THE HESSENBERG MATRIX AND THE RIEMANN MAPPING

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Abstract. We show in this paper that, if μ is a regular measure whose support is a Jordan arc or a connected finite union of Jordan arcs in the complex plane $\mathbb{C}$, then the limits of the elements of the diagonals of the Hessenberg matrix D of μ, whenever those limits exist, determine the coefficients of the Laurent series expansion of the Riemann mapping $\phi(z)$ which maps conformally the exterior of the unit disk onto the exterior of the support of the measure μ. Moreover, in the case of an arc of the unit circle, we use this result to show how to approximate the Riemann mapping of the support of μ from the entries of the Hessenberg matrix D.

1. Introduction

In this paper, we consider regular Borel measures μ defined on subsets of the complex plane that are Jordan arcs, or connected finite union of Jordan arcs, and we show how the entries of the Hessenberg matrix D associated with μ determines the Riemann mapping that takes the complement of the closed unit disk $\overline{\mathbb{D}}$ to the complement of the support of μ. In particular, the support of μ is determined by the Hessenberg matrix associated with μ.

The Riemann mapping theorem says, in its most common statement (see, for example, [1]), that given a simply connected domain $\Omega \subsetneq \mathbb{C}$ and given $z_0 \in \Omega$ there is a unique analytic function $\phi : \mathbb{D} \longrightarrow \Omega$ ($\mathbb{D}$ the open unit disk), such that $\phi(0) = z_0$ and $\phi'(z_0) > 0$, which defines a one-to-one mapping of $\mathbb{D}$ onto Ω. However, we will use an equivalent formulation for domains containing ∞ which can be found, for example, in [26, 20, 16]. In this case, the Riemann mapping theorem states that, for every $\Gamma \subset \mathbb{C}$ compact, which is not a point, such that $\mathbb{C}_\infty \setminus \Gamma$ is simply connected, there is a unique conformal mapping $\phi : \mathbb{C}_\infty \setminus \overline{\mathbb{D}} \to \mathbb{C}_\infty \setminus \Gamma$, such that $\phi(\infty) = \infty$ and $\phi'(\infty) > 0$, where $\phi'(\infty) = \operatorname{cap}(\Gamma)$ is the capacity of Γ. Moreover, if Γ is a simple Jordan curve, $\phi(z)$ is continuous in the unit circle $\mathbb{T}$.

There exists a well-known link between the Riemann mapping and the Green function, which has been described in the literature on potential theory (see, for example, [9]). If we denote by $\Phi(z)$ the inverse of $\phi(z)$ as defined in the previous paragraph, then the Green function for a compact set K with $\operatorname{cap}(K) > 0$, $g_K(z,\infty)$, can be obtained from the Riemann conformal mapping $\Phi(z)$ which takes $\mathbb{C}_\infty \setminus P_c(K)$ onto the exterior of $\overline{\mathbb{D}}$, where $P_c(K)$ is the polynomial convex hull of K. Moreover, if $\Phi(z) = \sum_{k=-1}^{\infty} c_{-k} z^{-k}$, $c_1 > 0$, in a neighborhood of ∞, then

$$g_K(z,\infty) = \begin{cases} \log|\Phi(z)| & \text{if } z \in \mathbb{C}_\infty \setminus P_c(K) \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $c_1 = \dfrac{1}{\mathrm{cap}(K)}$.

In the real case, the Hessenberg matrix agrees with the tridiagonal Jacobi matrix. Rakhmanov's theorem [21, 22] states that, if the support of a Borel measure is $[-1, 1]$ and $\mu' > 0$ almost everywhere in $[-1, 1]$, and

$$J = \begin{pmatrix} b_0 & a_1 & 0 & \dots \\ a_1 & b_1 & a_2 & \dots \\ 0 & a_2 & b_2 & \dots \\ 0 & 0 & a_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is the Jacobi matrix associated with μ, then $a_n \to \dfrac{1}{2}$ and $b_n \to 0$.

Note that in this case, the Riemann mapping of the interval $[-1, 1]$ is

$$\phi(z) = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{1}{2}z + 0 + \frac{1}{2}\frac{1}{z}$$

Moreover, under the conditions of Rakhmanov's theorem, if $\mathrm{supp}(\mu) = [a, b]$, then the above limits are $a_n \to \dfrac{b-a}{4}$ and $b_n \to \dfrac{a+b}{2}$. In this case, the Riemann mapping is

$$\phi(z) = \frac{b-a}{4}z + \frac{a+b}{2} + \frac{b-a}{4}\frac{1}{z}.$$

Conversely, P. Nevai established in [19] that, if $a_n \to a > 0$ and $b_n \to b$, then the support is $[-2a + b, b + 2a] \cup e$ (where e is at most a denumerable set of isolated points). Moreover, Nevai proved the equivalence between the existence of those limits and the ratio asymptotic of orthonormal polynomials.

Generalizations of Rakhmanov's theorem to orthogonal polynomials, and to orthogonal matrix polynomials on the unit circle, have been given in [18] and [28]. The case of orthogonal polynomials on an arc of circumference has been studied in [3].

As a final introductory motivating fact, we consider the Hessenberg matrix D associated with a measure μ on $\mathbb{T}$,

$$D = \begin{pmatrix} d_{1,1} & d_{1,2} & d_{1,3} & \dots \\ d_{2,1} & d_{2,2} & d_{2,3} & \dots \\ 0 & d_{3,2} & d_{3,3} & \dots \\ 0 & 0 & d_{4,3} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then, if μ is of Szegő class [12], then $\lim_{n\to\infty} d_{n+1,n} = 1$ and $\lim_{n\to\infty} d_{n-k,n} = 0$, for all $k \in \mathbb{N}$. On the other hand, $\phi(z) = z$.

In this paper we show a theorem relating the limits of the elements of the diagonals of the Hessenberg matrix of a measure μ, with the coefficients of the Laurent series expansion of the Riemann mapping of the support of μ. Specifically, we show that, if μ is a regular measure whose support is a Jordan arc or a connected union of Jordan arcs in the complex plane $\mathbb{C}$, then the limits of the values at the diagonals of the Hessenberg matrix D of μ, whenever those limits exist, determine the coefficients of the series expansion of the Riemann mapping $\phi(z)$ which applies conformally the exterior of the unit disk onto the exterior of the support of the measure.

