On Pseudo-BCI Algebras with Condition (pP)

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Abstract: As a generalization of BCI-algebras with condition (S) and pseudo-BCK algebras with condition (pP), the notion of pseudo-BCI algebras with condition (pP) is introduced. An example of pseudo-BCI algebras with condition (pP) is given, the result that every anti-grouped pseudo-BCI algebras is with condition (pP) is proved. Some properties of pseudo-BCI algebras with condition (pP) are investigated and the branch properties of pseudo-BCI algebras with condition (pP) are discussed.
Keywords:pseudo-BCK algebra;pseudo-BCI algebra;pseudo-BCI algebra with condition (pP)

1 Introduction

BCK/*BCI*-algebras with condition (S) is an important class of *BCK*/*BCI*-algebras which was first introduced by K. Iseki.

The notion of pseudo-*BCK* algebras with condition (pP) is introduced by A.Iorgulescu in [3], which is an extension of the notion of *BCK*-algebras with condition (S). In this paper, We introduce the notion of pseudo-*BCI* algebras with condition (pP), which is a generalization of *BCI*-algebras with condition (S) and pseudo-*BCK* algebras with condition (pP), then we studied some properties of pseudo-*BCI* algebras with condition (pP).

2 Preliminaries

Definition 2.1[1] A pseudo-*BCI* algebra is a structure $(X; \leq, \rightarrow, \rightsquigarrow, 1)$, where "$\leq$" is a binary relation on X, "$\rightarrow$" and "$\rightsquigarrow$" are binary operations on X and "1" is an element of X, verifying the axioms: for all $x, y, z\in X$,

(1) $y\rightarrow z\leq(z\rightarrow x)\rightsquigarrow(y\rightarrow x)$, $y\rightsquigarrow z\leq(z\rightsquigarrow x)\rightarrow(y\rightsquigarrow x)$;

(2) $x\leq(x\rightarrow y)\rightsquigarrow y$, $x\leq(x\rightsquigarrow y)\rightarrow y$;

(3) $x\leq x$;

(4) $x\leq y, y\leq x \Rightarrow x=y$;

(5) $x\leq y \Leftrightarrow x\rightarrow y =1 \Leftrightarrow x\rightsquigarrow y =1$.

If $(X; \leq, \rightarrow, \rightsquigarrow, 1)$ is a pseudo-*BCI* algebra satisfying $x\leq 1$ for all $x\in X$, then $(X; \leq, \rightarrow, \rightsquigarrow, 1)$ is a pseudo-*BCK* algebra.

If $(X; \leq, \rightarrow, \rightsquigarrow, 1)$ is a pseudo-*BCI* algebra satisfying $x\rightarrow y = x\rightsquigarrow y$ for all $x, y\in X$, then $(X; \rightarrow, 1)$ is a *BCI*-algebra.

Theorem 2.2[1, 10] Let $(X; \leq, \rightarrow, \rightsquigarrow, 1)$ be a pseudo-*BCI* algebra, then X satisfy the following properties $(\forall x, y, z\in X)$:

(1) $1\leq x \Rightarrow x=1$;

(2) $x\leq y \Rightarrow y\rightarrow z\leq x\rightarrow z$, $y\rightsquigarrow z\leq x\rightsquigarrow z$;

(3) $x\leq y, y\leq z \Rightarrow x\leq z$;

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(4) $x\rightsquigarrow(y\to z)=y\to(x\rightsquigarrow z)$;

(5) $x\le y\to z \Leftrightarrow y\le x\rightsquigarrow z$;

(6) $x\to y\le(z\to x)\to(z\to y)$, $x\rightsquigarrow y\le(z\rightsquigarrow x)\rightsquigarrow(z\rightsquigarrow y)$;

(7) $x\le y \Rightarrow z\to x\le z\to y$, $z\rightsquigarrow x\le z\rightsquigarrow y$;

(8) $1\to x=x$, $1\rightsquigarrow x=x$;

(9) $((y\to x)\rightsquigarrow x)\to x=y\to x$, $((y\rightsquigarrow x)\to x)\rightsquigarrow x=y\rightsquigarrow x$;

(10) $x\to y\le(y\to x)\rightsquigarrow 1$, $x\rightsquigarrow y \le(y\rightsquigarrow x)\to 1$;

(11) $(x\to y)\to 1=(x\to 1)\rightsquigarrow(y\to 1)$, $(x\rightsquigarrow y)\rightsquigarrow 1=(x\rightsquigarrow 1)\to(y\rightsquigarrow 1)$;

(12) $x\to 1=x\rightsquigarrow 1$.

