Spherical linear waves in de Sitter spacetime

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Abstract

With the Einstein-scalar field equations with positive cosmological constant in mind, we employ Christodoulou's framework, developed to study the vanishing cosmological constant case, to spherically symmetric solutions of the linear wave equation in de Sitter spacetime. We obtain an integro-differentiable evolution equation which we solve by taking initial data on a null cone. As a corollary we obtain elementary derivations of expected properties of linear waves in de Sitter spacetime: boundedness in terms of (characteristic) initial data, and a Price law establishing pointwise exponential decay, in Bondi time, to a constant.

1 Introduction

The study of the linear wave equation

$$\Box_g \phi = 0 \tag{1}$$

on fixed backgrounds (M, g) has been a stepping stone to the analysis of the nonlinearities of gravitation, from cosmic censorship to non-linear stability [6]. Here we revisit spherical linear waves in de Sitter spacetime as a prerequisite to the study of the Einstein-scalar field equations with positive cosmological constant and spherically symmetric initial data.

With this nonlinear problem in mind, we employ Christodoulou's framework, developed in the celebrated [3], to spherically symmetric solutions of the uncoupled equation (1). We obtain an integro-differentiable evolution equation which we solve by taking initial data on a null cone. As a corollary we obtain elementary derivations of expected properties of linear waves in de Sitter spacetime: boundedness in terms of (characteristic) initial data, and pointwise exponential decay, in Bondi time[1], to a constant[2]. Numerical evidence for such decay can be found in [2], and references therein, where higher spherical harmonics are also studied, as well as the non-linear system. Also, in [10], fundamental solutions of (1) in de Sitter spacetime are constructed, for smooth and compactly supported data; no symmetry assumptions are required, and exponential decay of certain homogeneous Sobolev L^p norms, $2 \leq p < \infty$, is proved. Also along these lines, Ringström [9] obtained exponential decay for non-linear perturbations of locally de Sitter cosmological models in the context of the Einstein-nonlinear scalar field system with a positive potential.

By comparison, the results presented here suffer from the requirement of symmetry. In fact, the methods used rely extensively on the assumption of spherical symmetry of solutions and on the existence of a regular center of symmetry, and it is not clear if it is possible to extend them to the study of higher spherical harmonics, or to the analysis of linear waves in other backgrounds, like Schwarzschild-de-Sitter[3]. Nonetheless, we believe that the relevance of this work goes beyond the fact that the method used is both elementary and presumably adaptable to the non-linear setting. First of all, although widely expected, we

[1]From which exponential decay to a constant in the usual static time coordinate easily follows.

[2]Such boundedness and decay results may be seen, respectively, as analogues of the Kay-Wald Theorem [7] and of a Price law [8, 4], both originally formulated for Cauchy data in a Schwarzschild background.

[3]For a thorough discussion of linear waves in black hole spacetimes see [6].

are unaware of a written proof of the results concerning the pointwise exponential decay (9)[4]. Secondly, the bounds in terms of characteristic initial data are, to our knowledge, original. Finally, our results apply to both the local and the cosmological regions, i.e. the past and the future of the cosmological horizon.

2 Christodoulou's framework for spherical waves

Bondi coordinates [3] (u, r, θ, φ) map the causal future of any point in de Sitter spacetime isometrically onto $\big([0,\infty)\times[0,\infty)\times S^2, g\big)$, where

$$g = -\left(1-\frac{\Lambda}{3}r^2\right)du^2 - 2dudr + r^2 d\Omega^2 \ , \tag{2}$$

with $d\Omega^2$ the round metric of the two-sphere (cf. Figure 1).

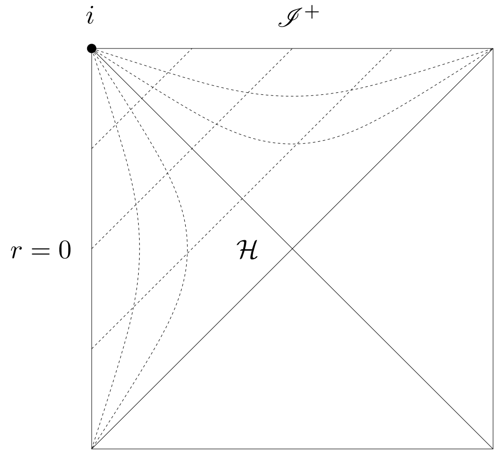

Figure 1: Penrose diagram of de Sitter spacetime. The lines $u =$ constant are the outgoing null geodesics starting at $r = 0$. The point i corresponds to $u = +\infty$, the cosmological horizon $\mathcal{H}$ to $r = \sqrt{\frac{3}{\Lambda}}$ and the future null infinity $\mathscr{I}^+$ to $r = \infty$.