There exist some previous results relating the properties of D and the support of μ. For example, if the Hessenberg matrix D defines a subnormal operator [15] in ℓ^2, then the closure of the convex hull of its numerical range agrees with the convex hull of its spectrum. On the other hand, the spectrum of the matrix D contains the spectrum of its minimal normal extension $N = \text{men}(D)$ which is precisely the support of the measure [7].

The organization of the paper is as follows: In Section 2 we prove the main theorem relating the limits of the elements of the diagonals of the Hessenberg matrix of a measure μ, with the coefficients of the Laurent series expansion of the Riemann mapping of the support of μ. In Section 3 we show that the Riemann mapping of the support of μ can be approximated from the entries of the Hessenberg matrix D. The last Section is devoted to several heuristic examples to illustrate the approximation results given in previous section.

For general information on the theory of orthogonal polynomials, we recommend the books [5, 26] by T. S. Chihara and G. Szegő, respectively, and the survey [14] by L. Golinskii and V. Totik.

2. The Diagonals Theorem

Let μ be a Borel probability measure in the complex plane, with support $\text{supp}(\mu)$, containing infinitely many points. Let $\mathcal{P}$ be the space of polynomials. The associated inner product is given by the expression

$$\langle Q(z), R(z)\rangle_\mu = \int_{\text{supp}(\mu)} Q(z)\overline{R(z)}d\mu(z),$$

for $Q, R \in \mathcal{P}$. Then there exists a unique orthonormal polynomials sequence (ONPS) $\{P_n(z)\}_{n=0}^{\infty}$ associated to the measure μ [5, 10, 26].

In the space $\mathcal{P}^2(\mu)$, closure of the polynomials space $\mathcal{P}$ in $L^2_\mu(\Omega)$, we consider the multiplication by z operator. Let $D = (d_{ij})_{i,j=1}^{\infty}$ be the infinite upper Hessenberg matrix of this operator in the basis of ONPS $\{P_n(z)\}_{n=0}^{\infty}$, hence

$$zP_n(z) = \sum_{k=0}^{n+1} d_{k+1,n+1}P_k(z), \quad n \geq 0, \tag{1}$$

with $P_0(z) = 1$.

It is a well-known fact that the monic polynomials are the characteristic polynomials of the finite sections of D.

In order to state our main result, we will require the measure μ to be regular with support a connected finite union of Jordan arcs, and we will also need to consider an auxiliary Toeplitz matrix. We next recall the definitions of all these notions.

A Jordan arc in $\mathbb{C}$ is any subset of $\mathbb{C}$ homeomorphic to the closed interval $[0, 1]$ on the real line.

A measure μ is regular if $\lim_{n\to\infty} \dfrac{1}{\sqrt[n]{\gamma_n}} = \text{cap}(\text{supp}(\mu))$, the capacity of the support of μ, where the γ_n are the leading coefficients of the orthonormal polynomials, i.e., $P_n(z) = \gamma_n z^n + \cdots$.

An infinite matrix $T = (a_{i,j})_{i,j=1}^{\infty}$ is a Toeplitz matrix if each descending diagonal from left to right is constant, i.e, there exists $(a_i)_{i\in\mathbb{Z}}$ such that

$$T = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \dots \\ a_1 & a_0 & a_{-1} & \dots \\ a_2 & a_1 & a_0 & \dots \\ a_3 & a_2 & a_1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Given a Toeplitz matrix T, the Laurent series whose coefficients are the entries a_i defines a function known as the symbol of T.

We are now in a position to state and prove the main result of the paper.

Theorem 1 (The diagonals theorem)**.** *Let $D = (d_{ij})_{i,j=1}^{\infty}$ be the Hessenberg matrix associated with a measure μ with compact support on the complex plane. Assume that:*

1. *The measure μ is regular with support $\operatorname{supp}(\mu)$ a Jordan arc or a connected finite union of Jordan arcs Γ such that $\mathbb{C}_\infty \setminus \Gamma$ is a simply connected set of the Riemann sphere $\mathbb{C}_\infty$.*
2. *There exists a Hessenberg-Toeplitz matrix T with its rows in ℓ^1, such that $D - T$ defines a compact operator in ℓ^2.*

Then, the symbol of T agrees with the Riemann mapping $\phi : \mathbb{C}_\infty \setminus \overline{\mathbb{D}} \to \mathbb{C}_\infty \setminus \Gamma$.

Proof. Suppose that

$$T = \begin{pmatrix} d_0 & d_{-1} & d_{-2} & \dots \\ d_1 & d_0 & d_{-1} & \dots \\ 0 & d_1 & d_0 & \dots \\ 0 & 0 & d_1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Hence the symbol of T is

$$d(z) = d_1 z + d_0 + d_{-1}\frac{1}{z} + d_{-2}\frac{1}{z^2} + \cdots = \sum_{k=-1}^{\infty} d_{-k} z^{-k}.$$

On the other hand, by the Riemann mapping theorem [20, 16], given $\Gamma \subset \mathbb{C}$ compact, since $\mathbb{C}_\infty \setminus \Gamma$ is simply connected, there is a unique conformal mapping $\phi : \mathbb{C}_\infty \setminus \overline{\mathbb{D}} \to \mathbb{C}_\infty \setminus \Gamma$, with the expression

$$\phi(z) = c_1 z + c_0 + c_{-1}\frac{1}{z} + c_{-2}\frac{1}{z^2} + \cdots = \sum_{k=-1}^{\infty} c_{-k} z^{-k},$$

such that $c_1 > 0$, with $c_1 = \operatorname{cap}(\Gamma)$ the capacity of Γ.

Therefore, in order to prove the theorem it suffices to show that $d(z)$ satisfies the properties that determine the Riemann mapping.

We will follow the next steps:

1. The symbol $d(z)$ is continuous in $\mathbb{C}_\infty \setminus \mathbb{D}$, analytic in $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}$, except at ∞, and $d(\mathbb{T}) = \Gamma$.
2. $d(z)$ is univalent in $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}$.
3. The first coefficient of $d(z)$ is the capacity of Γ, i. e. $d_1 = \operatorname{cap}(\Gamma)$.

