Sequential, successive, and simultaneous decoders for entanglement-assisted classical communication

Shen Chen Xu and Mark M. Wilde
School of Computer Science, McGill University,
Montreal, Quebec, H3A 2A7 Canada

August 17, 2011

Abstract

Bennett *et al.* showed that allowing shared entanglement between a sender and receiver before communication begins dramatically simplifies the theory of quantum channels, and these results suggest that it would be worthwhile to study other scenarios for entanglement-assisted classical communication. In this vein, the present paper makes several contributions to the theory of entanglement-assisted classical communication. First, we rephrase the Giovannetti-Lloyd-Maccone sequential decoding argument as a more general "packing lemma" and show that it gives an alternate way of achieving the entanglement-assisted classical capacity. Next, we show that a similar sequential decoder can achieve the Hsieh-Devetak-Winter region for entanglement-assisted classical communication over a multiple access channel. Third, we prove the existence of a quantum simultaneous decoder for entanglement-assisted classical communication over a multiple access channel with two senders. This result implies a solution of the quantum simultaneous decoding conjecture for unassisted classical communication over quantum multiple access channels with two senders, but the three-sender case still remains open (Sen recently and independently solved this unassisted two-sender case with a different technique). We then leverage this result to recover the known regions for unassisted and assisted quantum communication over a quantum multiple access channel, though our proof exploits a coherent quantum simultaneous decoder. Finally, we determine an achievable rate region for communication over an entanglement-assisted bosonic multiple access channel and compare it with the Yen-Shapiro outer bound for unassisted communication over the same channel.

Shared entanglement between a sender and receiver leads to surprises such as super-dense coding [5] and teleportation [2], and these protocols were the first to demonstrate that entanglement, classical bits, and quantum bits can interact in interesting ways. For this reason, one could argue that these protocols and their noisy generalizations [10, 29, 30] make quantum information theory [31, 37] richer than its classical counterpart [7]. A good way to think of the super-dense coding protocol is that it is a statement of resource conversion [10]: one noiseless qubit channel and one noiseless ebit are sufficient to generate two noiseless bit channels between a sender and receiver.

Bennett *et al.* explored a generalization of the super-dense coding protocol in which a sender and receiver are given noiseless entanglement in whatever form they wish and access to many independent uses of a noisy quantum channel, and the goal is to determine how many asymptotically perfect noiseless bit channels that the sender and receiver can simulate with the aforementioned resources [3, 4, 25]. The entanglement-assisted classical capacity theorem provides a beautiful answer to this question. The optimal rate at which they can communicate classical bits in the presence of free entanglement is equal to the mutual information of the channel [4, 25], defined as

$$I(\mathcal{N}) \equiv \max_{\phi^{AA'}} I(A;B)_{\rho},$$

where $\rho^{AB} \equiv \mathcal{N}^{A'\to B}(\phi^{AA'})$, $\mathcal{N}^{A'\to B}$ is the noisy channel connecting the sender to the receiver, and $\phi^{AA'}$ is a pure, bipartite state prepared at the sender's end of the channel. This result is the strongest statement that quantum information theorists have been able to make in the theory of quantum channels, because the above channel mutual information is additive as a function of any two channels $\mathcal{N}$ and $\mathcal{M}$ [1]:

$$I(\mathcal{N} \otimes \mathcal{M}) = I(\mathcal{N}) + I(\mathcal{M}),$$

and the mutual information $I(A;B)$ is concave in the input state when the channel is fixed [1] (these two properties imply that we can actually calculate the entanglement-assisted classical capacity of *any* quantum channel). Furthermore, this information measure is particularly robust in the sense that a quantum feedback channel from receiver to sender does not increase it—Bowen showed that the classical capacity of a quantum channel in the presence of unlimited quantum feedback communication is equal to the entanglement-assisted classical capacity [6]. For these reasons, the entanglement-assisted classical capacity of a quantum channel is the best formal analogy of Shannon's classical capacity of a classical channel [35].

The simplification that shared entanglement brings to the theory of quantum channels suggests that it might be fruitful to explore other scenarios in which communicating parties share entanglement, and this is precisely the goal of the present paper. Indeed, we explore five different scenarios for entanglement-assisted classical communication:

1. Sequential decoding for entanglement-assisted classical communication over a single-sender, single-receiver quantum channel.

2. Sequential and successive decoding for entanglement-assisted classical communication over a quantum multiple access channel (a two-sender, single-receiver channel).

3. Simultaneous decoding for classical communication over an entanglement-assisted quantum multiple access channel.

4. Coherent simultaneous decoding for assisted and unassisted quantum communication over a quantum multiple access channel.

5. Entanglement-assisted classical communication over a bosonic multiple access channel.

We briefly overview each of these scenarios in what follows.

Our first contribution is a sequential decoder for entanglement-assisted classical communication, meaning that the receiver performs a sequence of measurements with "yes/no" outcomes in order to determine the message that the sender transmits (the receiver performs these measurements on the channel outputs and his share of the entanglement). The idea of this approach is the same as the recent Giovannetti-Lloyd-Maccone (GLM) sequential decoder for unassisted classical communication [18] (which in turn bears similarities to the Feinstein approach [15, 32, 38]). In fact, our approach for proving that the sequential method works for the entanglement-assisted case is to rephrase their argument as a more general "packing lemma" [28, 37] and exploit the entanglement-assisted coding scheme of Hsieh *et al.* [28, 37].

Our next contribution is to extend this sequential decoding argument to a quantum multiple access channel. Winter [39] and Hsieh *et al.* [28] have already shown that successive decoding works well for unassisted and assisted transmission of classical information over a quantum multiple access channel, respectively. (Here, successive decoding means that the receiver first decodes one sender's message and follows by decoding the other sender's message). We show that a receiver can exploit a sequence of measurements with "yes/no" outcomes to determine the first sender's message, followed by a different sequence of "yes/no" measurements to determine the second sender's message. Thus, our decoder here is both sequential and successive and generalizes the GLM sequential decoding scheme.

Our third contribution is to prove that the receiver of an entanglement-assisted quantum multiple access channel can exploit a quantum simultaneous decoder to detect two messages sent by two respective senders. A simultaneous decoder is different from a successive decoder—it can detect the two senders' messages asymptotically faithfully as long as their transmission rates are within the pentagonal rate region of the multiple access channel [13, 39, 28]. A simultaneous decoder is more powerful than a successive decoder for two reasons:

1. A simultaneous decoder does not require the use of time-sharing in order to achieve the rate region of the multiple access channel (whereas a successive decoder requires the use of time-sharing). Thus, the technique should generalize well to the setting of "one-shot" information theory [9], where time-sharing does not apply because that theory is concerned with what is possible with a *single* use of a quantum channel.

2. Nearly every proof in classical network information theory exploits a simultaneous decoder [13]. Thus, a *quantum* simultaneous decoder would be of broad interest for a network theory of quantum information. In particular, the strategy for achieving the best known achievable rate region of the classical interference channel exploits a simultaneous decoder [22, 13]. (An interference channel has two senders and two receivers, and each sender is interested in communicating with one particular receiver.)

We should mention that Fawzi *et al.* could prove the existence of a quantum simultaneous decoder for certain quantum channels [14], but a proof for the general case remained missing and they did not address the entanglement-assisted case. Though, the results of this paper and recent work of Sen [34] give a quantum simultaneous decoder for unassisted communication over a two-sender multiple access channel and solve the conjecture from Ref. [14] for the two-sender case. It remains unclear how to prove the conjecture for the case of three senders. The results of this work might be useful for establishing an achievable rate region for a quantum interference channel setting in which sender-receiver pairs share entanglement before communication begins, but this remains the topic of future work.

We then leverage the above result to recover the known regions for assisted and unassisted quantum communication over a quantum multiple access channel [27, 40, 28]. We call the decoder a *coherent quantum simultaneous decoder* because we construct an isometry from the above simultaneous decoding POVM, and the isometry is what enables quantum communication between both senders and the receiver.

Our final contribution is to determine an achievable rate region for entanglement-assisted classical communication over the multiple access bosonic channel studied in Ref. [41]. This channel is simply a beamsplitter with two input ports, where the receiver obtains one output port and the environment of the channel obtains the other output port. The beamsplitter is a simplified model for light-based free-space communication in a multiple-access setting. In order to calculate the rate region for this setting, we apply the theorem of Hsieh *et al.* in Ref. [28] with both senders sharing a two-mode squeezed vacuum state [16] with the receiver. Since this state achieves the entanglement-assisted capacity of the single-mode lossy bosonic channel [20, 19, 26], we might suspect that it should do well in the multiple access setting. Though, it still remains open to determine whether this strategy is optimal.

1 Packing Argument for a Sequential Decoder

Giovannetti, Lloyd, and Maccone (GLM) offered a scheme for transmitting classical information over a quantum channel that exploits a sequential decoder [18]. In their sequential decoding scheme, the receiver tries to distinguish the transmitted message from a list of all possible messages one by one until the correct one is identified, by performing a sequence of projective measurements. We recast this procedure as a general packing argument in this section, and the next section demonstrates that the sequential decoding scheme works well for entanglement-assisted classical communication.

Theorem 1 (Sequential Packing) *Let* $\{p_X(x), \rho_x\}_{x\in\mathcal{X}}$ *be an ensemble of states indexed by letters in an alphabet* $\mathcal{X}$*. Each state* ρ_x *has the following spectral decomposition:*

$$\rho_x = \sum_y \lambda_{x,y} |\psi_{x,y}\rangle \langle\psi_{x,y}|, \tag{1.1}$$

and the expected density operator of the ensemble is as follows:

$$\rho \equiv \sum_{x\in\mathcal{X}} p_X(x)\, \rho_x. \tag{1.2}$$

Suppose there exists a code subspace projector Π *and codeword subspace projectors* $\{\Pi_x\}_{x\in\mathcal{X}}$ *such that the following properties hold for some* $D, d \geq 0$, $1/2 \geq \epsilon > 0$, *and for all* $x \in \mathcal{X}$*:*

$$\text{Tr}\{\Pi\rho_x\} \geq 1-\epsilon, \tag{1.3}$$
$$\text{Tr}\{\Pi_x\rho_x\} \geq 1-\epsilon, \tag{1.4}$$
$$\Pi_x\rho_x\Pi_x \geq \frac{1}{d}\Pi_x, \tag{1.5}$$
$$\Pi\rho\Pi \leq \frac{1}{D}\Pi, \tag{1.6}$$
$$[\Pi_x, \rho_x] = 0. \tag{1.7}$$

Then corresponding to a message set $\mathcal{M}$*, we can construct a random code* $\mathcal{C} = \{c_m\}_{m\in\mathcal{M}}$ *with* $c_m \in \mathcal{X}$ *such that the receiver can reliably distinguish between the states* $\{\rho_{c_m}\}_{m\in\mathcal{M}}$ *by performing a sequence of projective measurements using the projectors* Π *and* Π_x*. More precisely, suppose that our performance measure is the expectation of the average success probability where the expectation is with respect to all possible random choices of codes. Then we can bound this performance measure from below (as long as* $2 - \exp\{d\,|\mathcal{M}|\,/D\}$ *is positive):*

$$\mathbb{E}_{\mathcal{C}}\{\bar{p}_{succ}(\mathcal{C})\} \geq \left|(1-2\epsilon)\left(2 - e^{\frac{d}{D}|\mathcal{M}|}\right)\right|^2, \tag{1.8}$$

implying that the performance measure becomes arbitrarily close to one if D/d *is large,* $|\mathcal{M}| \ll D/d$*, and* ϵ *is arbitrarily small.*

Proof. The proof of this lemma is similar to the GLM proof, and we thus place it in Appendix A. ■

2 Sequential Decoding for Entanglement-Assisted Communication

In this section, we show an application of the GLM sequential decoding scheme to entanglement-assisted classical communication by exploiting the coding approach of Hsieh *et al.* [28]. The approach thus gives another way of achieving the entanglement-assisted classical capacity of a quantum channel.

Theorem 2 (Entanglement-Assisted Sequential Decoding) *The sequential decoding scheme can achieve the entanglement-assisted classical capacity of a quantum channel.*

Proof. Suppose that a quantum channel $\mathcal{N}^{A'\to A}$ connects Alice to Bob and that they share many copies of an arbitrary entangled pure state $|\phi\rangle^{A'A}$:

$$|\phi\rangle^{A'^n A^n} \equiv \left(|\phi\rangle^{A'A}\right)^{\otimes n} = |\phi\rangle^{A'A} \otimes |\phi\rangle^{A'A} \otimes \cdots \otimes |\phi\rangle^{A'A}, \tag{2.1}$$

where Alice has access to the system A' and Bob has access to the system A. Alice chooses a message from her message set $\mathcal{M}$ uniformly at random, applies a corresponding encoder to her shares A'^n of the entanglement, and sends the systems A'^n to Bob. Later in the analysis, we would like to be able to "pull" these encoding operations through the channel so that they are equivalent to some other operator acting at Bob's end. In order to do this, we can write the many copies of the shared entanglement as a direct sum of maximally entangled states [28, 37]. Starting from the Schmidt decomposition for one copy of the state $|\phi\rangle$

$$|\phi\rangle^{A'A} = \sum_z \sqrt{p_Z(z)}\,|z\rangle^{A'}|z\rangle^{A}, \tag{2.2}$$

we can derive the following using the method of types [7, 37]:

$$\begin{aligned}
|\phi\rangle^{A'^n A^n} &= \sum_{z^n} \sqrt{p_{Z^n}(z^n)}\, |z^n\rangle^{A'^n} |z^n\rangle^{A^n} && (2.3)\\
&= \sum_t \sum_{z^n \in T_t} \sqrt{p_{Z^n}(z^n)}\, |z^n\rangle^{A'^n} |z^n\rangle^{A^n} && (2.4)\\
&= \sum_t \sqrt{p_{Z^n}(z_t^n)\, d_t} \frac{1}{d_t} \sum_{z^n \in T_t} |z^n\rangle^{A'^n} |z^n\rangle^{A^n} && (2.5)\\
&= \sum_t \sqrt{p(t)}\, |\Phi_t\rangle^{A'^n A^n}, && (2.6)
\end{aligned}$$

where

$$p(t) \equiv p_{Z^n}(z_t^n)\, d_t, \tag{2.7}$$

T_t is a type class, d_t is the dimension of a type class subspace t, z_t^n is a representative sequence for the type class t, and each $|\Phi_t\rangle^{A'^n A^n}$ is maximally entangled on the type class subspace specified by t (see Refs. [28, 37] for more details on this approach). Thus, applying an operator acting on type class subspaces at Alice's end is equivalent to applying the transpose of the same operator at Bob's end. As in Refs. [28, 37], Alice constructs her encoders using the Heisenberg-Weyl set of operators $\{X(x_t)\, Z(z_t)\}_{x_t, z_t}$ that act on each of the type class subspaces

$$U(s) \equiv \bigoplus_t (-1)^{b_t} X(x_t)\, Z(z_t), \tag{2.8}$$

where b_t determines a phase that is applied to the operators in each subspace. We denote this unitary by $U(s)$ where s is some vector that contains all the needed indices x_t, z_t and b_t. Let $\mathcal{S}$ denote the set of all such possible vectors. We construct a random code $\{s_m\}_{m\in\mathcal{M}}$ where s_m is a vector chosen uniformly at random from $\mathcal{S}$ and the corresponding set of encoders is then $\{U(s_m)\}_{m\in\mathcal{M}}$. Since the "transpose trick" holds for each of these unitaries, we have that

$$U(s)^{A'^n} |\phi\rangle^{A'^n A^n} = U^T(s)^{A^n} |\phi\rangle^{A'^n A^n}. \tag{2.9}$$