In these coordinates the wave equation

$$\Box_g \phi = 0 \Leftrightarrow \partial_\mu\left(\sqrt{-\det(g)}\,\partial^\mu\phi\right) = 0 \ ,$$

for spherically symmetric functions, $\partial_\theta\phi = \partial_\varphi\phi = 0$, reads

$$-2r\frac{\partial}{\partial r}\left(\frac{\partial\phi}{\partial u}\right) - 2\frac{\partial\phi}{\partial u} + r\left(1-\frac{\Lambda}{3}r^2\right)\frac{\partial^2\phi}{\partial r^2} + \left(2-\frac{4}{3}\Lambda r^2\right)\frac{\partial\phi}{\partial r} = 0 \ . \tag{3}$$

Following Christodoulou [3] we consider the change of variable

$$h := \frac{\partial}{\partial r}(r\phi) \ .$$

If we assume that

$$\lim_{r\to 0} r\phi = 0 \ ,$$

it immediately follows that

$$\phi = \bar{h} := \frac{1}{r}\int_0^r h(u,s)\,ds \quad \text{ and } \quad \frac{\partial\phi}{\partial r} = \frac{\partial\bar{h}}{\partial r} = \frac{h-\bar{h}}{r} \ . \tag{4}$$

[4] For Schwarzschild-de-Sitter with non-vanishing mass, exponential pointwise decay in the local region up to the horizons follows from [5, 6]. See also [1].

Moreover, assuming that the crossed partial derivatives of $r\phi$ commute, we see that (3) is equivalent to

$$Dh = -\frac{\Lambda}{3} r(h - \bar{h}) \ , \tag{5}$$

where D is the differential operator given by

$$D := \frac{\partial}{\partial u} - \frac{1}{2}\left(1 - \frac{\Lambda}{3}r^2\right)\frac{\partial}{\partial r} \ .$$

3 Main result: statement and proof

Our main result is the following

Theorem 1. *Let $\Lambda > 0$. Given $h_0 \in \mathcal{C}^k([0,\infty))$, for some $k \geq 1$, the problem*

$$\begin{cases} Dh = -\frac{\Lambda}{3}r(h - \bar{h}) \\ h(0,r) = h_0(r) \end{cases} \tag{6}$$

has a unique solution $h \in \mathcal{C}^k([0,\infty)\times[0,\infty))$.

Moreover, if $\|h_0\|_{\mathcal{C}^0}$ is finite then

$$\|h\|_{\mathcal{C}^0} = \|h_0\|_{\mathcal{C}^0} \ . \tag{7}$$

Also, if $\|(1+r)^p\partial_r h_0\|_{\mathcal{C}^0}$ is finite for some $0 \leq p \leq 4$ and $H \leq 2\sqrt{\frac{\Lambda}{3}}$ then

$$\|(1+r)^p e^{Hu}\partial_r h\|_{\mathcal{C}^0} \lesssim \|(1+r)^p\partial_r h_0\|_{\mathcal{C}^0} \ , \tag{8}$$

and, consequently, there exits $\underline{h} \in \mathbb{R}$ such that

$$|h(u,r) - \underline{h}| \lesssim (1+r)^{n(p)}e^{-Hu} \ , \tag{9}$$

with

$$n(p) = \begin{cases} 0 & , \quad 2 < p \leq 4 \\ 2 & , \quad 0 \leq p \leq 2 \end{cases} \ . \tag{10}$$

