denotes the 0-Schur algebra over an algebraically closed field $\mathbb{F}$.

Given a finite dimensional $\mathbb{F}$-algebra A, by A-**mod** we denote the category of finite dimensional left A-modules. The algebra A is said to be representation-finite if up to isomorphism, there are only finitely many pairwise non-isomorphic indecomposable modules in A-**mod**. We refer to [25, 26] for the definition of tame and wild algebras. If $n = 1$ or $r = 1$, then $S_0(n,r)$ is clearly semisimple. Thus, in the following we always assume $n, r \geqslant 2$.

Let $H_0(r) = H_0(r)_{\mathbb{F}}$ be the 0-Hecke algebra of $\mathfrak{S} = \mathfrak{S}_r$ over $\mathbb{F}$. By [16], all the simple (right) $H_0(r)$-modules have dimension one[1]. More precisely, each subset $J \subseteq I$ gives rise to a simple $H_0(r)$-module $E_J = \mathbb{F}$ defined by

$$x \cdot T_i = \begin{cases} -x, & \text{if } i \in J; \\ 0, & \text{otherwise,} \end{cases} \tag{3.0.1}$$

where $x \in E_J$ and $i \in I$. Moreover, the E_J form a complete set of simple $H_0(r)$-modules. It follows that

$$H_0(r)/\mathrm{rad}\, H_0(r) \cong \underbrace{\mathbb{F} \times \cdots \times \mathbb{F}}_{2^{r-1}},$$

where $\mathrm{rad}\, H_0(r)$ is the Jacobson radical of $H_0(r)$; see [16, Th. 4.21]. Hence, the Gabriel quiver (or Ext-quiver) Γ of $H_0(r)$ has vertex set $\{v_J \mid J \subseteq I\}$, and the number of arrows from v_J to v_K, for $J, K \subseteq I$, equals to $\dim_{\mathbb{F}} \mathrm{Ext}^1_{H_0(r)}(E_J, E_K)$ which is described in [18, Th. 5.1] as follows.

[1]The $H_0(r)$-modules considered in [16, 17, 18] are left module. Since $H_0(r)$ admits an anti-automorphism $T_i \mapsto T_i$, all results there hold similarly for right $H_0(r)$-modules.

Lemma 3.1. *Suppose* $J, K \subseteq I$. *Then* $\dim \mathrm{Ext}^1_{H_0(r)}(E_J, E_K) = 1$ *if and only if* $J \nsubseteq K \nsubseteq J$ *and* $|j-k| \leqslant 1$ *for all* $j \in J \backslash K$ *and* $k \in K \backslash J$. *Otherwise, we have* $\mathrm{Ext}^1_{H_0(r)}(E_J, E_K) = 0$.

For each subset $J \subseteq I$, let P_J and Q_J denote the projective cover and injective hull of S_J, respectively. By [18, Prop. 4.5], $H_0(r)$ is selfinjective and, moreover, $P_J \cong Q_{\sigma(J)}$, where σ is a bijection $I \to I$ taking $i \mapsto r-i$. Without loss of generality, we set $P_J = e_J H_0(r)$ for an idempotent $e_J \in H_0(r)$. Then $\{e_J \mid J \subseteq I\}$ is a complete set of primitive orthogonal idempotents. By the lemma above, Γ has two isolated vertices $v_\emptyset$ and v_I, i.e., there are no arrows starting or ending at $v_\emptyset$ and v_I. This implies that

$$H_0(r) \cong \mathbb{F} \times \mathbb{F} \times \widehat{e} H_0(r) \widehat{e}, \tag{3.1.1}$$

where $\widehat{e} = \sum_J e_J$ with the sum taking over all proper subsets $J \subseteq I$.

Recall that for each $\lambda \in \Lambda(n,r)$, we have the element

$$x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} T_w \in H_0(r).$$

Applying (1.0.1) gives $x_\lambda^2 = x_\lambda$. Hence, the $H_0(r)$-module $T_0(n,r) := \bigoplus_{\lambda \in \Lambda(n,r)} x_\lambda H_0(r)$ is both projective and injective.

Suppose $n \geqslant r$. Then for

$$\lambda = (\underbrace{1, \ldots, 1}_{r}, 0, \ldots, 0) \in I(n,r),$$

we have $x_\lambda = 1$. Hence, if $n \geqslant r$, then $S_0(n,r) = \mathrm{End}_{H_0(r)}(T_0(n,r))$ is Morita equivalent to $H_0(r)$.

