On the Estimate for a Mean Value Relative to $\frac{4}{p} = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}$

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Abstract. For the positive integer n, let $f(n)$ denote the number of positive integer solutions $(n_1,\, n_2,\, n_3)$ of the Diophantine equation

$$\frac{4}{n} = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}.$$

For the prime number p, $f(p)$ can be split into $f_1(p) + f_2(p)$, where $f_i(p)(i = 1,\, 2)$ counts those solutions with exactly i of denominators $n_1,\, n_2,\, n_3$ divisible by p.

Recently Terence Tao proved that

$$\sum_{p<x} f_1(p) \ll x \exp\Big(\frac{c \log x}{\log\log x}\Big)$$

with other results. In this paper we shall improve it to

$$\sum_{p<x} f_1(p) \ll x \log^5 x \log\log^2 x.$$

1. Introduction

For the positive integer n, let $f(n)$ denote the number of positive integer solutions $(n_1,\, n_2,\, n_3)$ of the Diophantine equation

$$\frac{4}{n} = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}.$$

Erdös and Straus conjectured that for all $n \geq 2$, $f(n) > 0$. It is still an open problem now although there are some partial results.

In 1970, R. C. Vaughan[5] showed that the number of $n < x$ for which $f(n) = 0$ is at most $x \exp(-c \log^{\frac{2}{3}} x)$, where x is sufficiently large and c is a positive constant.

Recently Terence Tao[4] studied the situation in which n is the prime number p. He gave lower bound and upper bound for the mean value of

$f(p)$. Precisely, he split $f(p)$ into $f_1(p)+f_2(p)$, where $f_i(p)(i=1,\,2)$ counts those solutions with exactly i of denominators n_1, n_2, n_3 divisible by p. He proved that

$$x\log^2 x \ll \sum_{p<x} f_1(p) \ll x\exp(\frac{c\log x}{\log\log x}) \tag{1}$$

and

$$x\log^2 x \ll \sum_{p<x} f_2(p) \ll x\log^2 x\log\log x, \tag{2}$$

where p denotes the prime number, x is sufficiently large and c is a positive constant.

For the progress and some explanation on the estimate in (2), one can see [2]. In this paper we shall improve the upper bound in (1).

Theorem. Let p denote the prime number. Then for sufficiently large x, we have

$$\sum_{p<x} f_1(p) \ll x\log^5 x\log\log^2 x.$$

Throughout this paper, let p denote the prime number, c denote the positive constant, $p(n)$ be the least prime factor of n, $P(n)$ be the largest prime factor of n, $d(n)$ be the divisor function, $\varphi(n)$ be the Euler totient function, $\Omega(n)$ be the number of prime factors of n with multiplicity.

2. Some preliminaries

Lemma 1. Let

$$g(x) = a_n x^n + \cdots + a_1 x + a_0$$

be the polynomial in integer coefficients, $G(n)$ be the number of solutions to the congruence equation

$$g(x) \equiv 0 \ (\mathrm{mod}\, n).$$

Then $G(n)$ is a multiplicative function.

One can see page 34 of [1].

Lemma 2. Let $g(x)$ be the polynomial in integer coefficients. If

$$g(x) \equiv 0, \quad g'(x) \equiv 0 \ (\mathrm{mod}\, p)$$

have no common solution, then the number of solutions to

$$g(x) \equiv 0 \pmod{p^l}$$

is equal to that to

$$g(x) \equiv 0 \pmod{p}.$$

One can see page 36 of [1].

Lemma 3. For the fixed integer l, let $G(n)$ be the number of solutions to the congruence equation

$$4lx^2 + 1 \equiv 0 \pmod{n}.$$

Then

$$G(n) \leq d(n).$$

Proof. By Lemma 1, we know

$$G(p_1^{l_1} \cdots p_s^{l_s}) = G(p_1^{l_1}) \cdots G(p_s^{l_s}).$$

Write

$$g(x) = 4lx^2 + 1.$$

Then $g'(x) = 8lx$. It is obvious that

$$g(x) \equiv 0, \quad g'(x) \equiv 0 \pmod{p}$$

has no common solution. Thus Lemma 2 claims

$$G(p^l) = G(p).$$

It is easy to see that the congruence equation

$$4lx^2 + 1 \equiv 0 \pmod{p}$$

has at most two solutions. Therefore

$$G(p^l) = G(p) \leq 2 \leq d(p^l).$$

The conclusion of Lemma 3 follows.