Theorem 2.3[1] A structure $(X; \le, \to, \rightsquigarrow, 1)$ is a pseudo-*BCI* algebra if and only if it satisfies $(\forall x, y, z\in X)$:

(1) $y\to z\le(z\to x)\rightsquigarrow(y\to x)$, $y\rightsquigarrow z\le(z\rightsquigarrow x)\to(y\rightsquigarrow x)$;

(2) $x\le y, y\le x \Rightarrow x=y$;

(3) $x\le y \Leftrightarrow x\to y =1 \Leftrightarrow x\rightsquigarrow y =1$;

(4) $1\to x=x$, $1\rightsquigarrow x=x$.

Theorem 2.4[12] Let $(X; \le, \to, \rightsquigarrow, 1)$ be a pseudo-*BCI* algebra, denote

$K(X)=\{x\in X \mid x\le 1\}$.

We say $K(X)$ is the pseudo-*BCK* part of X.

Definition 2.5[9] A *BZ*-algebra is an algebra $(X; \to, 1)$ of type (2,0) in which the following axioms are satisfied:

(1) $(y\to z)\to((x\to y)\to(x\to z))=1$,

(2) $x\to x=1$,

(3) $1\to x=x$,

(4) if $x\to y= y\to x=1$, then $x=y$.

Theorem 2.6[10] A structure $(X; \to, \rightsquigarrow, 1)$ is a pseudo-*BCI* algebra if and only if it satisfies:

(1) $(X; \to, 1)$ is a *BZ*-algebra;

(2) $(X; \rightsquigarrow, 1)$ is a *BZ*-algebra;

(3) $\forall x, y\in X$, $x\to y=1 \Leftrightarrow x\rightsquigarrow y=1$;

(4) $\forall x, y, z\in X$, $x\rightsquigarrow(y\to z)=y\to(x\rightsquigarrow z)$.

The notion of branches in pseudo-*BCI* algebras was introduced in [13]. The set $V(a)=\{x\in X \mid x\le a\}$ is called a branch of pseudo-*BCI* algebra X, where a is a maximal element of X.

Theorem 2.7[13] Let a, b be two maximal elements of a pseudo-*BCI* algebra X. Then $(\forall x, y\in X)$:

(1) $x\in V(a), y\in V(b)\Rightarrow x\to y\in V(a\to b)$, $x\rightsquigarrow y\in V(a\rightsquigarrow b)$;

(2) $x\to y\in V(1)\Leftrightarrow y\to x\in V(1)$, $x\rightsquigarrow y\in V(1)\Leftrightarrow y\rightsquigarrow x\in V(1)$;

(3) $x\to y\in V(1)\Leftrightarrow x, y\in V(a)$; $x\rightsquigarrow y\in V(1)\Leftrightarrow x, y\in V(a)$, for some maximal element a;

(4) for $x\in V(b)$, $x\to a=b\to a$, $x\rightsquigarrow a=b\rightsquigarrow a$;

(5) for $a\neq b$, $V(a)\cap V(b)=\varnothing$.

Definition 2.8[3, 9] A pseudo-*BCK* algebra $(X;\le,\to,\rightsquigarrow,1)$ is called with condition (pP) (pseudo product) if it satisfies

(pP) for all $x, y\in X$, $\min\{z\in A\,|\, x\le y\to z\}=\min\{z\in A\,|\, y\le x\rightsquigarrow z\}$ exists (denoted by $x\otimes y$).

Definition 2.9[10] A pseudo-*BCI* algebra $(X;\le,\to,\rightsquigarrow,1)$ is called an anti-grouped pseudo-*BCI* algebra if it satisfies:

(G1) $\forall x, y, z\in X$, $(x\to y)\to(x\to z)=y\to z$.

(G2) $\forall x, y, z\in X$, $(x\rightsquigarrow y)\rightsquigarrow(x\rightsquigarrow z)=y\rightsquigarrow z$.