The induced ensemble at Bob's end is then

$$\left\{ \frac{1}{|\mathcal{S}|}, \sigma_s \right\}_{s\in\mathcal{S}}, \tag{2.10}$$

where

$$\begin{aligned}
\sigma_s &\equiv U^T(s)^{A^n} \rho^{A^n B^n} U^*(s)^{A^n}, && (2.11)\\
\rho^{A^n B^n} &\equiv \mathcal{N}^{A'^n \to B^n}\left(|\phi\rangle\langle\phi|^{A'^n A^n} \right). && (2.12)
\end{aligned}$$

Let $\overline{\sigma}$ denote the expected state of the ensemble:

$$\overline{\sigma} \equiv \frac{1}{|\mathcal{S}|} \sum_{s\in\mathcal{S}} \sigma_s. \tag{2.13}$$

We give Bob the following code subspace projector:

$$\Pi \equiv \Pi_\delta^{A^n} \otimes \Pi_\delta^{B^n}, \tag{2.14}$$

and the codeword subspace projectors:

$$\Pi_s \equiv U^T(s)^{A^n} \Pi_\delta^{A^n B^n} U^*(s)^{A^n}, \tag{2.15}$$

where $\Pi_\delta^{A^nB^n}$, $\Pi_\delta^{A^n}$, and $\Pi_\delta^{B^n}$ are the δ-typical projectors for many copies of the states $\rho^{A^nB^n}$, $\rho^{A^n} = \mathrm{Tr}_B\left\{\rho^{A^nB^n}\right\}$ and $\rho^{B^n} = \mathrm{Tr}_A\left\{\rho^{A^nB^n}\right\}$, respectively.

At this point we would like to apply our packing argument from Theorem 1 and we would like to have the following conditions hold:

$$\mathrm{Tr}\left\{\Pi\sigma_s\right\} \geq 1-\epsilon, \tag{2.16}$$

$$\mathrm{Tr}\left\{\Pi_s\sigma_s\right\} \geq 1-\epsilon, \tag{2.17}$$

$$\Pi\bar{\sigma}\Pi \leq 2^{-n\left(H(A)_\rho+H(B)_\rho-\eta(n,\delta)-\delta\right)}\Pi \tag{2.18}$$

$$\Pi_s\sigma_s\Pi_s \geq 2^{-n\left(H(AB)_\rho+\delta\right)}\Pi_s, \tag{2.19}$$

$$\left[\Pi_s,\sigma_s\right] = 0, \tag{2.20}$$

where the function $\eta(n,\delta)$ goes to zero as $n\to\infty$ and $\delta\to 0$. The first three conditions are shown in Refs. [28, 37]. The fourth condition follows from the equipartition property of typical subspaces [37] and the fact that $U^TU^* = I$ for any unitary operator U. The fifth condition follows from the fact that the projector Π_s commutes with the density operator σ_s. By our packing argument in Theorem 1 that gives a bound on the expectation of the average success probability, there exists a particular code, with which Alice can transmit messages from her set $\mathcal{M}$ and Bob can detect the transmitted state by performing a series of projective measurements, with its average success probability being greater than

$$\bar{p}_{\mathrm{succ}} \geq \left|(1-2\epsilon)\left(2-\exp\left\{2^{-n\left(H(A)_\rho+H(B)_\rho-H(AB)_\rho-\eta(n,\delta)-2\delta\right)}\left|\mathcal{M}\right|\right\}\right)\right|^2 \tag{2.21}$$

$$= \left|(1-2\epsilon)\left(2-\exp\left\{2^{-n\left(I(A\,;B)_\rho-\eta(n,\delta)-2\delta\right)}\left|\mathcal{M}\right|\right\}\right)\right|^2 \tag{2.22}$$

Therefore, Alice can pick the size of $\mathcal{M}$ to be $2^{n\left(I(A\,;B)_\rho-\eta(n,\delta)-3\delta\right)}$, and the rate of communication is then

$$C = \frac{1}{n}\log_2\left|\mathcal{M}\right| = I(A;B)_\rho - \eta(n,\delta) - 3\delta, \tag{2.23}$$

with the average success probability becoming greater than

$$\bar{p}_{\mathrm{succ}} \geq \left|(1-2\epsilon)\left(2-\exp\left\{2^{-n\delta}\right\}\right)\right|^2. \tag{2.24}$$

Thus, for sufficiently large n, the sequential decoding scheme achieves the entanglement-assisted classical capacity with arbitrarily high success probability.

As a final note, we should clarify a bit further: there is a codebook $\left\{U(s_m)\right\}_{m\in\mathcal{M}}$ for Alice with entanglement-assisted quantum codewords of the following form:

$$U^{A'^n}(s_m)\left|\phi\right\rangle^{A'^nA^n}. \tag{2.25}$$

If Alice sends message m, Bob performs a sequence of measurements in the following order (assuming a correct sequence of events):

$$\Pi \to I-\Pi_{s_1} \to \Pi \to I-\Pi_{s_2} \to \Pi \to \cdots \to \Pi \to \Pi_{s_m}, \tag{2.26}$$

with Π and Π_{s_i} of the form in (2.14) and (2.15), respectively. ■

3 Packing Argument for Sequential and Successive Decoding over a Multiple Access Channel

We now extend the packing argument from Section 1 to a multiple-access setting, in which there are two senders and one receiver. The resulting scheme is both sequential and successive—sequential in the above sense where the receiver linearly tests one codeword at a time and successive in the sense that the receiver first decodes one sender's message and follows by decoding the other sender's message. After doing so, we then briefly remark how this argument achieves the known strategies for both unassisted [39] and assisted classical communication [28].

Theorem 3 (Sequential and Successive Decoding) *Suppose there exists a doubly-indexed ensemble of quantum states, where two independent distributions generate the different indices x and y:*

$$\{p_X(x)\, p_Y(y),\rho_{x,y}\}. \tag{3.1}$$

Averaging with the distributions $p_X(x)$ and $p_Y(y)$ leads to the following states:

$$\rho_x \equiv \sum_y p_Y(y)\,\rho_{x,y}, \qquad \rho_y \equiv \sum_x p_X(x)\,\rho_{x,y}, \qquad \rho \equiv \sum_{x,y} p_X(x)\,p_Y(y)\,\rho_{x,y}. \tag{3.2}$$

Suppose that there exist projectors Π_x, Π_y, $\Pi_{x,y}$, and Π such that

$$\mathrm{Tr}\{\Pi\rho_x\} \geq 1-\epsilon, \tag{3.3}$$
$$\mathrm{Tr}\{\Pi_x\rho_x\} \geq 1-\epsilon, \tag{3.4}$$
$$\Pi_x\rho_x\Pi_x \geq \frac{1}{d_1^{(-)}}\Pi_x, \tag{3.5}$$
$$\Pi\rho\Pi \leq \frac{1}{D_1}\Pi, \tag{3.6}$$
$$[\Pi_x,\rho_x] = 0. \tag{3.7}$$

and

$$\mathrm{Tr}\{\Pi_x\rho_{x,y}\} \geq 1-\epsilon, \tag{3.8}$$
$$\mathrm{Tr}\{\Pi_{x,y}\rho_{x,y}\} \geq 1-\epsilon, \tag{3.9}$$
$$\Pi_{x,y}\rho_{x,y}\Pi_{x,y} \geq \frac{1}{d_2}\Pi_{x,y}, \tag{3.10}$$
$$\Pi_x\rho_x\Pi_x \leq \frac{1}{d_1^{(+)}}\Pi_x, \tag{3.11}$$
$$[\Pi_{x,y},\rho_{x,y}] = 0. \tag{3.12}$$

Suppose that $D_1/d_1^{(-)}$ is large, $|\mathcal{L}| \ll D_1/d_1^{(-)}$, $d_1^{(+)}/d_2$ is large, $|\mathcal{M}| \ll d_1^{(+)}/d_2$, and ϵ is arbitrarily small. Then there exists a sequential and successive decoding scheme for the receiver that succeeds with high probability, in the sense that the expectation of the average success probability is arbitrarily high:

$$\mathbb{E}_{\mathcal{C}}\{\bar{p}_{succ}(\mathcal{C})\} \geq \left|(1-2\epsilon)\left(2-e^{d_2|\mathcal{M}|/d_1^{(+)}}\right)\right|^2 - 2\sqrt{2(\epsilon+\epsilon')}, \tag{3.13}$$

with ϵ' chosen so that

$$2-e^{d_1^{(-)}|\mathcal{L}|/D_1} \geq 1-\epsilon'. \tag{3.14}$$

Proof. The random construction of the code is similar to that in the proof of Theorem 1. Given a message set $\mathcal{L} = \{1,2,\ldots,|\mathcal{L}|\}$, we construct a code $\mathcal{C}_1 \equiv \{x(l)\}_{l\in\mathcal{L}}$ for Alice randomly such that each $x(l)$ takes a value $x \in \mathcal{X}$ with probability $p_X(x)$. Similarly, given a message set $\mathcal{M} = \{1,2,\ldots,|\mathcal{M}|\}$, we construct a code $\mathcal{C}_2 \equiv \{y(m)\}_{m\in\mathcal{M}}$ for Bob randomly such that each $y(m)$ takes a value $y \in \mathcal{Y}$ with probability $p_Y(y)$. Using this code, Alice chooses a message l from the message set $\mathcal{L}$, Bob chooses a message m from the message set $\mathcal{M}$, and they encode their messages in the quantum codeword $\rho_{x(l),y(m)}$.

Suppose that the first sender Alice transmits message l and the second sender Bob transmits message m. Without loss of generality, the receiver first tries to recover the message that Alice transmits. In order to do so, he measures Π followed by $\Pi_{x(1)}$ to determine if the transmitted message corresponds to the first codeword of Alice, with $\Pi_{x(1)}$ corresponding to the outcome YES and $Q_{x(1)} \equiv I - \Pi_{x(1)}$ corresponding to the outcome NO. Suppose that the outcome is NO. He then measures Π to project the state back into the large subspace. Assuming a correct sequence of events, the receiver continues and measures $Q_{x(i)}$ and Π for $i \in \{2, \ldots, l-1\}$ until getting to the correct outcome $\Pi_{x(l)}$. Thus, the sequence of projectors measured is as follows, under the assumption of a correct sequence of events:

$$\Pi \rightarrow Q_{x(1)} \rightarrow \Pi \rightarrow Q_{x(2)} \rightarrow \Pi \rightarrow \cdots \rightarrow Q_{x(i)} \rightarrow \Pi \rightarrow \cdots \rightarrow \Pi \rightarrow \Pi_{x(l)}. \tag{3.15}$$

After receiving a YES outcome from $\Pi_{x(l)}$, the receiver assumes that the first sender transmitted message l. The receiver then tries to determine the codeword that Bob transmitted by exploiting the projectors $\Pi_{x(l)}$ and $\Pi_{x(l),y(j)}$. He does this in a similar fashion as above, proceeding in the following order (again under the assumption of a correct sequence of events):

$$Q_{x(l),y(1)} \rightarrow \Pi_{x(l)} \rightarrow Q_{x(l),y(2)} \rightarrow \Pi_{x(l)} \rightarrow \cdots \rightarrow Q_{x(l),y(i)} \rightarrow \Pi_{x(l)} \rightarrow \cdots \rightarrow \Pi_{x(l)} \rightarrow \Pi_{x(l),y(m)}. \tag{3.16}$$

The POVM corresponding to the above measurement strategy is as follows:

$$\Lambda_{l,m} \equiv M_{l,m}^{\dagger} M_{l,m}, \tag{3.17}$$

where

$$M_{l,m} \equiv \Pi_{x(l),y(m)} \overline{\overline{Q}}_{x(l),y(m-1)} \cdots \overline{\overline{Q}}_{x(l),y(1)} \Pi_{x(l)} \overline{Q}_{x(l-1)} \cdots \overline{Q}_{x(1)}, \tag{3.18}$$

$$\overline{\Theta} \equiv \Pi \Theta \Pi, \tag{3.19}$$

$$\overline{\overline{\Theta}} \equiv \Pi_{x(l)} \Theta \Pi_{x(l)}. \tag{3.20}$$

The average success probability of any particular code c is

$$\bar{p}_{\text{succ}}(c) \equiv \frac{1}{|\mathcal{L}|\,|\mathcal{M}|} \sum_{l,m} \text{Tr}\left\{\Lambda_{l,m} \rho_{x(l),y(m)}\right\}, \tag{3.21}$$

and the expectation of the average success probability is

$$\mathbb{E}_{X,Y}\left\{\bar{p}_{\text{succ}}(C)\right\} = \sum_{\substack{x(1),\ldots,x(|\mathcal{L}|),\\ y(1),\ldots,y(|\mathcal{M}|)}} p_X(x(1)) \cdots p_X(x(|\mathcal{L}|))\, p_Y(y(1)) \cdots$$

$$\cdots p_Y(y(|\mathcal{M}|)) \frac{1}{|\mathcal{L}|\,|\mathcal{M}|} \sum_{l,m} \text{Tr}\left\{\Lambda_{l,m} \rho_{x(l),y(m)}\right\}. \tag{3.22}$$

$$= \frac{1}{|\mathcal{L}|\,|\mathcal{M}|} \sum_{l,m} \sum_{x,y} p_X(x)\, p_Y(y)\, \text{Tr}\left\{\Psi_x^{m-1}\left(\Pi_x \Phi^{l-1}(\rho_{x,y})\, \Pi_x\right) \Pi_{x,y}\right\}, \tag{3.23}$$

where

$$\Phi(\cdot) \equiv \sum_x p_X(x)\, \overline{Q}_x(\cdot)\, \overline{Q}_x, \tag{3.24}$$

$$\Psi_x(\cdot) \equiv \sum_y p_Y(y)\, \overline{\overline{Q}}_{x,y}(\cdot)\, \overline{\overline{Q}}_{x,y}. \tag{3.25}$$

