FIDELITY OF STATES IN INFINITE DIMENSIONAL QUANTUM SYSTEMS

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Abstract. In this paper we discuss the fidelity of states in infinite dimensional systems, give an elementary proof of the infinite dimensional version of Uhlmann's theorem, and then, apply it to generalize several properties of the fidelity from finite dimensional case to infinite dimensional case. Some of them are somewhat different from those for finite dimensional case.

1. Introduction

In quantum mechanics, a quantum system is associated with a separable complex Hilbert space H, i.e., the state space. A quantum state is described as a density operator $\rho \in \mathcal{T}(H) \subseteq \mathcal{B}(H)$ which is positive and has trace 1, where $\mathcal{B}(H)$ and $\mathcal{T}(H)$ denote the von Neumann algebras of all bounded linear operators and the trace-class of all operators T with $\|T\|_{\rm Tr} = {\rm Tr}((T^\dagger T)^{\frac{1}{2}}) < \infty$, respectively. ρ is a pure state if $\rho^2 = \rho$; ρ is a mixed state if $\rho^2 \neq \rho$. Let us denote by $\mathcal{S}(H)$ the set of all states acting on H.

Recall also that the fidelity of states ρ and σ in $\mathcal{S}(H)$ is defined to be

$$F(\rho,\sigma) = {\rm Tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}}. \tag{1.1}$$

Fidelity is a very useful measure of closeness between two states and has several nice properties including the Uhlmann's theorem.

Uhlmann and co-workers developed Eq.(1.1) by the transition probability in the more general context of the representation theory of C*-algebras [1, 2, 3]. The result in [1] (also ref. [4]) implies that, if $\dim H < \infty$, then the equality

$$F(\rho,\sigma) = \max |\langle\psi|\phi\rangle|, \tag{1.2}$$

holds, where the maximization is over all purifications $|\psi\rangle$ of ρ and $|\phi\rangle$ of σ into a larger system of $H \otimes H$. This result is then referred as the Uhlmann's theorem. Eq.(1.2) does not provide a calculation tool for evaluating the fidelity, as does Eq.(1.1). However, in many instances, the properties of the fidelity are more easily deduced using Eq.(1.2) than Eq.(1.1). For example, Eq.(1.2) makes it clear that $0 \le F(\rho,\sigma) = F(\sigma,\rho) \le 1$; $F(\rho,\sigma) = 1$ if and only if $\rho = \sigma$.

In [5], Jozsa presented an elementary proof of the Uhlmann's theorem without involving the representation theory of C*-algebras. In this paper we will consider the fidelity of states in infinite dimensional systems, give an elementary proof of the infinite dimensional version of Uhlmann's theorem, and then, apply it to generalize several properties of the fidelity

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from finite dimensional case to infinite dimensional case. Of course, not all results for finite dimensional case can be generalized fully to infinite dimensional case. For example, in the finite dimensional case, it is known that $F(\rho,\sigma)=\min_{\{E_m\}}F(p_m,q_m)$, where the minimum is over all POVMs (positive operator-valued measure) $\{E_m\}$, and $p_m=\mathrm{Tr}(\rho E_m)$, $q_m=\mathrm{Tr}(\sigma E_m)$ are the probability distributions for ρ and σ corresponding to the POVM $\{E_m\}$. However, this is not true for infinite dimensional case. What we have is that $F(\rho,\sigma)=\inf_{\{E_m\}}F(p_m,q_m)$. The infimum attains the minimum if and only if ρ and σ meet certain condition.

Let H be a complex Hilbert space, $A\in\mathcal{B}(H)$ and $T\in\mathcal{T}(H)$. It is well known from the operator theory that $|\mathrm{Tr}(AT)|\leq\|AT\|_{\mathrm{Tr}}\leq\|A\|\|T\|_{\mathrm{Tr}}$. This fact will be used frequently in this paper.

2. Infinite dimensional version of the Uhlmann's theorem and an elementary proof

Recall that an operator $V\in\mathcal{B}(H)$ is called an isometry if $V^\dagger V=I$; is called a co-isometry if $VV^\dagger=I$. If $\dim H=\infty$ and $T\in\mathcal{B}(H)$, then, by the polar decomposition, there exists an isometry or a co-isometry V such that $T=V|T|$, where $|T|=(T^\dagger T)^{1/2}$. Generally speaking, V may not be unitary. In fact, there exists a unitary operator U such that $T=U|T|$ if and only if $\dim\ker T=\dim\ker T^\dagger$. However, the following lemma says that it is the case if T is a product of two positive operators.

Lemma 2.1. *Let H be a Hilbert space and $A,B\in\mathcal{B}(H)$. If $A\geq 0$ and $B\geq 0$, then there exists a unitary operator $V\in\mathcal{B}(H)$ such that $AB=V|AB|$.*

Proof. We need only to show that $\dim\ker AB=\dim\ker BA$ if both A and B are positive operators.

Note that, since $A\geq 0$ and $B\geq 0$, we have

$$\ker AB=\ker B\oplus\ker A\cap(\ker B)^\perp \tag{2.1}$$

and

$$\ker BA=\ker A\oplus\ker B\cap(\ker A)^\perp. \tag{2.2}$$

Obviously, if $\dim\ker A=\dim\ker B=\infty$, then $\dim\ker AB=\dim\ker BA=\infty$; if A (or B) is injective, then $\dim\ker AB=\dim\ker BA=\dim\ker B$ (or $\dim\ker AB=\dim\ker BA=\dim\ker A$).

Assume that $\dim\ker A<\infty$ and $\dim\ker B=\infty$. By Eqs.(2.1)-(2.2) we need only to check that $\dim\ker B\cap(\ker A)^\perp=\infty$. This is equivalent to show the following assertion.

Assertion. If $B\geq 0$ and $\dim\ker B=\infty$, then, for any subspace $M\subset H$ with $\dim M^\perp<\infty$, $\dim\ker P_MBP_M|_M=\infty$.