For each $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda(n,r)$, consider the subset J_λ of I defined by

$$J_\lambda = \{\lambda_1, \lambda_1 + \lambda_2, \ldots, \lambda_1 + \lambda_2 + \cdots \lambda_n\} \backslash \{r\}.$$

By [16, Cor. 4.14(2)],

$$x_\lambda H_0(r) \cong \bigoplus_{J \subseteq J_\lambda} P_J.$$

This gives a decomposition

$$T_0(n,r) = \bigoplus_{\lambda \in \Lambda(n,r)} x_\lambda H_0(r) = \bigoplus_{J \subseteq I} (P_J)^{d_J}, \tag{3.1.2}$$

where $d_J = |\{\lambda \in \Lambda(n,r) \mid J \subseteq J_\lambda\}|$.

Proposition 3.2. *For each* $J \subseteq I$, $d_J \neq 0$ *if and only if* $|J| \leqslant n-1$.

Proof. Suppose $d_J \neq 0$. Then there exist $\lambda \in \Lambda(n,r)$ such that $J \subseteq J_\lambda$. This implies that $|J| \leqslant |J_\lambda| \leqslant n-1$.

Conversely, suppose $|J| = m \leqslant n-1$. Write $J = \{i_1 < \cdots < i_m\}$ and define $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda(n, r)$ by setting

$$\lambda_1 = i_1, \lambda_2 = i_2 - i_1, \ldots, \lambda_m = i_m - i_{m-1}, \lambda_{m+1} = r - i_m, \lambda_{m+2} = \cdots = \lambda_n = 0.$$

Then $J = J_\lambda$. Therefore, $d_J \neq 0$. □

Remark 3.3. The $H_0(r)$-module $T_0(n, r) = \bigoplus_{\lambda \in \Lambda(n,r)} x_\lambda H_0(r)$ is also known as tensor space. More precisely, let $I(n, r) = \{\ \mathbf{i} = (i_1, \ldots, i_r) \mid 1 \leqslant i_j \leqslant n \text{ for all } 1 \leqslant j \leqslant r\ \}$. The symmetric group $\mathfrak{S}_r$ acts on $I(n, r)$ by place permutation:

$$\mathbf{i}w = (i_{w(1)}, i_{w(2)}, \ldots, i_{w(r)}) \text{ for all } \mathbf{i} \in I(n, r),\ w \in \mathfrak{S}_r.$$

Let Ω be an $\mathbb{F}$-vector space with basis $\{\omega_i \mid 1 \leqslant i \leqslant n\}$ whose r-fold tensor product $\Omega^{\otimes r}$ has basis $\{\omega_\mathbf{i} := \omega_{i_1} \otimes \cdots \otimes \omega_{i_r} \mid \mathbf{i} = (i_1, \ldots, i_r) \in I(n, r)\}$. Then $T_0(n, r)$ is isomorphic to $\Omega^{\otimes r}$ whose right $H_0(r)$-module structure is defined by

$$\omega_\mathbf{i} T_k = \begin{cases} \omega_{\mathbf{i}s_k}, & i_k < i_{k+1}; \\ 0, & i_k = i_{k+1}; \\ -\omega_\mathbf{i}, & i_k > i_{k+1}, \end{cases} \tag{3.3.1}$$

where $\mathbf{i} = (i_1, \ldots, i_r) \in I(n, r)$ and $k \in I$.

We claim that for each $J \subseteq I$, $d_J = |X_J|$, where

$$X_J = \{\mathbf{i} = (i_1, \ldots, i_r) \in I(n, r) \mid i_j > i_{j+1},\ i_k \geqslant i_{k+1} \text{ for all } j \in \sigma(J),\ k \in I \backslash \sigma(J)\}.$$

Indeed, since

$$\Omega^{\otimes r} \cong T_0(n, r) \cong \bigoplus_{J \subseteq I} (P_J)^{d_J} \cong \bigoplus_{J \subseteq I} (Q_{\sigma(J)})^{d_J},$$

it follows that

$$d_J = \dim_\mathbb{F} \mathrm{Hom}_{H_0(r)}(E_{\sigma(J)}, \mathrm{soc}\,\Omega^{\otimes r}) = \dim_\mathbb{F} \mathrm{Hom}_{H_0(r)}(E_{\sigma(J)}, \Omega^{\otimes r}).$$