Observe that we can rewrite the success probability in (3.22) as follows:

$$\sum_{\substack{x(1),\ldots,x(|\mathcal{L}|),\\ y(1),\ldots,y(|\mathcal{M}|)}} p_X(x(1)) \cdots p_X(x(|\mathcal{L}|))\, p_Y(y(1)) \cdots p_Y(y(|\mathcal{M}|)) \frac{1}{|\mathcal{L}|\,|\mathcal{M}|} \sum_{l,m} \text{Tr}\left\{\Gamma_{x,y,l,m} \omega_{x,y,l,m}\right\}, \tag{3.26}$$

where

$$\omega_{x,y,l,m} \equiv \Pi_{x(l)}\overline{Q}_{x(l-1)}\cdots\overline{Q}_{x(1)}\rho_{x(l),y(m)}\overline{Q}_{x(1)}\cdots\overline{Q}_{x(l-1)}\Pi_{x(l)}, \tag{3.27}$$

$$\Gamma_{x,y,l,m} \equiv \overline{\overline{Q}}_{x(l),y(1)}\cdots\overline{\overline{Q}}_{x(l),y(m-1)}\Pi_{x(l),y(m)}\overline{\overline{Q}}_{x(l),y(m-1)}\cdots\overline{\overline{Q}}_{x(l),y(1)}. \tag{3.28}$$

We can then obtain the following lower bound on (3.26):

$$\sum_{\substack{x(1),\ldots,x(|\mathcal{L}|),\\ y(1),\ldots,y(|\mathcal{M}|)}} p_X(x(1))\cdots p_X(x(|\mathcal{L}|))\,p_Y(y(1))\cdots \\ \cdots p_Y(y(|\mathcal{M}|))\frac{1}{|\mathcal{L}|\,|\mathcal{M}|}\sum_{l,m}\left[\mathrm{Tr}\left\{\Gamma_{x,y,l,m}\rho_{x(l),y(m)}\right\} - \left\|\rho_{x(l),y(m)} - \omega_{x,y,l,m}\right\|_1\right], \tag{3.29}$$

by exploiting the following inequality:

$$\mathrm{Tr}\left\{\Gamma_{x,y,l,m}\omega_{x,y,l,m}\right\} \geq \mathrm{Tr}\left\{\Gamma_{x,y,l,m}\rho_{x(l),y(m)}\right\} - \left\|\rho_{x(l),y(m)} - \omega_{x,y,l,m}\right\|_1, \tag{3.30}$$

which holds for all positive operators $\Gamma_{x,y,l,m}$, $\omega_{x,y,l,m}$, and $\rho_{x(l),y(m)}$ that have spectrum less than one. So it remains to show that both $\mathrm{Tr}\{\Gamma_{x,y,l,m}\rho_{x(l),y(m)}\}$ is arbitrarily close to one and $\left\|\rho_{x(l),y(m)} - \omega_{x,y,l,m}\right\|_1$ is arbitrarily small when averaging over all codewords and taking the expectation over random codes. We can apply Theorem 1 to obtain the following inequality:

$$\sum_{\substack{x(1),\ldots,x(|\mathcal{L}|),\\ y(1),\ldots,y(|\mathcal{M}|)}} p_X(x(1))\cdots p_X(x(|\mathcal{L}|))\,p_Y(y(1))\cdots p_Y(y(|\mathcal{M}|))\,\mathrm{Tr}\left\{\omega_{x,y,l,m}\right\} \tag{3.31}$$

$$\geq \left|(1-\epsilon)\left(2 - e^{d_1^{(-)}|\mathcal{L}|/D_1}\right)\right|^2 \tag{3.32}$$

$$\geq \left|(1-\epsilon)(1-\epsilon')\right|^2 \tag{3.33}$$

$$\geq 1 - 2(\epsilon+\epsilon'), \tag{3.34}$$

with ϵ' chosen as given in the statement of the theorem. We can then apply the Gentle Operator Lemma for ensembles (Lemma 9.4.3 in Ref. [37]) to prove the following inequality:

$$\sum_{\substack{x(1),\ldots,x(|\mathcal{L}|),\\ y(1),\ldots,y(|\mathcal{M}|)}} p_X(x(1))\cdots p_X(x(|\mathcal{L}|))\,p_Y(y(1))\cdots p_Y(y(|\mathcal{M}|))\frac{1}{|\mathcal{L}|\,|\mathcal{M}|}\left\|\rho_{x(l),y(m)} - \omega_{x,y,l,m}\right\|_1 \leq 2\sqrt{2(\epsilon+\epsilon')}. \tag{3.35}$$

Invoking Theorem 1 one more time gives us the following lower bound:

$$\sum_{\substack{x(1),\ldots,x(|\mathcal{L}|),\\ y(1),\ldots,y(|\mathcal{M}|)}} p_X(x(1))\cdots p_X(x(|\mathcal{L}|))\,p_Y(y(1))\cdots p_Y(y(|\mathcal{M}|))\frac{1}{|\mathcal{L}|\,|\mathcal{M}|}\sum_{l,m}\mathrm{Tr}\left\{\Gamma_{x,y,l,m}\rho_{x(l),y(m)}\right\} \\ \geq \left|(1-2\epsilon)\left(2 - e^{d_2|\mathcal{M}|/d_1^{(+)}}\right)\right|^2, \tag{3.36}$$

and this completes the proof of the theorem, by combining the above two inequalities with the lower bound in (3.29). ■

It is straightforward to apply this packing argument to either unassisted or assisted transmission of classical information over a quantum multiple access channel. For the unassisted case, one could exploit Winter's coding scheme with conditionally typical projectors [39], and we would pick the parameters as

$$D_1 = 2^{n[H(B)-\delta]}, \tag{3.37}$$

$$d_1^{(+)} = 2^{n[H(B|X)-\delta]}, \tag{3.38}$$

$$d_1^{(-)} = 2^{n[H(B|X)+\delta]}, \tag{3.39}$$

$$d_2 = 2^{n[H(B|XY)+\delta]}, \tag{3.40}$$

so that we would have

$$D_1/d_1^{(-)} = 2^{n[I(X;B)-2\delta]}, \tag{3.41}$$

$$d_1^{(+)}/d_2 = 2^{n[I(Y;B|X)-2\delta]}. \tag{3.42}$$

For the entanglement-assisted case, one could exploit the coding structure of Hsieh *et al.* [28] that we have discussed throughout this article, and we would pick the parameters as

$$D_1 = 2^{n[H(A)+H(B)+H(C)-\delta]}, \tag{3.43}$$

$$d_1^{(+)} = 2^{n[H(B)+H(AC)-\delta]}, \tag{3.44}$$

$$d_1^{(-)} = 2^{n[H(B)+H(AC)+\delta]}, \tag{3.45}$$

$$d_2 = 2^{n[H(ABC)+\delta]}, \tag{3.46}$$

so that we would have

$$D_1/d_1^{(-)} = 2^{n[I(A;C)-2\delta]}, \tag{3.47}$$

$$d_1^{(+)}/d_2 = 2^{n[I(B;AC)-2\delta]}. \tag{3.48}$$

4 Entanglement-Assisted Quantum Simultaneous Decoding

In this section, we prove the existence of a simultaneous decoder for entanglement-assisted classical communication over a quantum multiple access channel with two senders. A simultaneous decoder differs from a successive decoder in the sense that such a decoder allows for the receiver to reliably detect the messages of both senders with a single measurement as long as the rates are within the pentagonal rate region specified by Theorem 6 of Ref. [28] and Theorem 4 below (it might also be helpful to consult Ref. [13] to see the difference between classical successive and simultaneous decoders). The advantage of a simultaneous decoder over a successive decoder is that there is no need to invoke time-sharing in order to achieve the Hsieh-Devetak-Winter rate region of the entanglement-assisted multiple access channel in Ref. [28]. Also, an analogous classical decoder is required in order to achieve the Han-Kobayashi rate region for the classical interference channel [22] (though it requires a simultaneous decoder for three senders).

Concerning the quantum interference channel, Fawzi *et al.* made progress towards demonstrating that a quantized version of the classical Han-Kobayashi rate region is achievable for classical communication over a quantum interference channel [14], though they were only able to prove this result up to a conjecture regarding the existence of a quantum simultaneous decoder for general channels. The importance of this conjecture stems not only from the fact that it would allow for a quantization of the Han-Kobayashi rate region, but also more broadly from the fact that many coding theorems in classical network information theory exploit the simultaneous decoding technique [13]. Thus, having a general quantum simultaneous decoder for an arbitrary number of senders should allow for the wholesale import of much of classical network information theory into quantum network information theory.

Our result below applies only to channels with two senders, and the technique unfortunately does not generally extend to channels with three senders. Thus, this important case still remains open as a conjecture. Sen independently arrived at the results here by exploiting both the proof structure outlined below and a different technique as well [34].

Theorem 4 (Entanglement-Assisted Simultaneous Decoding) *Suppose that Alice and Charlie share many copies of an entangled pure state* $|\phi\rangle^{A'A}$ *where Alice has access to the system* A' *and Charlie has access to the system* A. *Similarly, let Bob and Charlie share many copies of an entangled pure state* $|\psi\rangle^{B'B}$. *Let* $\mathcal{N}^{A'B'\to C}$ *be a multiple access channel that connects Alice and Bob to Charlie, and let*

$$\rho^{ABC} \equiv \mathcal{N}^{A'B'\to C}\left(|\phi\rangle\langle\phi|^{A'A}\otimes|\psi\rangle\langle\psi|^{B'B}\right). \tag{4.1}$$

Then there exists an entanglement-assisted classical communication code with a corresponding quantum simultaneous decoder, such that the following rate region is achievable for $R_1, R_2 \geq 0$*:*

$$R_1 \leq I(A;C|B)_\rho, \tag{4.2}$$

$$R_2 \leq I(B;C|A)_\rho, \tag{4.3}$$

$$R_1 + R_2 \leq I(AB;C)_\rho, \tag{4.4}$$

where the entropies are with respect to the state in (4.1).

Proof. Suppose that Alice has a message set $\mathcal{L}$ and Bob has a message set $\mathcal{M}$ from which they will each choose a message $l \in \mathcal{L}$ and $m \in \mathcal{M}$ uniformly at random to send to Charlie. They construct random codes $C_1 \equiv \{s_1(l)\}_{l\in\mathcal{L}}$ and $C_2 \equiv \{s_2(m)\}_{m\in\mathcal{M}}$ in the same way as explained in the proof of Theorem 2. Both of them encode their messages by applying unitary encoders to their respective shares of the entanglement, giving rise to the following states after applying the transpose trick to each type class [28, 37]:

$$\left(U(s_1(l))^{A'^n} \otimes I^{A^n}\right)|\phi\rangle^{A'^n A^n} = \left(I^{A'^n} \otimes U^T(s_1(l))^{A^n}\right)|\phi\rangle^{A'^n A^n}, \tag{4.5}$$

$$\left(U(s_2(m))^{B'^n} \otimes I^{B^n}\right)|\psi\rangle^{B'^n B^n} = \left(I^{B'^n} \otimes U^T(s_2(m))^{B^n}\right)|\psi\rangle^{B'^n B^n}. \tag{4.6}$$

Then they both send their share of the state to Charlie over the multiple access channel $\mathcal{N}^{A'B'\to C}$, giving rise to a state $\sigma_{l,m}$ at Charlie's receiving end:

$$\sigma_{l,m} \equiv \left(U^T(s_1(l))^{A^n} \otimes U^T(s_2(m))^{B^n}\right)\rho^{A^nB^nC^n}\left(U^*(s_1(l))^{A^n} \otimes U^*(s_2(m))^{B^n}\right). \tag{4.7}$$

Charlie decodes with a simultaneous decoding POVM $\{\Lambda_{l,m}\}_{l\in\mathcal{L},m\in\mathcal{M}}$, defined as follows:

$$\Lambda_{l,m} \equiv \left(\sum_{l',m'}\Upsilon_{l',m'}\right)^{-\frac{1}{2}}\Upsilon_{l,m}\left(\sum_{l',m'}\Upsilon_{l',m'}\right)^{-\frac{1}{2}}, \tag{4.8}$$

where

$$\Upsilon_{l,m} \equiv U^T(s_1(l))^{A^n}\hat{\Pi}_3\hat{\Pi}_2U^T(s_2(m))^{B^n}\Pi^{A^nB^nC^n}U^*(s_2(m))^{B^n}\hat{\Pi}_2\hat{\Pi}_3U^*(s_1(l))^{A^n}, \tag{4.9}$$

and

$$\hat{\Pi}_1 \equiv \left(\Pi^{A^n} \otimes \Pi^{B^nC^n}\right), \tag{4.10}$$

$$\hat{\Pi}_2 \equiv \left(\Pi^{B^n} \otimes \Pi^{A^nC^n}\right), \tag{4.11}$$

$$\hat{\Pi}_3 \equiv \left(\Pi^{C^n} \otimes \Pi^{A^nB^n}\right). \tag{4.12}$$

The projectors Π^{A^n}, Π^{B^n}, Π^{C^n}, $\Pi^{A^nB^n}$, $\Pi^{A^nC^n}$, $\Pi^{B^nC^n}$ and $\Pi^{A^nB^nC^n}$ are δ-typical projectors for the state $\rho^{A^nB^nC^n}$ onto the specified systems after tracing out all other systems.