Remark 1. *The powers of $1+r$ obtained are far from optimal. Since we are mainly interested in understanding whether the decay in u obtained by this method is uniform in r, we were only careful in computing precise estimates for $2 < p \leq 4$, which is enough to establish uniform decay for $p > 2$ (if $p > 4$ the $p = 4$ result applies, and in fact it does not seem to be possible to obtain a stronger decay in r for $\partial_r h$). For $p \leq 2$ our method does not provide uniform decay, but it is not clear if this is an artifact of these techniques or an intrinsic property of spherical linear waves in de Sitter.*

Proof. For $h \in \mathcal{C}^0([0,\infty)\times[0,\infty))$, we have $r\bar{h} \in \mathcal{C}^0([0,\infty)\times[0,\infty))$, and so we can define $\mathcal{F}(h)$ to be the solution to the linear equation

$$\begin{cases} D(\mathcal{F}(h)) = -\frac{\Lambda}{3}r(\mathcal{F}(h) - \bar{h}) \\ \mathcal{F}(h)(0,r) = h_0(r) \end{cases} \ . \tag{11}$$

The integral lines of D (incoming light rays in de Sitter), which satisfy

$$\frac{dr}{du} = -\frac{1}{2}\left(1 - \frac{\Lambda}{3}r^2\right) \ , \tag{12}$$

are characteristics of the problem at hand. Integrating (11) along such characteristics we obtain

$$\mathcal{F}\left(h\right)(u_1, r_1) = h_0(r(0))e^{-\frac{\Lambda}{3}\int_0^{u_1} r(s)ds} + \frac{\Lambda}{3}\int_0^{u_1} r(v)\bar{h}(v, r(v))e^{-\frac{\Lambda}{3}\int_v^{u_1} r(s)ds}dv \ , \tag{13}$$

where, to simplify the notation, we denote the solution to (12) satisfying $r(u_1) = r_1$ simply by $s \mapsto r(s)$; we are dropping any explicit reference to the dependence on (u_1, r_1), but it should be noted, in particular, that $r(0)$ is an analytic function of (u_1, r_1).

Given $U, R > 0$, let $\mathcal{C}^0_{U,R}$ denote the Banach space $\left(\mathcal{C}^0\left([0,U]\times[0,R]\right), \|\cdot\|_{\mathcal{C}^0_{U,R}}\right)$, where

$$\|f\|_{\mathcal{C}^0_{U,R}} = \sup_{(u,r)\in[0,U]\times[0,R]} |f(u,r)| \ . \tag{14}$$

Let $r_c := \sqrt{\frac{3}{\Lambda}}$ be the unique non-negative root of $1 - \frac{\Lambda}{3}r^2$ (see (2) and (12)). The non-decreasing behavior of the characteristics satisfying $r_1 \geq r_c$ shows that the restriction of $\mathcal{F}$ to $\mathcal{C}^0_{U,R}$ is well defined for all $R \geq r_c$. In fact:

Lemma 1. *Given $U > 0$ and $R \geq r_c := \sqrt{\frac{3}{\Lambda}}$, $\mathcal{F}$ contracts in $\mathcal{C}^0_{U,R}$.*

Proof. Fix $U > 0$ and $R \geq r_c$. Then

$$\begin{aligned}
\|\mathcal{F}(h_1) - \mathcal{F}(h_2)\|_{\mathcal{C}^0_{U,R}} &= \sup_{(u_1,r_1)\in[0,U]\times[0,R]} |\mathcal{F}(h_1)(u_1,r_1) - \mathcal{F}(h_2)(u_1,r_1)| \\
&\leq \sup_{(u_1,r_1)\in[0,U]\times[0,R]} \left\{\frac{\Lambda}{3}\int_0^{u_1} r(v)\left|\bar{h}_1(v,r(v)) - \bar{h}_2(v,r(v))\right| e^{-\frac{\Lambda}{3}\int_v^{u_1} r(s)ds}dv\right\} \\
&\leq \sup_{(u_1,r_1)\in[0,U]\times[0,R]} \left\{\int_0^{u_1} \frac{\Lambda}{3}r(v)e^{-\frac{\Lambda}{3}\int_v^{u_1} r(s)ds}dv\right\} \cdot \|\bar{h}_1 - \bar{h}_2\|_{\mathcal{C}^0_{U,R}} \\
&\leq \sup_{(u_1,r_1)\in[0,U]\times[0,R]} \left\{\left[e^{-\frac{\Lambda}{3}\int_v^{u_1} r(s)ds}\right]_{v=0}^{u_1}\right\} \cdot \sup_{(u,r)\in[0,U]\times[0,R]} \left\{\frac{1}{r}\int_0^r |h_1(u,s) - h_2(u,s)|ds\right\} \\
&\leq \underbrace{\sup_{(u_1,r_1)\in[0,U]\times[0,R]} \left\{1 - e^{-\frac{\Lambda}{3}\int_0^{u_1} r(s)ds}\right\}}_{:=\sigma} \cdot \|h_1 - h_2\|_{\mathcal{C}^0_{U,R}} \ .
\end{aligned}$$