Theorem 2.10[10] Let $(X;\le,\to,\rightsquigarrow,1)$ be an anti-grouped pseudo-*BCI* algebra. Define

$\forall x, y\in X$, $x\cdot y=(x\rightsquigarrow 1)\rightsquigarrow y=(y\to 1)\to x$.

Then $(X;\cdot,1)$ is a non-commutative group.

Definition 2.11[12] Let $(X;\le,\to,\rightsquigarrow,1)$ be a pseudo-*BCI* algebra, denote

$AG(X)=\{x\in X\,|\,(x\to 1)\to 1=x\}$.

We say $AG(X)$ is the anti-grouped part of X.

3 Pseudo-*BCI* algebras with condition (pP)

Let $(X;\le,\to,\rightsquigarrow,1)$ be a pseudo-*BCI* algebra, we denote $A(x, y)=\{t\in X\,|\, y\le x\rightsquigarrow t\}=\{t\in X\,|\, x\le y\to t\}$, $\forall x, y\in X$.

Lemma 3.1 Let $(X;\le,\to,\rightsquigarrow,1)$ be a pseudo-*BCI* algebra and $a\in X$. Then the following conditions are equivalent:

(1) a is maximal;

(2) $(a\to 1)\to 1=a$;

(3) there is $x\in X$ such that $a=x\to 1$;

(4) $x\to a=(a\to 1)\rightsquigarrow(x\to 1)$;

(5) $(a\to x)\to 1=x\to a$.

Proof: (1)⇔(2)⇔(3) (see [11]).

(2)⇒(4). By (2) and Theorem 2.2, $(a\to 1)\rightsquigarrow(x\to 1)=x\to((a\to 1)\rightsquigarrow 1)=x\to((a\to 1)\to 1)=x\to a$.

(4)⇒(5). By (4) and Theorem 2.2, $(a\to x)\to 1=(a\to 1)\rightsquigarrow(x\to 1)=x\to a$.

(5)⇒(2). $(a\to 1)\to 1=1\to a=a$.

Theorem 3.2 Let x and y be two elements in a pseudo-*BCI* algebra $(X;\le,\to,\rightsquigarrow,1)$, then $a=(y\rightsquigarrow(x\rightsquigarrow 1))\to 1$ is the greatest element of $A(x, y)$ and $A(x, y)\subseteq V(a)$. Especially, if $K(X)$ is the pseudo-*BCK* part of X, then $\{1, x, y\}\subseteq A(x, y)\subseteq K(X)$, $\forall x, y\in K(X)$.

Proof: Since

$y\le(y\rightsquigarrow(x\rightsquigarrow 1))\to(x\rightsquigarrow 1)=x\rightsquigarrow((y\rightsquigarrow(x\rightsquigarrow 1))\to 1)=x\rightsquigarrow a$,

we have $a\in A(x, y)$. By $y\le x\rightsquigarrow a$, we get $y\rightsquigarrow(x\rightsquigarrow a)=1$, then $a\to 1=a\to(y\rightsquigarrow(x\rightsquigarrow a))=y\rightsquigarrow(x\rightsquigarrow 1)$.

$\forall t\in A(x, y)$, $y\le x\rightsquigarrow t$, that is $y\rightsquigarrow(x\rightsquigarrow t)=1$, then

$t\to 1= t\to(y\rightsquigarrow(x\rightsquigarrow t))= y\rightsquigarrow(x\rightsquigarrow 1)$.

So $a\to 1= t\to 1$. By Lemma 3.1 (3), a is a maximal element of X, applying Lamme 3.1 (4), we have

$t\to a=(a\to 1)\rightsquigarrow(t\to 1)=1$,

so $t\leq a$. Therefore, a is the greatest element of X, $t\in V(a)$ and $A(x, y)\subseteq V(a)$.

The next part of Theorem 3.2 is obviously true.

The following example shows that $A(x, y)$ may not has the least element.

Example 3.3 Let $X=\{a, b, c, d, 1\}$ with two binary operations as Table 1 and Table 2. Then $(X; \leq, \to, \rightsquigarrow, 1)$ is a pseudo-*BCI* algebra, where $x\leq y$ if and only if $x\to y =1$. $A(a, b)=\{a, b, 1\}$ has not any least element.

