Discrete Integrable Systems and Hodograph Transformations Arising from Motions of Discrete Plane Curves

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Abstract. We consider integrable discretizations of some soliton equations associated with the motions of plane curves: the Wadati-Konno-Ichikawa elastic beam equation, the complex Dym equation, and the short pulse equation. They are related to the modified KdV or the sine-Gordon equations by the hodograph transformations. Based on the observation that the hodograph transformations are regarded as the Euler-Lagrange transformations of the curve motions, we construct the discrete analogues of the hodograph transformations, which yield integrable discretizations of those soliton equations.

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1. Introduction

The study of discrete integrable systems has received considerable attention in the past decade (see, for example, [1]). Ablowitz and Ladik proposed a method of integrable discretizations of soliton equations, including the nonlinear Schrödinger equation and the modified KdV (mKdV) equation, based on the Lax form [2–4], and Hirota discretized various soliton equations such as the KdV, the mKdV, and the sine-Gordon equations by using the bilinear formalism [5–9]. Following the pioneering work of Ablowitz-Ladik and Hirota, Date, Jimbo and Miwa developed a unified algebraic approach from the point of view of the KP theory [10–16]. For other approaches to the discrete integrable systems, see, for example, [17, 18].

It is known that there is a class of soliton equations which admits loop, cusp, and peak soliton solutions. Among them, some soliton equations, e.g. the Wadati-Konno-Ichikawa

(WKI) elastic beam equation, the Camassa-Holm equation, the Degasperis-Procesi equation, the short pulse equation and the Hunter-Saxton equation, appear as model equations of various physical phenomena [19–26]. It should be noted that those equations are transformed to certain soliton equations which admit smooth soliton solutions through hodograph transformations [27–35]. For example, the WKI elastic beam equation is transformed to the potential mKdV equation [27].

Integrable discretization of those soliton equations has been regarded as a difficult problem until recently. Especially, a systematic treatment of hodograph transformations in discretizations has been unknown. Recently, some of the authors succeeded in integrable discretization of some equations in the above class of soliton equations by using the bilinear method, and it was confirmed that those integrable discrete equations work effectively on numerical computations of the above class of soliton equations as self-adaptive mesh schemes [36–39]. However, the method employed there is rather technical, so it is not easy to extract a fundamental structure of discretizations to apply this method to a broader class of nonlinear wave equations.

On the other hand, the discrete integrable systems have been applied to discretizations of curves and surfaces, and this area has been recently studied actively under the name of the discrete differential geometry [40,41]. In particular, there have been intensive studies in topics related to curve geometry after the pioneering work of Lamb and Goldstein-Petrich [42, 43], and then several frameworks for the motion of discrete curves have been proposed in various settings [44–49].

It is well known that the potential mKdV equation describes the motion of plane curves [43]. Recently, the authors considered continuous and discrete motion of discrete plane curves in the Euclidean plane and presented the explicit formula in terms of the τ-function [48, 49]. The hodograph transformation of the WKI elastic beam equation can be viewed as the Euler-Lagrange transformation of the motion of plane curves. From this fact, it may be possible to establish a discrete analogue of the hodograph transformation for the motion of discrete curves, and to discretize soliton equations having singularities naturally from a geometric point of view.

In the present paper, we construct discrete analogues of hodograph transformations by the Euler-Lagrange transformations of the motion of discrete plane curves in the Euclidean plane. Based on them, we construct semi-discrete and fully discrete analogues of the WKI elastic beam equation, the complex Dym equation, and the short pulse equation. In Section 2.1, we discuss the motion of plane curves described by the potential mKdV equation and the hodograph transformations for the WKI elastic beam equation and the complex Dym equation. We also discuss the relationship between the sine-Gordon equation and the short pulse equation.

In Section 2.2, we introduce the discrete hodograph transformation for the continuous motion of discrete plane curves which are described by the semi-discrete potential mKdV equation. Then we construct the semi-discrete WKI elastic beam equation and the semi-discrete complex Dym equation. Using the same technique to the semi-discrete sine-Gordon equation, we construct the semi-discrete short pulse equation.

In Section 2.3, we consider the discrete motion of discrete plane curves which are described by the discrete potential mKdV equation. Then we construct the discrete WKI elastic beam equation and the discrete complex Dym equation by using the discrete hodograph transformations. In a similar way, we construct the discrete short pulse equation from the discrete sine-Gordon equation.

In Section 3, we present a τ-function which gives soliton and breather solutions for these equations, and Hirota-type bilinear equations for the τ-function. Section 4 is devoted to the conclusion.

2. Motion of plane curves and hodograph transformations

2.1. Motion of smooth curves

Let $\gamma(s)$ be an arc-length parametrized curve in Euclidean plane $\mathbb{R}^2$. Then the tangent vector $\frac{\partial\gamma}{\partial s}$ satisfies

$$\left|\frac{\partial\gamma}{\partial s}\right| = 1. \tag{2.1}$$

Thus $\frac{\partial\gamma}{\partial s}$ admits the parametrization

$$\frac{\partial\gamma}{\partial s} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}. \tag{2.2}$$

The function $\theta = \theta(s)$ is called the angle function of γ which denotes the angle of $\frac{\partial\gamma}{\partial s}$ measured from the x-axis. We define the normal vector $\boldsymbol{N}$ by

$$\boldsymbol{N} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\frac{\partial\gamma}{\partial s} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}, \tag{2.3}$$

and introduce the Frenet frame

$$F = (\boldsymbol{T}, \boldsymbol{N}), \quad \boldsymbol{T} = \frac{\partial\gamma}{\partial s}, \tag{2.4}$$

which is the orthonormal basis attached to the curve. The Frenet equation is given by

$$\frac{\partial}{\partial s}F = F\begin{bmatrix} 0 & -\kappa \\ \kappa & 0 \end{bmatrix}, \tag{2.5}$$

where the function $\kappa = \frac{\partial\theta}{\partial s}$ is the curvature of γ. The angle function θ is also referred to as the potential function. Let us consider the following isoperimetric motion in time t:

$$\frac{\partial}{\partial t}F = F\begin{bmatrix} 0 & \kappa_{ss} + \frac{\kappa^3}{2} \\ -\kappa_{ss} - \frac{\kappa^3}{2} & 0 \end{bmatrix}. \tag{2.6}$$

In terms of $\frac{\partial\gamma}{\partial s}$, (2.5) and (2.6) can be expressed as

$$\frac{\partial^2\gamma}{\partial s^2} = \begin{bmatrix} 0 & -\kappa \\ \kappa & 0 \end{bmatrix}\frac{\partial\gamma}{\partial s}, \tag{2.7}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial\gamma}{\partial s}\right) = \begin{bmatrix} 0 & \kappa_{ss} + \frac{\kappa^3}{2} \\ -\kappa_{ss} - \frac{\kappa^3}{2} & 0 \end{bmatrix}\frac{\partial\gamma}{\partial s}, \tag{2.8}$$

respectively. Then the compatibility condition of (2.5) and (2.6), or (2.7) and (2.8) yields the mKdV equation for $\kappa = \kappa(s,t)$ [42, 43]

$$\kappa_t + \frac{3}{2}\kappa^2\kappa_s + \kappa_{sss} = 0, \tag{2.9}$$

or the potential mKdV equation for $\theta = \theta(s,t)$:

$$\theta_t + \frac{1}{2}(\theta_s)^3 + \theta_{sss} = 0. \tag{2.10}$$

The mKdV equation can be viewed as the governing equation of the Lagrangian description for the motion of the curves γ in terms of the arc-length parameter s. Let us consider the Eulerian description of the same motion of the curves. To this end, we introduce the Eulerian coordinates

$$\gamma(s,t) = \begin{bmatrix} x(s,t) \\ v(s,t) \end{bmatrix} = \int_0^s \begin{bmatrix} \cos\theta(s',t) \\ \sin\theta(s',t) \end{bmatrix} ds' + \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}, \tag{2.11}$$

and change the independent variables (s,t) to

$$(x,t') = \left(\int_0^s \cos\theta(s',t)\, ds' + x_0,\ t\right). \tag{2.12}$$

For simplicity we write t' as t without causing confusion. Let us write down the equation for v in terms of x and t. It can be easily shown that

$$s(x,t) = \int \sqrt{1+v_x^2}\, dx, \quad \kappa(x,t) = \frac{v_{xx}}{(1+v_x^2)^{\frac{3}{2}}}, \tag{2.13}$$

$$\boldsymbol{N} = \frac{1}{\sqrt{1+v_x^2}} \begin{bmatrix} -v_x \\ 1 \end{bmatrix}, \quad \boldsymbol{T} = \frac{1}{\sqrt{1+v_x^2}} \begin{bmatrix} 1 \\ v_x \end{bmatrix}. \tag{2.14}$$

Noticing

$$\frac{\partial}{\partial t}\gamma = -\kappa_s \mathbf{N} - \frac{1}{2}\kappa^2 \mathbf{T}, \tag{2.15}$$

it follows that

$$-\kappa_s = \gamma_t \cdot \mathbf{N} = \frac{v_t}{\sqrt{1+v_x^2}}, \tag{2.16}$$

by taking the inner product with $\boldsymbol{N}$ on both sides of (2.15). By using $\frac{ds}{dx} = \sqrt{1+v_x^2}$, we see that

$$v_t = -\kappa_s \sqrt{1+v_x^2} = -\kappa_x. \tag{2.17}$$

Thus we derive

$$v_t = -\left(\frac{v_{xx}}{(1+v_x^2)^{\frac{3}{2}}}\right)_x. \tag{2.18}$$

Introducing $u = v_x$, we obtain the WKI (Wadati-Konno-Ichikawa) elastic beam equation [19–22]

$$u_t = -\left(\frac{u_x}{(1+u^2)^{\frac{3}{2}}}\right)_{xx}. \tag{2.19}$$

Therefore, (2.18) or (2.19) can be viewed as the governing equation of the Eulerian description for the curve motions given by (2.5) and (2.6).