By the definition of $E_{\sigma(J)}$, we have an isomorphism of $\mathbb{F}$-spaces

$$\begin{aligned} &\mathrm{Hom}_{H_0(r)}(E_{\sigma(J)}, \Omega^{\otimes r}) \\ \cong &\{x \in \Omega^{\otimes r} \mid xT_j = -x \text{ for all } j \in \sigma(J),\ xT_k = 0 \text{ for all } k \in I \backslash \sigma(J)\} := V_J. \end{aligned}$$

It is easy to see that the coefficients of $x = \sum_\mathbf{i} x_\mathbf{i} \omega_\mathbf{i} \in V_J$ satisfy

$$\begin{cases} x_\mathbf{i} = 0, & \text{if there exists } j \in \sigma(J) \text{ such that } i_j \leqslant i_{j+1}; \\ x_{\mathbf{i}s_k} - x_\mathbf{i} = 0, & \text{if there exists } k \in I \backslash \sigma(J) \text{ such that } i_k < i_{k+1}. \end{cases} \tag{3.3.2}$$

By viewing (3.3.2) as a system of homogeneous linear equations with variables $x_{\mathbf{i}}$ for $\mathbf{i} \in I(n,r)$, we can identify V_J with the space S_J of solutions of (3.3.2). We conclude that all the $x_{\mathbf{i}}$ with $\mathbf{i} \in X_J$ form a set of free variables for (3.3.2). Consequently,

$$d_J = \dim_{\mathbb{F}} V_J = \dim_{\mathbb{F}} S_J = |X_J|.$$

By Proposition 3.2, for arbitrary positive integers n, r, $S_0(n,r)$ is Morita equivalent to

$$\mathrm{End}_{H_0(r)}\big(\bigoplus_{J\subseteq I,\,|J|\leqslant n-1} P_J\big) \cong eH_0(r)e, \tag{3.3.3}$$

where $e = \sum_{J\subseteq I,\,|J|\leqslant n-1} e_J$.

Proposition 3.4. *The algebra $S_0(n,r)$ is selfinjective.*

Proof. It is known that the usual duality $D = \mathrm{Hom}_{\mathbb{F}}(-,\mathbb{F})$ induces the Nakayama functor

$$\nu = D\mathrm{Hom}_{H_0(r)}(-, H_0(r)) : \textbf{mod-}H_0(r) \longrightarrow \textbf{mod-}H_0(r),$$

where $\textbf{mod-}H_0(r)$ denotes the category of finite dimensional right $H_0(r)$-modules. Since $P_J \cong Q_{\sigma(J)}$ for each subset $J \subseteq I$, we have

$$\nu(P_J) \cong Q_J \cong P_{\sigma(J)}.$$

Hence, the set $\{P_J \mid J \subseteq I,\, |J| \leqslant n-1\}$ is stable under ν, up to isomorphism. By [27, Lem. 2.2] and the selfinjectivity of $H_0(r)$, we infer that $eH_0(r)e$ is selfinjective. Consequently, $S_0(n,r)$ is selfinjective. □

Furthermore, from (3.3.3) it follows that $S_0(r-1,r)$ is Morita equivalent to $\mathbb{F} \times \widehat{e}H_0(r)\widehat{e}$. In conclusion, we obtain the following result which is a slight generalization of [5, §2.2(5)].

Proposition 3.5. *Suppose $n \geqslant r-1$. Then $S_0(n,r)$ and $H_0(r)$ have the same representation type.*

Combining the results above gives the following theorem.

Theorem 3.6. *Suppose $n \geqslant 3$. Then $S_0(n,r)$ is representation-finite (resp., tame, wild) if and only if $r \leqslant 3$ (resp., $r = 4$, $r \geqslant 5$).*

Proof. It is shown in [19, Th. 2.1] that the 0-Hecke algebra $H_0(r)$ is representation-finite (resp., tame, wild) if and only if $r \leqslant 3$ (resp., $r = 4$, $r \geqslant 5$).

Suppose $n \geqslant 3$. Then by Proposition 3.5, $S_0(n,r)$ and $H_0(r)$ have the same representation type in case $r \leqslant 4$. Therefore, $S_0(n,r)$ is representation-finite (resp., tame) if and only if $r \leqslant 3$ (resp., $r = 4$).