In fact, by the space decomposition $H=M\oplus M^\perp$, we may write $B=\begin{pmatrix} B_{11} & B_{12}\\ B_{12}^\dagger & B_{22}\end{pmatrix}$, where $B_{11}=P_MBP_M|_M$. Since $B\geq 0$, there exists some contractive operator D such that $B_{12}=B_{11}^{1/2}DB_{22}^{1/2}$ (for example, see [6]). Thus

$$\ker B=\ker B_{11}\oplus\ker B_{22}\oplus L,$$

where

$$L=\{|x\rangle\oplus|y\rangle:|x\rangle\in(\ker B_{11})^\perp,\ |y\rangle\in(\ker B_{22})^\perp,\ B_{11}|x\rangle+B_{12}|y\rangle=0\text{ and }B_{12}^\dagger|x\rangle+B_{22}|y\rangle=0\}.$$

Note that $\dim\ker B_{22} < \infty$ and $\dim L \leq \dim(\ker B_{22})^{\perp} < \infty$, we must have $\dim\ker B_{11} = \infty$.

Finally, assume that both $\ker A$ and $\ker B$ are finite dimensional. With respect to the space decomposition $H = (\ker A)^{\perp} \oplus \ker A$, we have

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^{\dagger} & B_{22} \end{pmatrix}.$$

As A_1 is injective with dense range,

$$AB = \begin{pmatrix} A_1B_{11} & A_1B_{12} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad BA = \begin{pmatrix} B_{11}A_1 & 0 \\ B_{12}^{\dagger}A_1 & 0 \end{pmatrix},$$

we see that

$$\begin{aligned} \ker AB = & \{|x\rangle \oplus |y\rangle : |x\rangle \in (\ker A)^{\perp},\ |y\rangle \in \ker A,\ B_{11}|x\rangle + B_{12}|y\rangle = 0\} \\ = & (\ker B_{11} \oplus \ker B_{12}) \\ & +\{|x\rangle \oplus |y\rangle : |x\rangle \in (\ker B_{11})^{\perp},\ |y\rangle \in (\ker B_{12})^{\perp},\ B_{11}|x\rangle + B_{12}|y\rangle = 0\} \end{aligned}$$

and

$$\ker BA = \ker A \oplus \{|x\rangle : |x\rangle \in (\ker A)^{\perp} \cap \ker B_{11}\} = \ker A \oplus \ker B_{11}.$$

Since $\dim\{|x\rangle \oplus |y\rangle : |x\rangle \in (\ker B_{11})^{\perp},\ |y\rangle \in (\ker B_{12})^{\perp},\ B_{11}|x\rangle + B_{12}|y\rangle = 0\} \leq \dim(\ker B_{12})^{\perp}$ and $\dim\ker B_{12} + \dim(\ker B_{12})^{\perp} = \dim\ker A$, one gets

$$\dim\ker AB \leq \dim\ker BA.$$

Symmetrically, we have $\dim\ker BA \leq \dim\ker AB$, and therefore, $\dim\ker AB = \dim\ker BA$. Complete the proof of the lemma. □

If $\dim H < \infty$, then, for any $T \in \mathcal{B}(H)$, we have $\|T\|_{\mathrm{Tr}} = \mathrm{Tr}(|T|) = \max_U\{\mathrm{Tr}(AU)\}$, where the maximum is over all unitary operators. This result is not valid even for trace-class operators if $\dim H = \infty$. The next lemma says that the above result is true if the operator is a product of two positive operators.

Lemma 2.2. *Let H be a complex Hilbert space and $A, B \in \mathcal{B}(H)$. If A, B are positive and $AB \in \mathcal{T}(H)$, then*

$$\|AB\|_{\mathrm{Tr}} = \mathrm{Tr}(|AB|) = \max\{\mathrm{Tr}(ABU) : U \in \mathcal{U}(H)\}, \tag{2.3}$$

where $\mathcal{U}(H)$ is the unitary group of all unitary operators in $\mathcal{B}(H)$.

Proof. For any unitary operator $U \in \mathcal{U}(H)$, we have

$$|\mathrm{Tr}(ABU)| \leq \|U\|\|AB\|_{\mathrm{Tr}} = \|AB\|_{\mathrm{Tr}} = \mathrm{Tr}(|AB|).$$

On the other hand, by Lemma 2.1, there exists a unitary operator V such that $AB = V|AB|$. Thus $|AB| = V^{\dagger}AB$ and

$$\|AB\|_{\mathrm{Tr}} = \mathrm{Tr}(|AB|) = \mathrm{Tr}(V^{\dagger}AB) = \mathrm{Tr}(ABV^{\dagger}).$$

Hence Eq.(2.3) holds. □

Lemma 2.3. *Let H, K be separable infinite dimensional complex Hilbert spaces and $A \in \mathcal{B}(H)$, $B \in \mathcal{B}(K)$. Let $\{|i\rangle\}_{i=1}^{\infty}$, $\{|i'\rangle\}_{i=1}^{\infty}$ be any orthonormal bases of H, K respectively, and U be the unitary operator defined by $U|i\rangle = |i'\rangle$. For each positive integer N, let $|m_N\rangle = \sum_{i=1}^{N} |i\rangle|i'\rangle$. If A or B is a trace-class operator, then,*

$$\lim_{N\to\infty} \langle m_N|A \otimes B|m_N\rangle = \mathrm{Tr}(UA^{\dagger}U^{\dagger}B).$$

Proof. Clearly, $UA^\dagger U^\dagger B \in \mathcal{T}(K)$ and

$$\mathrm{Tr}(UA^\dagger U^\dagger B) = \sum_{i,j} \langle i'|UA^\dagger U^\dagger|j'\rangle\langle j'|B|i'\rangle = \sum_{i,j}\langle i|A^\dagger|j\rangle\langle j'|B|i'\rangle,$$

which is absolutely convergent. Hence

$$\lim_{N\to\infty}\sum_{i,j=1}^{N}\langle i|A^\dagger|j\rangle\langle j'|B|i'\rangle = \mathrm{Tr}(UA^\dagger U^\dagger B). \tag{2.4}$$

On the other hand,

$$\langle m_N|A\otimes B|m_N\rangle = \sum_{i,j=1}^{N}\langle j|\langle j'|A\otimes B|i\rangle|i'\rangle = \sum_{i,j=1}^{N}\langle j|A|i\rangle\langle j'|B|i'\rangle = \sum_{i,j=1}^{N}\langle i|A^\dagger|j\rangle\langle j'|B|i'\rangle.$$

So, by Eq.(2.4), one obtains that

$$\lim_{N\to\infty}\langle m_N|A\otimes B|m_N\rangle = \mathrm{Tr}(UA^\dagger U^\dagger B),$$

as desired. □

The following is the infinite dimensional version of the Uhlmann's theorem. Recall that a unit vector $|\psi\rangle \in H\otimes K$ is said to be a purification of a state ρ on H if $\rho = \mathrm{Tr}_K(|\psi\rangle\langle\psi|)$.