Table 1 Definition of "$\to$"

$\to$	a	b	c	d	1
a	1	b	c	c	1
b	a	1	c	d	1
c	c	c	1	b	c
d	c	c	1	1	c
1	a	b	c	d	1

Table 2 Definition of "$\rightsquigarrow$"

$\rightsquigarrow$	a	b	c	d	1
a	1	b	c	d	1
b	a	1	c	c	1
c	c	c	1	a	c
d	c	c	1	1	c
1	a	b	c	d	1

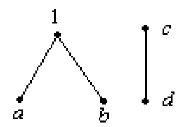

Figure 1 Hasse diagram

However there is a class of pseudo-*BCI* algebras such that any $A(x, y)$ has the least element. Let us introduce the following definition.

Definition 3.4 A pseudo-*BCI* algebra $(X; \leq, \to, \rightsquigarrow, 1)$ is called to be with condition (pP), if $A(x, y)$ has the least element, for all $x, y\in X$.

The least element of $A(x, y)$ is usually denoted by $x\otimes y$. By Definition 3.4, $x\otimes y$ is the element in X satisfies the following conditions:

(1) $y\leq x\rightsquigarrow(x\otimes y)$, $x\leq y\to(x\otimes y)$;

(2) $y\leq x\rightsquigarrow t\Rightarrow x\otimes y\leq t$, $x\leq y\to t\Rightarrow x\otimes y\leq t$. $\forall t\in X$.

Obviously, a pseudo-*BCI* algebra $(X; \leq, \to, \rightsquigarrow, 1)$ is with condition (pP) if and only if it satisfies one of the following conditions: $\forall x, y\in X$,

(1) $A(x, y)$ is bounded;

（2） min{$t\in X \mid y\leq x\rightsquigarrow t$} (or min{$t\in X \mid x\leq y\to t$}) exists;

（3） the inequality $y\leq x\rightsquigarrow u$ (or $x\leq y\to u$) with u as the unknown has the least solution.

Now, we will give an example of pseudo-*BCI* algebra with condition (pP).

Example 3.5 Let $X=\{a, b, c, d, 1\}$ with two binary operations as Table 3 and Table 4. Then $(X; \leq, \to, \rightsquigarrow, 1)$ is a pseudo-*BCI* algebra with condition (pP), where $x\leq y$ if and only if $x\to y=1$. The "⊗" multiplication table of X is as Table 5.

Table 3 Definition of "→"

→	*a*	*b*	*c*	*d*	1
a	1	*b*	*c*	*d*	1
b	1	1	*c*	*c*	1
c	*c*	*c*	1	*a*	*c*
d	*c*	*c*	1	1	*c*
1	*a*	*b*	*c*	*d*	1

Table 4 Definition of "⇝"

⇝	*a*	*b*	*c*	*d*	1
a	1	*b*	*c*	*c*	1
b	1	1	*c*	*c*	1
c	*c*	*c*	1	*b*	*c*
d	*c*	*c*	1	1	*c*
1	*a*	*b*	*c*	*d*	1

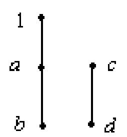

Figure 2 Hasse diagram

Table 5 Definition of "⊗"

→	*a*	*b*	*c*	*d*	1
a	*a*	*b*	*d*	*d*	*a*
b	*b*	*b*	*d*	*d*	*b*
c	*c*	*d*	*b*	*b*	*c*
d	*d*	*d*	*b*	*b*	*d*
1	*a*	*b*	*c*	*d*	1

By Theorem 3.2, we have the following result.

Theorem 3.6 If $(X; \leq, \to, \rightsquigarrow, 1)$ is a pseudo-*BCI* algebra with condition (pP), then the pseudo-*BCK* part of X is a pseudo-*BCK* algebra with condition (pP).

Lemma 3.7[11] Let $(X; \leq, \to, \rightsquigarrow, 1)$ be an anti-grouped pseudo-*BCI* algebra. Then ($\forall x, y\in X$):

（1） $x\leq y \Rightarrow x=y$;

（2） $x=(x\to y)\rightsquigarrow y$, $x=(x\rightsquigarrow y)\to y$.

Theorem 3.8 Every anti-grouped pseudo-*BCI* algebra $(X; \leq, \to, \rightsquigarrow, 1)$ is with condition (pP) in which $x\otimes y=(y\rightsquigarrow 1)\to x$ for all $x, y\in X$.