(1) We show first that $d(z) = d_1 z + \sum_{k=0}^{\infty} d_{-k} z^{-k}$ is continuous in $\mathbb{C}_\infty \setminus \mathbb{D}$ and analytic in $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}$.

We have, on the one hand, that the function given by the first summand $d_1 z$ is analytic in $\mathbb{C}_\infty$, except at ∞. On the other hand, consider $\widetilde{d}(z) = \sum_{k=0}^{\infty} d_{-k} z^{-k}$. If $|z| \geq 1$, then

$$\sum_{k=0}^{\infty} |d_{-k} z^{-k}| \leq \sum_{k=0}^{\infty} |d_{-k}| = M < \infty,$$

because the rows of T are in ℓ^1. Therefore, $\widetilde{d}(z)$ is continuous in $\mathbb{C}_\infty \setminus \mathbb{D}$ and analytic in $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}$ except at ∞.

Consider $D - T$ which, by hypothesis, defines a compact operator in ℓ^2. Then all its diagonals converge to zero [2] and hence, for every $k \in \{-1, 0, 1, 2, \dots\}$, the limits $\lim_{n\to\infty} d_{n-k,n}$ exist and $\lim_{n\to\infty} d_{n-k,n} = d_{-k}$.

Since the essential spectrum (see, for example, [7] for a definition) is invariant via compact perturbations [6], we have that $\sigma_{ess}(D) = \sigma_{ess}(T)$.

Since $\operatorname{supp}(\mu) = \Gamma$ is a compact set with empty interior and $\mathbb{C}_\infty \setminus \Gamma$ is connected, we can apply Merguelyan's theorem [11, p.97] to deduce that every continuous function in Γ can be uniformly approximated by polynomials.

Since the set of continuous functions *with compact support* is dense in $L^2_\mu(\Gamma)$, then $L^2_\mu(\Gamma) = P^2_\mu(\Gamma)$. Therefore, D defines a normal operator in ℓ^2, hence $\sigma(D) = \Gamma$, *see* [7] p. 41.

Since the support has no isolated points (for being a connected union of Jordan arcs) and

$$\sigma(D) \setminus \sigma_{ess}(D) = \{\lambda \mid \lambda \quad \text{isolated eigenvalue with finite multiplicity}\} = \emptyset,$$

then $\sigma(D) = \sigma_{ess}(D)$.

Finally, since $(d_1, d_0, d_{-1}, \dots) \in \ell^1$, then [4] $\sigma_{ess}(T) = d(\mathbb{T})$. Therefore,

$$d(\mathbb{T}) = \sigma_{ess}(T) = \sigma_{ess}(D) = \sigma(D) = \Gamma.$$

(2) We have shown that $d(z)$ is analytic in $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}$. Moreover, since $d(z)$ is continuous in $\mathbb{C}_\infty \setminus \mathbb{D}$, the set of limit points of $d(z)$ as $|z| \to 1$ agrees with $d(\mathbb{T}) = \Gamma$ which is bounded, without interior points and does not disconnect $\mathbb{C}_\infty$. Therefore $d(z)$ satisfies the hypothesis of Theorem 1.1 in [20] and we can conclude that $d : \mathbb{C}_\infty \setminus \overline{\mathbb{D}} \longrightarrow \mathbb{C}_\infty \setminus \Gamma$ is univalent, and being also analytic, is conformal in $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}$.

(3) We finally show that $d_1 = \operatorname{cap}(\Gamma)$.

The elements $d_{n+1,n}$ of the subdiagonal of the matrix D agree with the quotients γ_n/γ_{n+1}. Since $\lim_{n\to\infty} d_{n+1,n} = d_1$, then

$$d_1 = \lim_{n\to\infty} d_{n+1,n} = \lim_{n\to\infty} \frac{\gamma_n}{\gamma_{n+1}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{\gamma_n}}.$$

On the other hand, since μ is regular, then [25]

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{\gamma_n}} = \operatorname{cap}(\operatorname{supp}(\mu)).$$

Therefore, $d_1 = \mathrm{cap}(\mathrm{supp}(\mu)) = \mathrm{cap}(\Gamma)$. □

Remark 1. Note that if Γ is rectifiable then the rows of T are in ℓ^1.

Remark 2. As noted in the introduction, if Γ is the segment $[-1, 1]$ in $\mathbb{C}$, the Riemann mapping $\phi(z)$ which applies the exterior of the closed unit disk onto the exterior of Γ is

$$\phi(z) = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

On the other hand, by Rakhmanov's theorem [21, 22], if μ is a Borel measure in $[-1, 1]$ such that $\mu' > 0$ almost everywhere in $[-1, 1]$, and

$$J = \begin{pmatrix} b_0 & a_1 & 0 & \dots \\ a_1 & b_1 & a_2 & \dots \\ 0 & a_2 & b_2 & \dots \\ 0 & 0 & a_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is the Jacobi matrix associated with μ, then $a_n \to \dfrac{1}{2}$ and $b_n \to 0$. In this case there exists

$$T = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \dots \\ \frac{1}{2} & 0 & \frac{1}{2} & \dots \\ 0 & \frac{1}{2} & 0 & \dots \\ 0 & 0 & \frac{1}{2} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

such that $J - T$ defines a compact operator in ℓ^2 and its rows are in ℓ^1 as in the hypothesis of Theorem 1. Note that these limits agree (as predicted by Theorem 1) with the coefficients of the Riemann mapping $\phi(z)$.

As an illustration of Theorem 1 we consider the following example.