The average error probability when Alice and Bob choose their messages independently and uniformly at random is

$$\bar{p}_e \equiv \frac{1}{|\mathcal{L}|\cdot|\mathcal{M}|}\sum_{l,m}\text{Tr}\left\{(I - \Lambda_{l,m})\,\sigma_{l,m}\right\}. \tag{4.13}$$

We can upper bound this error probability from above[1] as

$$\bar{p}_e \leq \frac{1}{|\mathcal{L}| \cdot |\mathcal{M}|} \sum_{l,m} \text{Tr}\left\{(I - \Lambda_{l,m}) U^T(s_2(m)) \hat{\Pi}_1 U^*(s_2(m)) \sigma_{l,m} U^T(s_2(m)) \hat{\Pi}_1 U^*(s_2(m))\right\} + \left\| U^T(s_2(m)) \hat{\Pi}_1 U^*(s_2(m)) \sigma_{l,m} U^T(s_2(m)) \hat{\Pi}_1 U^*(s_2(m)) - \sigma_{l,m} \right\|_1 \tag{4.14}$$

$$\leq \frac{1}{|\mathcal{L}| \cdot |\mathcal{M}|} \sum_{l,m} \text{Tr}\{(I - \Lambda_{l,m}) \theta_{l,m}\} + 2\sqrt{\epsilon'}, \tag{4.15}$$

where we define

$$\theta_{l,m} \equiv U^T(s_2(m))^{B^n} \hat{\Pi}_1 U^*(s_2(m))^{B^n} \sigma_{l,m} U^T(s_2(m))^{B^n} \hat{\Pi}_1 U^*(s_2(m))^{B^n} \tag{4.16}$$

$$= U^T(s_2(m))^{B^n} \hat{\Pi}_1 U^T(s_1(l))^{A^n} \rho^{A^n B^n C^n} U^*(s_1(l))^{A^n} \hat{\Pi}_1 U^*(s_2(m))^{B^n}. \tag{4.17}$$

The first inequality follows from the inequality

$$\text{Tr}\{\Gamma\rho\} \leq \text{Tr}\{\Gamma\sigma\} + \|\rho - \sigma\|_1, \tag{4.18}$$

for any operators $0 \leq \Gamma, \rho, \sigma \leq I$ (Corollary 9.1.1 of Ref. [37]). The second inequality follows from the properties of quantum typicality, the Gentle Operator Lemma (Lemma 9.4.2 of Ref. [37]), and the inequality $\text{Tr}\{\hat{\Pi}_1 U^T(s_1(l))^{A^n} \rho^{A^n B^n C^n} U^*(s_1(l))^{A^n}\} \geq 1 - \epsilon'$ proved in Ref. [28]. We now recall the Hayashi-Nagaoka operator inequality [24] which holds for any positive operator S and T such that $0 \leq S \leq I$ and $T \geq 0$:

$$I - (S+T)^{-\frac{1}{2}} S (S+T)^{-\frac{1}{2}} \leq 2(I - S) + 4T. \tag{4.19}$$

Setting

$$S = \Upsilon_{l,m}, \tag{4.20}$$

$$T = \sum_{(l',m') \neq (l,m)} \Upsilon_{l',m'}, \tag{4.21}$$

and applying the Hayashi-Nagaoka operator inequality, we obtain the following upper bound on the error probability:

$$\bar{p}_e \leq \frac{1}{|\mathcal{L}| \cdot |\mathcal{M}|} \sum_{l,m} \left(2\text{Tr}\{(I - \Upsilon_{l,m}) \theta_{l,m}\} + 4 \sum_{(l',m') \neq (l,m)} \text{Tr}\{\Upsilon_{l',m'} \theta_{l,m}\} \right) + 2\sqrt{\epsilon}. \tag{4.22}$$

Considering the first term $\text{Tr}\{(I - \Upsilon_{l,m}) \theta_{l,m}\}$, we can prove that

$$\text{Tr}\{(I - \Upsilon_{l,m}) \theta_{l,m}\} \leq \epsilon'', \tag{4.23}$$

where ϵ'' approaches zero when n becomes large. This inequality follows from the following inequalities

$$\text{Tr}\left\{\hat{\Pi}_1 U^T(s_1(l))^{A^n} \rho^{A^n B^n C^n} U^*(s_1(l))^{A^n}\right\} \geq 1 - 2\epsilon, \tag{4.24}$$

$$\text{Tr}\left\{\hat{\Pi}_3 U^T(s_2(m))^{B^n} \rho^{A^n B^n C^n} U^*(s_2(m))^{B^n}\right\} \geq 1 - 2\epsilon, \tag{4.25}$$

$$\text{Tr}\left\{\hat{\Pi}_2 U^T(s_2(m))^{B^n} \rho^{A^n B^n C^n} U^*(s_2(m))^{B^n}\right\} \geq 1 - 2\epsilon, \tag{4.26}$$

$$\text{Tr}\left\{\Pi^{A^n B^n C^n} \rho^{A^n B^n C^n}\right\} \geq 1 - \epsilon, \tag{4.27}$$

[1]We are indebted to Pranab Sen for this observation [33] (c.f., versions 1 and 2 of this paper).

(which can be proved with the methods of Ref. [28]) and by applying "measurement on approximately close states" (Corollary 9.1.1 of Ref. [37]) and the Gentle Operator Lemma (Lemma 9.4.2 of Ref. [37]) several times.

In order to analyze the second term $\sum_{(l',m')\neq(l,m)} \text{Tr}\left\{\Upsilon_{l',m'}\theta_{l,m}\right\}$, we need to take the expectation over all random codes and make several observations about the behavior of the codeword states under the expectation. Note that the encoding unitaries after the transpose trick and the channel commute because they act on different systems, so we can apply the encoding unitaries first. To simplify the calculation, we first consider only applying a random encoding unitary to the system A^n:

$$\begin{aligned}
&\mathbb{E}_{\mathcal{C}_1}\left\{U^T(s)^{A^n}|\phi\rangle\langle\phi|^{A'^nA^n}U^*(s)^{A^n}\right\} \\
&=\frac{1}{|\mathcal{S}_1|}\sum_{s\in\mathcal{S}_1}U^T(s)^{A^n}\left(\sum_t\sqrt{p(t)}\,|\Phi_t\rangle^{A'^nA^n}\right)\left(\sum_{t'}\sqrt{p(t')}\,\langle\Phi_{t'}|^{A'^nA^n}\right)U^*(s)^{A^n} \qquad (4.28)\\
&=\sum_t p(t)\,\pi_t^{A'^n}\otimes\pi_t^{A^n} \qquad (4.29)
\end{aligned}$$

where π_t is the maximally mixed state on the type subspace t. To see why the last equality holds, we note that when $t=t'$, averaging over all elements in $\mathcal{S}_1$ gives rise to the state $\text{Tr}_{A^n}\left\{|\Phi_t\rangle\langle\Phi_t|^{A'^nA^n}\right\}\otimes\pi_t^{A^n}=\pi_t^{A'^n}\otimes\pi_t^{A^n}$; when $t\neq t'$, it can be shown that the whole expression sums up to zero [28, 37]. Now we can append the other state at Bob's side and send the overall state through the channel. Therefore, we have that

$$\begin{aligned}
\mathbb{E}_{\mathcal{C}_1}\left\{U^T(s)^{A^n}\rho^{A^nB^nC^n}U^*(s)^{A^n}\right\} &= \mathcal{N}^{A'^nB'^n\to C^n}\left(\sum_t p(t)\,\pi_t^{A'^n}\otimes\pi_t^{A^n}\otimes\psi^{B'^nB^n}\right) \qquad (4.30)\\
&=\sum_t p(t)\,\pi_t^{A^n}\otimes\mathcal{N}^{A'^nB'^n\to C^n}\left(\pi_t^{A^n}\otimes\psi^{B'^nB^n}\right). \qquad (4.31)
\end{aligned}$$

Now consider the above state sandwiched between the projectors $\hat{\Pi}_1$:

$$\begin{aligned}
&\hat{\Pi}_1\mathbb{E}_{\mathcal{C}_1}\left\{U^T(s)^{A^n}\rho^{A^nB^nC^n}U^*(s)^{A^n}\right\}\hat{\Pi}_1 \\
&=\left(\Pi^{A^n}\otimes\Pi^{B^nC^n}\right)\left(\sum_t p(t)\,\pi_t^{A^n}\otimes\mathcal{N}^{A'^nB'^n\to C^n}\left(\pi_t^{A^n}\otimes\psi^{B'^nB^n}\right)\right)\left(\Pi^{A^n}\otimes\Pi^{B^nC^n}\right) \qquad (4.32)\\
&=\sum_t p(t)\left(\Pi^{A^n}\pi_t^{A^n}\Pi^{A^n}\right)\otimes\left(\Pi^{B^nC^n}\mathcal{N}^{A'^nB'^n\to C^n}\left(\pi_t^{A^n}\otimes\psi^{B'^nB^n}\right)\Pi^{B^nC^n}\right) \qquad (4.33)
\end{aligned}$$

At this point, we note that $\pi_t^{A^n}=\Pi_t^{A^n}/\text{Tr}\left\{\Pi_t^{A^n}\right\}$, $\text{Tr}\left\{\Pi_t^{A^n}\right\}\geq 2^{n(H(A)-\eta(n,\delta))}$ for a typical type t, and $\Pi^{A^n}\Pi_t^{A^n}\Pi^{A^n}\leq\Pi^{A^n}$, where $\Pi_t^{A^n}$ is a projector onto the t^{th} type class subspace. Therefore, the above expression is bounded from above by the following one:

$$\begin{aligned}
&\leq 2^{-n\left(H(A)_\rho-\eta(n,\delta)\right)}\Pi^{A^n}\otimes\left(\Pi^{B^nC^n}\mathcal{N}^{A'^nB'^n\to C^n}\left(\sum_t p(t)\left(\pi_t^{A^n}\right)\otimes\psi^{B'^nB^n}\right)\Pi^{B^nC^n}\right) \\
&=2^{-n\left(H(A)_\rho-\eta(n,\delta)\right)}\Pi^{A^n}\otimes\left(\Pi^{B^nC^n}\mathcal{N}^{A'^nB'^n\to C^n}\left(\phi^{A'^n}\otimes\psi^{B'^nB^n}\right)\Pi^{B^nC^n}\right) \qquad (4.34)\\
&\leq 2^{-n\left(H(A)_\rho+H(BC)_\rho-\eta(n,\delta)-c\delta\right)}\hat{\Pi}_1. \qquad (4.35)
\end{aligned}$$

We also note that similar observations can be made when applying random encoding unitaries to the system B'^n alone or to both the systems A'^n and B'^n.

Now we proceed to bound the second term in the RHS of (4.22) from above by taking the expectation over the random codes $\mathcal{C}_1$ and $\mathcal{C}_2$:

$$\mathbb{E}_{\mathcal{C}_1,\mathcal{C}_2}\left\{\sum_{(l',m')\neq(l,m)}\operatorname{Tr}\left\{\Upsilon_{l',m'}\theta_{l,m}\right\}\right\} = \mathbb{E}_{\mathcal{C}_1,\mathcal{C}_2}\left\{\sum_{l'\neq l}\operatorname{Tr}\left\{\Upsilon_{l',m}\theta_{l,m}\right\} + \sum_{m'\neq m}\operatorname{Tr}\left\{\Upsilon_{l,m'}\theta_{l,m}\right\} + \sum_{l'\neq l,\ m'\neq m}\operatorname{Tr}\left\{\Upsilon_{l',m'}\theta_{l,m}\right\}\right\}.$$

We bound the first error on the RHS above, which corresponds to Charlie correctly identifying the message from Bob only:

$$\mathbb{E}_{\mathcal{C}_1,\mathcal{C}_2}\left\{\sum_{l'\neq l}\operatorname{Tr}\left\{\Upsilon_{l',m}\theta_{l,m}\right\}\right\}$$

$$= \sum_{l'\neq l}\mathbb{E}_{\mathcal{C}_2}\left\{\operatorname{Tr}\left\{\mathbb{E}_{\mathcal{C}_1}\left\{\Upsilon_{l',m}\right\}\mathbb{E}_{\mathcal{C}_1}\left\{\theta_{l,m}\right\}\right\}\right\} \tag{4.36}$$

$$= \sum_{l'\neq l}\mathbb{E}_{\mathcal{C}_2}\left\{\operatorname{Tr}\left\{\mathbb{E}_{\mathcal{C}_1}\left\{\Upsilon_{l',m}\right\}U^T\left(s_2\left(m\right)\right)^{B^n}\hat{\Pi}_1\mathbb{E}_{\mathcal{C}_1}\left\{U^T\left(s_1\left(l\right)\right)^{A^n}\rho^{A^nB^nC^n}U^*\left(s_1\left(l\right)\right)^{A^n}\right\}\hat{\Pi}_1U^*\left(s_2\left(m\right)\right)^{B^n}\right\}\right\} \tag{4.37}$$

$$\leq 2^{-n\left(H(A)_\rho+H(BC)_\rho-\eta(n,\delta)-c\delta\right)}\sum_{l'\neq l}\mathbb{E}_{\mathcal{C}_2}\left\{\operatorname{Tr}\left\{\mathbb{E}_{\mathcal{C}_1}\left\{\Upsilon_{l',m}\right\}U^T\left(s_2\left(m\right)\right)^{B^n}\hat{\Pi}_1U^*\left(s_2\left(m\right)\right)^{B^n}\right\}\right\} \tag{4.38}$$

$$\leq 2^{-n\left(H(A)_\rho+H(BC)_\rho-\eta(n,\delta)-c\delta\right)}\sum_{l'\neq l}\mathbb{E}_{\mathcal{C}_1,\mathcal{C}_2}\left\{\operatorname{Tr}\left\{\Pi^{A^nB^nC^n}\right\}\right\} \tag{4.39}$$

$$\leq 2^{-n\left(H(A)_\rho+H(BC)_\rho-H(ABC)_\rho-\eta(n,\delta)-2c\delta\right)}\left|\mathcal{L}\right| \tag{4.40}$$

$$= 2^{-n(I(A;C|B))_\rho-\eta(n,\delta)-2c\delta}\left|\mathcal{L}\right|. \tag{4.41}$$

The first equality follows from the fact that the codewords for messages l and l' are different and therefore independent (because of the way that we randomly selected the code). The first inequality follows from our observation in (4.35).