Now let $r \geqslant 5$. Then by Lemma 3.1, the Gabriel quiver Γ of $H_0(r)$ contains a full subquiver Σ of the following form

Σ:

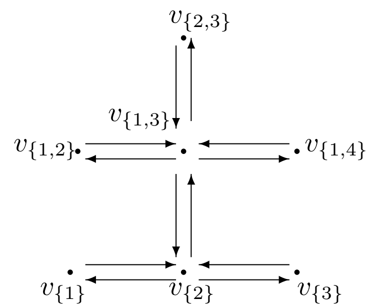

FIG.2. *Subquiver of the Gabriel quiver* Γ *of* $H_0(r)$

Since $n \geqslant 3$, all P_J with

$$J \in \big\{\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{1,4\},\{2,3\}\big\}$$

occur as direct summands in $T_0(n,r)$, it follows that the Gabriel quiver of $S_0(n,r)$ also contains a full subquiver of the form Σ. Hence, $S_0(n,r)$ is wild. $\square$

The rest of this section is devoted to determining the representation type of $S_0(2,r)$ with $r \geqslant 2$. For each $0 \leqslant i \leqslant r$, put $\lambda^{(i)} = (i, r-i) \in \Lambda(2,r)$. Then

$$x_{\lambda^{(0)}} H_0(r) \cong P_\emptyset \cong x_{\lambda^{(r)}} H_0(r), \quad x_{\lambda^{(i)}} H_0(r) \cong P_\emptyset \oplus P_{\{i\}} \ \text{ for } 1 \leqslant i < r.$$

Thus, $T_0(2,r) \cong (P_\emptyset)^{r+1} \oplus \big(\bigoplus_{i=1}^{r-1} P_{\{i\}}\big)$ and

$$S_0(2,r) \cong \mathbb{F}^{(r+1)\times(r+1)} \times \mathrm{End}_{\,H_0(r)}\Big(\bigoplus_{i=1}^{r-1} P_{\{i\}}\Big). \tag{3.6.1}$$

By Lemma 3.1, the Gabriel quiver of $A_0(r) := \mathrm{End}_{\,H_0(r)}\big(\bigoplus_{i=1}^{r-1} P_{\{i\}}\big)$ has the form

Δ_r:

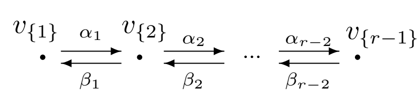

FIG.3. *Gabriel quiver of* $A_0(r)$

Hence, $A_0(r) \cong \mathbb{F}\Delta_r/\mathcal{I}_r$ for some admissible ideal $\mathcal{I}_r$ of the path algebra $\mathbb{F}\Delta_r$. Our next aim is to determine the ideal $\mathcal{I}_r$ by induction on r.

Recall from §2 that $S_0(2,r)$ has a basis $\{\zeta_A \mid A \in \Xi(2,r)\}$. By (1.1.2), for each $0 \leqslant i \leqslant r$, the idempotent $\zeta_{\lambda^{(i)}}$ is the composition

$$T_0(2,r) = \bigoplus_{j=0}^{r} x_{\lambda^{(j)}} H_0(r) \xrightarrow{\pi_i} x_{\lambda^{(i)}} H_0(r) \xrightarrow{\kappa_i} \bigoplus_{j=0}^{r} x_{\lambda^{(j)}} H_0(r) = T_0(2,r),$$

where π_i and κ_i denote the canonical projection and inclusion, respectively. Hence, $S_0(2,r)\zeta_{\lambda^{(0)}} \cong S_0(2,r)\zeta_{\lambda^{(r)}}$ is a simple projective module. In particular, $\zeta_{\lambda^{(0)}}$ and $\zeta_{\lambda^{(r)}}$ are primitive idempotents. For each $1 \leqslant i < r$, $\zeta_{\lambda^{(i)}}$ decomposes into a sum of orthogonal primitive idempotents $\zeta_{\lambda^{(i)}} = \zeta'_{\lambda^{(i)}} + \zeta''_{\lambda^{(i)}}$ such that $S_0(n,r)\zeta'_{\lambda^{(i)}} \cong S_0(n,r)\zeta_{\lambda^{(0)}}$. Consequently,