Example 1 (Arc of circle). We consider Γ an arc of the unit circle $\mathbb{T}$. In this case [13] (see also [23, 24]), there exists a regular measure for which the diagonals of the Hessenberg matrix stabilize from the second element on. The monic orthogonal polynomials associated to this measure satisfy $\Psi_0(0) = 1$ and $\Psi_n(0) = \dfrac{1}{a}$ $(a > 1)$, if $n \geq 1$, and the corresponding Hessenberg matrix is the following unitary matrix D

$$\begin{pmatrix} -\frac{1}{a} & -\frac{(a^2-1)^{1/2}}{a^2} & -\frac{(a^2-1)^{2/2}}{a^3} & -\frac{(a^2-1)^{3/2}}{a^4} & -\frac{(a^2-1)^{4/2}}{a^5} & \dots \\ \frac{(a^2-1)^{1/2}}{a} & -\frac{1}{a^2} & -\frac{(a^2-1)^{1/2}}{a^3} & -\frac{(a^2-1)^{2/2}}{a^4} & -\frac{(a^2-1)^{3/2}}{a^5} & \dots \\ 0 & \frac{(a^2-1)^{1/2}}{a} & -\frac{1}{a^2} & -\frac{(a^2-1)^{1/2}}{a^3} & -\frac{(a^2-1)^{2/2}}{a^4} & \dots \\ 0 & 0 & \frac{(a^2-1)^{1/2}}{a} & -\frac{1}{a^2} & \frac{(a^2-1)^{1/2}}{a^3} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Hence we know the limits of the diagonals and then the Toeplitz matrix T is

$$T = \begin{pmatrix} -\dfrac{1}{a^2} & -\dfrac{(a^2-1)^{1/2}}{a^3} & -\dfrac{(a^2-1)^{2/2}}{a^4} & \cdots \\ \dfrac{(a^2-1)^{1/2}}{a} & -\dfrac{1}{a^2} & -\dfrac{(a^2-1)^{1/2}}{a^3} & \cdots \\ 0 & \dfrac{(a^2-1)^{1/2}}{a} & -\dfrac{1}{a^2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that the first row is a geometric series of ratio $\dfrac{\sqrt{a^2-1}}{a} < 1$ (because $a > 1$). Hence the rows of T are in ℓ^1.

On the other hand,

$$D - T = \begin{pmatrix} -\dfrac{a-1}{a^2} & -\dfrac{(a^2-1)^{1/2}(a-1)}{a^3} & -\dfrac{(a^2-1)(a-1)}{a^4} & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is easy to check that $D - T$ is compact because it is Hilbert-Schmidt:

$$\sqrt{\sum_{i,j=1}^{\infty} |a_{ij}|^2} = \sqrt{\sum_{n=0}^{\infty} \left(\frac{(a-1)(a^2-1)^{n/2}}{a^{2+n}} \right)^2} = \frac{a-1}{a} < +\infty.$$

According to Theorem 1 we have that the expression of the Riemann mapping as a Laurent series is

$$\phi(z) = \frac{z\left(a - \sqrt{a^2-1}\, z\right)}{\sqrt{a^2-1} - az} = \frac{\sqrt{a^2-1}}{a} z - \frac{1}{a^2} - \frac{\sqrt{a^2-1}}{a^3 z} - O\left(\frac{1}{z^2}\right).$$

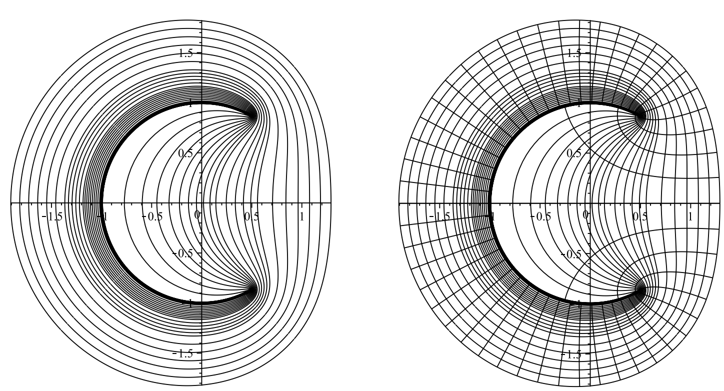

FIGURE 1. The Riemann mapping of the arc for $a = 2$

3. Approximation of the Riemann mapping

When the Hessenberg matrix D can not obtained as a closed form, and it is not possible to compute the limits of the elements of its diagonals, we may ask if it is still possible to compute approximations of the support of the measure μ, computing the image of the unit circle under suitable approximations of the Riemann mapping.

Specifically, since the coefficients of the Riemann mapping are the limits of the elements in each of the diagonals of the Hessenberg matrix, we may ask if the functions

$$h_n(z) = d_{n+1,n}z + d_{n,n} + \frac{d_{n-1,n}}{z} + \frac{d_{n-2,n}}{z^2} + \ldots + \frac{d_{1,n}}{z^{n-1}},$$

defined from the n-th column c_n of the Hessenberg matrix, are suitable approximations of the Riemann mapping $\phi(z)$.

We show in this section that this is indeed the case.

In what follows we will denote by Θ_n the norm in ℓ^2 the of n-th column of the matrix $D - T$ as a vector of ℓ^2, i.e.,

$$\Theta_n = \sqrt{\sum_{k=-1}^{n-1} |d_{-k} - d_{n-k,n}|^2}.$$

Lemma 1. *Suppose that D is bounded as an operator in ℓ^2 and that $D - T$ is compact. Then*

$$\lim_{n\to\infty} \Theta_n = 0.$$

Proof. If we apply $D - T$ to the vector e_{n-1}, we obtain the n-th column c_n of the matrix. Since $D - T$ defines a compact operator in ℓ^2 and e_n weakly converges to 0, then $(D - T)e_{n-1}$ converges strongly to 0.

Therefore, $\Theta_n = \|(D - T)e_{n-1}\|_2 \to 0$. □

Proposition 1. *Under the hypothesis of Theorem 1, the sequence of functions*

$$h_n(z) = d_{n+1,n}z + d_{n,n} + \frac{d_{n-1,n}}{z} + \frac{d_{n-2,n}}{z^2} + \ldots + \frac{d_{1,n}}{z^{n-1}}$$

converges uniformly to the Riemann mapping $\phi(z)$ on any compact set $K \subset \mathbb{C} \setminus \overline{\mathbb{D}}$.

Proof. Consider a compact subset $K \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ and consider $\varepsilon > 0$.