We note that (2.11) is the hodograph transformation between the potential mKdV equation (2.10) and the WKI elastic beam equation (2.18) found by Ishimori [27]. The above discussion shows that the hodograph transformation arises naturally as the transformation between the Lagrangian and Eulerian descriptions from the point of view of geometry of plane curves.

There is another equation related to the plane curve motions which is known as the complex Dym equation (this is often called the complex Harry Dym equation) [19,43,50–55]. Introducing the complex variables r and z, we consider the transformation

$$r(s,t) = e^{\sqrt{-1}\ \theta(s,t)}, \tag{2.20}$$

$$z(s,t) = \int_0^s e^{\sqrt{-1}\ \theta(s',t)}\, ds' + z_0, \quad t' = t. \tag{2.21}$$

Then the potential mKdV equation (2.10) is transformed to the complex Dym equation [52,53]

$$r_t = r^3 r_{zzz}. \tag{2.22}$$

Here we set $t' = t$ without causing confusion. The geometric meaning of the complex Dym equation may be described as follows. The variables r and z are expressed as

$$r = \cos\theta(s,t) + \sqrt{-1}\sin\theta(s,t) = x_s + \sqrt{-1}v_s, \quad z = x + \sqrt{-1}v + z_0. \tag{2.23}$$

Identifying the Euclidean plane $\mathbb{R}^2$ as $\mathbb{C}$, we see that z is the position vector of the curve γ, and r corresponds to the tangent vector $\frac{\partial\gamma}{\partial s}$. By treating the Eulerian coordinates of γ as a complex variable z, the complex Dym equation is nothing but the governing equation to describe the motion of the tangent vector $\frac{\partial\gamma}{\partial s}$.

It is well known that the sine-Gordon equation

$$\theta_{ys} = 4\sin\theta, \tag{2.24}$$

belongs to the the same hierarchy as the mKdV equation [56,57] and that it describes a certain motion of plane curves [58]. It is possible to derive the governing equation of curve motion in the Eulerian description in a similar manner to the case of the mKdV equation. In fact, applying the transformations

$$(x, y') = \left(\int_0^s \cos\theta(s',y)\, ds' + x_0,\ y\right), \tag{2.25}$$

$$v = \int_0^s \sin\theta(s',y)\, ds' + v_0, \tag{2.26}$$

we obtain the short pulse equation [25,59–62]

$$v_{xy} = 4v + \frac{2}{3}\left(v^3\right)_{xx}, \tag{2.27}$$

where we set $y' = y$ for simplicity. Again, we note that the short pulse equation (2.27) describes the same curve motions as the sine-Gordon equation by using the Eulerian description. The transformation (2.25) gives the hodograph transformation between them [33, 63–65].

2.2. Continuous motion of discrete curves

In this subsection we discuss the semi-discrete equations arising from the continuous motion of discrete plane curves and the hodograph transformations among them. A map $\gamma : \mathbb{Z} \to \mathbb{R}^2;\ l \mapsto \gamma_l$ is said to be a discrete curve of segment length a_l if

$$\left|\frac{\gamma_{l+1} - \gamma_l}{a_l}\right| = 1. \tag{2.28}$$

We introduce the angle function ψ_l of a discrete curve γ by

$$\frac{\gamma_{l+1} - \gamma_l}{a_l} = \begin{bmatrix} \cos\psi_l \\ \sin\psi_l \end{bmatrix}. \tag{2.29}$$

A discrete curve γ satisfies

$$\frac{\gamma_{l+1} - \gamma_l}{a_l} = R(\kappa_l)\ \frac{\gamma_l - \gamma_{l-1}}{a_{l-1}}, \tag{2.30}$$

for $\kappa_l = \psi_l - \psi_{l-1}$, where $R(\kappa_l)$ denotes the rotation matrix given by

$$R(\kappa_l) = \begin{pmatrix} \cos\kappa_l & -\sin\kappa_l \\ \sin\kappa_l & \cos\kappa_l \end{pmatrix}. \tag{2.31}$$

We set $a_l = \epsilon\,(> 0)$, and consider the following motion of discrete curves:

$$\frac{d\gamma_l}{d\zeta} = \frac{1}{\cos\frac{\kappa_l}{2}}\ R\left(-\frac{\kappa_l}{2}\right)\ \frac{\gamma_{l+1} - \gamma_l}{\epsilon}. \tag{2.32}$$

Then from the isoperimetric condition (2.28) and the compatibility condition of (2.30) and (2.32) , it follows that there exists a potential function θ_l characterized by

$$\psi_l = \frac{\theta_{l+1} + \theta_l}{2}, \quad \kappa_l = \frac{\theta_{l+1} - \theta_{l-1}}{2}, \tag{2.33}$$

and that θ_l satisfies the semi-discrete potential mKdV equation [2, 44–47, 66, 67]

$$\frac{d\theta_l}{d\zeta} = \frac{2}{\epsilon}\tan\left(\frac{\theta_{l+1} - \theta_{l-1}}{4}\right). \tag{2.34}$$

We note that $K_l = \frac{2}{\epsilon}\tan\frac{\kappa_l}{2}$ satisfies the semi-discrete mKdV equation

$$\frac{dK_l}{d\zeta} = \frac{2}{\epsilon}\left(1 + \frac{\epsilon^2}{4}K_l^2\right)(K_{l+1} - K_{l-1}). \tag{2.35}$$

It is possible to consider the Eulerian description of the curve motion defined by (2.30) and (2.32). Noticing (2.29) and (2.33), we introduce the Eulerian coordinates

$$\gamma_l(\zeta) = \begin{bmatrix} X_l(\zeta) \\ v_l(\zeta) \end{bmatrix} = \sum_{j=0}^{l-1} \begin{bmatrix} \epsilon\cos\left(\frac{\theta_{j+1}+\theta_j}{2}\right) \\ \epsilon\sin\left(\frac{\theta_{j+1}+\theta_j}{2}\right) \end{bmatrix} + \begin{bmatrix} X_0 \\ v_0 \end{bmatrix}. \tag{2.36}$$

Then from (2.29), (2.34) and (2.36), one can derive

$$\frac{d}{d\zeta}\delta_l = -\frac{v_{l+1} - v_l}{\epsilon}\left(\frac{\Delta_{l+1} - \Delta_l}{1 + \Delta_{l+1}\Delta_l} + \frac{\Delta_l - \Delta_{l-1}}{1 + \Delta_l\Delta_{l-1}}\right), \tag{2.37}$$

$$\frac{d}{d\zeta}\left(v_{l+1} - v_l\right) = \frac{\delta_l}{\epsilon}\left(\frac{\Delta_{l+1} - \Delta_l}{1 + \Delta_{l+1}\Delta_l} + \frac{\Delta_l - \Delta_{l-1}}{1 + \Delta_l\Delta_{l-1}}\right), \tag{2.38}$$

where

$$\delta_l = X_{l+1} - X_l, \quad \Delta_l = \mathrm{sgn}\,(v_{l+1} - v_l)\left|\frac{v_{l+1} - v_l}{\epsilon + \delta_l}\right| = \frac{v_{l+1} - v_l}{\epsilon + \delta_l}, \tag{2.39}$$

since $\epsilon + \delta_l = \epsilon(1 + \cos\psi_l) \geq 0$. Note that v_l and δ_l satisfy

$$\left(\frac{v_{l+1} - v_l}{\epsilon}\right)^2 + \left(\frac{\delta_l}{\epsilon}\right)^2 = 1. \tag{2.40}$$