$$S_0(2,r)/S_0(2,r)\zeta_{\lambda^{(0)}}S_0(2,r) \cong A_0(r) \cong S_0(2,r)/S_0(2,r)\zeta_{\lambda^{(r)}}S_0(2,r), \tag{3.6.2}$$

and for $1 \leqslant i < r$,

$$S_0(2,r)/S_0(2,r)\zeta_{\lambda^{(i)}}S_0(2,r) \cong A_0(r)/A_0(r)e_iA_0(r), \tag{3.6.3}$$

where e_i denotes the idempotent of $A_0(r)$ corresponding to the vertex $v_{\{i\}}$ of Δ_r. In other words, e_i is the composition of the canonical projection and inclusion

$$\bigoplus_{j=1}^{r-1} P_{\{j\}} \longrightarrow P_{\{i\}} \longrightarrow \bigoplus_{j=1}^{r-1} P_{\{j\}}.$$

Proposition 3.7. *Suppose $\lambda = (1, r-1)$ and $\mu = (0, r-1)$. Then there is an algebra isomorphism*

$$\phi : S_0(2,r)/S_0(2,r)\zeta_\lambda S_0(2,r) \longrightarrow S_0(2,r-1)/S_0(2,r-1)\zeta_\mu S_0(2,r-1).$$

Analogously, suppose $\rho = (r-1, 1)$ and $\tau = (r-1, 0)$, Then there is an algebra isomorphism

$$\psi : S_0(2,r)/S_0(2,r)\zeta_\rho S_0(2,r) \longrightarrow S_0(2,r-1)/S_0(2,r-1)\zeta_\tau S_0(2,r-1).$$

Proof. We only prove the first assertion. The second one can be proved similarly.

By Corollary 2.9, $S_0(2,r)$ has generators $\mathfrak{e}$, $\mathfrak{f}$ and ζ_ν $(\nu \in \Lambda(2,r))$ with relations:

(DS1) $\zeta_\nu\zeta_{\nu'} = \delta_{\nu,\nu'}\zeta_\nu$, $1 = \sum_{\nu\in\Lambda(2,r)} \zeta_\nu$;

(DS2) $\mathfrak{e}\zeta_\nu = \zeta_{\nu+\varepsilon_1-\varepsilon_2}\mathfrak{e}$ if $\nu_2 \geq 1$, $\mathfrak{e}\zeta_\nu = 0 = \zeta_\nu\mathfrak{f}$ if $\nu_2 = 0$;

(DS3) $\mathfrak{f}\zeta_\nu = \zeta_{\nu-\varepsilon_1+\varepsilon_2}\mathfrak{f}$ if $\nu_1 \geq 1$, $\mathfrak{f}\zeta_\nu = 0 = \zeta_\nu\mathfrak{e}$ if $\nu_1 = 0$;

(DS4) $\mathfrak{e}\mathfrak{f} - \mathfrak{f}\mathfrak{e} = \zeta_{(r,0)} - \zeta_{(0,r)}$.

While $S_0(2,r-1)$ has generators $\mathfrak{e}$, $\mathfrak{f}$ and ζ_θ $(\theta \in \Lambda(2,r-1))$ with similar relations. Write

$$\mathcal{K}_\lambda = S_0(2,r)\zeta_\lambda S_0(2,r) \text{ and } \mathcal{K}_\mu = S_0(2,r-1)\zeta_\mu S_0(2,r-1).$$

Consider the following elements in $S_0(2,r-1)/\mathcal{K}_\mu$:

$$\mathfrak{e}' = \mathfrak{e} + \mathcal{K}_\mu,\ \mathfrak{f}' = \mathfrak{f} + \mathcal{K}_\mu,\ \zeta'_\nu = \begin{cases} \zeta_{(\nu_1-1,\nu_2)} + \mathcal{K}_\mu, & \text{if } \nu_1 \geqslant 1; \\ 0, & \text{otherwise,} \end{cases}$$

where $\nu = (\nu_1, \nu_2) \in \Lambda(2, r)$. It is straightforward to check that $\mathfrak{e}', \mathfrak{f}', \zeta'_\nu$ satisfy the generating relations (DS1)–(DS4) for $S_0(2, r)$. Thus, there is a surjective algebra homomorphism

$$\widetilde{\phi} : S_0(2, r) \longrightarrow S_0(2, r-1)/\mathcal{K}_\mu$$

which takes $\mathfrak{e} \mapsto \mathfrak{e}'$, $\mathfrak{f} \mapsto \mathfrak{f}'$ and $\zeta_\nu \mapsto \zeta'_\nu$ for $\nu \in \Lambda(2, r)$. Since $\widetilde{\phi}(\zeta_\lambda) = 0$, it induces a surjective homomorphism

$$\phi : S_0(2, r)/\mathcal{K}_\lambda \longrightarrow S_0(2, r-1)/\mathcal{K}_\mu.$$