We now bound the second error term, which corresponds to Charlie correctly identifying the message from Alice only:

$$\mathbb{E}_{\mathcal{C}_1,\mathcal{C}_2}\left\{\sum_{m'\neq m}\operatorname{Tr}\left\{\Upsilon_{l,m'}\theta_{l,m}\right\}\right\}$$

$$= \sum_{m'\neq m}\mathbb{E}_{\mathcal{C}_1}\left\{\operatorname{Tr}\left\{\mathbb{E}_{\mathcal{C}_2}\left\{\Upsilon_{l,m'}\right\}\mathbb{E}_{\mathcal{C}_2}\left\{\theta_{l,m}\right\}\right\}\right\} \tag{4.42}$$

$$= \sum_{m'\neq m}\mathbb{E}_{\mathcal{C}_1}\left\{\operatorname{Tr}\left\{U^T\left(s_1\left(l\right)\right)^{A^n}\hat{\Pi}_3\hat{\Pi}_2\mathbb{E}_{\mathcal{C}_2}\left\{U^T\left(s_2\left(m'\right)\right)^{B^n}\Pi_{A^nB^nC^n}U^*\left(s_2\left(m'\right)\right)^{B^n}\right\}\hat{\Pi}_2\hat{\Pi}_3U^*\left(s_1\left(l\right)\right)^{A^n}\mathbb{E}_{\mathcal{C}_2}\left\{\theta_{l,m}\right\}\right\}\right\} \tag{4.43}$$

$$\leq 2^{n\left(H(ABC)_\rho+c\delta\right)}\sum_{m'\neq m}\mathbb{E}_{\mathcal{C}_1}\left\{\operatorname{Tr}\left\{\begin{array}{c}U^T\left(s_1\left(l\right)\right)^{A^n}\hat{\Pi}_3\hat{\Pi}_2\mathbb{E}_{\mathcal{C}_2}\left\{U^T\left(s_2\left(m'\right)\right)^{B^n}\rho_{A^nB^nC^n}U^*\left(s_2\left(m'\right)\right)^{B^n}\right\}\cdot\\ \hat{\Pi}_2\hat{\Pi}_3U^*\left(s_1\left(l\right)\right)^{A^n}\mathbb{E}_{\mathcal{C}_2}\left\{\theta_{l,m}\right\}\end{array}\right\}\right\} \tag{4.44}$$

$$\leq 2^{-n\left(H(B)_\rho+H(AC)_\rho-H(ABC)_\rho-\eta(n,\delta)-2c\delta\right)}\sum_{m'\neq m}\mathbb{E}_{\mathcal{C}_1}\left\{\operatorname{Tr}\left\{U^T\left(s_1\left(l\right)\right)^{A^n}\hat{\Pi}_3\hat{\Pi}_2\hat{\Pi}_3U^*\left(s_1\left(l\right)\right)^{A^n}\mathbb{E}_{\mathcal{C}_2}\left\{\theta_{l,m}\right\}\right\}\right\} \tag{4.45}$$

$$\leq 2^{-n\left(H(B)_\rho+H(AC)_\rho-H(ABC)_\rho-\eta(n,\delta)-2c\delta\right)}\sum_{m'\neq m}\operatorname{Tr}\left\{\mathbb{E}_{\mathcal{C}_1,\mathcal{C}_2}\left\{\theta_{l,m}\right\}\right\} \tag{4.46}$$

$$= 2^{-n\left(I(B;C|A)_\rho-\eta(n,\delta)-2c\delta\right)}\left|\mathcal{M}\right|. \tag{4.47}$$

The first equality follows because the codewords for messages m and m' are different and thus independent. The first inequality follows from

$$\Pi^{A^nB^nC^n}\leq 2^{n[H(ABC)+c\delta]}\Pi^{A^nB^nC^n}\rho^{A^nB^nC^n}\Pi^{A^nB^nC^n}\leq 2^{n[H(ABC)+c\delta]}\rho^{A^nB^nC^n},$$

and we applied a similar observation as in (4.35) to obtain the second inequality.

We now bound the third error term:

$$\mathbb{E}_{\mathcal{C}_1,\mathcal{C}_2}\left\{\sum_{l'\neq l,\ m'\neq m}\operatorname{Tr}\left\{\Upsilon_{l',m'}\theta_{l,m}\right\}\right\}$$

$$= \sum_{l'\neq l,\ m'\neq m}\operatorname{Tr}\left\{\mathbb{E}_{\mathcal{C}_1,\mathcal{C}_2}\left\{\Upsilon_{l',m'}\right\}\mathbb{E}_{\mathcal{C}_1,\mathcal{C}_2}\left\{\theta_{l,m}\right\}\right\} \tag{4.48}$$

$$= \sum_{\substack{l'\neq l\\ m'\neq m}}\operatorname{Tr}\left\{\mathbb{E}_{\mathcal{C}_1,\mathcal{C}_2}\left\{\Upsilon_{l',m'}\right\}\mathbb{E}_{\mathcal{C}_2}\left\{U^T\left(s_2\left(m\right)\right)^{B^n}\hat{\Pi}_1\mathbb{E}_{\mathcal{C}_1}\left\{U^T\left(s_1\left(l\right)\right)^{A^n}\rho^{A^nB^nC^n}U^*\left(s_1\left(l\right)\right)^{A^n}\right\}\hat{\Pi}_1U^*\left(s_2\left(m\right)\right)^{B^n}\right\}\right\} \tag{4.49}$$

Consider the following operator inequalities:

$$\begin{aligned}&\hat{\Pi}_1\mathbb{E}_{\mathcal{C}_1}\left\{U^T\left(s_1\left(l\right)\right)^{A^n}\rho^{A^nB^nC^n}U^*\left(s_1\left(l\right)\right)^{A^n}\right\}\hat{\Pi}_1\\ &\qquad\leq 2^{-n\left(H(A)_\rho-\eta(n,\delta)\right)}\Pi^{A^n}\otimes\Pi^{B^nC^n}\mathcal{N}\left(\phi^{A'^n}\otimes\psi^{B'^nB^n}\right)\Pi^{B^nC^n}\\ &\qquad\leq 2^{-n\left(H(A)_\rho-\eta(n,\delta)\right)}\Pi^{A^n}\otimes\mathcal{N}\left(\phi^{A'^n}\otimes\psi^{B'^nB^n}\right)\end{aligned} \tag{4.50}$$

The second inequality follows from the fact that the typical projector commutes with the state ρ and therefore $\Pi\rho\Pi=\sqrt{\rho}\Pi\sqrt{\rho}\leq\rho$. Thus the quantity in (4.49) is upper bounded by the following one:

$$\leq 2^{-n\left(H(A)_\rho-\eta(n,\delta)\right)}\sum_{l'\neq l,\ m'\neq m}\operatorname{Tr}\left\{\mathbb{E}_{\mathcal{C}_1,\mathcal{C}_2}\left\{\Upsilon_{l',m'}\right\}\Pi^{A^n}\otimes\mathbb{E}_{\mathcal{C}_2}\left\{U^T\left(s_2\left(m\right)\right)^{B^n}\mathcal{N}\left(\phi^{A'^n}\otimes\psi^{B'^nB^n}\right)U^*\left(s_2\left(m\right)\right)^{B^n}\right\}\right\} \tag{4.51}$$

$$= 2^{-n\left(H(A)_\rho-\eta(n,\delta)\right)}\sum_{l'\neq l,\ m'\neq m}\operatorname{Tr}\left\{\mathbb{E}_{\mathcal{C}_1,\mathcal{C}_2}\left\{\Upsilon_{l',m'}\right\}\Pi^{A^n}\otimes\left(\sum_t\pi_t^{B^n}\otimes\mathcal{N}\left(\phi^{A'^n}\otimes\pi_t^{B'^n}\right)\right)\right\} \tag{4.52}$$

$$= 2^{-n\left(H(A)_\rho-\eta(n,\delta)\right)}\sum_{\substack{l'\neq l\\ m'\neq m}}\operatorname{Tr}\left\{\mathbb{E}_{\mathcal{C}_1,\mathcal{C}_2}\left\{\Upsilon_{l',m'}\right\}\hat{\Pi}_3\left(\Pi^{A^n}\otimes\left(\sum_t\pi_t^{B^n}\otimes\mathcal{N}\left(\phi^{A'^n}\otimes\pi_t^{B'^n}\right)\right)\right)\hat{\Pi}_3\right\} \tag{4.53}$$

$$\leq 2^{-n\left(H(A)_\rho + H(B)_\rho - 2\eta(n,\delta)\right)} \sum_{\substack{l' \neq l \\ m' \neq m}} \text{Tr}\left\{ \mathbb{E}_{\mathcal{C}_1,\mathcal{C}_2}\left\{\Upsilon_{l',m'}\right\} \Pi^{A^n} \otimes \Pi^{B^n} \otimes \Pi^{C^n} \mathcal{N}\left(\phi^{A'^n} \otimes \sum_t \pi_t^{B'^n}\right) \Pi^{C^n} \right\} \tag{4.54}$$

$$\leq 2^{-n\left(H(A)_\rho + H(B)_\rho + H(C)_\rho - 2\eta(n,\delta) - c\delta\right)} \sum_{l' \neq l,\ m' \neq m} \text{Tr}\left\{\mathbb{E}_{\mathcal{C}_1,\mathcal{C}_2}\left\{\Upsilon_{l',m'}\right\}\right\} \tag{4.55}$$

$$\leq 2^{-n\left(H(A)_\rho + H(B)_\rho + H(C)_\rho - 2\eta(n,\delta) - c\delta\right)} \sum_{l' \neq l,\ m' \neq m} \text{Tr}\left\{\Pi^{A^n B^n C^n}\right\} \tag{4.56}$$

$$\leq 2^{-n\left(H(A)_\rho + H(B)_\rho + H(C)_\rho - H(ABC)_\rho - 2\eta(n,\delta) - 2c\delta\right)} |\mathcal{L}| \cdot |\mathcal{M}| \tag{4.57}$$

$$= 2^{-n\left(I(AB;C)_\rho - 2\eta(n,\delta) - 2c\delta\right)} |\mathcal{L}| \cdot |\mathcal{M}| \tag{4.58}$$

Thus, as long as we choose the message set sizes such that the corresponding rates obey the inequalities in the statement of the theorem, then this ensures the existence of a code with vanishing average error probability in the asymptotic limit of large blocklength n. ■

4.1 From Average to Maximal Error

The above scheme for entanglement-assisted classical communication satisfies an average error criterion (as specified in (4.13)), but we would like it to satisfy a stronger maximal error criterion, where we can guarantee that every message pair has a low error probability. In the single-sender single-receiver case, the standard argument is just to invoke Markov's inequality to demonstrate that throwing away half of the codewords ensures that the error for all codewords is less than 2ϵ if the original average error probability is less than ϵ [7]. This expurgation then only has a negligible impact on the rate of the code. We cannot employ such an argument for the multiple access case because the expurgation does not guarantee that the resulting expurgated codebook of message pairs decomposes as a product of two expurgated codebooks. Thus, the argument for average-to-maximal error needs to be a bit more clever.

Yard *et al.* introduced a straightforward scheme for constructing a code with low maximal error from one with low average error [40], based on some ideas in Ref. [8] and some further ideas of their own. The first idea from Ref. [8] is to suppose that the senders and receiver have access to uniform common randomness. That is, Alice and Charlie share some common randomness and so do Bob and Charlie. Let S denote the Alice-Charlie common randomness and let T denote the Bob-Charlie common randomness. Based on this common randomness, Alice and Bob each compute $l + S$ and $m + T$, where l is Alice's message and m is Bob's message and the addition is understood to be modulo the size of the respective message sets. Alice and Bob then encode according to $l + S$ and $m + T$ and Charlie decodes these messages. Using his share of the common randomness, he subtracts off S and T to obtain the intended messages l and m. Now, the expected error probability for when Alice and Bob transmit the message pair (l, m), where the expectation is with respect to the common randomness, becomes as follows:

$$\mathbb{E}_{S,T}\left\{\text{Tr}\left\{\left(I - \Lambda_{l+S,m+T}\right)\sigma_{l+S,m+T}\right\}\right\} = \frac{1}{LM}\sum_{s,t}\text{Tr}\left\{\left(I - \Lambda_{l+s,m+t}\right)\sigma_{l+s,m+T}\right\} \tag{4.59}$$

$$= \frac{1}{LM}\sum_{l,m}\text{Tr}\left\{\left(I - \Lambda_{l,m}\right)\sigma_{l,m}\right\}. \tag{4.60}$$

Thus, it becomes clear that the maximal error criterion for each message pair (l, m) is equivalent to the average error criterion if the senders and receiver have access to common randomness.

Yard *et al.* then take this argument further to show that preshared common randomness is not actually necessary. The main idea is to divide the overall number of channel uses into $N + 1$ blocks each of length n. For the first round, Alice and Bob use the channel n times to establish common randomness of respective sizes 2^{nR_1} and 2^{nR_2} with Charlie. Since the common randomness is uniformly distributed and our protocol works well for the uniform distribution, this round fails with probability no larger than ϵ. Alice, Bob, and Charlie then use this established common randomness and the randomized protocol given above for the next N rounds (the key point is that they can use the same common randomness from the first round for all of the subsequent N rounds). By choosing $N = 1/\sqrt{\epsilon}$, the first round establishes common randomness at the negligible rates

$$\frac{1}{nN}\log 2^{nR_i} = \sqrt{\epsilon}R_i, \tag{4.61}$$

while ensuring that the subsequent rounds have an error probability no larger than $N\epsilon = \sqrt{\epsilon}$. Now, the actual distribution resulting from the first round is ϵ-close to perfect common randomness, but this only results in an error probability of $2\sqrt{\epsilon}$ for the N-blocked protocol. The resulting achievable rates for classical communication become $(1-\sqrt{\epsilon})\,R_i$ for $i \in \{1,2\}$. Thus, this blocked scheme shows how to convert a protocol with low average error probability to one with low maximal error probability.

4.2 Unassisted Simultaneous Decoding

A simple corollary of Theorem 4 is a simultaneous decoder for unassisted classical communication. The proof is virtually identical to the above proof, but it takes advantage of the proof technique in Section III-B of Ref. [28] (thus we omit the details of the proof). The below result implies a complete solution of the "strong interference" case for transmitting classical data over a quantum interference channel [13], if the encoders are restricted to product-state inputs. Sen independently obtained a proof of the below corollary with a different technique [34].

Corollary 5 (Unassisted Simultaneous Decoding) *Let $\mathcal{N}^{A'B'\to C}$ be a multiple access channel that connects Alice and Bob to Charlie, and let*

$$\rho^{XYC} \equiv \sum_{x,y} p_X(x)\, p_Y(y)\, |x\rangle\langle x|^X \otimes |y\rangle\langle y|^Y \otimes \mathcal{N}^{A'B'\to C}\left(\rho_x^{A'} \otimes \sigma_y^{B'}\right). \tag{4.62}$$

Then there exists a classical communication code with a corresponding quantum simultaneous decoder, such that the following rate region is achievable for $R_1, R_2 \geq 0$:

$$R_1 \leq I(X;C|Y)_\rho, \tag{4.63}$$

$$R_2 \leq I(Y;C|X)_\rho, \tag{4.64}$$

$$R_1 + R_2 \leq I(XY;C)_\rho, \tag{4.65}$$

where the entropies are with respect to the state in (4.62).

5 Quantum communication over a quantum multiple access channel

In this section, we recover the previously known achievable rate regions for assisted and unassisted quantum communication over a multiple access channel [28, 27, 40]. We do so by employing a coherent version of the protocol from Section 4 (many researchers have often employed this approach in quantum Shannon theory [23, 10, 28, 37]). Different from prior work, we show that this region can be achieved without the need for time sharing—the simultaneous nature of our decoding scheme guarantees this. Our scheme below achieves quantum communication by employing the blocked protocol from Section 4.1 (this is again related to the average versus maximal error issue).