Throughout, to obtain estimates, and in particular to estimate σ, one needs to consider three (causally) separate regions, naturally corresponding to the bifurcations of (12): the local region ($r < r_c$), the cosmological horizon ($r = r_c$), and the cosmological region ($r > r_c$). However, since the computations are similar we will only present the details concerning the most delicate case, $r > r_c$.

The solution to (12) satisfying $r_1 = r(u_1) > r_c := \sqrt{\frac{3}{\Lambda}}$, is given by

$$r(u) = \sqrt{\frac{3}{\Lambda}}\coth\left(\frac{1}{2}\sqrt{\frac{\Lambda}{3}}(c-u)\right) , \tag{15}$$

for an appropriate choice of $c = c(u_1, r_1)$, from which it follows that

$$\begin{aligned}
-\frac{\Lambda}{3}\int_0^{u_1} r(s)ds &= \int_0^{u_1} -\sqrt{\frac{\Lambda}{3}}\coth\left(\frac{1}{2}\sqrt{\frac{\Lambda}{3}}(c-s)\right)ds \\
&= \int_0^{u_1} 2\frac{d}{ds}\ln\left[\sinh\left(\frac{1}{2}\sqrt{\frac{\Lambda}{3}}(c-s)\right)\right]ds = \ln\left[\frac{\sinh\left(\frac{1}{2}\sqrt{\frac{\Lambda}{3}}(c-u_1)\right)}{\sinh\left(\frac{1}{2}\sqrt{\frac{\Lambda}{3}}c\right)}\right]^2 ,
\end{aligned}$$

and consequently

$$\begin{aligned}
e^{-\frac{\Lambda}{3}\int_0^{u_1} r(s)ds} &= \frac{\sinh^2\left(\frac{1}{2}\sqrt{\frac{\Lambda}{3}}(c-u_1)\right)}{\sinh^2\left(\frac{1}{2}\sqrt{\frac{\Lambda}{3}}c\right)} = \frac{\cosh^2\left(\frac{1}{2}\sqrt{\frac{\Lambda}{3}}(c-u_1)\right)}{\sinh^2\left(\frac{1}{2}\sqrt{\frac{\Lambda}{3}}c\right)\coth^2\left(\frac{1}{2}\sqrt{\frac{\Lambda}{3}}(c-u_1)\right)} \\
&= \left[\frac{\cosh\left(\alpha(c-u_1)\right)}{\sinh\left(\alpha c\right)}\frac{1}{2\alpha r_1}\right]^2 = \left[\frac{e^{\alpha(c-u_1)} + e^{-\alpha(c-u_1)}}{e^{\alpha c} - e^{-\alpha c}}\right]^2 \frac{1}{4\alpha^2 r_1^2} \\
&\geq \frac{e^{-2\alpha u_1}}{4\alpha^2 r_1^2},
\end{aligned}$$

where $\alpha := \frac{1}{2}\sqrt{\frac{\Lambda}{3}}$. Define

$$\begin{aligned}\sigma_{cosm}(U,R) &:= \sup_{(u_1,r_1)\in[0,U]\times(r_c,R]} \left(1-e^{-\frac{\Lambda}{3}\int_0^{u_1} r(s)ds}\right) \\ &\leq \sup_{(u_1,r_1)\in[0,U]\times(r_c,R]} \left(1-\frac{e^{-2\alpha u_1}}{4\alpha^2 r_1^2}\right) \leq \left(1-\frac{3}{\Lambda}\frac{e^{-\sqrt{\frac{\Lambda}{3}}U}}{R^2}\right) < 1 \ .\end{aligned}$$

