On Tor-tilting Modules[1]

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Abstract

Let R be a ring. A right R-module U is called *Tor-tilting* if $\mathrm{Cogen}(U^+) = U^{\cdot\cdot}$, where $U^+ = \mathrm{Hom}_{\mathbf{Z}}(U, \mathbf{Q}/\mathbf{Z})$ and $U^{\cdot\cdot} = \mathrm{KerTor}_1^R(U, -)$. Some characterizations of Tor-tilting modules are given. Among others, it is shown that U_R is Tor-tilting if and only if U^+ is cotitling. Moreover, both tilting modules and completely faithful flat modules are proved to be Tor-tilting. Some properties of torsion theories induced by a Tor-tilting module are also investigated.

Keywords: cotilting module, tilting module, Tor-tilting module, torsion theory.

1 Introduction

The relationship among projective modules, injective modules and flat modules is well known in homological algebra theory. Projective generators, injective cogenerators and completely faithful flat modules can be regarded jointly from a similar aspect. Tilting modules and cotilting modules, which generalize the projective generators and injective cogenerators respectively, have drawn more and more academic interest in representation theory and homological algebra theory. The reader is referred to [1] and [2] for some fundamental theory on the above mentioned objects.

Now we have the following three "similar triangles"

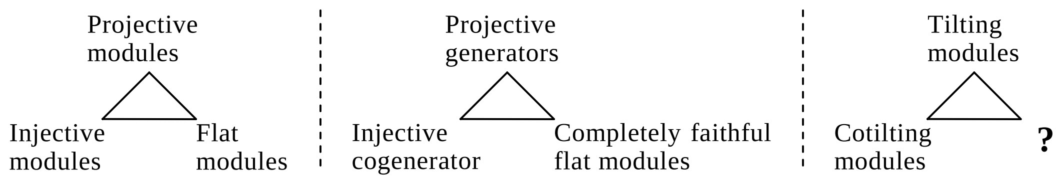

Figure 1: Three "similar triangles"

The motivation of the present paper is the "**?**" in the last triangle in Figure 1. We shall introduce the notion of Tor-tilting modules which play the role of "**?**".

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Recall from [2] that a right R-module T is said to be *tilting* in case it satisfies the following three equivalent conditions:

(1) GenT_R = $T^{\perp}$, where GenT_R denotes the class of modules generated by T_R and

$$T^{\perp} = \mathrm{KerExt}_R^1(T, -) = \{M \in \text{Mod-}R \mid \mathrm{Ext}_R^1(T, M) = 0\}.$$

(2) (i) proj.dim$T_R \le 1$; (ii) $\mathrm{Ext}_R^1(T, T^{(\alpha)}) = 0$ $(\forall \alpha)$; (iii) $\mathrm{KerHom}_R(T, -)$ ' $T^{\perp} = \{0\}$.

(3) (i) proj.dim $T_R \le 1$; (ii) $\mathrm{Ext}_R^1(T, T^{(\alpha)}) = 0$ $(\forall \alpha)$; (iii) there exists an exact sequence

$$0 \to R_R \to T_0 \to T_1 \to 0$$

where T_0, $T_1 \in \mathrm{Add}T_R$ = {$M \in$ Mod-R | M is isomorphic to a direct summand of a direct sum of copies of T_R}.

If the tilting module T_R is finitely presented, then "$\mathrm{Ext}_R^1(T, T^{(\alpha)}) = 0$" in the above conditions can be replaced by "$\mathrm{Ext}_R^1(T, T) = 0$", simultaneously, "Add$T_R$" can be replaced by "addT_R", which indicates the class of modules isomorphic to a direct summand of a finite direct sum of copies of T_R. In this case, T_R is called a *classical tilting module*.

Dually, a left R-module $_RW$ is said to be *cotilting* provided the following three equivalent conditions are satisfied:

(1) Cogen $_RW = {}^{\perp}W$, where Cogen $_RW$ is the class of modules cogenerated by $_RW$ and

$${}^{\perp}W = \mathrm{KerExt}_R^1(-, W) = \{M \in R\text{-Mod} \mid \mathrm{Ext}_R^1(M, W) = 0\}.$$

(2) (i) inj.dim $_RW \le 1$; (ii) $\mathrm{Ext}_R^1(W^{\alpha}, W) = 0$ $(\forall \alpha)$; (iii) $\mathrm{KerHom}_R(-, W)$ ' ${}^{\perp}W = \{0\}$.

(3) (i) inj.dim $_RW \le 1$; (ii) $\mathrm{Ext}_R^1(W^{\alpha}, W) = 0$ $(\forall \alpha)$; (iii) there exists an exact sequence

$$0 \to W_1 \to W_0 \to C \to 0$$

where W_0, $W_1 \in \mathrm{Prod}_RW$ = {$M \in R$-Mod | M is isomorphic to a direct summand of a direct product of copies of $_RW$} and $_RC$ is an injective cogenerator of R-Mod.