Theorem 2.4. *Let H and K be separable infinite dimensional complex Hilbert spaces. For any states ρ and σ on H, we have*

$$F(\rho,\sigma) = \max\{|\langle\psi|\phi\rangle| : |\psi\rangle\in\mathcal{P}_\rho,\ |\phi\rangle\in\mathcal{P}_\sigma\},$$

where $\mathcal{P}_\rho = \{|\psi\rangle \in H\otimes K : |\psi\rangle$ is a purification of $\rho\}$.

Proof. Assume that $\rho,\sigma\in\mathcal{S}(H)$. Then there exist orthonormal bases of H, $\{|i_H\rangle\}_{i=1}^\infty$ and $\{|i'_H\rangle\}_{i=1}^\infty$ such that $\rho=\sum_{i=1}^\infty p_i|i_H\rangle\langle i_H|$ and $\sigma=\sum_{i=1}^\infty q_i|i'_H\rangle\langle i'_H|$ with $\sum_{i=1}^\infty p_i=\sum_{i=1}^\infty q_i=1$. If $|\psi\rangle,|\phi\rangle\in H\otimes K$ are purifications of ρ, σ, respectively, then there exist orthonormal sets $\{|i_K\rangle\}_{i=1}^\infty$ and $\{|i'_K\rangle\}_{i=1}^\infty$ in K such that $|\psi\rangle=\sum_{i=1}^\infty\sqrt{p_i}|i_H\rangle|i_K\rangle$ and $|\phi\rangle=\sum_{i=1}^\infty\sqrt{q_i}|i'_H\rangle|i'_K\rangle$.

Pick any orthonormal bases $\{|i''_H\rangle\}_{i=1}^\infty$ of H and $\{|i''_K\rangle\}_{i=1}^\infty$ of K. Let U_H,U_K,V_H,V_K be partial isometries defined by respectively

$$U_H|i''_H\rangle=|i_H\rangle,\ U_K|i''_K\rangle=|i_K\rangle,\ V_H|i''_H\rangle=|i'_H\rangle,\ V_K|i''_K\rangle=|i'_K\rangle \tag{2.5}$$

for each $i=1,2,\ldots$. For any integer $N>0$, let

$$|m_N\rangle=\sum_{i=1}^N|i''_H\rangle|i''_K\rangle.$$

Then

$$|\psi_N\rangle=\sum_{i=1}^N\sqrt{p_i}|i_H\rangle|i_K\rangle=\sum_{i=1}^N\sqrt{\rho}(U_H\otimes U_K)|i''_H\rangle|i''_K\rangle=(\sqrt{\rho}U_H\otimes U_K)|m_N\rangle$$

and

$$|\phi_N\rangle=\sum_{i=1}^N\sqrt{q_i}|i'_H\rangle|i'_K\rangle=\sum_{i=1}^N\sqrt{\sigma}(V_H\otimes V_K)|i''_H\rangle|i''_K\rangle=(\sqrt{\sigma}V_H\otimes V_K)|m_N\rangle.$$

It follows from Lemma 2.3 that

$$\begin{array}{rl} |\langle\psi|\phi\rangle| = & \lim_{N\to\infty}|\langle\psi_N|\phi_N\rangle| = \lim_{N\to\infty}|\langle m_N|U_H^\dagger\sqrt{\rho}\sqrt{\sigma}V_H\otimes U_K^\dagger V_K|m_N\rangle| \\ = & |\mathrm{Tr}(UV_H^\dagger\sqrt{\sigma}\sqrt{\rho}U_HU^\dagger U_K^\dagger V_K)| \leq \|U_HU^\dagger U_K^\dagger V_KUV_H^\dagger\|\mathrm{Tr}(|\sqrt{\sigma}\sqrt{\rho}|) \\ \leq & \mathrm{Tr}(|\sqrt{\sigma}\sqrt{\rho}|) = \mathrm{Tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}} = F(\rho,\sigma), \end{array} \tag{2.6}$$

where U is the unitary operator defined by $U|i''_H\rangle = |i''_K\rangle$. Therefore, we have proved that

$$F(\rho,\sigma) \geq \sup\{|\langle\psi|\phi\rangle| : |\psi\rangle\in\mathcal{P}_\rho,\ |\phi\rangle\in\mathcal{P}_\sigma\}.$$

Now, to complete the proof, it suffices to find $|\psi\rangle\in\mathcal{P}_\rho$ and $|\phi\rangle\in\mathcal{P}_\sigma$ such that $|\langle\psi|\phi\rangle| = F(\rho,\sigma)$.

By applying Lemma 2.1, we see that $\sqrt{\sigma}\sqrt{\rho}$ has a polar decomposition $\sqrt{\sigma}\sqrt{\rho} = U_0|\sqrt{\sigma}\sqrt{\rho}|$ with U_0 a unitary operator.

Let $\{|i_K\rangle\}_{i=1}^\infty$ be an orthonormal basis of K and let $|\psi\rangle = \sum_{i=1}^\infty\sqrt{p_i}|i_H\rangle|i_K\rangle$ and $|\phi\rangle = \sum_{i=1}^\infty\sqrt{q_i}|i'_H\rangle|i_K\rangle$. Then $|\psi\rangle\in\mathcal{P}_\rho$ and $|\phi\rangle\in\mathcal{P}_\sigma$. Let $|i''_H\rangle = |i_H\rangle$, $|i''_K\rangle = |i_K\rangle$, $i = 1,2\ldots$. Then, by Eq.(2.5), $U_H = I$, $U_K = I$, V_H is a unitary operator determined by $V_H|i_H\rangle = |i'_H\rangle$. Take $|i'_K\rangle$ so that $V_K = UU_0^\dagger V_HU^\dagger$. Then for such choice of $|\psi\rangle$ and $|\phi\rangle$ we have

$$\begin{array}{rl} |\langle\psi|\phi\rangle| = & \lim_{N\to\infty}|\langle\psi_N|\phi_N\rangle| = \lim_{N\to\infty}|\langle m_N|\sqrt{\rho}\sqrt{\sigma}V_H\otimes V_K|m_N\rangle| \\ = & |\mathrm{Tr}(UV_H^\dagger\sqrt{\sigma}\sqrt{\rho}U^\dagger V_K)| = |\mathrm{Tr}(U^\dagger V_KUV_H^\dagger U_0|\sqrt{\sigma}\sqrt{\rho}|)| \\ = & |\mathrm{Tr}(|\sqrt{\sigma}\sqrt{\rho}|)| = F(\rho,\sigma), \end{array}$$

completing the proof. □

By checking the proof of Theorem 2.4, it is easily seen that the following holds.