Proof: $\forall x, y\in X$, by Theorem 3.2, $a=(y\rightsquigarrow(x\rightsquigarrow 1))\to 1$ is the greatest element of $A(x, y)$. Since X is an anti-grouped pseudo-*BCI* algebra, every element in X is maximal.By Lemma 3.1**,** $a=(y\rightsquigarrow(x\rightsquigarrow 1))\to 1=(y\rightsquigarrow(x\to 1))\to 1=(x\to(y\rightsquigarrow 1))\to 1=(y\rightsquigarrow 1)\to x$. By Lemma 3.7 (1)**,** $A(x, y)=\{t\in X \mid y\leq x\rightsquigarrow t\}=\{t\in X \mid y=x\rightsquigarrow t\}$, we have $a=(y\rightsquigarrow 1)\to x=((x\rightsquigarrow t)\rightsquigarrow 1)\to x=((x\rightsquigarrow 1)\to(t\rightsquigarrow 1))\to x=$

$(t\rightsquigarrow((x\rightsquigarrow 1)\to 1))\to x=(t\rightsquigarrow x)\to x=t$, thus $A(x, y)=\{a\}$.

Hence a is the least element of $A(x, y)$. Therefore, X is with condition (pP) and $x\otimes y=(y\rightsquigarrow 1)\to x$.

Theorem 3.9 Let $(X; \leq, \to, \rightsquigarrow, 1)$ be a pseudo-*BCI* algebra with condition (pP), then the following holds: for all $x, y, z\in X$.

(1) $x\otimes y\leq z \Leftrightarrow x\leq y\to z \Leftrightarrow y\leq x\rightsquigarrow z$;

(2) $(x\to y)\otimes x\leq y$; $x\otimes(x\rightsquigarrow y)\leq y$;

(3) $z\to(y\to x)=(z\otimes y)\to x$; $z\rightsquigarrow(y\rightsquigarrow x)=(y\otimes z)\rightsquigarrow x$;

(4) $x\leq y \Rightarrow x\otimes z\leq y\otimes z$; $z\otimes x\leq z\otimes y$;

(5) $(x\otimes y)\otimes z= x\otimes(y\otimes z)$;

(6) $(y\to z)\otimes(x\to y)\leq x\to z$; $(x\rightsquigarrow y)\otimes(y\rightsquigarrow z)\leq x\rightsquigarrow z$;

(7) $x\otimes 1=1\otimes x=x$;

(8) $x\otimes(y\to z)\leq y\to(x\otimes z)$; $(y\rightsquigarrow z)\otimes x\leq y\rightsquigarrow(z\otimes x)$;

(9) $x\to y\leq(x\otimes z)\to(y\otimes z)$; $x\rightsquigarrow y\leq(z\otimes x)\rightsquigarrow(z\otimes y)$.

Proof: (1) By Definition 3.4.

(2) By the above (1) and $x\to y\leq x\to y$, we have $(x\to y)\otimes x\leq y$. Similarly, we can proof $x\otimes(x\rightsquigarrow y)\leq y$.

(3) $z\leq (z\to(y\to x))\rightsquigarrow(y\to x)= y\to((z\to(y\to x))\rightsquigarrow x)$, By the above (1), $z\otimes y\leq(z\to(y\to x))\rightsquigarrow x$, hence $z\to(y\to x)\leq((z\to(y\to x))\rightsquigarrow x)\to x\leq(z\otimes y)\to x$. On the other hand, $z\leq y\to(z\otimes y)\leq((z\otimes y)\to x)\rightsquigarrow(y\to x)$, hence $(z\otimes y)\to x\leq(((z\otimes y)\to x)\rightsquigarrow(y\to x))\to(y\to x)\leq z\to(y\to x)$. Therefore, $z\to(y\to x)=(z\otimes y)\to x$. Similarly, we can proof $z\rightsquigarrow(y\rightsquigarrow x)=(y\otimes z)\rightsquigarrow x$.

(4) If $x\leq y$, then $z\leq y\rightsquigarrow(y\otimes z)\leq x\rightsquigarrow(y\otimes z)$, so we have $x\otimes z\leq y\otimes z$. Similarly, we can proof $x\leq y \Rightarrow z\otimes x\leq z\otimes y$.