From (2.37) and (2.38), we obtain

$$\frac{d}{d\zeta}\left(\frac{v_{l+1} - v_l}{X_{l+1} - X_l}\right) = \frac{1}{\epsilon}\left(1 + \left(\frac{v_{l+1} - v_l}{X_{l+1} - X_l}\right)^2\right)\left(\frac{\Delta_{l+1} - \Delta_l}{1 + \Delta_{l+1}\Delta_l} + \frac{\Delta_l - \Delta_{l-1}}{1 + \Delta_l\Delta_{l-1}}\right). \tag{2.41}$$

The system of (2.37), (2.38) and (2.39) is nothing but the semi-discrete WKI elastic beam equation. We remark that (2.36) can be regarded as the hodograph transformation between (2.34) and the semi-discrete WKI elastic beam equation. Note that the angle function $\psi_l = \frac{\theta_{l+1}+\theta_l}{2}$ satisfies

$$\cos\psi_l = \frac{X_{l+1} - X_l}{\epsilon}, \qquad \sin\psi_l = \frac{v_{l+1} - v_l}{\epsilon}, \qquad \tan\psi_l = \frac{v_{l+1} - v_l}{X_{l+1} - X_l}. \tag{2.42}$$

Thus (2.41) can be rewritten as

$$\frac{d}{d\zeta}\psi_l = \frac{1}{\epsilon}\left(\tan\frac{\psi_{l+1} - \psi_l}{2} + \tan\frac{\psi_l - \psi_{l-1}}{2}\right). \tag{2.43}$$

Equation (2.43) with the discrete hodograph transformation

$$X_l(\zeta) = \sum_{j=0}^{l-1} \epsilon\cos\psi_j(\zeta) + X_0, \quad v_l(\zeta) = \sum_{j=0}^{l-1} \epsilon\sin\psi_l(\zeta) + v_0, \tag{2.44}$$

can be also regarded as the semi-discrete WKI elastic beam equation. In the continuous limit $\epsilon \to 0$ with $s = \epsilon l + \zeta$ and $t = -\frac{\epsilon^2}{6}\zeta$, (2.43) and (2.44) converge to

$$\theta_t + \frac{1}{2}(\theta_s)^3 + \theta_{sss} = 0, \tag{2.45}$$

and

$$x(s,t) = \int_0^s \cos\theta(s',t)ds' + x_0, \quad v(s,t) = \int_0^s \sin\theta(s',t)ds' + v_0, \tag{2.46}$$

which give the (potential) WKI elastic beam equation (2.18) (see Appendix).

One can construct a semi-discrete version of the complex Dym equation as follows. In view of (2.23), (2.33) and (2.36), it is natural to introduce the complex variables $r_l(\zeta)$ and $Z_l(\zeta)$ by

$$r_l(\zeta) = e^{\sqrt{-1}\psi_l} = e^{\sqrt{-1}\frac{\theta_{l+1}+\theta_l}{2}}, \quad Z_l(\zeta) = X_l(\zeta) + \sqrt{-1}v_l(\zeta) = \sum_{j=0}^{l-1} \epsilon e^{\sqrt{-1}\,\frac{\theta_{j+1}+\theta_j}{2}} + Z_0. \tag{2.47}$$

Then we have from (2.34) and (2.47)

$$\frac{dr_l}{d\zeta} = \frac{r_l}{\epsilon}\left(\frac{r_{l+1} - r_l}{r_{l+1} + r_l} + \frac{r_l - r_{l-1}}{r_l + r_{l-1}}\right), \quad \frac{Z_{l+1} - Z_l}{\epsilon} = r_l, \tag{2.48}$$

which is the semi-discrete complex Dym equation. The geometric meaning of (2.48) can be described as follows: under the identification of $\mathbb{R}^2$ as $\mathbb{C}$, Z_l is the position vector of the curve

γ_l, and r_l corresponds to the segment vector $\frac{\gamma_{l+1}-\gamma_l}{\epsilon}$. Then (2.48) is the governing equation describing the motion of the segment vector in the Eulerian coordinates of γ_l. To take the continuous limit, we use the angle function ψ_l. Then the semi-discrete complex Dym equation (2.48) is rewritten as

$$\frac{d}{d\zeta}\psi_l = \frac{1}{\epsilon}\left(\tan\frac{\psi_{l+1}-\psi_l}{2} + \tan\frac{\psi_l-\psi_{l-1}}{2}\right). \tag{2.49}$$

and

$$Z_l(\zeta) = \sum_{j=0}^{l-1} \epsilon e^{\sqrt{-1}\psi_j(\zeta)} + Z_0\,, \quad r_l(\zeta) = e^{\sqrt{-1}\psi_l(\zeta)}\,. \tag{2.50}$$

In the continuous limit $\epsilon \to 0$ with $s = \epsilon l + \zeta$ and $t = -\frac{\epsilon^2}{6}\zeta$, (2.49) and (2.50) converge to

$$\theta_t + \frac{1}{2}(\theta_s)^3 + \theta_{sss} = 0\,, \tag{2.51}$$

and

$$z(s,t) = \int_0^s e^{\sqrt{-1}\theta(s',t)}ds' + z_0\,, \quad r(s,t) = e^{\sqrt{-1}\theta(s,t)}\,, \tag{2.52}$$

which give the complex Dym equation (2.22) (see Appendix).

Now we construct the semi-discrete short pulse equation. To this end, we consider the semi-discrete sine-Gordon equation

$$\frac{d}{dy}(\theta_{l+1}-\theta_l) = 4\epsilon\sin\left(\frac{\theta_{l+1}+\theta_{l-1}}{2}\right). \tag{2.53}$$

Similar to the continuous case, the semi-discrete sine-Gordon equation (2.53) can be regarded as describing a certain motion of discrete plane curves. Therefore, we may expect that the application of the same transformation as the case of the semi-discrete WKI equation to the semi-discrete sine-Gordon equation (2.53) yields the semi-discrete analogue of the short pulse equation. In fact, by using the transformation

$$\gamma_l(y) = \begin{bmatrix} X_l(y) \\ v_l(y) \end{bmatrix} = \sum_{j=0}^{l-1}\begin{bmatrix} \epsilon\cos\left(\frac{\theta_{j+1}+\theta_j}{2}\right) \\ \epsilon\sin\left(\frac{\theta_{j+1}+\theta_j}{2}\right) \end{bmatrix} + \begin{bmatrix} X_0 \\ v_0 \end{bmatrix}, \tag{2.54}$$

we obtain the semi-discrete short pulse equation

$$\frac{d}{dy}(X_{l+1}-X_l) = -2(v_{l+1}^2 - v_l^2), \tag{2.55}$$

$$\frac{d}{dy}(v_{l+1}-v_l) = 2(X_{l+1}-X_l)(v_{l+1}+v_l). \tag{2.56}$$

We note that the following relation also holds from (2.54)

$$\left(\frac{v_{l+1}-v_l}{\epsilon}\right)^2 + \left(\frac{X_{l+1}-X_l}{\epsilon}\right)^2 = 1. \tag{2.57}$$

From (2.55) and (2.56), we obtain

$$\frac{d}{dy}\left(\frac{v_{l+1}-v_l}{X_{l+1}-X_l}\right) = 2(v_{l+1}+v_l) + 2\left(\frac{v_{l+1}-v_l}{X_{l+1}-X_l}\right)^2(v_{l+1}+v_l)\,. \tag{2.58}$$

In order to take the continuous limit, we assume the boundary condition $X_l = v_l = 0$ for $l < 0$, which is consistent with (2.54). Then the continuous limit $\epsilon \to 0$ (i.e., $X_{l+1} - X_l \to 0$) gives

$$\frac{v_{l+1} - v_l}{X_{l+1} - X_l} \to \frac{\partial v}{\partial x}, \qquad \frac{v_{l+1} + v_l}{2} \to v,$$

$$\frac{\partial X_l}{\partial y} = \frac{\partial X_0}{\partial y} + \sum_{j=0}^{l-1} \frac{\partial (X_{j+1} - X_j)}{\partial y} = \frac{\partial X_0}{\partial y} - 2\sum_{j=0}^{l-1}(v_{j+1}^2 - v_j^2) = -2v_l^2 \to \frac{\partial x}{\partial y} = -2v^2,$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial y'} + \frac{\partial x}{\partial y}\frac{\partial}{\partial x} = \frac{\partial}{\partial y'} - 2v_l^2\frac{\partial}{\partial x} \to \frac{\partial}{\partial y'} - 2v^2\frac{\partial}{\partial x}.$$

Consequently, (2.58) converges to

$$(\partial_{y'} - 2v^2\partial_x)v_x = 4v + 4vv_x^2, \tag{2.59}$$

which is nothing but the short pulse equation (2.27).