Corollary 2.5. *Let H and K be separable infinite dimensional complex Hilbert spaces. For any states ρ and σ on H, we have*

$$F(\rho,\sigma) = \max\{|\langle\psi_0|\phi\rangle| : |\phi\rangle\in\mathcal{P}_\sigma\} = \max\{|\langle\psi|\phi_0\rangle| : |\psi\rangle\in\mathcal{P}_\rho\},$$

where $|\psi_0\rangle$ is any fixed purification of ρ of the form $|\psi_0\rangle = \sum_{i=1}^\infty\sqrt{p_i}|i_H\rangle|i_K\rangle$ with $\{|i_K\rangle\}$ an orthonormal basis of K and $|\phi_0\rangle$ is any fixed purification of σ of the form $|\phi_0\rangle = \sum_{i=1}^\infty\sqrt{q_i}|i'_H\rangle|i'_K\rangle$ with $\{|i'_K\rangle\}$ an orthonormal basis of K.

The fidelity is not a distance because it does not meet the triangular inequality. However, like to the finite dimensional case, by use of Theorem 2.4 and Corollary 2.5, one can show that the are-cosine of fidelity is a distance.

Corollary 2.6. *$A(\rho,\sigma) =: \arccos F(\rho,\sigma)$ is a distance on $\mathcal{S}(H)$.*

Several remarkable properties of fidelity in finite dimensional case are still valid for infinite dimensional case. For instance,

Monotonicity of the fidelity For any quantum channel $\mathcal{E}$, we have

$$F(\mathcal{E}(\rho),\mathcal{E}(\sigma)) \geq F(\rho,\sigma). \tag{2.7}$$

Recall that a quantum channel is a completely positive and trace preserving linear map from $\mathcal{T}(H)$ into $\mathcal{T}(K)$.

Strong concavity of the fidelity Let p_i and q_i be probability distributions over the same index set, and ρ_i and σ_i states also indexed by the same index set. Then

$$F(\sum_i p_i\rho_i, \sum_i q_i\sigma_i) \geq \sum_i\sqrt{p_iq_i}F(\rho_i,\sigma_i). \tag{2.8}$$

3. Connection to the classical fidelity and trace distance

If $\dim H < \infty$, the quantum fidelity is related to the classical fidelity by considering the probability distributions induced by a measurement. In fact [7, pp. 412]

$$F(\rho,\sigma) = \min_{\{E_m\}} F(p_m, q_m), \tag{3.1}$$

where the minimum is over all POVMs (positive operator-valued measure) $\{E_m\}$, and $p_m = \mathrm{Tr}(\rho E_m)$, $q_m = \mathrm{Tr}(\sigma E_m)$ are the probability distributions for ρ and σ corresponding to the POVM $\{E_m\}$.

It is natural to ask whether or not Eq.(3.1) is true if $\dim H = \infty$? The following result is our answer.

For a positive operator $A \in \mathcal{B}(H)$, with respect to the space decomposition $H = (\ker A)^\perp \oplus \ker A$, $A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}$, where $A_1 : (\ker A)^\perp \to (\ker A)^\perp$ is injective and hence A_1^{-1} makes sense. In this paper, we always denote $A^{[-1]}$ for the may unbounded densely defined positive operator defined by $A^{[-1]} = \begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}$ with domain $\mathcal{D}(A^{[-1]}) = \mathrm{ran}(A) \oplus \ker A$. Here $\mathrm{ran}(A)$ stands for the range of A.

Theorem 3.1. *Let H be a separable infinite dimensional complex Hilbert space. Then, for any states $\rho,\sigma \in \mathcal{S}(H)$, we have*

$$F(\rho,\sigma) = \inf_{\{E_m\}} F(p_m, q_m), \tag{3.2}$$

where the infimum is over all POVMs $\{E_m\}$, and $p_m = \mathrm{Tr}(\rho E_m)$, $q_m = \mathrm{Tr}(\sigma E_m)$ are the probability distributions for ρ and σ corresponding to the POVM $\{E_m\}$. Moreover, the infimum attains the minimum if and only if the operator $M = \rho^{[-1/2]}\sqrt{\rho^{1/2}\sigma\rho^{1/2}}\rho^{[-1/2]}$ (may unbounded) is diagonal.

Firstly, we need a lemma.

Lemma 3.2. *Let H be an infinite dimensional complex Hilbert space. Assume that $A \in \mathcal{T}(H)$ and $\{T_n\}_{n=0}^\infty \subset \mathcal{B}(H)$. If SOT-$\lim_{n\to\infty} T_n = T_0$, then $\lim_{n\to\infty} \mathrm{Tr}(T_n A) = \mathrm{Tr}(T_0 A)$. Here SOT means the strong operator topology.*

Proof. As T_n converges to T_0 under the strong operator topology, there is a constant $d > 0$ such that $\sup_n \|T_n\| \leq d$. For any $\varepsilon > 0$, there exists a finite rank projection P_ε such that $\|A - P_\varepsilon A P_\varepsilon\|_{\mathrm{Tr}} < \varepsilon/(2d+1)$ because $A \in \mathcal{T}(H)$. On the other hand, P_ε is of finite rank, together with SOT-$\lim_{n\to\infty} T_n = T_0$, implies that

$$\lim_{n\to\infty} \|P_\varepsilon(T_n - T_0)P_\varepsilon\| = 0.$$

So, for above $\varepsilon > 0$, there exists some N such that

$$\|P_\varepsilon(T_n - T_0)P_\varepsilon\| < \frac{1}{(2d+1)\|A\|_{\mathrm{Tr}}}\varepsilon$$

whenever $n > N$. Thus we have

$$\begin{aligned}|\mathrm{Tr}((T_n - T_0)A)| &\leq |\mathrm{Tr}((T_n - T_0)(A - P_\varepsilon A P_\varepsilon))| + |\mathrm{Tr}((T_n - T_0)P_\varepsilon A P_\varepsilon)| \\ &\leq \|T_n - T_0\|\|A - P_\varepsilon A P_\varepsilon\|_{\mathrm{Tr}} + \|P_\varepsilon(T_n - T_0)P_\varepsilon\|\|A\|_{\mathrm{Tr}} \\ &< 2d\|A - P_\varepsilon A P_\varepsilon\|_{\mathrm{Tr}} + \tfrac{\varepsilon}{2d+1} \\ &< \tfrac{2d}{2d+1}\varepsilon + \tfrac{\varepsilon}{2d+1} = \varepsilon.\end{aligned}$$