It is natural to consider right R-modules U with flat.dim$_RU \le 1$ such that $\mathrm{Tor}_1^R(U, (U^{(\alpha)})^+) = 0$ (for all cardinals α) and Ker($U \otimes_R$ -) ' $U^{¨} = 0$, where $U^{¨}$ = $\mathrm{KerTor}_1^R(U, -) = \{M \in R\text{-Mod} \mid \mathrm{Tor}_1^R(U, M) = 0\}$.

Throughout R is an associative ring with identity and all modules are unitary. R-Mod and Mod-R indicate the category of left and right R-modules, respectively. The projective, injective and flat dimension of a module M are denoted, respectively, by prod.dim$M$, inj.dim$M$ and flat.dim$M$. The reader is also referred to [1] and [2] for undefined terms and notations.

2 Main results

Let us start with the following definition.

Definition. A right R-module U is said to be *Tor-tilting* provided Cogen(U^+) = $\mathrm{KerTor}_1^R(U, -)$.

Next, we give some characterizations for a Tor-tilting module U_R.

Theorem 1. *The following are equivalent for a right R-module U.*

(1) *U_R is Tor-tilting.*

(2) U^{+} *is cotilting*.

(3) U_R *satisfies the following three conditions*:

(i) flat.dim $_RU \leq 1$;

(ii) $\mathrm{Tor}_1^R(U, (U^{(\alpha)})^{+}) = 0$ *for all cardinals* α; *and*

(iii) $\mathrm{Ker}(U\otimes_R -)\,'\,\ \mathrm{KerTor}_1^R(U, -) = \{0\}$.

(4) U_R *satisfies the following three conditions*:

(i) flat.dim $_RU \leq 1$;

(ii) $\mathrm{Tor}_1^R(U, (U^{(\alpha)})^{+}) = 0$ *for all cardinals* α; *and*

(iii) *There exists an exact sequence* $0\to V_1 \to V_0 \to C \to 0$, *where* $V_0, V_1 \in \mathrm{Prod}_R(U^{+})$ *and* $_RC$ *is an injective cogenerator of* R-Mod.

(5) $\mathrm{Tor}_1^R(U, U^{+}) = 0$ *and* $(\mathrm{Ker}(U\otimes_R -), \mathrm{KerTor}_1^R(U, -))$ *is a torsion theory*.

Proof. (1)⇔(2). Note that there is a natural isomorphism

$$(\mathrm{Tor}_1^R(U, M))^{+} \cong \mathrm{Ext}_R^1(M, U^{+})$$

for every left R-module M (see [9, p. 360]). It follows that $\mathrm{KerTor}_1^R(U, -) = \mathrm{KerExt}_R^1(-, U^{+})$ which guarantees (1)⇔(2).

(2)⇔(3)⇔(4) follows by the natural isomorphisms

$$(U\otimes_R M)^{+} \cong \mathrm{Hom}_R(M, U^{+}) \text{ and } (\mathrm{Tor}_1^R(U, M))^{+} \cong \mathrm{Ext}_R^1(M, U^{+}).$$

(1)⇒(5). Suppose U_R is Tor-tilting. It follows that $\mathrm{Tor}_1^R(U, U^{+}) = 0$ by the equivalence of (1) and (3). Moreover, $\mathrm{Cogen}(U^{+})$ is always closed under submodules and products. Simultaneously, $\mathrm{Cogen}(U^{+})$ is closed under extensions since $\mathrm{Cogen}(U^{+}) = \mathrm{KerTor}_1^R(U, -)$. Thus, $\mathrm{Cogen}(U^{+})$ is a torsion-free class in R-Mod. One the other hand, we have

$$\mathrm{Hom}_R(M, \mathrm{Cogen}(U^{+})) = 0 \Leftrightarrow \mathrm{Hom}_R(M, U^{+}) = 0 \Leftrightarrow (U\otimes_R M)^{+} = 0 \Leftrightarrow U\otimes_R M = 0 \Leftrightarrow M \in \mathrm{Ker}(U\otimes_R -).$$

By [2, Proposition 1.4.2(2)], $(\mathrm{Ker}(U\otimes_R -), \mathrm{KerTor}_1^R(U, -)) = (\mathrm{Ker}(U\otimes_R -), \mathrm{Cogen}(U^{+}))$ is a torsion theory.