Similar computations give

$$\sigma_{loc} := \sup_{(u_1,r_1)\in[0,U]\times[0,r_c)} \left(1-e^{-\frac{\Lambda}{3}\int_0^{u_1} r(s)ds}\right) \leq \left(1-\frac{e^{-\sqrt{\frac{\Lambda}{3}}U}}{4}\right) < 1 \ ,$$

for the local region, and

$$\sigma_{hor} := \sup_{u_1\in[0,U]} \left(1-e^{-\frac{\Lambda}{3}\int_0^{u_1} r_c ds}\right) \leq 1-e^{-\sqrt{\frac{\Lambda}{3}}U} < 1 \ ,$$

along the cosmological horizon. Finally $\sigma = \max\{\sigma_{loc},\sigma_{hor},\sigma_{cosm}\} < 1$, and the statement of the lemma follows.

□

By the contraction mapping theorem, given $U > 0$ and $R \geq r_c$, there exits a unique fixed point $h_{U,R} \in \mathcal{C}^0_{U,R}$ of $\mathcal{F}$. Uniqueness guarantees that in the intersection of two rectangles $[0,U_1]\times[0,R_1]\cap[0,U_2]\times[0,R_2]$ the corresponding h_{U_1,R_1} and h_{U_2,R_2} coincide. Consequently

$$h(u,r) := h_{u+1,r+r_c}(u,r) \ ,$$

is well defined in $[0,\infty)\times[0,\infty)$; it is also clearly continuous and satisfies $h = \mathcal{F}(h)$, i.e.,

$$h(u_1,r_1) = h_0(r(0))e^{-\frac{\Lambda}{3}\int_0^{u_1} r(s)ds} + \frac{\Lambda}{3}\int_0^{u_1} r(v)\bar{h}(v,r(v))e^{-\frac{\Lambda}{3}\int_v^{u_1} r(s)ds}dv \ , \tag{16}$$

in $[0,\infty)\times[0,\infty)$. Continuity of h implies continuity of $r\bar{h}$, so we are allowed to differentiate (16) in the direction of D, which proves that h is in fact a ($\mathcal{C}^0$) solution of (6). Existence and uniqueness in $\mathcal{C}^0([0,\infty)\times[0,\infty))$ follows.

To see that a solution of (6) is as regular as its initial condition assume that $h_0 \in \mathcal{C}^{k+1}$, $k \geq 0$, and start by noticing that if $h \in \mathcal{C}^k$ then $r\bar{h}$ and $\partial_r(r\bar{h})$ are also in $\mathcal{C}^k$. In particular for $h \in \mathcal{C}^0$ we can differentiate (16) with respect to u_1 to obtain

$$\begin{aligned}\frac{\partial h}{\partial u_1} &= \frac{\partial}{\partial u_1}\left(h_0(r(0))e^{-\frac{\Lambda}{3}\int_0^{u_1} r(s)ds}\right) + \frac{\Lambda}{3}\left(r\bar{h}\right)(u_1,r_1) \\ &\quad + \frac{\Lambda}{3}\int_0^{u_1} \frac{\partial(r\bar{h})}{\partial r}\frac{\partial r}{\partial u_1}(v,r(v))e^{-\frac{\Lambda}{3}\int_v^{u_1} r(s)ds}dv \\ &\quad + \frac{\Lambda}{3}\int_0^{u_1} r(v)\bar{h}(v,r(v))\frac{\partial}{\partial u_1}\left(e^{-\frac{\Lambda}{3}\int_v^{u_1} r(s)ds}\right) \ .\end{aligned} \tag{17}$$

This last expression shows that $h_0 \in \mathcal{C}^{k+1}$ and $h \in \mathcal{C}^k$ implies $\frac{\partial h}{\partial u_1} \in \mathcal{C}^k$. The same reasoning works for the derivative with respect to r_1. Consequently, if $h_0 \in C^{k+1}$ and $h \in \mathcal{C}^k$, then h is in fact in C^{k+1} and the regularity statement follows by induction.