Therefore, $\lim_{n\to\infty} \mathrm{Tr}(T_n A) = \mathrm{Tr}(T_0 A)$. □

Proof of Theorem 3.1. Let $\{E_m\}$ be a POVM. Then $E_m \geq 0$ and $\sum_m E_m = I$, here the series converges under the strong operator topology. By Lemma 2.1, there exists a unitary operator U such that $\sqrt{\rho^{1/2}\sigma\rho^{1/2}} = U\sqrt{\sigma}\sqrt{\rho}$. Thus, by the Cauchy-Schwarz inequality and Lemma 3.2,

$$\begin{aligned}F(\rho,\sigma) = &\ \mathrm{Tr}(U\sqrt{\sigma}\sqrt{\rho}) = \textstyle\sum_m \mathrm{Tr}(U\sqrt{\sigma}\sqrt{E_m}\sqrt{E_m}\sqrt{\rho}) \\ \leq &\ \textstyle\sum_m \sqrt{\mathrm{Tr}(\rho E_m)\mathrm{Tr}(\sigma E_m)} = \sum_m \sqrt{p_m q_m} = F(p_m, q_m).\end{aligned} \tag{3.3}$$

Hence we have

$$F(\rho,\sigma) \leq \inf_{\{E_m\}} F(p_m, q_m).$$

Next we show that the equality holds in the above inequality, that is, Eq.(3.2) holds. By the spectral decomposition, there is an orthonormal basis $\{|i\rangle\}_{i=1}^{\infty}$ of H such that $\rho = \sum_i r_i |i\rangle\langle i|$ with $\sum_i r_i = 1$. For any positive integer n, let H_n be the n-dimensional subspace spanned by $|1\rangle, |2\rangle, \ldots, |n\rangle$, and P_n be the projection from H onto H_n. Define $\rho_n = \alpha_n^{-1} P_n \rho P_n$ and $\sigma_n = \beta_n^{-1} P_n \sigma P_n$, where $\alpha_n = \mathrm{Tr}(P_n \rho P_n)$ and $\beta_n = \mathrm{Tr}(P_n \sigma P_n)$. Clearly, $\lim_{n\to\infty} \alpha_n = 1$, $\lim_{n\to\infty} \beta_n = 1$, SOT-$\lim_{n\to\infty} \rho_n =$SOT-$\lim_{n\to\infty} P_n \rho P_n = \rho$ and SOT-$\lim_{n\to\infty} \sigma_n =$SOT-$\lim_{n\to\infty} P_n \sigma P_n = \sigma$. By [8], we see that $\lim_{n\to\infty} \rho_n = \rho$ and $\lim_{n\to\infty} \sigma_n = \sigma$ under the trace norm topology. It follows that $\lim_{n\to\infty} \sqrt{\rho_n^{1/2}\sigma_n\rho_n^{1/2}} = \sqrt{\rho^{1/2}\sigma\rho^{1/2}}$ under the trace norm topology, which implies that $\lim_{n\to\infty} F(\rho_n, \sigma_n) = F(\rho, \sigma)$. So, for any $\varepsilon > 0$, there exists some N_1 such that

$$|F(\rho,\sigma) - \alpha_n\beta_n F(\rho_n, \sigma_n)| < \varepsilon/2 \tag{3.4}$$

whenever $n > N_1$.

On the other hand, note that $\lim_{n\to\infty} \sqrt{\alpha_n\beta_n}\mathrm{Tr}(\rho P_n) = 1$ and $\lim_{n\to\infty} \sqrt{\alpha_n\beta_n}\mathrm{Tr}(\sigma P_n) = 1$. Thus, for the above $\varepsilon > 0$, there exists some N_2 such that

$$|1 - \sqrt{\alpha_n\beta_n}\mathrm{Tr}(\rho P_n)| < \varepsilon/2 \quad \text{and} \quad |1 - \sqrt{\alpha_n\beta_n}\mathrm{Tr}(\sigma P_n)| < \varepsilon/2 \tag{3.5}$$

whenever $n > N_2$.

Now, consider ρ_n and σ_n for $n \geq \max\{N_1, N_2\}$. With respect to the space decomposition $H = H_n \oplus H_n^{\perp}$, we have $\rho_n = \begin{pmatrix} \rho_0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\sigma_n = \begin{pmatrix} \sigma_0 & 0 \\ 0 & 0 \end{pmatrix}$, where $\rho_0, \sigma_0 \in \mathcal{S}(H_n)$. Applying Eq.(3.1) to ρ_0 and σ_0, there exists POVM $\{E'_m\} \subseteq \mathcal{B}(H_n)$ with $\sum_{m=1}^{n} E'_m = I_n$ such that

$$F(\rho_0, \sigma_0) = \sum_{m=1}^{n} \sqrt{\mathrm{Tr}(\rho_0 E'_m)\mathrm{Tr}(\sigma_0 E'_m)}.$$

Let $E_m = E'_m \oplus 0$ and $E_{n+1} = I - P_n$. It is obvious that $\sum_{m=1}^{n+1} E_m = I$ and

$$F(\rho_n, \sigma_n) = F(\rho_0, \sigma_0) = \sum_{m=1}^{n} \sqrt{\mathrm{Tr}(\rho_0 E'_m)\mathrm{Tr}(\sigma_0 E'_m)} = \sum_{m=1}^{n+1} \sqrt{\mathrm{Tr}(\rho_n E_m)\mathrm{Tr}(\sigma_n E_m)}.$$