2.3. Discrete motion of discrete curves

Now let us recall the following discrete motion of discrete plane curve γ_n^m introduced by Matsuura [49]:

$$\left|\frac{\gamma_{n+1}^m - \gamma_n^m}{a_n}\right| = 1, \tag{2.60}$$

$$\frac{\gamma_{n+1}^m - \gamma_n^m}{a_n} = R(\kappa_n^m)\,\frac{\gamma_n^m - \gamma_{n-1}^m}{a_{n-1}}, \tag{2.61}$$

$$\frac{\gamma_n^{m+1} - \gamma_n^m}{b_m} = R(\omega_n^m)\,\frac{\gamma_{n+1}^m - \gamma_n^m}{a_n}, \tag{2.62}$$

where a_n and b_m are arbitrary functions in n and m, respectively. Compatibility of the system (2.60)–(2.62) implies the existence of the potential function θ_n^m defined by

$$\omega_m^n = \frac{\theta_n^{m+1} - \theta_{n+1}^m}{2}, \quad \kappa_n^m = \frac{\theta_{n+1}^m - \theta_{n-1}^m}{2}, \tag{2.63}$$

and it follows that θ_n^m satisfies the discrete potential mKdV equation [68]:

$$\tan\left(\frac{\theta_{n+1}^{m+1} - \theta_n^m}{4}\right) = \frac{b_m + a_n}{b_m - a_n}\,\tan\left(\frac{\theta_n^{m+1} - \theta_{n+1}^m}{4}\right). \tag{2.64}$$

Note that the functions ψ_n^m and ϕ_n^m can be expressed as

$$\psi_n^m = \frac{\theta_{n+1}^m + \theta_n^m}{2}, \quad \phi_n^m = \frac{\theta_n^{m+1} + \theta_n^m}{2}. \tag{2.65}$$

Note also

$$\frac{\gamma_{n+1}^m - \gamma_n^m}{a_n} = \begin{bmatrix} \cos\psi_n^m \\ \sin\psi_n^m \end{bmatrix}, \quad \frac{\gamma_n^{m+1} - \gamma_n^m}{b_m} = \begin{bmatrix} \cos\phi_n^m \\ \sin\phi_n^m \end{bmatrix}, \tag{2.66}$$

and

$$\gamma_n^m = \begin{bmatrix} X_n^m \\ v_n^m \end{bmatrix} = \sum_{j=0}^{n-1} \begin{bmatrix} X_{j+1}^m - X_j^m \\ v_{j+1}^m - v_j^m \end{bmatrix} + \begin{bmatrix} X_0^m \\ v_0^m \end{bmatrix} = \sum_{j=0}^{n-1} a_j \begin{bmatrix} \cos\psi_j^m \\ \sin\psi_j^m \end{bmatrix} + \begin{bmatrix} X_0^m \\ v_0^m \end{bmatrix}. \tag{2.67}$$

From the discrete potential mKdV equation (2.64) and the hodograph transformation (2.67), we obtain

$$\frac{\Delta_n^{m+1}-\Gamma_n^m}{1+\Delta_n^{m+1}\Gamma_n^m}=\frac{b_m+a_n}{b_m-a_n}\frac{\Gamma_n^m-\Delta_n^m}{1+\Gamma_n^m\Delta_n^m}\,, \tag{2.68}$$

where

$$\Delta_n^m=\mathrm{sgn}(v_{n+1}^m-v_n^m)\left|\frac{v_{n+1}^m-v_n^m}{a_n+(X_{n+1}^m-X_n^m)}\right|=\frac{v_{n+1}^m-v_n^m}{a_n+(X_{n+1}^m-X_n^m)}\,, \tag{2.69}$$

$$\Gamma_n^m=\mathrm{sgn}(v_n^{m+1}-v_n^m)\left|\frac{v_n^{m+1}-v_n^m}{b_m+(X_n^{m+1}-X_n^m)}\right|=\frac{v_n^{m+1}-v_n^m}{b_m+(X_n^{m+1}-X_n^m)}\,, \tag{2.70}$$

since $a_n+(X_{n+1}^m-X_n^m)=a_n(1+\cos\psi_n^m)\geq 0$ and $b_m+(X_n^{m+1}-X_n^m)=b_m(1+\cos\phi_n^m)\geq 0$. We note that v_n^m and X_n^m satisfy the following relations

$$\left(\frac{X_{n+1}^m-X_n^m}{a_n}\right)^2+\left(\frac{v_{n+1}^m-v_n^m}{a_n}\right)^2=1\,, \tag{2.71}$$

$$\left(\frac{X_n^{m+1}-X_n^m}{b_m}\right)^2+\left(\frac{v_n^{m+1}-v_n^m}{b_m}\right)^2=1\,. \tag{2.72}$$

To construct an explicit form of the discrete WKI elastic beam equation, we consider an identity

$$e^{\sqrt{-1}\psi_n^{m+1}}e^{\sqrt{-1}\psi_n^m}=e^{\sqrt{-1}\phi_{n+1}^m}e^{\sqrt{-1}\phi_n^m}\,. \tag{2.73}$$

Substituting

$$e^{\sqrt{-1}\psi_n^m}=\cos\psi_n^m+\sqrt{-1}\sin\psi_n^m=\frac{X_{n+1}^m-X_n^m}{a_n}+\sqrt{-1}\,\frac{v_{n+1}^m-v_n^m}{a_n}\,,$$

$$e^{\sqrt{-1}\phi_n^m}=\cos\phi_n^m+\sqrt{-1}\sin\phi_n^m=\frac{X_n^{m+1}-X_n^m}{b_m}+\sqrt{-1}\,\frac{v_n^{m+1}-v_n^m}{b_m}\,,$$

into (2.73), we obtain the system of two discrete equations from the real and imaginary parts, respectively, which should be considered together with the constraints (2.71) and (2.72). To incorporate (2.71) and (2.72), an easy way is to consider the identity

$$e^{\sqrt{-1}\psi_n^{m+1}}e^{-\sqrt{-1}\phi_n^m}=e^{\sqrt{-1}\phi_{n+1}^m}e^{-\sqrt{-1}\psi_n^m}\,, \tag{2.74}$$

instead of (2.73). We then obtain the following system of two discrete equations

$$\begin{aligned}&(X_{n+1}^{m+1}-X_n^{m+1})(X_n^{m+1}-X_n^m)+(v_{n+1}^{m+1}-v_n^{m+1})(v_n^{m+1}-v_n^m)\\&\qquad=(X_{n+1}^{m+1}-X_{n+1}^m)(X_{n+1}^m-X_n^m)+(v_{n+1}^{m+1}-v_{n+1}^m)(v_{n+1}^m-v_n^m)\,,\end{aligned} \tag{2.75}$$

$$\begin{aligned}&(X_{n+1}^{m+1}-X_n^{m+1})(v_n^{m+1}-v_n^m)-(X_n^{m+1}-X_n^m)(v_{n+1}^{m+1}-v_n^{m+1})\\&\qquad=(X_{n+1}^{m+1}-X_{n+1}^m)(v_{n+1}^m-v_n^m)-(X_{n+1}^m-X_n^m)(v_{n+1}^{m+1}-v_{n+1}^m)\,,\end{aligned} \tag{2.76}$$

which is simplified to

$$\begin{aligned}&(X_{n+1}^{m+1}-X_{n+1}^m-X_n^{m+1}+X_n^m)(X_{n+1}^m-X_n^{m+1})\\&\qquad+(v_{n+1}^{m+1}-v_{n+1}^m-v_n^{m+1}+v_n^m)(v_{n+1}^m-v_n^{m+1})=0\,,\end{aligned} \tag{2.77}$$

$$\begin{aligned}&(X_{n+1}^{m+1}+X_{n+1}^m-X_n^{m+1}-X_n^m)(v_{n+1}^{m+1}-v_{n+1}^m+v_n^{m+1}-v_n^m)\\&\qquad-(X_{n+1}^{m+1}-X_{n+1}^m+X_n^{m+1}-X_n^m)(v_{n+1}^{m+1}+v_{n+1}^m-v_n^{m+1}-v_n^m)=0\,.\end{aligned} \tag{2.78}$$

Note that the second equation (2.78) is further simplified to

$$(X_{n+1}^m - X_n^{m+1})(v_{n+1}^{m+1} - v_n^m) - (X_{n+1}^{m+1} - X_n^m)(v_{n+1}^m - v_n^{m+1}) = 0\,. \tag{2.79}$$