(5)⇒(3). Suppose (5) then $\mathrm{Ker}(U\otimes_R -)\,'\,\ \mathrm{KerTor}_1^R(U, -) = \{0\}$ and $\mathrm{KerTor}_1^R(U, -)$ is closed under submodules and products. Now, for any left R-module M, we have an exact sequence

$$0\to K \to F \to M \to 0,$$

where $_RF$ is free and hence $\mathrm{KerTor}_1^R(U, F) = 0$. This implies $\mathrm{KerTor}_1^R(U, K) = 0$ since $\mathrm{KerTor}_1^R(U, -)$ is closed under submodules. The following long exact sequence

$$\ldots \to \mathrm{Tor}_2^R(U, F) \to \mathrm{Tor}_2^R(U, M) \to \mathrm{Tor}_1^R(U, K) \to \ldots$$

forces $\mathrm{Tor}_2^R(U, M) = 0$. This shows flat.dim $_RU \leq 1$. Moreover, (3)(ii) follows since $\mathrm{KerTor}_1^R(U, -)$ is closed under products and the hypothesis that $\mathrm{Tor}_1^R(U, U^{+}) = 0$. ■

Given a right R-module U_R and a left R-module ${}_RM$, let

$$\mathrm{Ann}_M(U) = \{m \in M \mid u \otimes m = 0 \in U \otimes_R M \text{ for all } w \in W\}.$$

Recall that U_R is said to be *completely faithful* in case $\mathrm{Ann}_M(U) = 0$ for every left R-module M. It is well known that a flat right R-module V is completely faithful if and only if $V \otimes_R M \neq 0$ whenever ${}_RM \neq 0$. We refer the reader to [1, Exercise 19.18-21] for details. The following Proposition is an immediate consequence of Theorem 1.

Proposition 1. *Every completely faithful flat module is Tor-tilting.*

Let P be a right R-module with endomorphism ring A. Take an arbitrary cogenerator Q_R of the Mod-R and put $K_A = \mathrm{Hom}_R(P, Q)$. Denote by T_P the covariant functor $-\otimes_A P$ and H_P the covariant functor $\mathrm{Hom}_R(P, -)$. Recall from [5] that P_R is called a **-module* if the pair (T_P, H_P) defines an equivalence:

$$T_P\colon \mathrm{Cogen}K_A \text{ F } \mathrm{Gen}P_R\colon H_P.$$

It is proved by Colpi and Menini [5, Proposition 1.2(2)] that $\mathrm{Cogen}K_A = \{L \in \text{Mod-}A \mid \mathrm{Tor}_1^A(L, P) = 0\}$ in case P_R is a *-module. So we have

Proposition 2. *Let P be an abelian group and $R = \mathrm{End}_{\mathbf{Z}}P$. If ${}_{\mathbf{Z}}P$ is a *-module then P_R is Tor-tilting.*

Combining Theorem 1, Proposition 2 and the following result, one can see the similarity among the “triangles” in Figure 1.

Theorem 2. *Every tilting module is Tor-tilting.*

Proof. Suppose that T_R is a tilting module. Then $\text{flat.dim}T_R \leq \text{proj.dim}T_R \leq 1$. Moreover, if

$${}_RM \in \mathrm{Ker}(T \otimes_R -) \text{ ' } \mathrm{KerTor}_1^R(T, -)$$

then $M^+ \in \mathrm{KerHom}_R(T, -) \text{ ' } \mathrm{KerExt}_R^1(T, -) = 0$ and hence $M = 0$. This shows that

$$\mathrm{Ker}(T \otimes_R -) \text{ ' } \mathrm{KerTor}_1^R(T, -) = 0.$$

Now, let $0 \to K \to P \to T \to 0$ be a projective resolution of T_R, where P and K are projective. Then we have the following commutative diagram with the first row exact

$$\begin{array}{ccccccc}
\mathrm{Tor}_1^R(P,(T^{(\alpha)})^+) \to & \mathrm{Tor}_1^R(T,(T^{(\alpha)})^+) & \longrightarrow & K \otimes_R (T^{(\alpha)})^+ & \xrightarrow{f} & P \otimes_R (T^{(\alpha)})^+ \\
 & & & \downarrow g & & \downarrow \\
 & & & (\mathrm{Hom}(K,T^{(\alpha)}))^+ & \xrightarrow{h} & (\mathrm{Hom}(P,T^{(\alpha)}))^+
\end{array}$$

where $\mathrm{Tor}_1^R(P, (T^{(\alpha)})^+) = 0$ since P is projective. We complete the proof by showing that f is a monomorphism. Note that the following exact sequence

$$0 \to \mathrm{Hom}_R(T, T^{(\alpha)}) \to \mathrm{Hom}_R(P, T^{(\alpha)}) \to \mathrm{Hom}_R(K, T^{(\alpha)}) \to \mathrm{Ext}_R^1(T, T^{(\alpha)}) = 0$$

guarantees that h is a monomorphism. In addition, we claim that g is a monomorphism. Indeed, let $K \oplus L = R^{(\beta)}$ for some cardinal β without loss of generality. Then we have the following commutative diagram