Now define $F_m = \sqrt{\alpha_n\beta_n}P_nE_mP_n$ for $m = 1, 2, \cdots, n+1$ and $F_0 = I - \sqrt{\alpha_n\beta_n}P_n$. It is clear that $\{F_m\}$ is a POVM. Furthermore

$$\begin{array}{rcl}\sum_{m=1}^{n+1}\sqrt{\mathrm{Tr}(\rho F_m)\mathrm{Tr}(\sigma F_m)} & = & \sum_{m=1}^{n+1}\sqrt{\alpha_n\beta_n\mathrm{Tr}(P_n\rho P_nE_m)\mathrm{Tr}(P_n\sigma P_nE_m)} \\ & = & \sum_{m=1}^{n+1}\sqrt{\alpha_n\beta_n}\sqrt{\alpha_n\mathrm{Tr}(\rho_nE_m)\beta_n\mathrm{Tr}(\sigma_nE_m)} \\ & = & \sum_{m=1}^{n+1}\alpha_n\beta_n\sqrt{\mathrm{Tr}(\rho_nE_m)\mathrm{Tr}(\sigma_nE_m)} \\ & = & \alpha_n\beta_nF(\rho_n, \sigma_n).\end{array} \tag{3.6}$$

Hence, by Eqs.(3.4)-(3.6), we get

$$\begin{array}{rl} & |F(\rho,\sigma) - \sum_{m=0}^{n+1}\sqrt{\mathrm{Tr}(\rho F_m)\mathrm{Tr}(\sigma F_m)}| \\ \leq & |F(\rho,\sigma) - \sum_{m=1}^{n+1}\sqrt{\mathrm{Tr}(\rho F_m)\mathrm{Tr}(\sigma F_m)}| + \sqrt{\mathrm{Tr}(\rho F_0)\mathrm{Tr}(\sigma F_0)} \\ = & |F(\rho,\sigma) - \alpha_n\beta_nF(\rho_n,\sigma_n)| + \sqrt{(1-\sqrt{\alpha_n\beta_n}\mathrm{Tr}(\rho P_n))(1-\sqrt{\alpha_n\beta_n}\mathrm{Tr}(\sigma P_n))} \\ < & \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{array}$$

Thus we have proved that, for any $\varepsilon > 0$, there exists some POVM $\{F_m\}$ such that

$$F(p_m, q_m) < F(\rho,\sigma) + \varepsilon,$$

where $p_m = \mathrm{Tr}(\rho F_m)$, $q_m = \mathrm{Tr}(\sigma F_m)$ are the probability distributions for ρ and σ corresponding to the POVM $\{F_m\}$. So Eq.(3.2) is true.

It is clear that the infimum of Eq.(3.2) attains the minimum if and only if there exists a POVM $\{E_m\}$ such that the Cauchy-Schwarz inequality is satisfied with equality for each term in the sum of Eq.(3.3), that is,

$$\sqrt{E_m}\sqrt{\rho} = \lambda_m\sqrt{E_m}\sqrt{\sigma}U^\dagger \tag{3.7}$$

for some set of numbers $\lambda_m \geq 0$. Note that $\sqrt{\rho^{1/2}\sigma\rho^{1/2}} = U\sqrt{\sigma}\sqrt{\rho} = \sqrt{\rho}\sqrt{\sigma}U^\dagger$, we get that the range of $\sqrt{\rho^{1/2}\sigma\rho^{1/2}}$ is contained in the range of $\rho^{1/2}$ and hence

$$\sqrt{\sigma}U^\dagger = \rho^{[-1/2]}\sqrt{\rho^{1/2}\sigma\rho^{1/2}}. \tag{3.8}$$

Substituting Eq.(3.8) into Eq.(3.7), we find that

$$\sqrt{E_m}\sqrt{\rho} = \lambda_m\sqrt{E_m}\rho^{[-1/2]}\sqrt{\rho^{1/2}\sigma\rho^{1/2}} \tag{3.9}$$

for each m. It follows that $\sqrt{E_m}\sqrt{\rho} \neq 0 \Rightarrow \lambda_m \neq 0$. While, if $\sqrt{E_m}\sqrt{\rho} = 0$, one may take $\lambda_m = 0$. Let $H_0 = \mathrm{span}\{\mathrm{ran}(\sqrt{E_m}) : \sqrt{E_m}\sqrt{\rho} = 0\}$, and P_0 be the projection onto H_0. Then, Eq.(3.9) implies that Eq.(3.7) is equivalent to

$$\sqrt{E_m}(I - P_0 - \lambda_mM) = 0 \tag{3.10}$$

holds for all m, where $M = \rho^{[-1/2]}\sqrt{\rho^{1/2}\sigma\rho^{1/2}}\rho^{[-1/2]}$ (may unbounded). Now it is easily seen that the closure of $\mathrm{ran}(\sqrt{E_m})$ reduces M to the scalar operator λ_m^{-1} if $\sqrt{E_m}\sqrt{\rho} \neq 0$, and $\ker M = H_0$. Thus $0, \lambda_m^{-1} \in \sigma_p(M)$, the point spectrum (i.e., eigenvalues) of M. Since $\sum_m E_m = I$, we see that $\sum_m \mathrm{ran}(\sqrt{E_m}) = H$ and the spectrum of M, $\sigma(M) \subseteq \mathrm{cl}\ \{0, \lambda_m^{-1}\} = \mathrm{cl}\ \sigma_p(M)$. So M must be diagonal. Conversely, if M is diagonal, say $M = \sum_m \gamma_m|m\rangle\langle m|$ with $\{|m\rangle\}$ an orthonormal basis of H. Let $\lambda_m = \gamma_m^{-1}$ if $\gamma_m \neq 0$; $\lambda_m = 0$ if $\gamma_m = 0$. Then the POVM $\{E_m = |m\rangle\langle m|\}$ satisfies Eq.(3.10) and thus Eq.(3.9). Hence $F(\rho,\sigma) = \sum_m\sqrt{\mathrm{Tr}(\rho E_m)\mathrm{Tr}(\sigma E_m)} = F(p_m, q_m)$. This completes the proof. □

Remark 3.3. There do exist some ρ and σ such that there is no POVM $\{E_m\}$ satisfying $F(\rho,\sigma) = \sum_m\sqrt{\mathrm{Tr}(\rho E_m)\mathrm{Tr}(\sigma E_m)}$. For example, let $H = L_2([0,1])$ and M_t the operator defined by $(M_t f)(t) = tf(t)$ for any $f \in H$. Then M_t is positive and is not diagonal because $\sigma(M_t) = [0, 1]$ and the point spectrum $\sigma_p(M_t) = \emptyset$. Let $\rho \in \mathcal{S}(H)$ be injective as an operator. Then $d = \mathrm{Tr}(M_t^2 \rho) \neq 0$. Let $M = d^{-1} M_t$ and $\sigma = M\rho M$. As $\mathrm{Tr}(M^2\rho) = 1$, σ is a state. Now it is clear that $M = \rho^{-1/2}\sqrt{\rho^{1/2}\sigma\rho^{1/2}}\rho^{-1/2}$, which is not diagonal. Thus by Theorem 3.1, the infimum in Eq.(3.2) does not attain the minimum.