We remark that the system (2.77) and (2.78) can be solved explicitly in terms of v_{n+1}^{m+1} and X_{n+1}^{m+1} in the form of rational functions of v_n^m, v_{n+1}^m, v_{n+1}^{m+1}, X_n^m, X_{n+1}^m and X_n^{m+1}. Therefore the system (2.77) and (2.78) (or (2.79)) can be regarded as the explicit form of the discrete WKI elastic beam equation. Note that we can obtain (2.68), (2.69) and (2.70) by replacing X_n^m by $X_n^m = X_n^m + \sum_{j=0}^{n-1} a_j + \sum_{j=0}^{m-1} b_j$ in (2.75) and (2.76) and then dividing (2.76) by (2.75). By using the potential function θ_n^m, the discrete WKI elastic beam equation can be written as

$$\tan\left(\frac{\theta_{n+1}^{m+1} - \theta_n^m}{4}\right) = \frac{b_m + a_n}{b_m - a_n}\ \tan\left(\frac{\theta_n^{m+1} - \theta_{n+1}^m}{4}\right), \tag{2.80}$$

$$X_n^m = \sum_{j=0}^{n-1} a_j \cos\left(\frac{\theta_{j+1}^m + \theta_j^m}{2}\right) + X_0^m\,. \tag{2.81}$$

Setting

$$\zeta = (n+m)\delta\,, \quad l = n - m\,, \quad a_n = a\,, \quad b_m = b\,, \quad \delta = \frac{a+b}{2}\,, \quad \epsilon = \frac{a-b}{2}\,, \tag{2.82}$$

and taking the continuous limit $\delta \to 0$, (2.80) and (2.81) are reduced to

$$\frac{d\theta_l}{d\zeta} = \frac{2}{\epsilon} \tan\left(\frac{\theta_{l+1} - \theta_{l-1}}{4}\right), \tag{2.83}$$

$$X_l(\zeta) = \sum_{j=0}^{l-1} a_j \cos\left(\frac{\theta_{j+1}(\zeta) + \theta_j(\zeta)}{2}\right) + X_0(\zeta)\,, \tag{2.84}$$

which are transformed to the semi-discrete WKI elastic beam equation (2.37), (2.38) and (2.39).

Let us consider a discrete analogue of the complex Dym equation. Introducing

$$r_n^m = e^{\sqrt{-1}\frac{\theta_{n+1}^m + \theta_n^m}{2}}\,, \tag{2.85}$$

$$\begin{aligned} Z_n^m = X_n^m + \sqrt{-1} v_n^m &= \sum_{j=0}^{n-1} a_j \cos\psi_j^m + \sqrt{-1}\sum_{j=0}^{n-1} a_j \sin\psi_j^m + Z_0^m \\ &= \sum_{j=0}^{n-1} a_j r_j^m + Z_0^m\,, \end{aligned} \tag{2.86}$$

and using the discrete potential mKdV equation (2.64), we derive the discrete analogue of the complex Dym equation

$$\frac{r_n^{m+1} - \rho_n^m}{r_n^{m+1} + \rho_n^m} = \frac{b_m + a_n}{b_m - a_n}\,\frac{\rho_n^m - r_n^m}{\rho_n^m + r_n^m}, \qquad \frac{r_n^{m+1}}{\rho_n^m} = \frac{\rho_{n+1}^m}{r_n^m}, \tag{2.87}$$

$$Z_{n+1}^m - Z_n^m = a_n r_n^m\,, \tag{2.88}$$

where ρ_n^m is an auxiliary variable defined by

$$\rho_n^m = e^{\sqrt{-1}\frac{\theta_n^{m+1} + \theta_n^m}{2}}\,. \tag{2.89}$$

Similar to the semi-discrete case, the system of (2.87) and (2.88) describes the motion of segment vector of the curve γ_n^m in the Eulerian coordinates of γ_n^m. It should be noted that by introducing Q_n^m by

$$Q_n^m = e^{\sqrt{-1}\theta_n^m}, \tag{2.90}$$

we have an alternate form of the discrete complex Dym equation

$$r_n^m = \sqrt{Q_{n+1}^m Q_n^m}, \tag{2.91}$$

$$\frac{\sqrt{Q_{n+1}^{m+1}} - \sqrt{Q_n^m}}{\sqrt{Q_{n+1}^{m+1}} + \sqrt{Q_n^m}} = \frac{b_m + a_n}{b_m - a_n} \frac{\sqrt{Q_n^{m+1}} - \sqrt{Q_{n+1}^m}}{\sqrt{Q_n^{m+1}} + \sqrt{Q_{n+1}^m}}, \tag{2.92}$$

$$Z_{n+1}^m - Z_n^m = a_n r_n^m. \tag{2.93}$$

Using θ_n^m, the discrete complex Dym equation can be written as

$$\tan\left(\frac{\theta_{n+1}^{m+1} - \theta_n^m}{4}\right) = \frac{b_m + a_n}{b_m - a_n} \tan\left(\frac{\theta_n^{m+1} - \theta_{n+1}^m}{4}\right), \tag{2.94}$$

$$Z_n^m = \sum_{j=0}^{n-1} a_j e^{\sqrt{-1}\frac{\theta_{j+1}^m + \theta_j^m}{2}} + Z_0^m. \tag{2.95}$$

Setting

$$\zeta = (n+m)\delta, \quad l = n - m, \quad a_n = a, \quad b_m = b, \quad \delta = \frac{a+b}{2}, \quad \epsilon = \frac{a-b}{2}, \tag{2.96}$$

and taking the continuous limit $\delta \to 0$, (2.94) and (2.95) become

$$\frac{d\theta_l}{d\zeta} = \frac{2}{\epsilon} \tan\left(\frac{\theta_{l+1} - \theta_{l-1}}{4}\right), \tag{2.97}$$

$$Z_l(\zeta) = \sum_{j=0}^{l-1} a_j e^{\sqrt{-1}\frac{\theta_{j+1}(\zeta) + \theta_j(\zeta)}{2}} + Z_0(\zeta), \tag{2.98}$$

which are transformed to the semi-discrete complex Dym equation (2.48).

We next construct the discrete short pulse equation. Consider the following discrete motion of plane discrete curve γ_n^k [48]

$$\left|\frac{\gamma_{n+1}^k - \gamma_n^k}{a_n}\right| = 1, \tag{2.99}$$

$$\frac{\gamma_{n+1}^k - \gamma_n^k}{a_n} = R(\kappa_n^k) \frac{\gamma_n^k - \gamma_{n-1}^k}{a_{n-1}}, \tag{2.100}$$

$$\gamma_n^{k+1} - S\gamma_n^k = \frac{1}{c_k} S R(-\sigma_n^k) \frac{\gamma_{n+1}^k - \gamma_n^k}{a_n}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.101}$$

where a_n and c_k are arbitrary functions in n and k, respectively. Compatibility of this system implies the existence of the potential function θ_n^k defined by

$$\kappa_n^k = \frac{\theta_{n+1}^k - \theta_{n-1}^k}{2}, \quad \sigma_n^k = \frac{\theta_n^{k+1} + \theta_{n+1}^k}{2}, \tag{2.102}$$

and it follows that θ_n^k satisfies the discrete sine-Gordon equation

$$\sin\left(\frac{\theta_{n+1}^{k+1}-\theta_{n+1}^{k}-\theta_{n}^{k+1}+\theta_n^k}{4}\right)=a_nc_k\ \sin\left(\frac{\theta_{n+1}^{k+1}+\theta_{n+1}^{k}+\theta_{n}^{k+1}+\theta_n^k}{4}\right). \tag{2.103}$$

Note that the functions ψ_n^k and φ_n^k can be expressed as

$$\psi_n^k=\frac{\theta_{n+1}^k+\theta_n^k}{2},\quad \varphi_n^k=\frac{\theta_n^{k+1}-\theta_n^k}{2}. \tag{2.104}$$

Note also

$$\frac{\gamma_{n+1}^k-\gamma_n^k}{a_n}=\begin{bmatrix}\cos\psi_n^k\\ \sin\psi_n^k\end{bmatrix},\quad \frac{\gamma_n^{k+1}-S\,\gamma_n^k}{\frac{1}{c_k}}=\begin{bmatrix}\cos\varphi_n^k\\ \sin\varphi_n^k\end{bmatrix}, \tag{2.105}$$

and

$$\gamma_n^k=\begin{bmatrix}X_n^k\\ v_n^k\end{bmatrix}=\sum_{j=0}^{n-1}\begin{bmatrix}X_{j+1}^k-X_j^k\\ v_{j+1}^k-v_j^k\end{bmatrix}+\begin{bmatrix}X_0^k\\ v_0^k\end{bmatrix}=\sum_{j=0}^{n-1}a_j\begin{bmatrix}\cos\psi_j^k\\ \sin\psi_j^k\end{bmatrix}+\begin{bmatrix}X_0^k\\ v_0^k\end{bmatrix}. \tag{2.106}$$