We first recall the resource inequality formalism of Devetak *et al.* [10]. We denote one noiseless classical bit channel from Alice to Bob as $[c \to c]_{AB}$, one noiseless qubit channel from Alice to Bob as $[q \to q]_{AB}$, and one ebit of entanglement shared between Alice and Bob as $[qq]_{AB}$. We will also be using a coherent channel from Alice to Bob, which is defined to implement the map [23]:

$$|i\rangle^{A} \rightarrow |i\rangle^{A} |i\rangle^{B} . \tag{5.1}$$

We denote this communication resource as $[q \rightarrow qq]_{AB}$. Note that a coherent channel is a stronger resource than a classical channel because it can simulate a classical channel if Alice only sends computational basis states through it. Furthermore, let $\langle \mathcal{N} \rangle$ denote one use of a multiple access channel $\mathcal{N}^{A'B' \rightarrow C}$. Resource inequalities describe ways of consuming some communication resources in order to create others. For example, the protocol described in Section 4 implements the following resource inequality:

$$\langle \mathcal{N} \rangle + H(A)_{\rho} [qq]_{AC} + H(B)_{\rho} [qq]_{BC} \geq R_1 [c \rightarrow c]_{AC} + R_2 [c \rightarrow c]_{BC} . \tag{5.2}$$

We upgrade our protocol for entanglement-assisted classical communication from the previous section to one for entanglement-assisted coherent communication with two senders and one receiver.

Theorem 6 *The following resource inequality corresponds to an achievable coherent simultaneous decoding protocol for entanglement-assisted coherent communication over a noisy multiple access quantum channel* $\mathcal{N}$*:*

$$\langle \mathcal{N} \rangle + H(A)_{\rho} [qq]_{AC} + H(B)_{\rho} [qq]_{BC} \geq R_1 [q \rightarrow qq]_{AC} + R_2 [q \rightarrow qq]_{BC} , \tag{5.3}$$

where

$$\rho^{ABC} \equiv \mathcal{N}^{A'B' \rightarrow C} \left(\phi^{A'A} \otimes \psi^{B'B} \right) , \tag{5.4}$$

as long as

$$R_1 \leq I(A;C|B)_{\rho} , \tag{5.5}$$

$$R_2 \leq I(B;C|A)_{\rho} , \tag{5.6}$$

$$R_1 + R_2 \leq I(AB;C)_{\rho} . \tag{5.7}$$

The entropies are with respect to the state in (5.4).

Proof. We again exploit the blocked protocol from Section 4.1. We assume that Alice, Bob, and Charlie have already established their common randomness, and we describe how the protocol operates for the first of the N rounds (the round after the one that establishes common randomness). Let S denote the Alice-Charlie common randomness, and let T denote the Bob-Charlie common randomness.

Suppose that Alice shares a state with a reference system R_A:

$$\sum_{j,l=1}^{L} \alpha_{j,l} |j\rangle^{R_A} |l\rangle^{A_1} , \tag{5.8}$$

where $\{|j\rangle\}$ and $\{|l\rangle\}$ are some orthonormal bases for R_A and A_1 respectively. Similarly, Bob also shares a state with a reference system R_B:

$$\sum_{k,m=1}^{M} \beta_{k,m} |k\rangle^{R_B} |m\rangle^{B_1} , \tag{5.9}$$

where $\{|k\rangle\}$ and $\{|m\rangle\}$ are some orthonormal bases for R_B and B_1 respectively. The parameters L and M for these states are chosen such that

$$R_1 \approx \frac{1}{n} \log L \leq I(A;C|B)_{\rho} , \tag{5.10}$$

$$R_2 \approx \frac{1}{n} \log M \leq I(B;C|A)_{\rho} , \tag{5.11}$$

$$R_1 + R_2 \approx \frac{1}{n} \log (LM) \leq I(AB;C)_{\rho} . \tag{5.12}$$

Alice would like to simulate the action of a coherent channel on her system A_1 to a system A_2 for Charlie:

$$\sum_{j,l} \alpha_{j,l} \left|j\right\rangle^{R_A} \left|l\right\rangle^{A_1} \to \sum_{j,l} \alpha_{j,l} \left|j\right\rangle^{R_A} \left|l\right\rangle^{A_1} \left|l\right\rangle^{A_2}, \tag{5.13}$$

and Bob would like to do the same. We demand that they simulate these resources with vanishing error in the limit of many channel uses. As before, Alice and Charlie share many copies of a pure entangled state $\left|\phi\right\rangle^{AA'}$, Bob and Charlie share many copies of $\left|\psi\right\rangle^{BB'}$, and they all have access to many uses of a noisy multiple access channel $\mathcal{N}^{A'B'\to C}$.

They have their encoding unitaries $\{U\left(s_1\left(l\right)\right)^{A'^n}\}_l$ and $\{U\left(s_2\left(m\right)\right)^{B'^n}\}_m$ as described in Section 4, and they employ them now as the following controlled unitaries that act on the systems A_1 and B_1 and their shares of the entanglement:

$$\sum_{l} \left|l\right\rangle \left\langle l\right|^{A_1} \otimes U\left(s_1\left(l\right)\right)^{A'^n}, \tag{5.14}$$

$$\sum_{m} \left|m\right\rangle \left\langle m\right|^{B_1} \otimes U\left(s_2\left(m\right)\right)^{B'^n}. \tag{5.15}$$

The resulting global state after applying these unitaries and the transpose trick is as follows:

$$\left(\sum_{j,l} \alpha_{j,l} \left|j\right\rangle^{R_A} \left|l\right\rangle^{A_1} U^T\left(s_1\left(l\right)\right)^{A^n} \left|\phi\right\rangle^{A^n A'^n}\right) \otimes \left(\sum_{k,m} \beta_{k,m} \left|k\right\rangle^{R_B} \left|m\right\rangle^{B_1} U^T\left(s_2\left(m\right)\right)^{B^n} \left|\psi\right\rangle^{B^n B'^n}\right). \tag{5.16}$$

Alice and Bob both then apply the respective unitaries $U\left(s_1\left(S\right)\right)^{A'^n}$ and $U\left(s_2\left(T\right)\right)^{B'^n}$, conditional on their common randomness shared with Charlie. The resulting state is

$$\left(\sum_{j,l} \alpha_{j,l} \left|j\right\rangle^{R_A} \left|l\right\rangle^{A_1} U^T\left(s_1\left(l+S\right)\right)^{A^n} \left|\phi\right\rangle^{A^n A'^n}\right) \otimes \left(\sum_{k,m} \beta_{k,m} \left|k\right\rangle^{R_B} \left|m\right\rangle^{B_1} U^T\left(s_2\left(m+T\right)\right)^{B^n} \left|\psi\right\rangle^{B^n B'^n}\right), \tag{5.17}$$

where the addition $l+S$ and $m+T$ is modulo L and M, respectively. They both then send their shares of the states over the multiple access channel $\mathcal{N}^{A'B'\to C}$, whose isometric extension is $U_{\mathcal{N}}^{A'B'\to CE}$ and acts on $\left|\phi\right\rangle^{A^n A'^n} \otimes \left|\psi\right\rangle^{B^n B'^n}$ as follows:

$$\left|\varphi\right\rangle^{A^n B^n C^n E^n} \equiv U_{\mathcal{N}}^{A'^n B'^n \to C^n E^n} \left(\left|\phi\right\rangle^{A^n A'^n} \otimes \left|\psi\right\rangle^{B^n B'^n}\right). \tag{5.18}$$

After the transmission, the overall state becomes

$$\sum_{j,k,l,m} \alpha_{j,l}\beta_{k,m} \left|j\right\rangle^{R_A} \left|l\right\rangle^{A_1} \left|k\right\rangle^{R_B} \left|m\right\rangle^{B_1} \left(U^T\left(s_1\left(l+S\right)\right)^{A^n} \otimes U^T\left(s_2\left(m+T\right)\right)^{B^n}\right) \left|\varphi\right\rangle^{A^n B^n C^n E^n}. \tag{5.19}$$

Charlie performs the following coherent measurement constructed from the POVM $\{\Lambda_{p,q}\}$ of Section 4:

$$\Upsilon = \sum_{p,q} \left(\sqrt{\Lambda_{p,q}}\right)^{A^n B^n C^n} \otimes \left|p\right\rangle^{A_2} \otimes \left|q\right\rangle^{B_2}. \tag{5.20}$$

Given that the original POVM is good on average, in the sense that

$$\frac{1}{LM} \sum_{l,m} \operatorname{Tr}\left\{\Lambda_{l,m}\sigma_{l,m}\right\} \geq 1-\epsilon, \tag{5.21}$$

for all $\epsilon > 0$ and sufficiently large n (where each $\sigma_{l,m}$ is an entanglement-assisted quantum codeword as before), this coherent measurement also has little effect on the received state while coherently copying the basis states in registers A_1 and B_1. That is, the expected fidelity overlap between the states

$$\Upsilon^{A^n B^n C^n A_2 B_2} \left|\omega\right\rangle^F , \tag{5.22}$$

and

$$\sum_{j,k,l,m} \alpha_{j,l}\beta_{k,m} \left|j\right\rangle^{R_A} \left|l\right\rangle^{A_1} \left|k\right\rangle^{R_B} \left|m\right\rangle^{B_1} \left(U^T\left(s_1\left(l+S\right)\right)^{A^n} \otimes U^T\left(s_2\left(m+T\right)\right)^{B^n}\right) \left|\varphi\right\rangle^{A^n B^n C^n E^n} \left|l+S\right\rangle^{A_2} \left|m+T\right\rangle^{B_2} \tag{5.23}$$

is larger than $1-\epsilon$, where $\left|\omega\right\rangle$ denotes the state in (5.19), the system F denotes all the systems $A_1 B_1 A^n B^n C^n E^n$, and the expectation is with respect to the common randomness S and T. To see why this is true, consider the following chain of inequalities:

$$\begin{aligned}
&\frac{1}{LM}\sum_{s,t}\sum_{j,k,l,m} \alpha_{j,l}^{*}\beta_{k,m}^{*} \left\langle j\right|^{R_A} \left\langle l\right|^{A_1} \left\langle k\right|^{R_B} \left\langle m\right|^{B_1} \left\langle\varphi\right| \left(U^*\left(s_1\left(l+s\right)\right)^{A^n} \otimes U^*\left(s_2\left(m+t\right)\right)^{B^n}\right) \left\langle l+s\right|^{A_2} \left\langle m+t\right|^{B_2} \\
&\left(\sum_{p,q} \left(\sqrt{\Lambda_{p,q}}\right)^{A^n B^n C^n} \left|p\right\rangle^{A_2} \left|q\right\rangle^{B_2}\right) \\
&\sum_{j',k',l',m'} \alpha_{j',l'}\beta_{k',m'} \left|j'\right\rangle^{R_A} \left|l'\right\rangle^{A_1} \left|k'\right\rangle^{R_B} \left|m'\right\rangle^{B_1} \left(U^T\left(s_1\left(l'+s\right)\right)^{A^n} \otimes U^T\left(s_2\left(m'+t\right)\right)^{B^n}\right) \left|\varphi\right\rangle \\
&= \frac{1}{LM}\sum_{s,t}\sum_{j,k,l,m} \left|\alpha_{j,l}\right|^2 \left|\beta_{k,m}\right|^2 \left\langle\varphi\right| \left(U^*\left(s_1\left(l+s\right)\right)^{A^n} \otimes U^*\left(s_2\left(m+t\right)\right)^{B^n}\right) \sqrt{\Lambda_{l+s,m+t}} \times \\
&\qquad \left(U^T\left(s_1\left(l+s\right)\right)^{A^n} \otimes U^T\left(s_2\left(m+t\right)\right)^{B^n}\right) \left|\varphi\right\rangle && (5.24)\\
&= \sum_{j,k,l,m} \left|\alpha_{j,l}\right|^2 \left|\beta_{k,m}\right|^2 \frac{1}{LM}\sum_{s,t} \left\langle\varphi\right| \left(U^*\left(s_1\left(l+s\right)\right)^{A^n} \otimes U^*\left(s_2\left(m+t\right)\right)^{B^n}\right) \sqrt{\Lambda_{l+s,m+t}} \times \\
&\qquad \left(U^T\left(s_1\left(l+s\right)\right)^{A^n} \otimes U^T\left(s_2\left(m+t\right)\right)^{B^n}\right) \left|\varphi\right\rangle && (5.25)\\
&\geq \sum_{j,k,l,m} \left|\alpha_{j,l}\right|^2 \left|\beta_{k,m}\right|^2 \times \\
&\frac{1}{LM}\sum_{s,t} \mathrm{Tr}\left\{\left(U^T\left(s_1\left(l+s\right)\right)^{A^n} \otimes U^T\left(s_2\left(m+t\right)\right)^{B^n}\right) \left|\varphi\right\rangle\left\langle\varphi\right| \left(U^*\left(s_1\left(l+s\right)\right)^{A^n} \otimes U^*\left(s_2\left(m+t\right)\right)^{B^n}\right) \Lambda_{l+s,m+t}\right\} && (5.26)\\
&\geq \sum_{j,k,l,m} \left|\alpha_{j,l}\right|^2 \left|\beta_{k,m}\right|^2 \left(1-\epsilon\right) && (5.27)\\
&= 1-\epsilon && (5.28)
\end{aligned}$$

where the last inequality follows from (5.21). Thus, the resulting state is $2\sqrt{\epsilon}$-close in expected trace distance to the following state:

$$\sum_{j,k,l,m} \alpha_{j,l}\beta_{k,m} \left|j\right\rangle^{R_A} \left|l\right\rangle^{A_1} \left|k\right\rangle^{R_B} \left|m\right\rangle^{B_1} \left(U^T\left(s_1\left(l+S\right)\right)^{A^n} \otimes U^T\left(s_2\left(m+T\right)\right)^{B^n}\right) \left|\varphi\right\rangle^{A^n B^n C^n E^n} \left|l+S\right\rangle^{A_2} \left|m+T\right\rangle^{B_2} . \tag{5.29}$$

Now Charlie performs the following controlled unitary:

$$\sum_{l,m} \left|l\right\rangle\left\langle l\right|^{A_2} \otimes \left|m\right\rangle\left\langle m\right|^{B_2} \otimes \left(U^*\left(s_1\left(l\right)\right)^{A^n} \otimes U^*\left(s_2\left(m\right)\right)^{B^n}\right), \tag{5.30}$$

and the resulting state is as follows:

$$\sum_{j,k,l,m} \alpha_{j,l}\beta_{k,m} \left|j\right\rangle^{R_A} \left|l\right\rangle^{A_1} \left|k\right\rangle^{R_B} \left|m\right\rangle^{B_1} \left|\varphi\right\rangle^{A^n B^n C^n E^n} \left|l+S\right\rangle^{A_2} \left|m+T\right\rangle^{B_2} \tag{5.31}$$

Charlie then performs the generalized Pauli shifts $X^{A_2}(-S)$ and $X^{B_2}(-T)$ (based on his common randomness) to produce the state

$$\left(\sum_{j,l} \alpha_{j,l} \left|j\right\rangle^{R_A} \left|l\right\rangle^{A_1} \left|l\right\rangle^{A_2}\right) \otimes \left(\sum_{k,m} \beta_{k,m} \left|k\right\rangle^{R_B} \left|m\right\rangle^{B_1} \left|m\right\rangle^{B_2}\right) \otimes \left|\varphi\right\rangle^{A^n B^n C^n E^n}, \tag{5.32}$$

so that Alice and Bob have successfully generated coherent channels with the receiver Charlie for this round.