To establish (7) first note that:

Lemma 2. *If $\|h_0\|_{\mathcal{C}^0} \leq y_0$ and $\|h\|_{\mathcal{C}^0} \leq y_0$, for some $y_0 \geq 0$, then $\|\mathcal{F}(h)\|_{\mathcal{C}^0} \leq y_0$.*

Proof. From (13) we see that

$$\begin{aligned}|\mathcal{F}(h)(u_1,r_1)| &\leq \|h_0\|_{\mathcal{C}^0}\, e^{-\frac{\Lambda}{3}\int_0^{u_1} r(s)ds} + \|\bar{h}\|_{\mathcal{C}^0}\,\frac{\Lambda}{3}\int_0^{u_1} r(v)e^{-\frac{\Lambda}{3}\int_v^{u_1} r(s)ds}dv \\ &\leq y_0 \underbrace{\left(e^{-\frac{\Lambda}{3}\int_0^{u_1} r(s)ds} + \frac{\Lambda}{3}\int_0^{u_1} r(v)e^{-\frac{\Lambda}{3}\int_v^{u_1} r(s)ds}dv\right)}_{\equiv 1} = y_0 \ .\end{aligned}$$

The last step follows by a direct computation, as before, or by noticing that since $h \equiv 1$ is a solution to (6), with $h_0 \equiv 1$, one has $\mathcal{F}(1) \equiv 1$.

□

Now consider the sequence

$$\begin{cases} h_0(u,r) = h_0(r) \\ h_{n+1} = \mathcal{F}(h_n) \end{cases} .$$

We have already established that, for any $U > 0$ and $R \geq r_c$, h_n converges in $\mathcal{C}^0_{U,R}$ to h, the solution of (6). Lemma 2 then tells us that

$$\|h_n\|_{\mathcal{C}^0_{U,R}} \leq \|h_n\|_{\mathcal{C}^0} \leq \|h_0\|_{\mathcal{C}^0} , \quad \text{and so} \quad \|h\|_{\mathcal{C}^0_{U,R}} = \lim_{n\to\infty} \|h_n\|_{\mathcal{C}^0_{U,R}} \leq \|h_0\|_{\mathcal{C}^0} .$$

Since this holds for arbitrarily large U and R, the bound (7) follows.

We will now show that the estimate (8) holds. First of all if $h \in \mathcal{C}^1$ we see that Dh and $\partial_r Dh$ are both continuous, and consequently $D\partial_r h$ exists and its equal to $\partial_r Dh + [D, \partial_r]h$ [5]. Using this last fact and equations (4) while differentiating (5) with respect to r we obtain an evolution equation for $\partial_r h$:

$$D\partial_r h = -2\frac{\Lambda}{3} r\, \partial_r h . \tag{18}$$

Integrating the last equation along the (ingoing) characteristics, as before, yields

$$\partial_r h(u_1, r_1) = \partial_r h_0(r_0) e^{-\frac{2\Lambda}{3}\int_0^{u_1} r(s)ds}. \tag{19}$$

It is then clear that initial data controls the supremum norm of $\partial_r h$. In fact, let

$$d_0 = \|(1+r)^p \partial_r h_0\|_{\mathcal{C}^0} .$$

In the cosmological region ($r > r_c$), one has, after recalling (15),

$$\begin{aligned} \left|(1+r_1)^p e^{Hu_1} \partial_r h(u_1, r_1)\right| &= \left|(1+r_1)^p e^{Hu_1} \partial_r h_0(r_0) e^{-\frac{2\Lambda}{3}\int_0^{u_1} r(s)ds}\right| \\ &\leq d_0 \left(\frac{1+r_1}{1+r_0} \frac{\sinh(\alpha(c-u_1))}{\sinh(\alpha c)}\right)^p e^{Hu_1} \left(\frac{\sinh(\alpha(c-u_1))}{\sinh(\alpha c)}\right)^{4-p} , \end{aligned} \tag{20}$$

where $\alpha = \frac{1}{2}\sqrt{\frac{\Lambda}{3}}$ as before. Now, since $c - u_1 \leq c$, then $e^{-2\alpha(c-u_1)} \geq e^{-2\alpha c}$, and

$$\begin{aligned} \frac{\sinh(\alpha(c-u_1))}{\sinh(\alpha c)} &= \frac{e^{\alpha(c-u_1)} - e^{-\alpha(c-u_1)}}{e^{\alpha c} - e^{-\alpha c}} \\ &= e^{-\alpha u_1} \frac{1 - e^{-2\alpha(c-u_1)}}{1 - e^{-2\alpha c}} \\ &\leq e^{-\alpha u_1} . \end{aligned} \tag{21}$$