From the discrete sine-Gordon equation (2.103) and the hodograph transformation (2.106), we obtain

$$\Delta_n^{k+1}-\Delta_n^k=a_nc_k(\Gamma_n^{k+1}+\Gamma_n^k), \tag{2.107}$$

where

$$\Delta_n^k=\mathrm{sgn}(v_{n+1}^k-v_n^k)\left|\frac{v_{n+1}^k-v_n^k}{a_n+(X_{n+1}^k-X_n^k)}\right|=\frac{v_{n+1}^k-v_n^k}{a_n+(X_{n+1}^k-X_n^k)}, \tag{2.108}$$

$$\Gamma_n^k=\mathrm{sgn}(v_n^{k+1}+v_n^k)\left|\frac{v_n^{k+1}+v_n^k}{\frac{1}{c_k}+(X_n^{k+1}-X_n^k)}\right|=\frac{v_n^{k+1}+v_n^k}{\frac{1}{c_k}+(X_n^{k+1}-X_n^k)}, \tag{2.109}$$

since $a_n+(X_{n+1}^k-X_n^k)=a_n(1+\cos\psi_n^k)\geq 0$ and $\frac{1}{c_k}+(X_n^{k+1}-X_n^k)=\frac{1}{c_k}(1+\cos\varphi_n^k)\geq 0$. We note that v_n^k and X_n^k satisfy the following relations

$$\left(\frac{v_{n+1}^k-v_n^k}{a_n}\right)^2+\left(\frac{X_{n+1}^k-X_n^k}{a_n}\right)^2=1, \tag{2.110}$$

$$\left(\frac{v_n^{k+1}+v_n^k}{\frac{1}{c_k}}\right)^2+\left(\frac{X_n^{k+1}-X_n^k}{\frac{1}{c_k}}\right)^2=1. \tag{2.111}$$

We now construct an explicit form of the discrete short pulse equation. Similar to the case of discrete WKI elastic beam equation, we consider the identity

$$e^{\sqrt{-1}\psi_n^{k+1}}e^{-\sqrt{-1}\varphi_n^k}=e^{\sqrt{-1}\psi_n^k}e^{\sqrt{-1}\varphi_{n+1}^k}, \tag{2.112}$$

with

$$e^{\sqrt{-1}\psi_n^k}=\cos\psi_n^k+\sqrt{-1}\sin\psi_n^k=\frac{X_{n+1}^k-X_n^k}{a_n}+\sqrt{-1}\,\frac{v_{n+1}^k-v_n^k}{a_n},$$

$$e^{\sqrt{-1}\varphi_n^k}=\cos\varphi_n^k+\sqrt{-1}\sin\varphi_n^k=\frac{X_n^{k+1}-X_n^k}{\frac{1}{c_k}}+\sqrt{-1}\,\frac{v_n^{k+1}+v_n^k}{\frac{1}{c_k}}.$$

We then obtain the following system of two discrete equations

$$(X^{k+1}_{n+1} - X^{k}_{n+1} - X^{k+1}_{n} + X^{k}_{n})(X^{k}_{n+1} - X^{k+1}_{n}) \\ - (v^{k+1}_{n+1} + v^{k}_{n+1} - v^{k+1}_{n} - v^{k}_{n})(v^{k}_{n+1} + v^{k+1}_{n}) = 0\,, \tag{2.113}$$

$$(X^{k+1}_{n+1} - X^{k}_{n+1} + X^{k+1}_{n} - X^{k}_{n})(v^{k+1}_{n+1} - v^{k}_{n+1} - v^{k+1}_{n} + v^{k}_{n}) \\ - (X^{k+1}_{n+1} + X^{k}_{n+1} - X^{k+1}_{n} - X^{k}_{n})(v^{k+1}_{n+1} + v^{k}_{n+1} + v^{k+1}_{n} + v^{k}_{n}) = 0\,. \tag{2.114}$$

Note that the second equation (2.114) is further simplified to

$$(X^{k}_{n+1} - X^{k+1}_{n})(v^{k+1}_{n+1} + v^{k}_{n}) + (X^{k+1}_{n+1} - X^{k}_{n})(v^{k}_{n+1} + v^{k+1}_{n}) = 0\,. \tag{2.115}$$

Replacing X^k_n by $X^k_n + \sum_{j=0}^{k-1} \frac{1}{c_j}$, (2.113) and (2.114) become

$$(X^{k+1}_{n+1} - X^{k}_{n+1} - X^{k+1}_{n} + X^{k}_{n})\left(X^{k}_{n+1} - X^{k+1}_{n} - \frac{1}{c_k}\right) \\ - (v^{k+1}_{n+1} + v^{k}_{n+1} - v^{k+1}_{n} - v^{k}_{n})(v^{k}_{n+1} + v^{k+1}_{n}) = 0\,, \tag{2.116}$$

$$\left(\frac{2}{c_k} + X^{k+1}_{n+1} - X^{k}_{n+1} + X^{k+1}_{n} - X^{k}_{n}\right)(v^{k+1}_{n+1} - v^{k}_{n+1} - v^{k+1}_{n} + v^{k}_{n}) \\ - (X^{k+1}_{n+1} + X^{k}_{n+1} - X^{k+1}_{n} - X^{k}_{n})(v^{k+1}_{n+1} + v^{k}_{n+1} + v^{k+1}_{n} + v^{k}_{n}) = 0\,. \tag{2.117}$$

Note that this form was obtained in [37] by using the bilinear method. Taking the continuous limit $c_k \to 0$ of (2.116) and (2.117), we obtain the semi-discrete short pulse equation (2.55) and (2.56).

3. τ-function and soliton type solutions

In this section, we list the τ-function and the bilinear equations which give rise to the soliton and breather type solutions to the equations and curve motions discussed in Section 2. Although they have been already discussed in [37, 48, 69], we collect and present the results for completeness and the convenience of readers. It should be remarked that all the solutions can be expressed in terms of one τ-function.

The solutions can be expressed in the following form:

$$\theta^{m,k}_{n,l}(s,t,\zeta,y) = \frac{2}{\sqrt{-1}} \log \frac{\tau^{m,k}_{n,l}}{\tau^{*\,m,k}_{n,l}}\,, \tag{3.1}$$

$$\gamma^{m,k}_{n,l}(s,t,\zeta,y) = \begin{bmatrix} -\frac{1}{2}(\log \tau^{m,k}_{n,l}\tau^{*\,m,k}_{n,l})_y \\ \frac{1}{2\sqrt{-1}}\left(\log \frac{\tau^{m,k}_{n,l}}{\tau^{*\,m,k}_{n,l}}\right)_y \end{bmatrix}. \tag{3.2}$$

Here, the τ-function $\tau^{m,k}_{n,l}(s,t,\zeta,y)$ is given by [48]:

$$\tau^{m,k}_{n,l}(s,t,\zeta,y) = \exp\left[-\left(s + \zeta + \epsilon l + \sum_{n'}^{n-1} a_{n'} + \sum_{m'}^{m-1} b_{m'} + \sum_{k'}^{k-1} \frac{1}{c_{k'}}\right) y\right] \det\left(f^{(i)}_{j-1}\right)_{i,j=1,\ldots,N}\,, \tag{3.3}$$

$$f^{(i)}_j = e^{\eta_i} + e^{\mu_i}\,, \tag{3.4}$$

$$\left\{\begin{aligned} e^{\eta_i} &= \alpha_i p_i^j (1-\epsilon p_i)^{-l} \prod_{n'}^{n-1}(1-a_{n'}p_i)^{-1} \prod_{m'}^{m-1}(1-b_{m'}p_i)^{-1} \prod_{k'}^{k-1}\left(1-\frac{c_{k'}}{p_i}\right)^{-1} e^{p_i s - 4p_i^3 t + \frac{p_i}{1-\epsilon^2 p_i^2}\zeta + \frac{1}{p_i} y}, \\ e^{\mu_j} &= \beta_i (-p_i)^j (1+\epsilon p_i)^{-l} \prod_{n'}^{n-1}(1+a_{n'}p_i)^{-1} \prod_{m'}^{m-1}(1+b_{m'}p_i)^{-1} \prod_{k'}^{k-1}\left(1+\frac{c_{k'}}{p_i}\right)^{-1} e^{-p_i s + 4p_i^3 t - \frac{p_i}{1-\epsilon^2 p_i^2}\zeta - \frac{1}{p_i} y}. \end{aligned}\right. \tag{3.5}$$

The parameters are chosen as follows:

(i) **N-soliton solution:**

$$p_i,\ \alpha_i \in \mathbb{R}, \quad \beta_i \in \sqrt{-1}\mathbb{R} \quad (i = 1, \ldots, N). \tag{3.6}$$

(ii) **M-breather solution:**

$$\begin{aligned} &N = 2M, \quad p_i,\ \alpha_i,\ \beta_i \in \mathbb{C} \quad (i = 1, \ldots, 2M), \\ &p_{2j} = p_{2j-1}^*, \quad \alpha_{2j} = \alpha_{2j-1}^*, \quad \beta_{2j} = -\beta_{2j-1}^* \quad (j = 1, \ldots, M). \end{aligned} \tag{3.7}$$

The bilinear equations which are necessary to recover the equations and curve motions are given as follows. Note that we only show the relevant independent variables, and other variables can be regarded as parameters.