For two states ρ and σ, recall that the trace distance of them is defined by $D(\rho, \sigma) = \frac{1}{2}\|\rho - \sigma\|_{\mathrm{Tr}}$. By use of Uhlmann's theorem and Eq.(3.1), it holds for finite dimensional case that

$$1 - F(\rho, \sigma) \leq D(\rho, \sigma) \leq \sqrt{1 - F(\rho, \sigma)^2}. \tag{3.11}$$

This reveals that the trace distance and the fidelity are qualitatively equivalent measures of closeness for quantum states. Now Theorem 3.1 allows us to establish the same relationship between fidelity measure and trace distance measure for states of infinite dimensional systems.

Theorem 3.4. *Let H be an infinite dimensional separable complex Hilbert space. Then for any states $\rho, \sigma \in \mathcal{S}(H)$, the inequalities in Eq.(3.11) hold.*

Proof. Firstly, it is obvious that if both $\rho = |a\rangle\langle a|$ and $\sigma = |b\rangle\langle b|$ are pure states, then $D(\rho, \sigma) = D(|a\rangle, |b\rangle) = \sqrt{1 - F(|a\rangle, |b\rangle)^2} = \sqrt{1 - F(\rho, \sigma)^2}$. (Ref. [7, pp. 415] for a proof that is valid for both finite and infinite dimensional cases.)

Let ρ and σ be any two states, and let $|\psi\rangle$ and $|\phi\rangle$ be purifications chosen such that $F(\rho, \sigma) = |\langle\psi|\phi\rangle|$ by Theorem 2.4. Since the trace distance is non-increasing under the partial trace, we see that

$$D(\rho, \sigma) \leq D(|\psi\rangle, |\phi\rangle) = \sqrt{1 - F(|\psi\rangle, |\phi\rangle)^2} = \sqrt{1 - F(\rho, \sigma)^2}.$$

This establishes the inequality

$$D(\rho, \sigma) \leq \sqrt{1 - F(\rho, \sigma)^2}. \tag{3.12}$$

To see the other inequality of Eq.(3.11) is true, Theorem 3.1 is needed.

For any given $\varepsilon > 0$, by Theorem 3.1, we may take a POVM $\{E_m\}$ such that

$$F(\rho, \sigma) \leq F(p_m, q_m) = \sum_m \sqrt{p_m q_m} < F(\rho, \sigma) + \varepsilon, \tag{3.13}$$

where $p_m = \mathrm{Tr}(\rho E_m)$ and $q_m = \mathrm{Tr}(\sigma E_m)$ are the probabilities for obtaining outcome m for the states ρ and σ, respectively. Observe that, for both finite and infinite dimensional cases, we have

$$D(\rho, \sigma) = \max_{\{E_m\}} D(p_m, q_m), \tag{3.14}$$

where $D(p_m, q_m) = \frac{1}{2}\sum_m |p_m - q_m|$ and the maximun is over all POVM $\{E_m\}$. It follows from Eq.(3.14) and

$$\sum_m (\sqrt{p_m} - \sqrt{q_m})^2 = \sum_m p_m + \sum_m q_m - 2F(p_m, q_m) = 2(1 - F(p_m, q_m)),$$

that

$$\begin{aligned} 2(1 - F(\rho, \sigma)) - 2\varepsilon < {} & 2(1 - F(p_m, q_m)) = \textstyle\sum_m (\sqrt{p_m} - \sqrt{q_m})^2 \\ \leq {} & \textstyle\sum_m |\sqrt{p_m} - \sqrt{q_m}|(\sqrt{p_m} + \sqrt{q_m}) = \sum_m |p_m - q_m| \\ = {} & 2D(p_m, q_m) \leq 2D(\rho, \sigma). \end{aligned}$$

Thus we have proved that

$$(1 - F(\rho, \sigma)) - \varepsilon < D(\rho, \sigma)$$

holds for any $\varepsilon > 0$. This forces that

$$1 - F(\rho, \sigma) \leq D(\rho, \sigma),$$

which, combining the inequality (3.12), completes the proof of the theorem. □

4. Fidelities connected to channels

For finite dimensional case, ensemble average fidelity and entanglement fidelity are two kinds of important fidelities connected to a quantum channel. In this section we give the definitions of ensemble average fidelity and entanglement fidelity connected to a quantum channel for an infinite dimensional system, and discuss their relationship.

Let H be an infinite dimensional separable complex Hilbert space. Recall that a quantum channel $\mathcal{E} : \mathcal{T}(H) \to \mathcal{T}(H)$ is a trace preserving completely positive linear map. Like the finite dimensional case, for such quantum channel $\mathcal{E}$ and a given ensemble $\{p_j, \rho_j\}_{j=1}^{\infty}$, one can define ensemble average fidelity by

$$\overline{F} = \sum_j p_j F(\rho_j, \mathcal{E}(\rho_j))^2. \tag{4.1}$$

Similarly, for a state ρ, one can define the entanglement fidelity by

$$\begin{aligned} F(\rho, \mathcal{E}) = & \ F(|\psi\rangle, (\mathcal{E} \otimes I)(|\psi\rangle\langle\psi|))^2 \\ = & \ \langle\psi|(\mathcal{E} \otimes I)(|\psi\rangle\langle\psi|)|\psi\rangle, \end{aligned} \tag{4.2}$$

where $|\psi\rangle \in H \otimes H$ is a purification of ρ. Note that the definition $F(\rho, \mathcal{E})$ does not depend on the choices of purifications. To see this, let $|\psi\rangle = \sum_j \sqrt{p_j}|j\rangle|\mu_j\rangle$ be any purification, where $\{j\}$ is an orthonormal basis and $\{\mu_j\}$ is an orthonormal set of H. By [9], there exists a sequence of operators $\{E_i\} \subseteq \mathcal{B}(H)$ with $\sum_i E_i^\dagger E_i = I$ such that

$$\mathcal{E}(\sigma) = \sum_i E_i \sigma E_i^\dagger \quad \text{for all} \quad \sigma \in \mathcal{S}(H).$$

Thus

$$\begin{aligned} F(\rho, \mathcal{E}) = & \ \textstyle\sum_i \langle\psi|(E_i \otimes I)(|\psi\rangle\langle\psi|)(E_i^\dagger \otimes I)|\psi\rangle \\ = & \ \textstyle\sum_i \langle\psi| \sum_{j,k} \sqrt{p_j p_k}(E_i \otimes I)(|j\rangle|\mu_j\rangle\langle k|\langle\mu_k|)(E_i^\dagger \otimes I)|\psi\rangle \\ = & \ \textstyle\sum_i \sum_{j,k} p_j p_k \langle j|E_i|j\rangle\langle k|E_i^\dagger|k\rangle \\ = & \ \textstyle\sum_i |\mathrm{Tr}(E_i\rho)|^2, \end{aligned} \tag{4.3}$$

which is dependent only to ρ and $\mathcal{E}$.