Lemma 4. For $x \geq 2$, we have

$$\sum_{n \leq x} \frac{d^2(n)}{n} \ll \log^4 x.$$

Proof. Theorem 2 in [3] asserts that

$$\sum_{n \leq x} d^2(n) \ll x \log^3 x.$$

Then

$$\begin{aligned}\sum_{n \leq x} \frac{d^2(n)}{n} &\leq \sum_{i \leq \log_2 x} \sum_{2^i \leq n < 2^{i+1}} \frac{d^2(n)}{n} \\ &\ll \sum_{i \leq \log_2 x} i^3 \\ &\ll \log^4 x.\end{aligned}$$

Lemma 5. Let

$$\Psi(x,\, y) = \sum_{\substack{n \leq x \\ P(n) \leq y}} 1.$$

Then for $x \geq 10$, we have

$$\Psi(x,\, \log x \log\log x) \ll \exp(\frac{3 \log x}{(\log\log x)^{\frac{1}{2}}}).$$

This is Lemma 1 in [3].

Lemma 6. The estimate

$$\sum_{\substack{Z^{\frac{1}{2}} \leq n \\ P(n) \leq Z^{\frac{1}{r}}}} \frac{d^2(n)}{n} \ll \exp(\sum_{p \leq Z} \frac{d^2(p)}{p} - \frac{r}{10} \log r)$$

holds true for $1 \leq r \leq \frac{\log Z}{\log\log Z}$ uniformly.

This is a special case of Lemma 4 in [3].

3. The proof of Theorem

According to the discussion in the beginning of section 3 of [4], in order to estimate

$$\sum_{p < x} f_1(p),$$

it is enough to estimate

$$\sum_{\substack{a,\,l\\ al\le x}} \frac{xd(4la^2+1)}{\varphi(4al)\log(1+\frac{x}{al})}. \tag{3}$$

Using the bound

$$\varphi(n) \gg \frac{n}{\log\log n},$$

we should estimate

$$x\log\log x \sum_{\substack{a,\,l\\ al\le x}} \frac{d(4la^2+1)}{al\log(1+\frac{x}{al})}$$

or

$$x\log\log x \sum_{i\le \log_2 x}\ \sum_{j\le \log_2 x - i} \frac{1}{1+\log_2 x - i - j}\cdot$$
$$\cdot \frac{1}{2^{i+j}} \sum_{2^i<a\le 2^{i+1}}\ \sum_{2^j<l\le 2^{j+1}} d(4la^2+1).$$

Now we consider the estimate for the sum

$$\sum_{V<l\le 2V}\ \sum_{W<a\le 2W} d(4la^2+1). \tag{4}$$

We shall use some ideas from [3].

Firstly assume that $V \le W$. Let

$$Z = W^{\frac{1}{20}}. \tag{5}$$

Write n uniquely as

$$n = p_1^{s_1}\cdots p_j^{s_j}\cdot p_{j+1}^{s_{j+1}}\cdots p_r^{s_r}, \qquad p_1<\cdots<p_j<p_{j+1}<\cdots<p_r,$$

where

$$p_1^{s_1}\cdots p_j^{s_j} \le Z < p_1^{s_1}\cdots p_j^{s_j}p_{j+1}^{s_{j+1}}.$$

We can decompose $4la^2+1$ as

$$4la^2+1 = (p_1^{s_1}\cdots p_j^{s_j})(p_{j+1}^{s_{j+1}}\cdots p_r^{s_r}) = b(l,\,a)c(l,\,a), \tag{6}$$

where

$$b(l,\,a)\le Z, \qquad (b(l,\,a),\,c(l,\,a))=1.$$

We shall discuss in four cases as in [3].

Case I. $p(c(l,\, a)) > Z^{\frac{1}{2}}$.