$$\begin{array}{l}
(K \oplus L) \otimes (T^{(\alpha)})^+ = R^{(\beta)} \otimes (T^{(\alpha)})^+ \cong ((T^{(\alpha)})^+)^{(\beta)} \hookrightarrow ((T^{(\alpha)})^+)^{\beta} \cong ((T^{(\alpha)})^{(\beta)})^+ \\
\qquad\qquad \downarrow \sigma \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow \\
(\mathrm{Hom}(K \oplus L, T^{(\alpha)}))^+ = (\mathrm{Hom}(R^{(\beta)}, T^{(\alpha)}))^+ \cong ((\mathrm{Hom}(R, T^{(\alpha)}))^{\beta})^+ \cong ((T^{(\alpha)})^{\beta})^+
\end{array}$$

where σ is the canonical homomorphism. It is easy to see that σ is injective and, consequently, g is a monomorphism. ■

Recall that a torsion theory (T, F) is *cohereditary* [8] if F is closed under factor modules. For a Tor-tilting module U_R, we have a torsion theory

$$(\mathrm{Ker}(U\otimes_R -), \mathrm{KerTor}_1^R(U, -))$$

by Theorem 1. Now for any module $M \in \mathrm{KerTor}_1^R(U, -)$ with $K \leq M$, we have an exact sequence

$$0 \to K \to M \to M/K \to 0,$$

which induces the long exact sequence

$$0 = \mathrm{Tor}_1^R(U, M) \to \mathrm{Tor}_1^R(U, M/K) \to U\otimes_R K \to U\otimes_R M \to U\otimes_R (M/K) \to 0$$

It is easy to see that $\mathrm{Tor}_1^R(U, M/K)$ if and only if $U\otimes_R K \to U\otimes_R M$ is monic. Therefore, we have the following

Proposition 3. *The torsion theory* $(\mathrm{Ker}(U\otimes_R -), \mathrm{KerTor}_1^R(U, -))$ *induced by a Tor-tilting module* U_R *is cohereditary if and only if* U_R *is M-flat for each* $M \in \mathrm{KerTor}_1^R(U, -)$.

Given a torsion theory $\tau = (\mathsf{T}, \mathsf{F})$. Recall from [7] that a module M is said to be *τ-finitely generated* if $M/K \in \mathsf{T}$ for some finitely generated submodule K of M. M is said to be *τ-finitely presented* if there is an exact sequence

$$0 \to K \to F \to M \to 0$$

where F is finitely generated free module and K is τ-finitely.

Let ${}_SU_R$ be a bimodule such that U_R is Tor-tilting and M is τ-finitely generated with respect to

$$\tau = (\mathrm{Ker}(U\otimes_R -), \mathrm{KerTor}_1^R(U, -)).$$

Then $U\otimes(M/K) = 0$ for some finitely generated submodule K of M. So we have an exact sequence

$$R^n \to M \to M/K \to 0$$

where n is a positive integer. But this yields an epimorphism $U\otimes_R R^n \to U\otimes_R M$ of left S-modules. This means that $U\otimes_R M$ is finitely generated by ${}_SU$.

If there is an exact sequence $0 \to K \to F \to M \to 0$, where K has a finitely generated submodule K_1 such that $K/K_1 \in \mathrm{Ker}(U\otimes_R -)$. Then we have the following commutative exact diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & K/K_1 & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \to & K_1 & \to & F & \to & N & \to & 0\\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \to & K & \longrightarrow & F & \to & M & \to & 0\\
 & & \downarrow & & & & \downarrow & & \\
 & & K/K_1 & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

where $N \cong F/K_1$ is finitely presented and the right column guarantees $U \otimes N \cong U \otimes M$ since $U \otimes (K/K_1) = 0$. Consequently, we have the following result

Proposition 4. *Let* ${}_SU_R$ *be a bimodule such that* U_R *is Tor-tilting and* $\tau = (\mathrm{Ker}(U \otimes_R -), \mathrm{KerTor}_1^R(U, -))$ *be the torsion theory induced by* U_R.

(1) *If* ${}_RM$ *is* τ*-finitely generated then the left S-module* $U \otimes_R M$ *is finitely generated by* ${}_SU$.

(2) *If* ${}_RM$ *is* τ*-finitely presented then* $U \otimes N \cong U \otimes M$ *for some finitely presented left R-module* ${}_RN$.

3 Final instructions

It would be interesting to investigate whether a Tor-tilting module induces equivalence and duality for some module categories. On the other hand, one may be interested to characterize a ring R which is τ-coherent [7] with respect to a torsion theory $(\mathrm{Ker}(U \otimes_R -), \mathrm{KerTor}_1^R(U, -))$ induced by a Tor-tilting module U_R.

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