In the sequel we will give some properties of entanglement fidelity for infinite dimensional systems.

Firstly note that, by monotonicity of the fidelity Eq.(2.7), it is easily checked that

$$F(\rho, \mathcal{E}) \leq [F(\rho, \mathcal{E}(\rho))]^2. \tag{4.4}$$

Proposition 4.1. *Let H be an infinite dimensional separable complex Hilbert space. Assume that $\mathcal{E} : \mathcal{T}(H) \to \mathcal{T}(H)$ is a quantum channel and $\rho \in \mathcal{S}(H)$. Then the entanglement fidelity $F(\rho, \mathcal{E})$ is a convex function of ρ.*

Proof. Take any states $\rho_1, \rho_2 \in \mathcal{S}(H)$. Define a real function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) \equiv F(x\rho_1 + (1-x)\rho_2, \mathcal{E}), \quad \forall x \in \mathbb{R}.$$

By using of Eq.(4.3) and elementary calculus, one sees that the second derivative of f is

$$f''(x) = \sum_i |\mathrm{Tr}((\rho_1 - \rho_2)E_i)|^2.$$

Hence $f''(x) \geq 0$, which implies that $F(\rho, \mathcal{E})$ is convex, as desired. □

Proposition 4.2. *Let H be an infinite dimensional separable complex Hilbert space. Assume that $\mathcal{E} : \mathcal{T}(H) \to \mathcal{T}(H)$ is a quantum channel. Then for any given ensemble $\{p_j, \rho_j\}$, we have $F(\sum_j p_j\rho_j, \mathcal{E}) \leq \overline{F}$.*

Proof. For any $k \in \mathbb{N}$, let $\lambda_k = \sum_{j=1}^k p_j$. Then, by Proposition 4.1, we have

$$\begin{aligned} F(\textstyle\sum_j p_j\rho_j, \mathcal{E}) = &\ F(\lambda_k(\textstyle\sum_{j=1}^k \frac{p_j}{\lambda_k}\rho_j) + (1-\lambda_k)(\sum_{j=k+1}^\infty \frac{p_j}{1-\lambda_k}\rho_j), \mathcal{E}) \\ \leq &\ \lambda_k F(\textstyle\sum_{j=1}^k \frac{p_j}{\lambda_k}\rho_j, \mathcal{E}) + (1-\lambda_k)F(\sum_{j=k+1}^\infty \frac{p_j}{1-\lambda_k}\rho_j, \mathcal{E}) \\ \leq &\ \lambda_k \textstyle\sum_{j=1}^k \frac{p_j}{\lambda_k} F(\rho_j, \mathcal{E}) + (1-\lambda_k)F(\sum_{j=k+1}^\infty \frac{p_j}{1-\lambda_k}\rho_j, \mathcal{E}) \\ = &\ \textstyle\sum_{j=1}^k p_j F(\rho_j, \mathcal{E}) + (1-\lambda_k)F(\sum_{j=k+1}^\infty \frac{p_j}{1-\lambda_k}\rho_j, \mathcal{E}). \end{aligned} \tag{4.5}$$

Note that $0 \leq F(\rho, \mathcal{E}) \leq 1$ and $\lim_{k\to\infty} \lambda_k = \sum_{j=1}^\infty p_j = 1$. So

$$\lim_{k\to\infty}(1-\lambda_k)F(\sum_{j=k+1}^\infty \frac{p_j}{1-\lambda_k}\rho_j, \mathcal{E}) = 0.$$

Thus, for any $\varepsilon > 0$, there exists some N such that

$$(1-\lambda_k)F(\sum_{j=k+1}^\infty \frac{p_j}{1-\lambda_k}\rho_j, \mathcal{E}) < \varepsilon \tag{4.6}$$

whenever $k > N$. It follows from Eq.(4.5) that

$$F(\sum_j p_j\rho_j, \mathcal{E}) < \sum_{j=1}^\infty p_j F(\rho_j, \mathcal{E}) + \varepsilon.$$

By the arbitrariness of ε and Eq.(4.4), we obtain that

$$\begin{aligned} F(\textstyle\sum_j p_j\rho_j, \mathcal{E}) \leq &\ \textstyle\sum_{j=1}^\infty p_j F(\rho_j, \mathcal{E}) \\ \leq &\ \textstyle\sum_{j=1}^\infty p_j F(\rho_j, \mathcal{E}(\rho_j))^2 = \overline{F}, \end{aligned}$$

Completing the proof. □

5. Conclusion

In this paper we prove the infinite dimensional version of the Uhlmann's theorem by an elementary approach, which states that the fidelity of states ρ and σ is larger than or equal to the absolute value of the inner product of any purifications $|\psi\rangle$ and $|\phi\rangle$ of ρ and σ, i.e., $F(\rho, \sigma) \geq |\langle\psi|\phi\rangle|$; moreover, there exist some purifications such that the equality holds. This allows us to generalize a large part of the results concerning fidelity for finite dimensional systems to that for infinite dimensional systems. We also discuss the relationship between quantum fidelity and classical fidelity and show that $F(\rho, \sigma) = \inf_{\{E_m\}} F(p_m, q_m)$. Not like to that of finite dimensional case, the infimum can not attain the minimum in general. We give a necessary and sufficient condition for the infimum attains the minimum. Using this result, we find that the fidelity and the trace distance are equivalent in describing the closeness of states. The concepts of ensemble average fidelity and entanglement fidelity for a channel are generalized to infinite dimensional case. The relationship of such two fidelities is discussed.

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