Also

$$\begin{aligned} \frac{1+r_1}{1+r_0} \frac{\sinh(\alpha(c-u_1))}{\sinh(\alpha c)} &= \frac{1 + \frac{1}{2\alpha}\coth(\alpha(c-u_1))}{1 + \frac{1}{2\alpha}\coth(\alpha c)} \frac{\sinh(\alpha(c-u_1))}{\sinh(\alpha c)} \\ &= \frac{\sinh(\alpha(c-u_1)) + \frac{1}{2\alpha}\cosh(\alpha(c-u_1))}{\sinh(\alpha(c)) + \frac{1}{2\alpha}\cosh(\alpha c)} \\ &\leq \frac{1 + \frac{1}{2\alpha}}{\frac{1}{2\alpha}} \cdot \frac{\cosh(\alpha(c-u_1))}{\cosh(\alpha c)} \\ &\leq (2\alpha + 1)\, 2e^{-\alpha u_1}. \end{aligned} \tag{22}$$

[5] Here we are using the following generalized version of the Schwarz Lemma: if X and Y are two nonvanishing $\mathcal{C}^1$ vector fields in $\mathbb{R}^2$ and f is a $\mathcal{C}^1$ function such that $X \cdot (Y \cdot f)$ exists and is continuous then $Y \cdot (X \cdot f)$ also exists and is equal to $X \cdot (Y \cdot f) - [X, Y] \cdot f$.

Therefore, if $0 \le p \le 4$ and $H \le 4\alpha = 2\sqrt{\Lambda/3}$, we plug (21) and (22) into (20) to obtain

$$\begin{aligned} \sup_{(u_1,r_1)\in[0,U]\times[r_c,R]} \left|(1+r_1)^p e^{Hu_1}\partial_r h(u_1,r_1)\right| &\le d_0 \sup_{(u_1,r_1)\in[0,U]\times[r_c,R]} \left|2^p(2\alpha+1)^p e^{(H-4\alpha)u_1}\right| \\ &\le 2^p(2\alpha+1)^p d_0 \ . \end{aligned} \tag{23}$$

Similar, although simpler, computations yield

$$\sup_{(u_1,r_1)\in[0,U]\times[0,r_c]} \left|(1+r_1)^p e^{Hu_1}\partial_r h(u_1,r_1)\right| \le 16 \sup_{r_1\in[0,r_c]} |(1+r_1)^p \partial_r h_0(r_1)| \le 16 d_0 \tag{24}$$

for the local region. This proves (8).

To finish the proof of Theorem 1 all is left is to establish the pointwise decay statement (9). Start with

$$\begin{aligned} \left|h(u,r)-\bar h(u,r)\right| &\le \frac{1}{r}\int_0^r |h(u,r)-h(u,s)|\,ds \\ &\le \frac{1}{r}\int_0^r\int_s^r |\partial_\rho h(u,\rho)|\,d\rho\,ds \\ &\lesssim \frac{1}{r}\int_0^r\int_s^r \frac{e^{-Hu}}{(1+\rho)^p}d\rho\,ds \lesssim \begin{cases} \frac{e^{-Hu}}{1+r} & , \quad 2<p\le 4 \\ re^{-Hu} & , \quad 0\le p\le 2 \end{cases} . \end{aligned}$$

These estimates for $2 < p \le 4$ follow from a lengthy, though straightforward, computation; they seem to be the optimal results which follow from this method. The remaining cases, with the exception of $p = 0$, are far from optimal. In fact, since we are mainly interested in a qualitative analysis, namely if the decay obtained is or not uniform in r (see Remark 1), then, to avoid further computations, the results for $p \le 2$ were obtained from the estimate for $p = 0$.