We now prove that ϕ is an isomorphism by a dimension comparison. By (3.6.1) and (3.6.2),

$$\dim S_0(2, r-1)/\mathcal{K}_\mu = \dim S_0(2, r-1) - r^2 = \binom{r+2}{3} - r^2.$$

On the other hand, for each $0 \leqslant i \leqslant r$, put $\lambda^{(i)} = (i, r-i) \in \Lambda(2, r)$ as above. Note that $\lambda = \lambda^{(1)}$. Since $\sum_{i=0}^{r} \zeta_{\lambda^{(i)}} = 1$, we obtain a decomposition

$$\mathcal{K}_\lambda = \zeta_{\lambda^{(0)}}\mathcal{K}_\lambda \oplus \zeta_{\lambda^{(1)}}\mathcal{K}_\lambda \oplus \cdots \oplus \zeta_{\lambda^{(r)}}\mathcal{K}_\lambda.$$

We are going to compute the dimensions of

$$\zeta_{\lambda^{(i)}}\mathcal{K}_\lambda\zeta_{\lambda^{(j)}} = (\zeta_{\lambda^{(i)}} S_0(2, r)\zeta_\lambda) \cdot (\zeta_\lambda S_0(2, r)\zeta_{\lambda^{(j)}}).$$

A direct calculation shows that $\zeta_{\lambda^{(0)}} S_0(2, r)\zeta_\lambda = \mathbb{F}\zeta_{A_0}$ and $\zeta_{\lambda^{(r)}} S_0(2, r)\zeta_\lambda = \mathbb{F}\zeta_{A_r}$, where

$$A_0 = \begin{pmatrix} 0 & 0 \\ 1 & r-1 \end{pmatrix} \text{ and } A_r = \begin{pmatrix} 1 & r-1 \\ 0 & 0 \end{pmatrix}.$$

For $1 \leqslant i < r$, $\zeta_{\lambda^{(i)}} S_0(2, r)\zeta_\lambda$ has a basis $\{\zeta_{A_1^{(i)}}, \zeta_{A_2^{(i)}}\}$, where

$$A_1^{(i)} = \begin{pmatrix} 1 & i-1 \\ 0 & r-i \end{pmatrix} \text{ and } A_2^{(i)} = \begin{pmatrix} 0 & i \\ 1 & r-i-1 \end{pmatrix}.$$

Similarly, $\zeta_\lambda S_0(2, r)\zeta_{\lambda^{(0)}} = \mathbb{F}\zeta_{B_0}$ and $\zeta_\lambda S_0(2, r)\zeta_{\lambda^{(r)}} = \mathbb{F}\zeta_{B_r}$, where

$$B_0 = \begin{pmatrix} 0 & 1 \\ 0 & r-1 \end{pmatrix} \text{ and } B_r = \begin{pmatrix} 1 & 0 \\ r-1 & 0 \end{pmatrix}.$$

For $1 \leqslant i < r$, $\zeta_\lambda S_0(2, r)\zeta_{\lambda^{(i)}}$ has a basis $\{\zeta_{B_1^{(i)}}, \zeta_{B_2^{(i)}}\}$, where

$$B_1^{(i)} = \begin{pmatrix} 1 & 0 \\ i-1 & r-i \end{pmatrix} \text{ and } B_2^{(i)} = \begin{pmatrix} 0 & 1 \\ i & r-i-1 \end{pmatrix}.$$

Furthermore, $\zeta_{\lambda^{(0)}}\mathcal{K}_\lambda$ has a basis

$$\{\zeta_A \mid A = \begin{pmatrix} 0 & 0 \\ i & r-i \end{pmatrix} \text{ for } 0 \leqslant i \leqslant r\},$$

and $\zeta_{\lambda^{(r)}}\mathcal{K}_\lambda$ has a basis

$$\{\zeta_A \mid A = \begin{pmatrix} i & r-i \\ 0 & 0 \end{pmatrix} \text{ for } 0 \leqslant i \leqslant r\}.$$

Hence, $\dim \zeta_{\lambda^{(0)}}\mathcal{K}_\lambda = \dim \zeta_{\lambda^{(r)}}\mathcal{K}_\lambda = r + 1$.