Since $p(c(l,\, a)) > Z^{\frac{1}{2}}$, $d(c(l,\, a)) = O(1)$. Thus

$$d(4la^2+1) = d(b(l,\, a))d(c(l,\, a)) \ll d(b(l,\, a)).$$

Lemmas 3 and 4 yield that

$$\begin{aligned}\sum\nolimits_{\mathrm{I}} &\ll \sum_{b\le Z} d(b) \sum_{V<l\le 2V} \sum_{\substack{W<a\le 2W\\ 4la^2+1\equiv 0 \,(\mathrm{mod}\, b)}} 1\\ &\ll \sum_{b\le Z} d(b) \sum_{V<l\le 2V} \frac{W}{b} \sum_{\substack{a=1\\ 4la^2+1\equiv 0 \,(\mathrm{mod}\, b)}}^{b} 1\\ &\ll VW \sum_{b\le Z} \frac{d^2(b)}{b}\\ &\ll VW \log^4(2W).\end{aligned}$$

Case II. $p(c(l,\, a)) \le Z^{\frac{1}{2}}$, $b(l,\, a) \le Z^{\frac{1}{2}}$.

Write $p = p(c(l,\, a))$. Then $p^s \| 4la^2+1$, $p \le Z^{\frac{1}{2}}$. The fact that $b(l,\, a) \le Z^{\frac{1}{2}}$, $b(l,\, a)p^s > Z$ yields $p^s > Z^{\frac{1}{2}}$. Let s_p be the smallest s such that $p^s > Z^{\frac{1}{2}}$. Thus $s_p \ge 2$. On the other hand, $p^{\frac{s_p}{2}} \le p^{s_p-1} \le Z^{\frac{1}{2}} \Longrightarrow p^{s_p} \le Z$. Now we have

$$\frac{1}{p^{s_p}} \le \min(\frac{1}{Z^{\frac{1}{2}}},\, \frac{1}{p^2}).$$

Hence

$$\begin{aligned}\sum_{p\le Z^{\frac{1}{2}}} \frac{1}{p^{s_p}} &\le \sum_{p\le Z^{\frac{1}{4}}} \frac{1}{Z^{\frac{1}{2}}} + \sum_{Z^{\frac{1}{4}}<p} \frac{1}{p^2}\\ &\ll Z^{-\frac{1}{4}}.\end{aligned}$$

Lemmas 3 yields that

$$\sum\nolimits_{\mathrm{II}} \ll W^{\varepsilon} \sum_{p\le Z^{\frac12}} \sum_{V<l\le 2V} \sum_{\substack{W<a\le 2W\\ 4la^2+1\equiv 0 \,(\mathrm{mod}\, p^{s_p})}} 1$$
$$\ll W^{\varepsilon} \sum_{p\le Z^{\frac12}} \sum_{V<l\le 2V} \frac{W}{p^{s_p}} \sum_{\substack{a=1\\ 4la^2+1\equiv 0 \,(\mathrm{mod}\, p^{s_p})}}^{p^{s_p}} 1$$
$$\ll VW^{1+\varepsilon} \sum_{p\le Z^{\frac12}} \frac{1}{p^{s_p}}$$
$$\ll VW^{1-\frac{1}{80}+\varepsilon} \ll VW,$$

where $p^{s_p} \le Z$ works.

Case III. $p(c(l,\, a)) \le \log W \log\log W$, $b(l,\, a) > Z^{\frac12}$.

We have $p(c(l,\, a)) \le \log W \log\log W \Longrightarrow P(b(l,\, a)) < \log W \log\log W$. Then Lemmas 3 and 5 yield that

$$\sum\nolimits_{\mathrm{III}} \ll W^{\varepsilon} \sum_{\substack{Z^{\frac12}<b\le Z\\ P(b)<\log W \log\log W}} \sum_{V<l\le 2V} \sum_{\substack{W<a\le 2W\\ 4la^2+1\equiv 0 \,(\mathrm{mod}\, b)}} 1$$
$$\ll W^{\varepsilon} \sum_{\substack{Z^{\frac12}<b\le Z\\ P(b)<\log W \log\log W}} \frac{d(b)}{b} VW$$
$$\ll VW^{1+2\varepsilon} Z^{-\frac12} \sum_{\substack{b\le Z\\ P(b)<\log W \log\log W}} 1$$
$$\ll VW^{1-\frac{1}{40}+2\varepsilon} \Psi(W,\, \log W \log\log W)$$
$$\ll VW^{1-\frac{1}{40}+3\varepsilon} \ll VW.$$

Case IV. $\log W \log\log W < p(c(l,\, a)) \le Z^{\frac12}$, $b(l,\, a) > Z^{\frac12}$.