For every $z \in K$, we have

$$\begin{aligned} |h_n(z) - \phi(z)| = \\ &= \Bigg|(d_1 - d_{n+1,n})z + (d_0 - d_{n,n}) + \frac{d_{-1} - d_{n-1,n}}{z} + \cdots \\ &\quad \cdots + \frac{d_{1-n} - d_{1,n}}{z^{n-1}} + \sum_{k=n}^{\infty} d_{-k}\frac{1}{z^{-k}}\Bigg| \\ &\le \sum_{k=-1}^{n-1} \left|\frac{d_{-k} - d_{n-k,n}}{z^k}\right| + \sum_{k=n}^{\infty}\left|\frac{d_{-k}}{z^k}\right| \le \sum_{k=-1}^{n-1} \left|\frac{d_{-k} - d_{n-k,n}}{z^k}\right| + \sum_{k=n}^{\infty} |d_{-k}|. \end{aligned} \tag{2}$$

The second summand in the last inequality is the end of a vector in ℓ^1. Therefore, given $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that, for every $n < N_0$, $\sum_{k=n}^{\infty} |d_{-k}| < \frac{\varepsilon}{2}$.

On the other hand, for the first summand, if we apply Cauchy-Schwarz inequality, we have

$$\sum_{k=-1}^{n-1}\left|\frac{d_{-k}}{z^k}\right| \leq \sqrt{\sum_{k=-1}^{n-1}|d_{-k}-d_{n-k,n}|^2}\sqrt{\sum_{k=-1}^{n-1}|z|^{-2k}} = \Theta_n\sqrt{\sum_{k=-1}^{n-1}|z|^{-2k}}.$$

Since K is compact, there exist $r, R \in \mathbb{R}$ such that $1 < r \leq |z| \leq R$ for every $z \in K$. Then, for every $z \in K$,

$$\sum_{k=-1}^{n-1}|z|^{-2k} \leq R^2 + \sum_{k=0}^{\infty} r^{-2k} = R^2 + \frac{1}{r^2-1} = C.$$

On the other hand, since Θ_n converges to 0, there exists $N_1 \in \mathbb{N}$ such that, for every $n > N_1$, $\Theta_n < \dfrac{\varepsilon}{2C}$.

Taking $N = \max\{N_0, N_1\}$ we have that

$$|h_n(z) - \phi(z)| < \epsilon$$

for every $z \in K$, for every $n > N$. □

Note that, in Proposition 1, the condition $(d_1, d_0, d_{-1}, \dots) \in \ell^1$ is not necessary, since in (2) we can apply Cauchy-Schwarz inequality, and we have

$$\sum_{k=n}^{\infty}\left|\frac{d_{-k}}{z^k}\right| \leq \sqrt{\sum_{k=n}^{\infty}|d_{-k}|^2}\sqrt{\sum_{k=n}^{\infty} r^{-2k}},$$

which is a product of the ends of two convergent series, hence it converges to 0, very fast as n diverges to ∞. In fact, even when we have not the sequence $d_1, d_0, d_{-1}, \dots,$ explicitly, we can assure, since the second series is geometric, that the order of this summand is lower than $\mathrm{O}\left(1/r^{(n-1)}\right)$.

Remark 3. To obtain a bound for

$$|h_n(z) - \phi(z)| \leq \sum_{k=-1}^{n-1}\left|\frac{d_{-k}-d_{n-k,n}}{z^k}\right| + \sum_{k=n}^{\infty}\left|\frac{d_{-k}}{z^k}\right|, \tag{3}$$

we need to bound the two summands on the right. The second one goes to zero for all z with $|z| \geq 1$ since $(d_1, d_0, d_{-1}, \dots) \in \ell^1$

Consider the sequence $\{c_n\}_{n=1}^{\infty}$ of column vectors of the matrix $D-T$. Since every c_n has at most n non null elements, we can calculate its norm in ℓ^1 and in ℓ^2. We denote these norms by θ_n and Θ_n:

$$\theta n = \|c_n\|_1 = \sum_{k=-1}^{n}|d_{n-k,n} - d_{-k}|,$$

$$\Theta n = \|c_n\|_2 = \sqrt{\sum_{k=-1}^{n}|d_{n-k,n} - d_{-k}|^2}.$$

Take any $r \geq 1$ and consider $z = re^{i\theta}$. In this case, the first summand on the right of inequality (3) can be expressed as

$$\sum_{k=-1}^{n-1} |d_{-k} - d_{n-k,n}| r^{-k} \leq r|d_1 - d_{n+1,n}| + \sum_{k=0}^{n-1} |d_{-k} - d_{n-k,n}| = \theta_n + (r-1)|d_1 - d_{n+1,n}|.$$

where $|d_1 - d_{n+1,n}|$ converges to 0 as $n \to \infty$. Then,

i) If $\theta_n \to 0$ we have that $|h_n(z) - \phi(z)| \to 0$ for all z with $|z| \geq 1$. The convergence of θ_n will depend on the degree of compactness of $D - T$ or, in other words, of the Schatten-von Neumann class to which it belongs.

ii) If $\theta_n \nrightarrow 0$, we can apply Cauchy-Schwarz inequality and we have

$$\sum_{k=-1}^{n-1} \left| \frac{d_{-k} - d_{n-k,n}}{z^k} \right| \leq \Theta_n \sqrt{\sum_{k=-1}^{\infty} r^{-2k}} \leq \Theta_n \frac{1}{r^2 - 1},$$

where the above is valid only when $r > 1$. Then, $|h_n(z) - \phi(z)| \to 0$ for all z with $|z| > 1$.

Corollary 1. *Under the hypothesis of Theorem 1, for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $r \in (1, \delta + 1)$ there is a natural number N, such that for all $n \geq N$ we have*

$$|h_n(re^{i\theta}) - \phi(e^{i\theta})| \leq \varepsilon,$$

for every $\theta \in [0, 2\pi]$.

Proof. Consider $\varepsilon > 0$. Consider the compact set $K = \{z : 1 \leq |z| \leq 2\}$. Since the function $\phi(z)$ is uniformly continuous in K, there exists $\delta > 0$, such that for every $r \in [1, 1 + \delta)$, we have

$$|\phi(re^{i\theta}) - \phi(e^{i\theta})| < \frac{\varepsilon}{2},$$

for every $\theta \in [0, 2\pi]$.