The above scheme constitutes the first of the N blocks after establishing the common randomness. Alice, Bob, and Charlie perform the same scheme for the next $N-1$ blocks, and they use the same common randomness for each round. For similar reasons as given at the end of Section 4.1, this scheme works well if we set the number N of rounds equal to $\epsilon^{-1/4}$ (we require $\epsilon^{-1/4}$ this time because each round disturbs the state by $2\sqrt{\epsilon}$ so that the overall disturbance for all N rounds is no larger than $N(2\sqrt{\epsilon}) = 2\epsilon^{1/4}$). ■

The coherent communication identity is a helpful tool in quantum Shannon theory, and it results from the protocols coherent teleportation and coherent super-dense coding [23, 37]. It states that two coherent channels are equivalent to a noiseless quantum channel and noiseless entanglement:

$$2\log d\,[q \to qq] = \log d\,[q \to q] + \log d\,[qq], \tag{5.33}$$

where d is the dimension of the underlying systems. Employing this identity gives us the following achievable rate region for entanglement-assisted quantum communication:

Corollary 7 *There exists an entanglement-assisted quantum communication protocol with a coherent quantum simultaneous decoder if the rates $\widetilde{R}_1$ and $\widetilde{R}_2$ of quantum communication satisfy the following inequalities:*

$$\widetilde{R}_1 \leq \frac{1}{2} I(A;C|B)_\rho, \tag{5.34}$$

$$\widetilde{R}_2 \leq \frac{1}{2} I(B;C|A)_\rho, \tag{5.35}$$

$$\widetilde{R}_1 + \widetilde{R}_2 \leq \frac{1}{2} I(AB;C)_\rho. \tag{5.36}$$

Proof. We simply recall the resource inequality from the previous theorem and apply the coherent communication identity:

$$\langle \mathcal{N} \rangle + H(A)_\rho [qq]_{AC} + H(B)_\rho [qq]_{BC} \geq R_1 [q \to qq]_{AC} + R_2 [q \to qq]_{BC} \tag{5.37}$$

$$\geq \frac{1}{2} R_1 [qq]_{AC} + \frac{1}{2} R_1 [q \to q]_{AC} + \frac{1}{2} R_2 [qq]_{BC} + \frac{1}{2} R_2 [q \to q]_{BC}. \tag{5.38}$$

Throughout out the rest of this section, we will assume that R_1 and R_2 satisfy the conditions (5.5)-(5.7). If we allow catalytic protocols, that is we allow the use of some resources for free, provided that they are returned at the end of the protocol, then we obtain a protocol for entanglement-assisted quantum communication over a multiple access channel that implements the following resource inequality:

$$\langle \mathcal{N} \rangle + \left(H(A)_\rho - \frac{1}{2} R_1\right) [qq]_{AC} + \left(H(B)_\rho - \frac{1}{2} R_2\right) [qq]_{BC} \geq \frac{1}{2} R_1 [q \to q]_{AC} + \frac{1}{2} R_2 [q \to q]_{BC}. \tag{5.39}$$

■

Combining the above protocol further with entanglement distribution $[q \to q] \geq [qq]$ gives the following corollary:

Corollary 8 *There exists a catalytic quantum communication protocol (that consumes no net entanglement) with a coherent quantum simultaneous decoder if the rates S_1 and S_2 of quantum communication satisfy the following inequalities:*

$$S_1 \leq I(A\rangle C|B)_\rho, \tag{5.40}$$

$$S_2 \leq I(B\rangle C|A)_\rho, \tag{5.41}$$

$$S_1 + S_2 \leq I(AB\rangle C)_\rho. \tag{5.42}$$

Proof. The protocol from the above corollary in turn leads to a proof of an achievable rate region for unassisted quantum communication over a multiple access channel:

$$\langle \mathcal{N} \rangle + \left(H(A)_\rho - \frac{1}{2}R_1 \right) [qq]_{AC} + \left(H(B)_\rho - \frac{1}{2}R_2 \right) [qq]_{BC}$$

$$\geq \frac{1}{2}R_1 [q \to q]_{AC} + \frac{1}{2}R_2 [q \to q]_{BC} \tag{5.43}$$

$$\geq \left(R_1 - H(A)_\rho \right) [q \to q]_{AC} + \left(R_2 - H(B)_\rho \right) [q \to q]_{BC} + \left(H(A)_\rho - \frac{1}{2}R_1 \right) [qq]_{AC} + \left(H(B)_\rho - \frac{1}{2}R_2 \right) [qq]_{BC}. \tag{5.44}$$

The second inequality follows from the fact that we can perform entanglement distribution using noiseless quantum channels. After resource cancellation, this leads to

$$\langle \mathcal{N} \rangle \geq \left(R_1 - H(A)_\rho \right) [q \to q]_{AC} + \left(R_2 - H(B)_\rho \right) [q \to q]_{BC} \tag{5.45}$$

$$= S_1 [q \to q]_{AC} + S_1 [q \to q]_{BC}, \tag{5.46}$$

where

$$S_1 \leq I(A\rangle B|C)_\rho, \tag{5.47}$$

$$S_2 \leq I(B\rangle A|C)_\rho, \tag{5.48}$$

$$S_1 + S_2 \leq I(AB\rangle C)_\rho. \tag{5.49}$$

Again, both of these two capacity regions can be achieved without time sharing, thanks to our simultaneous decoder. ■

6 Entanglement-Assisted Bosonic Multiple Access Channel

This final section details our last contribution—an achievable rate region for entanglement-assisted classical communication over a bosonic multiple access channel (see Refs. [36, 12] for a nice review of bosonic channels). Perhaps the simplest model for this channel is the following beamsplitter transformation (Yen and Shapiro [41] considered unassisted communication over such a channel):

$$\hat{c} = \sqrt{\eta}\hat{a} + \sqrt{1-\eta}\hat{b}, \tag{6.1}$$

$$\hat{e} = -\sqrt{1-\eta}\hat{a} + \sqrt{\eta}\hat{b}, \tag{6.2}$$

where $\hat{a}$ is the annihilation operator representing the first sender Alice's input signal, $\hat{b}$ is the annihilation operator representing the second sender Bob's input signal, $\hat{c}$ is the annihilation operator for the receiver's output, and $\hat{e}$ is the annihilation operator for an inaccessible environment output of the channel. We prove the following theorem:

Theorem 9 *Suppose that Alice is allowed a mean photon number N_{S_a} at her transmitter and Bob is allowed a mean photon number N_{S_b} at his transmitter. Then the following rate region is achievable for entanglement-assisted transmission of classical information over the beamsplitter quantum multiple access channel:*

$$R_1 \leq g(N_{S_a}) + g\left(\left(\lambda_{BC}^{+}+1\right)/2-1\right) + g\left(\left(\lambda_{BC}^{-}+1\right)/2-1\right) - g\left(\eta N_{S_b} + (1-\eta) N_{S_a}\right), \tag{6.3}$$

$$R_2 \leq g(N_{S_b}) + g\left(\left(\lambda_{AC}^{+}+1\right)/2-1\right) + g\left(\left(\lambda_{AC}^{-}+1\right)/2-1\right) - g\left(\eta N_{S_b} + (1-\eta) N_{S_a}\right), \tag{6.4}$$

$$R_1 + R_2 \leq g(N_{S_a}) + g(N_{S_b}) + g\left(\eta N_{S_a} + (1-\eta) N_{S_b}\right) - g\left(\eta N_{S_b} + (1-\eta) N_{S_a}\right), \tag{6.5}$$

where

$$g(N) \equiv (N+1)\log(N+1) - N\log N, \tag{6.6}$$

$$\lambda_{AC}^{(\pm)} = (1-\eta)\left|N_{S_a} - N_{S_b}\right| \pm \sqrt{(1-\eta)^2 \left(N_{S_a} - N_{S_b}\right)^2 + 2(1-\eta)\left(2N_{S_a}N_{S_b} + N_{S_a} + N_{S_b}\right) + 1}, \tag{6.7}$$

$$\lambda_{BC}^{(\pm)} = \eta\left|N_{S_a} - N_{S_b}\right| \pm \sqrt{\eta^2 \left(N_{S_a} - N_{S_b}\right)^2 + 2\eta\left(2N_{S_a}N_{S_b} + N_{S_a} + N_{S_b}\right) + 1}. \tag{6.8}$$

(Observe that $\lambda_{AC}^{(\pm)}$ and $\lambda_{BC}^{(\pm)}$ are related by the substitution $\eta \leftrightarrow 1-\eta$.)

Proof. We assume the most natural entangled states that Alice and Charlie and Bob and Charlie can share: a two-mode squeezed vacuum [16, 36]. This state has the following form:

$$\sum_{n=0}^{\infty} \sqrt{\frac{N_S^n}{(N_S+1)^{n+1}}} \left|n\right\rangle \left|n\right\rangle, \tag{6.9}$$

where N_S is the average number of photons in one mode (after tracing over the other), Alice or Bob has the first mode, and Charlie has the second mode. The covariance matrix for such a state is as follows [36]:

$$V_{\text{TMS}}(N_S) \equiv \begin{bmatrix} 2N_S+1 & 0 & 2\sqrt{N_S(N_S+1)} & 0 \\ 0 & 2N_S+1 & 0 & -2\sqrt{N_S(N_S+1)} \\ 2\sqrt{N_S(N_S+1)} & 0 & 2N_S+1 & 0 \\ 0 & -2\sqrt{N_S(N_S+1)} & 0 & 2N_S+1 \end{bmatrix}. \tag{6.10}$$

The covariance matrix for the overall state before the channel acts is as follows:

$$V^{AA'BB'} \equiv V_{\text{TMS}}(N_{S_a}) \oplus V_{\text{TMS}}(N_{S_b}), \tag{6.11}$$

where N_{S_a} is the average number of photons in one share of the state that Alice shares with Charlie and N_{S_b} is the average number of photons in one share of the state that Bob shares with Charlie.

The symplectic operator for a beamsplitter unitary is as follows [36]:

$$S_{\text{BS}}^{A'B'} \equiv \begin{bmatrix} \sqrt{\eta} I & \sqrt{1-\eta} I \\ -\sqrt{1-\eta} I & \sqrt{\eta} I \end{bmatrix}, \tag{6.12}$$

and the covariance matrix of the state resulting from the beamsplitter interaction is

$$V^{ACBE} \equiv \left(S_{\text{BS}}^{A'B'} \oplus I^{AB}\right) V^{AA'BB'} \left((S_{\text{BS}}^{A'B'})^T \oplus I^{AB}\right), \tag{6.13}$$

where modes C and E emerge from the output ports of the beamsplitter (with input ports A' and B').

Hsieh *et al.* proved that the following rate region is achievable for entanglement-assisted communication over a quantum multiple access channel $\mathcal{M}$:

$$R_1 \leq I(A;BC)_\rho, \tag{6.14}$$

$$R_2 \leq I(B;AC)_\rho, \tag{6.15}$$

$$R_1 + R_2 \leq I(AB;C)_\rho, \tag{6.16}$$

where ρ^{ABC} is a state of the following form:

$$\rho^{ABC} \equiv \mathcal{M}^{A'B'\to C}(\phi^{AA'} \otimes \psi^{BB'}), \tag{6.17}$$

and $\phi^{AA'}$ and $\psi^{BB'}$ are pure, bipartite states [28]. Their theorem applies to finite-dimensional systems, but nevertheless, we apply their theorem to the infinite-dimensional setting by means of a limiting argument.[2] By inspecting the above theorem, it becomes clear that it is necessary to compute just seven entropies in order to determine the achievable rate region: $H(A)_\rho$, $H(B)_\rho$, $H(C)_\rho$, $H(AB)_\rho$, $H(AC)_\rho$, $H(BC)_\rho$, and $H(ABC)_\rho$. Observe that $H(ABC)_\rho = H(E)_\rho$ if we define E as the environment of the channel. In order to determine these entropies, we just need to figure out the covariance matrices for each of the seven different systems corresponding to these entropies because the entropies are a function of the symplectic eigenvalues of these covariance matrices. These seven different covariance matrices are as follows:

$$V^E = \begin{bmatrix} 2(\eta N_{S_b} + (1-\eta) N_{S_a}) + 1 & 0 \\ 0 & 2(\eta N_{S_b} + (1-\eta) N_{S_a}) + 1 \end{bmatrix}, \tag{6.18}$$

$$V^A = \begin{bmatrix} 2N_{S_a} + 1 & 0 \\ 0 & 2N_{S_a} + 1 \end{bmatrix}, \tag{6.19}$$

$$V^B = \begin{bmatrix} 2N_{S_b} + 1 & 0 \\ 0 & 2N_{S_b} + 1 \end{bmatrix}, \tag{6.20}$$

$$V^C = \begin{bmatrix} 2(\eta N_{S_a} + (1-\eta) N_{S_b}) + 1 & 0 \\ 0 & 2(\eta N_{S_a} + (1-\eta) N_{S_b}) + 1 \end{bmatrix}, \tag{6.21}$$

$$V^{AB} = \begin{bmatrix} 2N_{S_a} + 1 & 0 & 0 & 0 \\ 0 & 2N_{S_a} + 1 & 0 & 0 \\ 0 & 0 & 2N_{S_b} + 1 & 0 \\ 0 & 0 & 0 & 2N_{S_b} + 1 \end{bmatrix}, \tag{6.22}$$

$$V^{AC} = \begin{bmatrix} 2N_{S_a} + 1 & 0 & 2\sqrt{\eta}\sqrt{N_{S_a}(N_{S_a}+1)} & 0 \\ 0 & 2N_{S_a} + 1 & 0 & -2\sqrt{\eta}\sqrt{N_{S_a}(N_{S_a}+1)} \\ 2\sqrt{\eta}\sqrt{N_{S_a}(N_{S_a}+1)} & 0 & 2(\eta N_{S_a} + \overline{\eta} N_{S_b}) + 1 & 0 \\ 0 & -2\sqrt{\eta}\sqrt{N_{S_a}(N_{S_a}+1)} & 0 & 2(\eta N_{S_a} + \overline{\eta} N_{S_b}) + 1 \end{bmatrix}, \tag{6.23}$$