Let

$$r_0 = [\frac{\log Z}{\log(\log W \log\log W)}].$$

Since

$$\log W \log\log W > Z^{\frac{1}{r_0+1}},$$

for $2 \le r \le r_0$, we consider these $(l,\, a)$ which satisfy

$$Z^{\frac{1}{r+1}} < p(c(l,\, a)) \le Z^{\frac1r}$$

so that

$$P(b(l,\, a)) < p(c(l,\, a)) \le Z^{\frac{1}{r}}.$$

We have

$$\Omega(c(l,\, a)) \le \frac{3\log W}{\log p(c(l,\, a))} \le \frac{3(r+1)\log W}{\log Z} \le 60(r+1) \le 120r$$

so that

$$d(c(l,\, a)) \le A^r,$$

where A is a positive constant.

Lemmas 3, 4 and 6 yield that

$$\begin{aligned}
\sum\nolimits_{\mathrm{IV}} &\ll \sum_{2\le r\le r_0} A^r \sum_{\substack{Z^{\frac{1}{2}}<b\le Z\\ P(b)<Z^{\frac{1}{r}}}} d(b) \sum_{V<l\le 2V} \sum_{\substack{W<a\le 2W\\ 4la^2+1\equiv 0\,(\mathrm{mod}\, b)}} 1\\
&\ll VW \sum_{2\le r\le r_0} A^r \sum_{\substack{Z^{\frac{1}{2}}<b\le Z\\ P(b)<Z^{\frac{1}{r}}}} \frac{d^2(b)}{b}\\
&\ll VW \sum_{2\le r\le r_0} A^r \exp\Big(\sum_{p\le Z} \frac{d^2(p)}{p} - \frac{r}{10}\log r\Big)\\
&= VW \sum_{2\le r\le r_0} A^r \exp\Big(\sum_{p\le Z} \frac{4}{p} - \frac{r}{10}\log r\Big)\\
&\ll VW \sum_{2\le r\le r_0} A^r \exp\Big(4\log\log Z - \frac{r}{10}\log r\Big)\\
&\ll VW \log^4 Z \sum_{r=2}^{\infty} A^r \exp\Big(-\frac{r}{10}\log r\Big)\\
&\ll VW \log^4(2W),
\end{aligned}$$

where the series is convergent.

Then assume that $W < V$. In this situation, we shall change the role of l and r and shall consider the linear congruence equation

$$4a^2 l + 1 \equiv 0\,(\mathrm{mod}\, n)$$

for the fixed a. This situation is simpler than previous one. We can get the similar estimate as above.

Combining all of above, we get

$$\sum_{V<l\leq 2V}\sum_{W<a\leq 2W} d(4la^2+1) \ll VW\log^4(2W).$$

Hence,

$$\begin{aligned}
& x\log\log x \sum_{i\leq \log_2 x}\sum_{j\leq \log_2 x-i} \frac{1}{1+\log_2 x-i-j}\cdot \\
& \quad\cdot \frac{1}{2^{i+j}} \sum_{2^i<a\leq 2^{i+1}}\sum_{2^j<l\leq 2^{j+1}} d(4la^2+1) \\
& \ll x\log^4 x\log\log x \sum_{i\leq \log_2 x}\sum_{j\leq \log_2 x-i} \frac{1}{1+\log_2 x-i-j} \\
& \ll x\log^4 x\log\log x \sum_{i\leq \log_2 x}\sum_{1\leq h\leq \log_2 x-i+1} \frac{1}{h} \\
& \ll x\log^4 x\log\log x \sum_{i\leq \log_2 x} \log(\log_2 x-i+2) \\
& \ll x\log^5 x\log\log^2 x.
\end{aligned}$$

So far the proof of Theorem is finished.

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