(5) $\forall b\in X$, By the above (1) and (3), $(x\otimes y)\otimes z\leq b\Leftrightarrow x\otimes y\leq z\to b\Leftrightarrow x\leq y\to(z\to b)\Leftrightarrow x\leq(y\otimes z)\to b\Leftrightarrow x\otimes(y\otimes z)\leq b$, hence $(x\otimes y)\otimes z= x\otimes(y\otimes z)$.

(6) Since $y\to z\leq(x\to y)\to(x\to z)$, so $(y\to z)\otimes(x\to y)\leq x\to z$. Similarly, $(x\rightsquigarrow y)\otimes(y\rightsquigarrow z)\leq x\rightsquigarrow z$.

(7) By the above (2), $1\otimes x=(x\to x)\otimes x\leq x$. In addition, $x\leq 1\rightsquigarrow(1\otimes x)=1\otimes x$. Therefore, $1\otimes x=x$. Similarly, $x\otimes 1= x$.

(8) By the above (2), $(y\to z)\otimes y\leq z$. By the above (4), $x\otimes((y\to z)\otimes y)\leq x\otimes z$. By the above (5), $(x\otimes(y\to z))\otimes y\leq x\otimes z$, thus $x\otimes(y\to z)\leq y\to(x\otimes z)$. Similarly, $(y\rightsquigarrow z)\otimes x\leq y\rightsquigarrow(z\otimes x)$.

(9) By the above (2), $(x\to y)\otimes x\leq y$. By the above (4), $((x\to y)\otimes x)\otimes z\leq y\otimes z$. By the above (5), $(x\to y)\otimes(x\otimes z)\leq y\otimes z$. Therefore, $x\to y\leq(x\otimes z)\to(y\otimes z)$. Similarly, we can proof $x\rightsquigarrow y\leq(z\otimes x)\rightsquigarrow(z\otimes y)$.

By Theorem 3.9 (4), (5), (7), we know that if $(X; \leq, \to, \rightsquigarrow, 1)$ is a pseudo-*BCI* algebra with condition (pP), then $(X; \leq, \otimes, 1)$ is a partially ordered monoid with 1 as the unit element.

Theorem 3.10 Let $(X; \leq, \to, \rightsquigarrow, 1)$ be a pseudo-*BCI* algebra with condition (pP), then X is a *BCI*-algebra if and only if $x\otimes y= y\otimes x$, for all $x, y\in X$.

Proof: $\forall b\in X$, $x\otimes y\le b \Leftrightarrow x\le y\to b \Leftrightarrow x\le y\rightsquigarrow b \Leftrightarrow y\otimes x\le b$, hence $x\otimes y= y\otimes x$, for all $x, y\in X$.

Conversely, for all $x, y, z\in X$, $x\le y\to z \Leftrightarrow x\otimes y\le z\Leftrightarrow y\otimes x\le z \Leftrightarrow x\le y\rightsquigarrow z$, thus $y\to z=y\rightsquigarrow z$. Therefore, X is a *BCI*-algebra.

We now give a characterization of a pseudo-*BCI* algebra with condition (pP).

Theorem 3.11 A structure $(X; \otimes, \le, \to, \rightsquigarrow, 1)$ is a pseudo-*BCI* algebra with condition (pP) if and only if the following hold ($\forall x, y, z\in X$):

(1) $y\to z\le(z\to x)\rightsquigarrow(y\to x)$, $y\rightsquigarrow z\le(z\rightsquigarrow x)\to(y\rightsquigarrow x)$;

(2) $x\le y$, $y\le x \Rightarrow x=y$;

(3) $x\le y \Leftrightarrow x\to y =1 \Leftrightarrow x\rightsquigarrow y =1$;

(4) $1\to x=x$, $1\rightsquigarrow x=x$;

(5) $x\to(y\to z)=(x\otimes y)\to z$; $x\rightsquigarrow(y\rightsquigarrow z)=(y\otimes x)\rightsquigarrow z$.

Proof: By Theorem 2.3 and Theorem 3.9 (3), we only need to show the sufficiency. By the above (5), we have

$x\to(y\to(x\otimes y))=(x\otimes y)\to (x\otimes y)=1$,

and $x\otimes y$ is a solution of the inequality $x\le y\to u$ with u as the unknown. Also, for any solution t of $x\le y\to u$, since

$(x\otimes y)\to t= x\to(y\to t)=1$,

we obtain $x\otimes y\le t$. Hence $x\otimes y$ is the least solution of $x\le y\to u$. Therefore, X is a pseudo-*BCI* algebra with condition (pP)

It is easy to proof the following lemma.