We consider now, for every $r \in (1, 1 + \delta)$, the compact set $K_r = \{z \in \mathbb{C} : |z| = r\}$. By the uniform convergence of h_n on compact sets of $\mathbb{C} \setminus \overline{\mathbb{D}}$, established in Proposition 1, there exists $n \in \mathbb{N}$ such that, for every $n \geq N$, we have

$$|h_n(re^{i\theta}) - \phi(re^{i\theta})| < \frac{\varepsilon}{2}$$

for all $\theta \in [0, 2\pi]$. Then,

$$|h_n(re^{i\theta}) - \phi(e^{i\theta})| \leq |h_n(re^{i\theta}) - \phi(re^{i\theta})| + |\phi(re^{i\theta}) - \phi(e^{i\theta})| \leq \varepsilon$$

for every $\theta \in [0, 2\pi]$ □

We will use this result, in the last section, to approximate the support of the measure by equipotential curves of the function $h_n(re^{i\theta})$, for suitables n and r.

Example 1 (Arc of circle revisited). Consider again the arc of circumference Γ of Example 1, where we proved that this case satisfies the hypothesis of Theorem 1 and hence

$$\phi(z) = \frac{z(z\sqrt{a^2 - 1} - a)}{za - \sqrt{a^2 - 1}} = \frac{\sqrt{a^2 - 1}}{a} z - \frac{1}{a^2} - \frac{\sqrt{a^2 - 1}}{a^3 z} - \frac{\sqrt{(a^2 - 1)^2}}{a^4 z^2} - \cdots .$$

We also showed that, in this case, the n-th column of $D - T$ reduces to a single element and then

$$\Theta_n = \theta_n = \frac{a-1}{a^2}\left(\sqrt{1-\frac{1}{a^2}}\right)^n .$$

Note that, since $a > 1$, $\sqrt{1-\frac{1}{a^2}} < 1$.

Then, for every $z \in \mathbb{C} \setminus \mathbb{D}$,

$$\begin{aligned}|h_n(z) - \phi(z)| &\le \sum_{k=-1}^{n-1} \left|\frac{d_{-k} - d_{n-k,n}}{z^k}\right| + \sum_{k=n}^{\infty} \left|\frac{d_{-k}}{z^k}\right| \\ &\le \theta_n + \sum_{k=n}^{\infty} \left|\frac{d_{-k}}{z^k}\right| = \theta_n + \frac{\frac{1}{a^2}\left(\frac{\sqrt{a^2-1}}{a}\right)^n}{1-\frac{\sqrt{a^2-1}}{a}}.\end{aligned}$$

In particular, if $a = 2$,

$$|h_n(z) - \phi(z)| \le \theta_n + \frac{\frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)^n}{1-\frac{\sqrt{3}}{2}} = \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)^n + \frac{\frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)^n}{1-\frac{\sqrt{3}}{2}} = \frac{5+2\sqrt{3}}{4}\left(\frac{\sqrt{3}}{2}\right)^n ,$$

if $|z| \ge 1$.

This inequality allows us to calculate the value of n necessary to obtain a desired approximation of supp(μ). Some values can be seen in the following table:

Bound of $\|\|h_n\|_{\mathbb{T}} - \phi\|_{\mathbb{T}}\|\|_\infty$	n
0.2	17
0.1	22
0.01	38
0.001	54
0.0001	70

In the following figure we show the graphical result of approximating $supp(\mu)$ using $h_n(\mathbb{T})$.

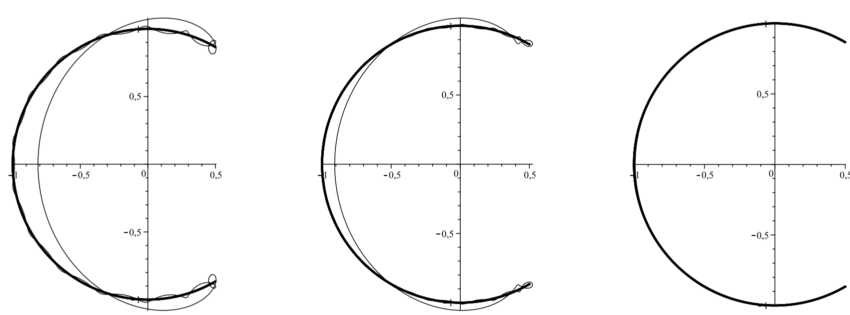

FIGURE 2. $h_n(\mathbb{T})$ for $n = 16$, $n = 21$ and $n = 37$, respectively

In the following figure we show the graphical result of approximating the equipotential lines of $\phi(z)$:

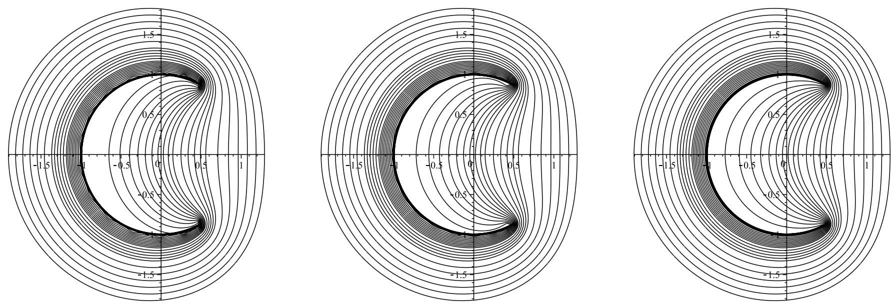

FIGURE 3. $h_n(S_r)$ for some values of $r \in (1, 1.5]$, for $n = 16$, $n = 21$ and $n = 37$, respectively

As can be seen, the difference between $\phi(z)$ and $h_n(z)$ decreases as $|z|$ increases.

4. Numerical examples

In this section, we present some numerical experiments using the results from the previous sections on the approximation of the Riemann mapping.