Using (5) we then see that

$$\begin{aligned} |\partial_u h| &= \left|Dh + \frac{1}{2}\left(1-\frac{\Lambda}{3}r^2\right)\partial_r h\right| \\ &\le \left|-\frac{\Lambda}{3}r\left(h-\bar h\right)\right| + \frac{1}{2}\left|\left(1-\frac{\Lambda}{3}r^2\right)\partial_r h\right| \lesssim (1+r)^{n(p)}e^{-Hu} \ , \end{aligned}$$

with $n(p)$ as in the statement of the theorem.

Now since $\partial_u h$ is integrable with respect to u, by the fundamental theorem of calculus, we see that there exists

$$\lim_{u\to\infty} h(u,r) = \underline{h}(r) \ .$$

But

$$\begin{aligned} |\underline{h}(r_2)-\underline{h}(r_1)| &= \lim_{u\to\infty} |h(u,r_2)-h(u,r_1)| \\ &\le \lim_{u\to\infty}\left|\int_{r_1}^{r_2} |\partial_r h(u,r)|dr\right| \\ &\lesssim \lim_{u\to\infty} |r_2-r_1|e^{-Hu} = 0 \ , \end{aligned}$$

and, consequently, there exists $\underline{h}\in\mathbb{R}$ such that

$$\underline{h}(r) \equiv \underline{h} \ .$$

Finally

$$\begin{aligned} |h(u,r)-\underline{h}| &\le \int_u^\infty |\partial_v h(v,r)|\,dv \\ &\lesssim \int_u^\infty (1+r)^{n(p)}e^{-Hv}dv \lesssim (1+r)^{n(p)}e^{-Hu} \ . \end{aligned}$$

□

Remark 2. *The same calculation shows that given $R > 0$ the solutions of* (6) *satisfy $|h(u,r)-\underline{h}| \lesssim e^{-Hu}$ uniformly for $r \in [0,R]$, even if $\|(1+r)^p\partial_r h_0\|_{C^0}$ is not finite.*

4 Boundedness and exponential pointwise decay for spherical linear waves in de Sitter

We now translate part of the results in Theorem (1) back into results concerning linear waves in de Sitter.

Theorem 2. *Let (M,g) be de Sitter spacetime with cosmological constant Λ and (u,r,θ,φ) Bondi coordinates as in Section 2. Let $\phi=\phi(u,r)\in\mathcal{C}^2([0,\infty)\times[0,\infty))$ be a solution [6] to*

$$\Box_g\phi = 0\ .$$

Then

$$|\phi| \leq \sup_{r\geq 0}\left|\partial_r\left(r\phi(0,r)\right)\right|\ . \tag{25}$$

Moreover, if for some $p\geq 0$

$$\sup_{r\geq 0}\left|(1+r)^p\frac{\partial^2}{\partial r^2}\left(r\phi(0,r)\right)\right| < \infty\ , \tag{26}$$

then there exists $\underline{\phi}\in\mathbb{R}$ such that, for $H\leq 2\sqrt{\frac{\Lambda}{3}}$,

$$\left|\phi(u,r)-\underline{\phi}\right| \lesssim (1+r)^{n(p)}e^{-Hu}\quad,\ \textit{if}\quad 0\leq p\leq 4\ , \tag{27}$$

where

$$n(p)=\begin{cases} 0 & ,\quad 2<p\leq 4 \\ 2 & ,\quad 0\leq p\leq 2 \end{cases}\ . \tag{28}$$

Proof. Since ϕ is a spherically symmetric $\mathcal{C}^2$ solution of (1) we saw in Section 2 that $h=\partial_r(r\phi)$ satisfies (5), with $\phi=\bar{h}$. Applying Theorem (1) the results easily follows. □

Remark 3. *Note once more that the powers of r obtained are far from optimal, see Remark 1.*

Remark 4. *It should be emphasized that the boundedness and decay results are logically independent. In fact* (25) *follows from* (7)*, which in turn is a consequence of a fortunate trick (see proof of Lemma 2) relying on the non-positivity of the factor of the zeroth order term in* (5) *(here, non-negativity of Λ) and the fact that $\mathcal{F}$* (13) *is a contraction in appropriate function spaces; in some sense one is required to prove existence and uniqueness of* (6) *in the process. That is no longer the case for obtaining* (9)*, from which pointwise decay of ϕ follows.*

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