Lemma 3.12 Let $(X; \le, \to, \rightsquigarrow, 1)$ be a pseudo-*BCI* algebra, $K(X)$ and $AG(X)$ respectively the pseudo-*BCK* part and anti-grouped part of X. Then

（1） $K(X)\cap AG(X)=\{1\}$;

（2） X is a pseudo-*BCK* algebra if and only if $AG(X)=\{1\}$;

（3） X is an anti-grouped pseudo-*BCI* algebra if and only if $K(X)=\{1\}$.

Theorem 3.13 Let $(X; \le, \to, \rightsquigarrow, 1)$ be a pseudo-*BCI* algebra with condition (pP). Then $(X; \otimes, 1)$ is a non-commutative group if and only if $(X; \le, \to, \rightsquigarrow, 1)$ is an anti-grouped pseudo-*BCI* algebra.

Proof: Let $(X; \otimes, 1)$ be a non-commutative group and x is an element in the pseudo-*BCK* part $K(X)$ of X. Denoe x^{-1} for the inverse element of x. Since 1 is the unit element of X, we have $x\otimes x^{-1}=1$. By Theorem 3.9 (4) (7) and $x\le 1$, we obtain

$1= x\otimes x^{-1}\le 1\otimes x^{-1}= x^{-1}$, that is $x^{-1}=1$. Hence

$x=x\otimes 1= x\otimes x^{-1}=1$.

Therefore, $K(X)=\{1\}$. By Lemma 3.12 (3), X is an anti-grouped pseudo-*BCI* algebra.

Conversely, Let $(X; \le, \to, \rightsquigarrow, 1)$ be an anti-grouped pseudo-*BCI* algebra. By Theorem 2.10, $(X; \cdot, 1)$ is a non-commutative group, where $x\cdot y= (x\rightsquigarrow 1)\rightsquigarrow y= (y\to 1)\to x$. $\forall x, y\in X$. By Theorem 3.8, $x\otimes y=(y\rightsquigarrow 1)\to x=(y\to 1)\to x$. Hence $x\cdot y= x\otimes y$. Therefore, $(X; \otimes, 1)$ is just the non-commutative group $(X; \cdot, 1)$.

Theorem 3.14 Suppose that $V(a)$ is a branch of a pseudo-*BCI* algebra $(X; \le, \to, \rightsquigarrow, 1)$ with condition (pP). If $V(a)$ is bounded, so is every branch $V(b)$ of X.

Proof: Let m_a be the least element of $V(a)$ and let

$m_b= m_a\otimes(a\to 1)\otimes b$.

$\forall x\in V(b)$, since $b\in V(b)$, by Theorem 2.7 (3), we have $b\to x\in V(1)$. Since $a\in V(a)$, $1\in V(1)$, by Theorem 2.7 (1), $a\to 1\in V(a\to 1)$. Applying Theorem 2.7 (1) again, we have

$(a\to 1)\to(b\to x)\in V((a\to 1)\to 1)= V(a)$.

Then $m_a\to((a\to 1)\to(b\to x))=1$ by m_a being the least element of $V(a)$. by Theorem 3.9 (3), we can get

$m_b\to x=(m_a\otimes(a\to 1)\otimes b)\to x= m_a\to((a\to 1)\to(b\to x))=1$.

Hence $m_b\leq x$. As a special case, we have $m_b\leq b$, thus $m_b\in V(b)$. Hence m_b is the least element of $V(b)$. Therefore, $V(b)$ is bounded.

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具有条件(pP)的伪-BCI 代数

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摘要：作为具有条件(S)的 BCI-代数和具有条件(pP)的伪-BCK 代数的推广，引入了具有条件(pP)的伪-BCI 代数的概念。给出了具体的例子，证明了群逆伪-BCI 代数都是具有条件(pP)的伪-BCI 代数，研究了具有条件(pP)的伪-BCI 代数的性质和它的分支的性质。

关键词：伪-BCK 代数；伪-BCI 代数；具有条件(pP)的伪-BCI 代数

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