Example 2. Let Γ be a cross-like set formed by the intervals $[-a, a]$ y $[-bi, bi]$, with $a, b \in (0, \infty)$, and let μ be the uniform measure on Γ. The Riemann mapping deduced from ([17], pg 118), is

$$\phi(z) = \frac{\sqrt{a^2(z^2+1)^2 + b^2(z^2-1)^2}}{2z}.$$

In the particular case of $a = b$,

$$\phi(z) = \frac{a\sqrt{2}}{2z}\sqrt{z^4+1}.$$

The Laurent series expansion of $\phi(z)$ in a neighborhood of infinity is

$$\phi(z) = \frac{\sqrt{a^2+b^2}}{2}\, z + \frac{-2\,b^2 + 2\,a^2}{4\,\sqrt{a^2+b^2}}\,\frac{1}{z} + \frac{\sqrt{a^2+b^2}\left(\frac{1}{2} - \frac{(-2\,b^2+2\,a^2)^2}{8\,(a^2+b^2)^2}\right)}{2\,z^3} + \mathrm{O}\left(\frac{1}{z^5}\right).$$

Note that the first coefficient $\dfrac{\sqrt{a^2+b^2}}{2}$ agrees with the capacity of the support.

If $a = b = 1$, the series expansion is

$$\phi(z) = \frac{1}{2}\sqrt{2}z + \frac{1}{4}\frac{\sqrt{2}}{z^3} - \frac{1}{16}\frac{\sqrt{2}}{z^7} + \frac{1}{32}\frac{\sqrt{2}}{z^{11}} - \frac{5}{256}\frac{\sqrt{2}}{z^{15}} + \frac{7}{512}\frac{\sqrt{2}}{z^{19}} - \frac{21}{2048}\frac{\sqrt{2}}{z^{23}} + \mathrm{O}\left(\frac{1}{z^{27}}\right).$$

In the following image we represent the Riemann mapping of the cross.

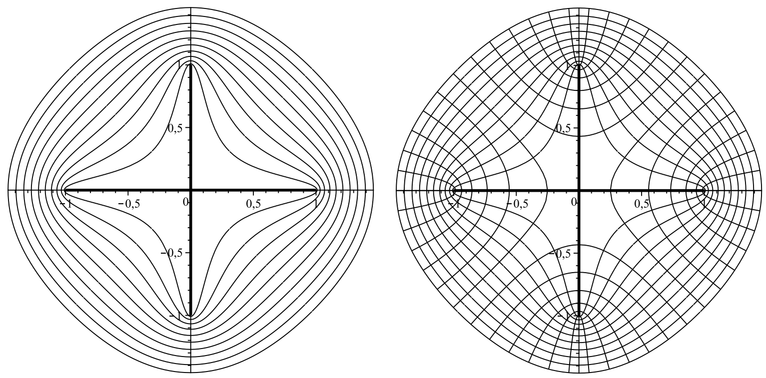

FIGURE 4. The Riemann mapping $\phi(z)$ for a cross-like set

The 9-th section of the Hessenberg matrix of μ, obtained from the moment matrix is

$$\begin{pmatrix}
0 & 0 & 0 & \frac{\sqrt{7}}{5} & 0 & 0 & 0 & -\frac{2\sqrt{15}}{45} & 0 \\
\frac{\sqrt{3}}{3} & 0 & 0 & 0 & \frac{2\sqrt{3}}{5} & 0 & 0 & 0 & -\frac{4\sqrt{3}\sqrt{17}}{231} \\
0 & \frac{\sqrt{5}\sqrt{3}}{5} & 0 & 0 & 0 & \frac{2\sqrt{5}\sqrt{11}}{45} & 0 & 0 & 0 \\
0 & 0 & \frac{\sqrt{7}\sqrt{5}}{7} & 0 & 0 & 0 & \frac{2\sqrt{7}\sqrt{13}}{77} & 0 & 0 \\
0 & 0 & 0 & \frac{4\sqrt{7}}{15} & 0 & 0 & 0 & \frac{19\sqrt{15}}{195} & 0 \\
0 & 0 & 0 & 0 & \frac{15\sqrt{11}}{77} & 0 & 0 & 0 & \frac{12\sqrt{11}\sqrt{17}}{385} \\
0 & 0 & 0 & 0 & 0 & \frac{7\sqrt{13}\sqrt{11}}{117} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{3\sqrt{15}\sqrt{13}}{55} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{88\sqrt{17}\sqrt{15}}{1989} & 0
\end{pmatrix}$$

In this case, a closed form for the Hessenberg matrix it is not known and it is not easy to compute the limits of the diagonals of D. Although, it is still possible to compute approximations of the support of the measure μ, computing the image of the unit circle under suitable approximations of the Riemann mapping. Specifically, since the coefficients of the Riemann mapping are the limits of the elements in the diagonals of the Hessenberg matrix, we may consider, as approximations of the Riemann mapping $\phi(z)$, the functions $h_n(z)$.

As opposed to the case of the arc in Example 1, we have not here an explicit formula for Θ_n and θ_n, used there to estimate the degree of approximation obtained using $h_n(z)$ instead of $\phi(z)$ (see Remark 1). In the following table we give a list of values of Θ_n and θ_n:

n	Θ_n	θ_n	n	Θ_n	θ_n
4	0.1756039179	0.1771699698	52	0.1435839520e-1	0.374355145e-1
8	0.8706648269e-1	0.1081557877	56	0.1335920853e-1	0.359786966e-1
12	0.5894618764e-1	0.846332410e-1	60	0.1249073290e-1	0.346766415e-1
16	0.4475241502e-1	0.716638451e-1	64	0.1172882241e-1	0.335039558e-1
20	0.3613474685e-1	0.631649554e-1	68	0.1105494437e-1	0.324406982e-1
24	0.3032967468e-1	0.570537158e-1	72	0.1045463567e-1	0.314709643e-1
28	0.2614682972e-1	0.523932383e-1	76	0.9916442395e-2	0.305818978e-1
32	0.2298656524e-1	0.486911124e-1	80	0.9431174829e-2	0.297629768e-1
36	0.2051319544e-1	0.456605530e-1	84	0.8991373805e-2	0.290054994e-1
40	0.1852386296e-1	0.431218007e-1	88	0.8590921072e-2	0.283021988e-1
44	0.1688863030e-1	0.409557583e-1	92	0.8224750644e-2	0.276469516e-1
48	0.1552035810e-1	0.390800153e-1	96	0.7888631635e-2	0.270345579e-1

We consider values of n that are multiples of 4 because for these values of n the approximations are worse since the matrix $D - T$ has three of every four diagonals nulls.