Continuous case [37, 48]: $\tau = \tau(s, t, y)$,

$$\frac{1}{2} D_s D_y\, \tau \cdot \tau = -\left(\tau^*\right)^2, \tag{3.8}$$

$$D_s^2\, \tau \cdot \tau^* = 0, \tag{3.9}$$

$$\left(D_s^3 + D_t\right)\, \tau \cdot \tau^* = 0. \tag{3.10}$$

Semi-discrete case [37, 69]: $\tau = \tau_l(\zeta, y)$,

$$D_\zeta\, \tau_l \cdot \tau_l^* = \frac{1}{2\epsilon}\left(\tau_{l-1}^* \tau_{l+1} - \tau_{l+1}^* \tau_{l-1}\right), \tag{3.11}$$

$$\tau_l \tau_l^* = \frac{1}{2}\left(\tau_{l-1}^* \tau_{l+1} + \tau_{l+1}^* \tau_{l-1}\right), \tag{3.12}$$

$$\frac{1}{2} D_\zeta D_y\, \tau_l \cdot \tau_l = -\tau_{l+1}^* \tau_{l-1}^*, \tag{3.13}$$

$$D_y\, \tau_{l+1} \cdot \tau_l = -\epsilon \tau_{l+1}^* \tau_l^*. \tag{3.14}$$

Discrete case [37, 48]: $\tau = \tau_n^{m,k}(y)$,

$$D_y\, \tau_{n+1}^{m,k} \cdot \tau_n^{m,k} = -a_n {\tau^*}_{n+1}^{m,k} {\tau^*}_n^{m,k}, \tag{3.15}$$

$$D_y\, \tau_n^{m+1,k} \cdot \tau_n^{m,k} = -b_m {\tau^*}_{n+1}^{m,k} {\tau^*}_n^{m,k}, \tag{3.16}$$

$$D_y\, \tau_n^{m,k+1} \cdot {\tau^*}_n^{m,k} = -\frac{1}{c_k} {\tau^*}_n^{m,k+1} \tau_n^{m,k}, \tag{3.17}$$

$$b_m {\tau^*}_n^{m+1,k} \tau_{n+1}^{m,k} - a_n {\tau^*}_{n+1}^{m,k} \tau_n^{m+1,k} + (a_n - b_m) {\tau^*}_{n+1}^{m+1,k} \tau_n^{m,k} = 0. \tag{3.18}$$

4. Conclusions

In this paper, we have discretized several soliton equations which admit loop type soliton solutions through the discrete analogues of the hodograph transformations based on the geometry of plane curves. More concretely, we have constructed semi-discrete and fully discrete versions of the WKI elastic beam equation, the complex Dym equation and the short pulse equation, and presented the τ-function which gives rise to the soliton and breather solutions.

Geometric consideration is effective for discretization of soliton equations which admit soliton solutions with singularities, and it may be also applicable to other soliton equations. For example, the soliton equations arising from the curve motions in the Minkowski plane are of so-called "defocusing type" which have nonlinear terms with different signs compared to the equations discussed in this paper, and it is known that the structure and behaviour of the solutions are quite different. The geometric consideration may also be useful to discretize this class of equations. This problem will be reported in a forthcoming paper.

Appendix: Hodograph transformations

The WKI elastic beam equation:
A conservation law of the potential mKdV equation (2.10) is given by

$$(\cos\theta)_t + \left(\frac{1}{2}(\theta_s)^2\cos\theta - \theta_{ss}\sin\theta\right)_s = 0\,. \tag{A.1}$$

Consider the hodograph transformation [27]

$$x(s,t) = \int_0^s \cos\theta(s',t)ds' + x_0\,, \quad t'(s,t) = t\,, \tag{A.2}$$

which leads to

$$\frac{\partial}{\partial s} = \cos\theta\frac{\partial}{\partial x}\,, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} + \left(\theta_{ss}\sin\theta - \frac{1}{2}(\theta_s)^2\cos\theta\right)\frac{\partial}{\partial x}\,. \tag{A.3}$$

Applying (A.3) to (A.1), we obtain

$$\theta_{t'} + \cos^2\theta(\sin\theta)_{xxx} = 0\,, \tag{A.4}$$

which can be rewritten as

$$(\tan\theta)_{t'} + (\sin\theta)_{xxx} = 0\,. \tag{A.5}$$

Introducing a new dependent variable $v(s,t) = \int_0^s \sin\theta(s',t)ds' + v_0$ (note $\tan\theta = v_x$ and $\sin\theta = v_x/\sqrt{1+(v_x)^2}$), (A.5) is transformed to

$$v_{t'x} + \left(\frac{v_x}{\sqrt{1+(v_x)^2}}\right)_{xxx} = 0\,, \tag{A.6}$$

which is the (potential) WKI elastic beam equation (2.18).

The complex Dym equation:
A conservation law of the potential mKdV equation (2.10) is given by

$$(e^{\sqrt{-1}\theta})_t + \left(\frac{1}{2}(\theta_s)^2 e^{\sqrt{-1}\theta} + \sqrt{-1}\theta_{ss}e^{\sqrt{-1}\theta}\right)_s = 0\,. \tag{A.7}$$

Consider the hodograph transformation [52, 53]

$$z(s,t) = \int_0^s e^{\sqrt{-1}\theta(s',t)}ds' + z_0\,, \quad t'(s,t) = t\,, \tag{A.8}$$

which leads to

$$\frac{\partial}{\partial s} = e^{\sqrt{-1}\theta}\frac{\partial}{\partial z}\,, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} + \left(-\frac{1}{2}(\theta_s)^2 e^{\sqrt{-1}\theta} - \sqrt{-1}\theta_{ss}e^{\sqrt{-1}\theta}\right)\frac{\partial}{\partial z}\,. \tag{A.9}$$

Applying (A.9) to (A.7) and introducing a new dependent variable $r = e^{\sqrt{-1}\theta}$, we obtain the complex Dym equation

$$r_{t'} + r^3(r)_{zzz} = 0\,. \tag{A.10}$$

Note that

$$z = x + \sqrt{-1}v\,, \qquad r = \frac{\partial z}{\partial s}\,. \tag{A.11}$$

The short pulse equation:

A conservation law of the sine-Gordon equation (2.24) is given by

$$(\cos\theta)_y + \left(\frac{(\theta_y)^2}{8}\right)_s = 0\,. \tag{A.12}$$

Consider the hodograph transformation [33, 63–65]

$$x(s,y) = \int_0^s \cos\theta(s',y)ds' + x_0\,, \quad y'(s,y) = y\,, \tag{A.13}$$

which leads to

$$\frac{\partial}{\partial s} = \cos\theta\frac{\partial}{\partial x}\,, \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial y'} - \frac{(\theta_y)^2}{8}\frac{\partial}{\partial x}\,. \tag{A.14}$$

Introduce a new dependent variable

$$v(s,y) = \int_0^s \sin\theta(s',y)ds' + v_0 = \int_0^s \frac{\theta_{ys'}(s',y)}{4}ds' + v_0 = \frac{1}{4}\theta_y\,, \tag{A.15}$$

then it follows

$$v_x = \tan\theta\,.$$

Applying (A.14) to (A.12), we obtain

$$\left(\frac{\partial}{\partial y'} - 2v^2\frac{\partial}{\partial x}\right)\cos\theta = -4vv_x\cos\theta\,. \tag{A.16}$$

this can be rewritten as

$$\left(\frac{\partial}{\partial y'} - 2v^2\frac{\partial}{\partial x}\right)\frac{1}{\cos^2\theta} = 8vv_x\frac{1}{\cos^2\theta}\,. \tag{A.17}$$