Now fix $1 \leqslant i \leqslant r-1$. Then $\zeta_{\lambda^{(i)}}\mathcal{K}_\lambda\zeta_{\lambda^{(0)}}$ (resp., $\zeta_{\lambda^{(i)}}\mathcal{K}_\lambda\zeta_{\lambda^{(r)}}$) has a basis ζ_C (resp., ζ_D) with

$$C = \begin{pmatrix} 0 & i \\ 0 & r-i \end{pmatrix} \text{ (resp., } D = \begin{pmatrix} i & 0 \\ r-i & 0 \end{pmatrix}).$$

By [7, Th. 13.18(1)] and [28, Th. 3.2], for $1 \leqslant j \leqslant r-1$, we have

$$\begin{aligned}
\zeta_{A_1^{(i)}}\zeta_{B_1^{(j)}} &= \sum_{m=0}^{\min\{i-1,j-1\}} \zeta_{X_m} \text{ with } X_m = \begin{pmatrix} 1+m & i-1-m \\ j-1-m & r-j-i+m+1 \end{pmatrix}, \\
\zeta_{A_1^{(i)}}\zeta_{B_2^{(j)}} &= \zeta_{A_2^{(i)}}\zeta_{B_1^{(j)}} = \zeta_Y \text{ with } Y = \begin{pmatrix} 0 & i \\ j & r-j-i \end{pmatrix}, \\
\zeta_{A_2^{(i)}}\zeta_{B_2^{(j)}} &= \sum_{m=0}^{\min\{i,j\}} \zeta_{X_{m-1}} = \zeta_{A_1^{(i)}}\zeta_{B_1^{(j)}} + \zeta_{A_1^{(i)}}\zeta_{B_2^{(j)}}.
\end{aligned}$$

Since

$$\zeta_Y \neq 0 \Longleftrightarrow r-j-i \geqslant 0 \Longleftrightarrow j \leqslant r-i,$$

it follows that

$$\dim \zeta_{\lambda^{(i)}}\mathcal{K}_\lambda\zeta_{\lambda^{(j)}} = \begin{cases} 2, & \text{if } 1 \leqslant j \leqslant r-i; \\ 1, & \text{if } r-i < j \leqslant r-1. \end{cases}$$

Hence,

$$\dim \zeta_{\lambda^{(i)}}\mathcal{K}_\lambda = \sum_{j=0}^{r} \dim \zeta_{\lambda^{(i)}}\mathcal{K}_\lambda\zeta_{\lambda^{(j)}} = 2r+1-i.$$

Consequently, we obtain that

$$\begin{aligned}
\dim \mathcal{K}_\lambda &= \sum_{i=0}^{r} \dim \zeta_{\lambda^{(i)}}\mathcal{K}_\lambda = 2(r+1) + \sum_{i=1}^{r-1}(2r+1-i) = \frac{3r^2+3r+2}{2} \\
&= \dim S_0(2,r) - \dim S_0(2,r-1)/\mathcal{K}_\mu.
\end{aligned}$$

This together with the surjectivity of ϕ implies that ϕ is an isomorphism. □

The proposition above together with (3.6.2) and (3.6.3) implies that ϕ induces a surjective algebra homomorphism

$$\bar{\phi} : A_0(r) \longrightarrow A_0(r-1)$$

taking $e_1 \mapsto 0$ and $e_i \mapsto e_{i-1}$, for $2 \leqslant i \leqslant r-1$ with $\operatorname{Ker} \bar{\phi} = A_0(r)e_1A_0(r)$. Similarly, ψ induces a surjective algebra homomorphism

$$\bar{\psi} : A_0(r) \longrightarrow A_0(r-1)$$

taking $e_{r-1} \mapsto 0$ and $e_i \mapsto e_i$, for $1 \leqslant i \leqslant r-2$ with $\operatorname{Ker} \bar{\psi} = A_0(r)e_{r-1}A_0(r)$.