Some result of approximating $\mathrm{supp}(\mu)$ and the Riemann mapping using this method, are shown in the followings figures.

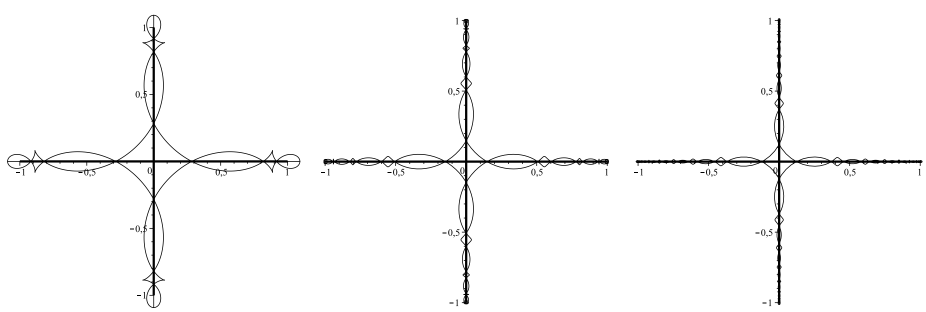

FIGURE 5. $h_n(\mathbb{T})$ for $n = 12$, $n = 32$, and $n = 60$, respectively

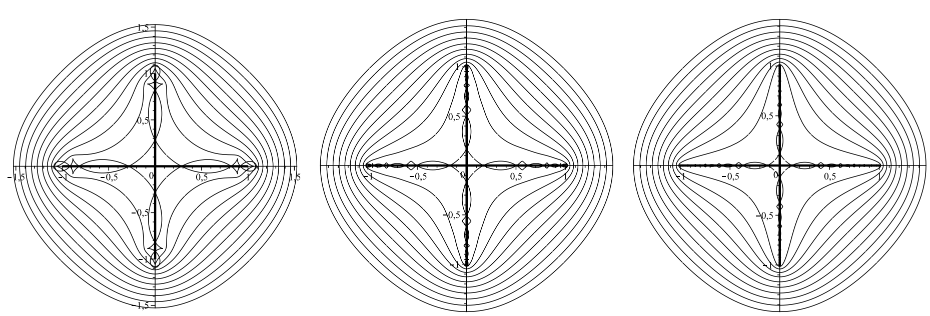

FIGURE 6. Approximations of the Riemann mapping using $h_n(re^{i\theta})$, $n = 12, 32, 60$, $r \in [1, 2]$

As can be seen, the difference between $\phi(z)$ and $h_n(z)$ decreases as $|z|$ increases. In the following figures we show some close-ups of the above figures.

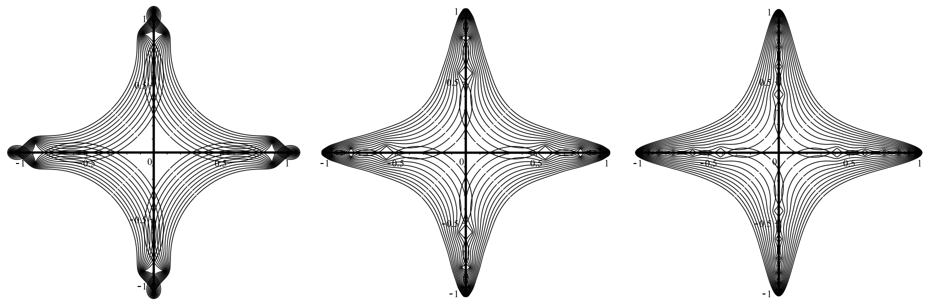

FIGURE 7. Approximations of the Riemann mapping using $h_n(re^{i\theta})$, $n = 12, 32, 60$, $r \in [1, 1.1]$

Example 3. In the following example we take Γ as the half part of a drop-like set of parametric equation

$$z(t) = \frac{(e^{it})^2}{1 + 2e^{it}}, t \in [0, \pi],$$

and μ the uniform measure on Γ.

Although in this case we do not know if the matrix $D - T$ defines a compact operator, the following figures seem to indicate that the convergence of $h_n(\mathbb{T})$ is very fast to Γ. In the following figure we show several approximations of the support of μ using this method.

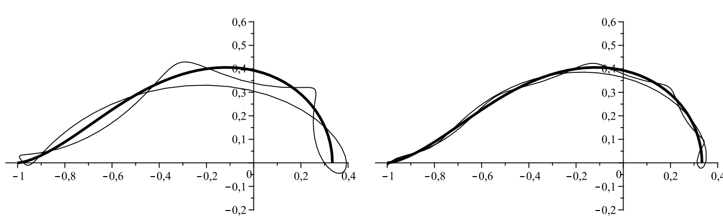

FIGURE 8. $h_n(\mathbb{T})$ for $n = 5$, $n = 8$ and $n = 11$, respectively

Example 4. For the last example we take Γ as the spiral with parametric equation

$$z(t) = t\frac{e^{it}}{6}, t \in [0, 2\pi]$$

and we consider μ the uniform measure on Γ.

Although, in this case we do not know if the matrix $D - T$ defines a compact operator, the following figures seem to indicate that the convergence of $h_n(\mathbb{T})$ to Γ is worse than in the previous example. In the following figure we show several approximations of the support of μ using this method.

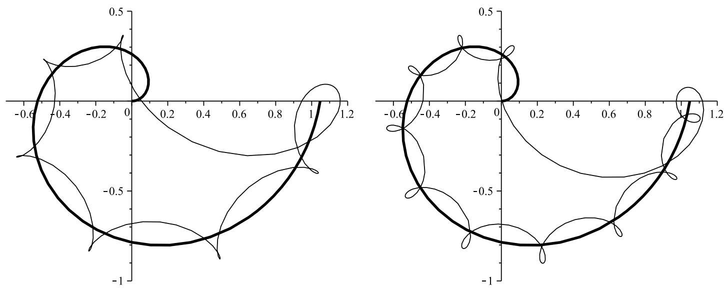

FIGURE 9. $h_n(\mathbb{T})$ for $n = 7$ and $n = 11$, respectively

Acknowledgements

The authors have been supported by Comunidad Autónoma de Madrid and Universidad Politécnica de Madrid (UPM-CAM Q061010133).

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