From

$$\frac{1}{\cos^2\theta} = 1 + \tan^2\theta = 1 + v_x^2\,,$$

it follows that

$$\left(\frac{\partial}{\partial y'} - 2v^2\frac{\partial}{\partial x}\right)(1 + v_x^2) = 8vv_x(1 + v_x^2)\,, \tag{A.18}$$

which is nothing but the short pulse equation

$$v_{xy'} = 4v + \frac{2}{3}\left(v^3\right)_{xx}\,. \tag{A.19}$$

References

[1] Grammaticos B, Kosmann-Schwarzbach Y, Tamizhmani T (ed) 2004 *Discrete Integrable Systems*, *Lecture Notes in Physics* **644** (Springer-Verlag, Berlin).
[2] Ablowitz M J and Ladik J F 1975 *J. Math. Phys.* **16**, 598–603.
[3] Ablowitz M J and Ladik J F 1977 *Stud. Appl. Math.* **57**, 1–12.
[4] Ablowitz M J, Prinari B and Trubatch A D 2004 *Discrete and continuous nonlinear Schrödinger systems*, *London Mathematical Society Lecture Note Series* **302**, (Cambridge University Press, Cambridge).
[5] Hirota R 1977 *J. Phys. Soc. Jpn.* **43**, 4116–4124.
[6] Hirota R 1977 *J. Phys. Soc. Jpn.* **43**, 2074–2078.
[7] Hirota R 1977 *J. Phys. Soc. Jpn.* **43**, 2079–2086.
[8] Hirota R 1978 *J. Phys. Soc. Jpn.* **45**, 321–332.
[9] Hirota R 1979 *J. Phys. Soc. Jpn.* **46**, 312–319.
[10] Date E, Jimbo M and Miwa T 1982 *J. Phys. Soc. Jpn.* **51**, 4116–4124.
[11] Date E, Jimbo M and Miwa T 1982 *J. Phys. Soc. Jpn.* **51**, 4125–4131.
[12] Date E, Jimbo M and Miwa T 1983 *J. Phys. Soc. Jpn.* **53**, 388–393.
[13] Date E, Jimbo M and Miwa T 1983 *J. Phys. Soc. Jpn.* **53**, 761–765.
[14] Date E, Jimbo M and Miwa T 1983 *J. Phys. Soc. Jpn.* **53**, 766–771.
[15] Jimbo M and Miwa T 1983 *Publ. RIMS* **19**, 943-1001.
[16] Miwa T 1982 *Proc. Japan Acad. Ser. A Math. Sci.* **58**, 9–12.
[17] Suris Y B 2003 *The Problem of Integrable Discretization: Hamiltonian Approach* (Birkhäuser, Basel).
[18] Nijhoff F J and Capel H 1995 *Acta Appl. Math.* **39**, (1995) 133–158.
[19] Wadati M, Konno K and Ichikawa Y 1979 *J. Phys. Soc. Jpn* **47**, 1698–1700.
[20] Konno K, Ichikawa Y and Wadati M 1981 *J. Phys. Soc. Jpn* **50**, 1025–1026.
[21] Ichikawa Y, Konno K and Wadati M 1981 *J. Phys. Soc. Jpn* **50**, 1799–1802.
[22] Konno K and Jeffrey A 1983 *J. Phys. Soc. Jpn* **52**, 1–3.
[23] Camassa R and Holm D 1993 *Phys. Rev. Lett.* **71**, 1661–1664.
[24] Degasperis A and Procesi M 1999 in *Symmetry and Perturbation Theory* ed. A. Degasperis and G. Gaeta (Singapore: World Scientific) 22–37.
[25] Schäfer T and Wayne C E 2004 *Physica D* **196**, 90–105.
[26] Hunter J K and Saxton R A 1991 *SIAM J. Appl. Math.* **51**, 1498–1521.
[27] Ishimori Y 1981 *J. Phys. Soc. Jpn.* **50**, 2471–2472.
[28] Ishimori Y 1982 *J. Phys. Soc. Jpn.* **51**, 3036–3041.
[29] Shiff J 1998 *Physica D* **121**, 24–43.
[30] Matsuno Y 2005 *J. Phys. Soc. Jpn.,* **74**, 1983–1987.
[31] Matsuno Y 2005 *Inverse Problems* **21**,1553.
[32] Matsuno Y 2005 *Inverse Problems* **21**,2085.
[33] Matsuno Y 2007 *J. Phys. Soc. Jpn.* **76**, 084003.
[34] Matsuno Y 2006 *Phys. Lett. A* **359**, 451–457.
[35] Rogers C and Schief W K 2002 *Bäcklund and Darboux Transformations. Geometry and Modern Applications in Soliton Theory*, *Cambridge Texts in Applied Mathematics* (Cambridge University Press, Cambridge).
[36] Ohta Y, Maruno K and Feng B-F 2008 *J. Phys. A: Math. Theor.* **41**, 355205.
[37] Feng B-F, Maruno K and Ohta Y 2010 *J. Phys. A: Math. Theor.* **43**, 085203.
[38] Feng B-F, Maruno K and Ohta Y 2010 *J. Phys. A: Math. Theor.* **43**, 265202.
[39] Feng B-F, Maruno K and Ohta Y 2010 *J. Comp. Appl. Math.* **235**, 229-243.
[40] Bobenko A I and Seiler R 1999 *Discrete Integrable Geometry and Physics*, *Oxford Lecture Series in Mathematics and its Applications* **16**, (Oxford University Press, New York)
[41] Bobenko A I and Suris Y B 2008 *Discrete Differential Geometry*, *Graduate Studies in Mathematics* **98**, (AMS, Rhode Island)
[42] Lamb Jr. G 1976 *Phys. Rev. Lett.* **37**, 235–237.

[43] Goldstein R E and Petrich D M 1991 *Phys. Rev. Lett.* **67**, 3203–3206.

[44] Doliwa A and Santini P M 1995 *J. Math. Phys.* **36**, 1259–1273.

[45] Doliwa A and Santini P M 1996 in *Symmetries and Integrability of Difference Equations*, D. Levi, L. Vinet and P. Winternitz (eds.), (AMS, Providence), 91–102.

[46] Doliwa A and Santini P M 1999 in *Discrete Integrable Geometry and Physics*, A. Bobenko and R. Seiler (eds.), (Oxford University Press, Oxford), 139–154.

[47] Hoffmann T and Kutz N 2004 *Stud. Appl. Math.* **113**, 31–55.

[48] Inoguchi J, Kajiwara K, Matsuura N and Ohta Y 2011 Motion and Bäcklund transformations of discrete plane curves, *Kyushu J. Math.* , in press (arXiv:1008.2808).

[49] Matsuura N 2011 Discrete KdV and Discrete Modified KdV Equations Arising from Motions of Planar Discrete Curves, *Int. Math. Res. Notices.* , in press.

[50] Kruskal M D 1975 *Dynamical Systems, Theory and Applications, Lecture Note in Physics* **38** (Springer-Verlag New York), 313.

[51] Kadanoff L P 1990 *Phys. Rev. Lett.* **65**, 2986–2988.

[52] Kawamoto S 1985 *J. Phys. Soc. Jpn.* **54**, 2055–2056.

[53] Dmitrieva L A 1994 *J. Phys. A: Math. Gen.* **27**, 8197–8205.

[54] Hereman W, Banerjee P P and Chatterjee M R 1989 *J. Phys. A: Math. Gen.* **22**, 241-255.

[55] Wadati M, Ichikawa Y and Shimizu T 1980 *Prog. Theor. Phys.* **64**, 1659–1967.

[56] Ablowitz M J, Kaup D J, Newell A C and Segur H 1974 *Stud. Appl. Math.* **53**, 249–315.

[57] Ablowitz M J and Seger H 1981 *Solitons and Inverse Scattering Transform* (SIAM, Philadelphia).

[58] Nakayama K, Segur H and Wadati M 1992 *Phys. Rev. Lett.* **69**, 2603–2606.

[59] Rabelo M L 1989 *Stud. Appl. Math.* **81**, 221–248.

[60] Beals R, Rabelo M and Tenenblat K 1989 *Stud. Appl. Math.* **81**, 125–151.

[61] Brunelli J C 2005 *J. Math. Phys.* **46**, 123507.

[62] Brunelli J C 2006 *Phys. Lett. A* **353**, 475–478.

[63] Sakovich A and Sakovich S 2005 *J. Phys. Soc. Jpn.* **74**, 239–241.

[64] Sakovich A and Sakovich S 2006 *J. Phys. A* **39**, L361–367.

[65] Rogers C, Schief W K and Szereszewski A 2010 *J. Phys. A* **43**, 385210.

[66] Hirota R 1973 *J. Phys. Soc. Jpn.* **35**, 289–294.

[67] Hoffmann T 2009 Discrete differential geometry of curves and surfaces, COE lecture Notes, Vol. 18, Kyushu University.

[68] Hirota R 1998 *J. Phys. Soc. Jpn.* **67**, 2234–2236.

[69] Inoguchi J, Kajiwara K, Matsuura N and Ohta Y 2011 Explicit solutions to semi-discrete modified KdV equation and motion of discrete plane curves, in preparation.