Corollary 3.8. *Suppose $r \geqslant 2$. Then $A_0(r) \cong \mathbb{F}\Delta_r/\mathcal{I}_r$, where $\mathcal{I}_r$ is the ideal of $\mathbb{F}\Delta_r$ generated by $\{\beta_1\alpha_1,\ \alpha_{r-2}\beta_{r-2},\ \beta_i\alpha_i - \alpha_{i-1}\beta_{i-1}$ for $2 \leqslant i \leqslant r-2\}$.*

Proof. Obviously, $A_0(2) \cong \mathbb{F} \cong \mathbb{F}\Delta_1$. By the proof of [19, Th. 2.1], we have

$$A_0(3) \cong \mathbb{F}\Delta_2/\langle \beta_1\alpha_1, \alpha_1\beta_1\rangle \text{ and } A_0(4) \cong \mathbb{F}\Delta_3/\langle \beta_1\alpha_1, \alpha_1\beta_1 - \beta_2\alpha_2, \alpha_2\beta_2\rangle.$$

Hence, the assertion is true for $r = 2, 3, 4$. Applying the surjective homomorphisms $\bar{\phi}$ and $\bar{\psi}$ together with an induction on r proves the assertion for all $r \geqslant 2$. □

Theorem 3.9. *Let $r \geqslant 2$. Then $S_0(2, r)$ is representation-finite (resp., tame, wild) if $r \leqslant 5$ (resp., $r = 6$, $r \geqslant 7$).*

Proof. By (3.6.1), $S_0(2, r)$ and $A_0(r)$ have the same representation type. It is clear that $A_0(r)$ is representation-finite for $r \leqslant 4$. By applying the covering technique developed in [29, 30], we obtain that there are 40 isolasses of indecomposable $A_0(5)$-modules. Hence, $A_0(5)$ is also representation-finite.

By [31] and [32, Th. 4.2], $A_0(6)$ is a selfinjective algebra of tubular type which is tame.

It remains to show that $A_0(r)$ is wild for $r \geqslant 7$. The universal cover of $A_0(r)$ has the following form

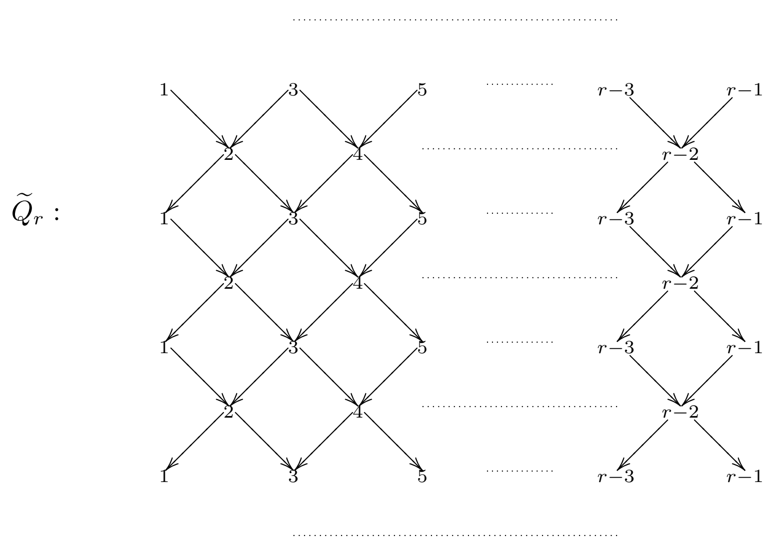

FIG.4. *Universal cover of $A_0(r)$*

with all squares commutative and all paths $1 \longrightarrow 2 \longrightarrow 1$ and $r-1 \longrightarrow r-2 \longrightarrow r-1$ being zero. The quiver $\widetilde{Q}_r$ contains a full subquiver of the form

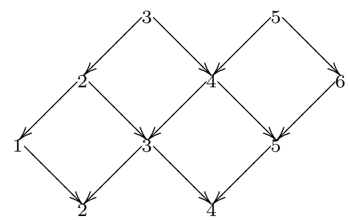

FIG.5. *Subquiver of* $\widetilde{Q}_r$

with all squares commutative which gives rise to a wild algebra. Hence, $A_0(r)$ is wild for $r \geqslant 7$. □

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