$$V^{BC} = \begin{bmatrix} 2(\eta N_{S_a} + \overline{\eta} N_{S_b}) + 1 & 0 & 2\sqrt{\overline{\eta}}\sqrt{N_{S_b}(N_{S_b}+1)} & 0 \\ 0 & 2(\eta N_{S_a} + \overline{\eta} N_{S_b}) + 1 & 0 & -2\sqrt{\overline{\eta}}\sqrt{N_{S_b}(N_{S_b}+1)} \\ 2\sqrt{\overline{\eta}}\sqrt{N_{S_b}(N_{S_b}+1)} & 0 & 2N_{S_b} + 1 & 0 \\ 0 & -2\sqrt{\overline{\eta}}\sqrt{N_{S_b}(N_{S_b}+1)} & 0 & 2N_{S_b} + 1 \end{bmatrix}, \tag{6.24}$$

where $\overline{\eta} \equiv 1 - \eta$. The five entropies $H(A)_\rho$, $H(B)_\rho$, $H(C)_\rho$, $H(E)_\rho$, and $H(AB)_\rho$ are straightforward to compute because their covariance matrices all correspond to those for thermal states:

$$H(A) = g(N_{S_a}), \tag{6.25}$$

$$H(B) = g(N_{S_b}), \tag{6.26}$$

$$H(C) = g(\eta N_{S_a} + (1-\eta) N_{S_b}), \tag{6.27}$$

$$H(ABC) = H(E) = g(\eta N_{S_b} + (1-\eta) N_{S_a}), \tag{6.28}$$

$$H(AB) = g(N_{S_a}) + g(N_{S_b}). \tag{6.29}$$

[2]The argument is similar to those appearing Refs. [41, 21], for example, and is simply that an infinite-dimensional Hilbert space with a mean photon-number constraint is effectively identical to a finite-dimensional Hilbert space. Suppose that we truncate the Hilbert space at the channel input so that it is spanned by the Fock number states $\{|0\rangle, |1\rangle, \ldots, |K\rangle\}$ where $K \gg N_S$. Thus, all coherent states, squeezed states, and thermal states become truncated to this finite-dimensional Hilbert space. Applying the Hsieh-Devetak-Winter theorem to squeezed states in this truncated Hilbert space gives a capacity region which is strictly an inner bound to the region in (6.3-6.5). As we let K grow without bound, the entropies given by the Hsieh-Devetak-Winter theorem converge to the entropies in (6.3-6.5).

We can calculate the other entropies $H(AC)$ and $H(BC)$ by computing the symplectic eigenvalues of the covariance matrices in (6.23) and (6.24), respectively:

$$\lambda_{AC}^{(\pm)} = (1-\eta)\left|N_{S_a} - N_{S_b}\right| \pm \sqrt{(1-\eta)^2 \left(N_{S_a} - N_{S_b}\right)^2 + 2(1-\eta)\left(2N_{S_a}N_{S_b} + N_{S_a} + N_{S_b}\right) + 1}, \quad (6.30)$$

$$\lambda_{BC}^{(\pm)} = \eta\left|N_{S_a} - N_{S_b}\right| \pm \sqrt{\eta^2 \left(N_{S_a} - N_{S_b}\right)^2 + 2\eta\left(2N_{S_a}N_{S_b} + N_{S_a} + N_{S_b}\right) + 1}. \quad (6.31)$$

Recall that we find the symplectic eigenvalues of a matrix V by computing the eigenvalues of the matrix $|iJV|$ [36] where

$$J \equiv \bigoplus_{i=1}^{n} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad (6.32)$$

and n is the number of modes. These symplectic eigenvalues lead to the following values for the entropies:

$$H(AC) = g\left(\left(\lambda_{AC}^{+} + 1\right)/2 - 1\right) + g\left(\left(\lambda_{AC}^{-} + 1\right)/2 - 1\right), \quad (6.33)$$

$$H(BC) = g\left(\left(\lambda_{BC}^{+} + 1\right)/2 - 1\right) + g\left(\left(\lambda_{BC}^{-} + 1\right)/2 - 1\right), \quad (6.34)$$

by exploiting the fact that the entropy of a Gaussian state ρ is the following function of its symplectic eigenvalues $\{\nu_k\}$ [36]:

$$H(\rho) = \sum_k g\left(\left(\nu_k + 1\right)/2 - 1\right). \quad (6.35)$$

Thus, an achievable rate region for the entanglement-assisted bosonic multiple access channel is as stated in the theorem. ■

Figure 1 plots several achievable rate regions given by Theorem 9 as the transmissivity parameter η varies from 0 to 1. The first plot has Alice's mean photon number much higher than Bob's, while the second plot sets them equal.

6.1 Comparison with the Unassisted Bosonic Multiple Access Rate Region

We would also like to compare the achievable rate region given by Theorem 9 to the Yen-Shapiro outer bound for unassisted classical communication over the beamsplitter bosonic multiple access channel [41]. Consider that the Yen-Shapiro outer bound is as follows:

$$R_1 \leq g\left(N_{S_a}\right), \quad (6.36)$$

$$R_2 \leq g\left(N_{S_b}\right), \quad (6.37)$$

$$R_1 + R_2 \leq g\left(\eta N_{S_a} + (1-\eta) N_{S_b}\right). \quad (6.38)$$

They derived this outer bound with two straightforward arguments. First, if N_{S_a} and N_{S_b} are the respective mean photon numbers at the channel input, then the mean photon number at the output is $\eta N_{S_a} + (1-\eta) N_{S_b}$, and the Holevo quantity can never exceed $g\left(\eta N_{S_a} + (1-\eta) N_{S_b}\right)$ [17]. The individual rate bounds follow by assuming that the receiver gets access to both output ports of the channel. The best strategy would then be simply to invert the beamsplitter, and the rate bounds follow from a similar argument (that the Holevo quantity for mean photon number constraints N_{S_a} and N_{S_b} cannot exceed $g\left(N_{S_a}\right)$ and $g\left(N_{S_b}\right)$, respectively).

It is straightforward to demonstrate that the sum rate bound in Theorem 9 always exceeds the sum rate bound in (6.38). Consider that the difference between these two sum rate bounds is

$$g\left(N_{S_a}\right) + g\left(N_{S_b}\right) - g\left(\eta N_{S_b} + (1-\eta) N_{S_a}\right), \quad (6.39)$$

and this quantity is always positive because $g(x)$ is positive and monotone increasing for $x \geq 0$ (i.e., supposing WLOG that $N_{S_a} \geq N_{S_b}$, it follows that $N_{S_a} \geq \eta N_{S_b} + (1-\eta) N_{S_a}$ and thus $g\left(N_{S_a}\right) \geq g\left(\eta N_{S_b} + (1-\eta) N_{S_a}\right)$). The individual rate bounds are incomparable as Figure 2 demonstrates—there are examples of channels and photon number constraints for which the assisted region contains or does not contain the Yen-Shapiro unassisted outer bound.

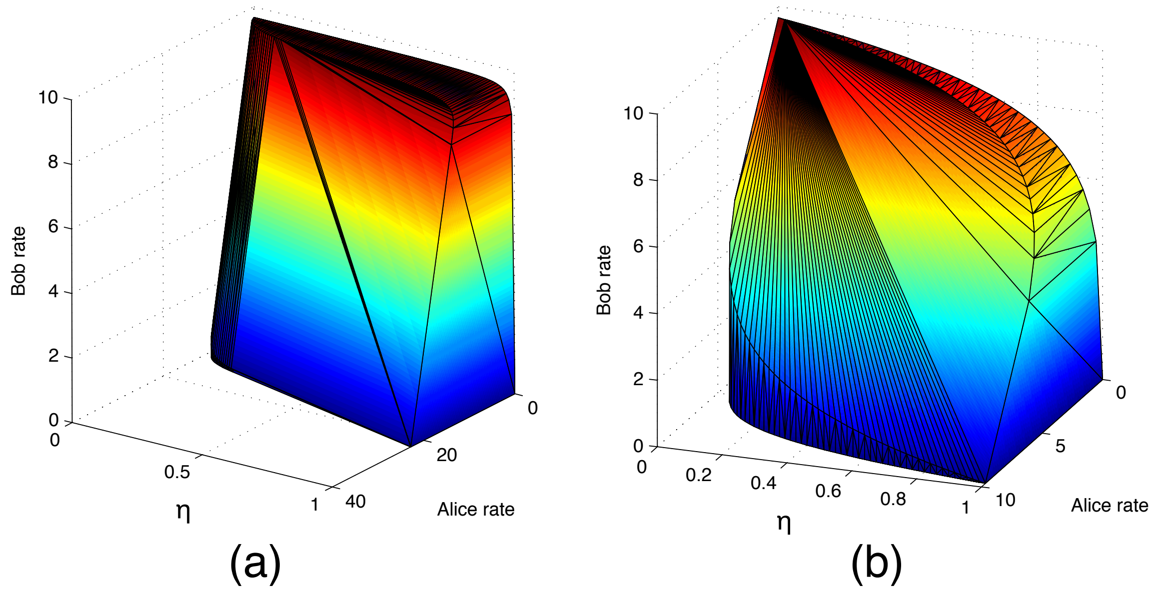

Figure 1: The figure displays the achievable rate region from Theorem 9 for the beamsplitter multiple access channel as the beamsplitter transmissivity η varies from 0 to 1. (a) The region as η varies when Alice and Bob's mean input photon number are fixed at $N_{S_a} = 1000$ and $N_{S_b} = 10$, respectively. (b) The region as η varies when $N_{S_a} = 10$ and $N_{S_b} = 10$.

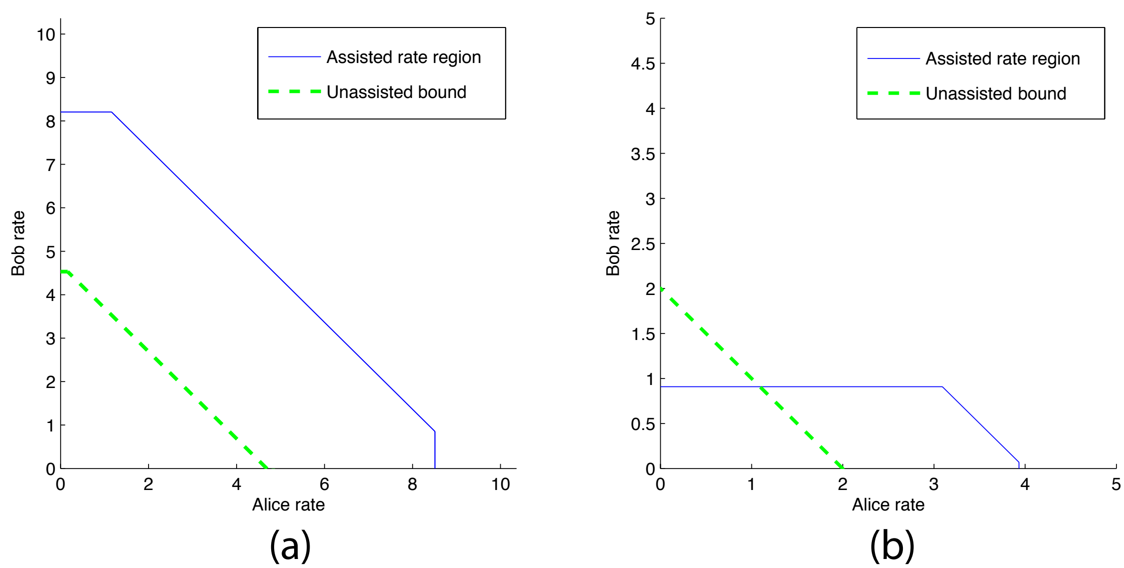

Figure 2: The figure compares the achievable rate region from Theorem 9 with the Yen-Shapiro outer bound on the unassisted region [41] for two examples. (a) The first example shows that our assisted achievable rate region contains the Yen-Shapiro outer bound on the unassisted region when $N_{S_a} = 10$, $N_{S_b} = 8$, and $\eta = 1/2$. (b) The second example shows that our assisted achievable rate region does *not* contain the Yen-Shapiro outer bound on the unassisted region when $N_{S_a} = 1$, $N_{S_b} = 1$, and $\eta = 0.95$.

7 Conclusion

We have discussed five different scenarios for entanglement-assisted classical communication: sequential decoding for a single-sender, single-receiver channel, sequential and successive decoding for a multiple access channel, simultaneous decoding, coherent simultaneous decoding, and communication over a bosonic channel. Our third contribution gives further progress toward proving the quantum simultaneous decoding conjecture from Ref. [14] (see Appendix A in the thesis of Dutil for a different manifestation of this conjecture in distributed compression [11]).

Several open questions remain. It would of course be good to prove that the quantum simultaneous decoding conjecture holds in the general case for entanglement-assisted classical communication or even to broaden the classes of channels or the conditions for which it holds. It would be worthwhile to determine whether our strategy for entanglement-assisted classical communication over a bosonic multiple access channel is optimal.

We are grateful to Vittorio Giovannetti for suggesting the idea of extending the GLM sequential decoder to the entanglement-assisted case, and we thank Pranab Sen for sharing his results in Ref. [34] and for pointing out that a slight modification of our proof technique from the first version of this article solves the quantum simultaneous decoding conjecture for two senders. MMW acknowledges useful discussions with Omar Fawzi, Patrick Hayden, Ivan Savov, and Pranab Sen during the development of Ref. [14]. MMW acknowledges financial support from the MDEIE (Québec) PSR-SIIRI international collaboration grant.

A Appendix

Proof of the Sequential Packing Lemma. Our proof below is essentially identical to the proof given in Ref. [18], with the exception that it extracts only the most basic conditions needed (these conditions are given in the statement of the theorem). Given a message set $\mathcal{M} = \{1, 2, \ldots, |\mathcal{M}|\}$, we construct a code $\mathcal{C} \equiv \{c_m\}_{m\in\mathcal{M}}$ randomly such that each c_m takes a value in $\mathcal{X}$ with probability $p_X(c_m)$. Using this code, Alice chooses a message m from the message set $\mathcal{M}$ and encodes it in the quantum codeword ρ_{c_m}. To decode the message m, Bob performs the following steps:

1. Starting from $k = 1$, Bob tries to determine if he received the kth message.
2. Bob first makes a projective measurement with the code subspace projector Π to determine if the received state is in the code subspace.
3. If the answer is NO, then an error has occurred and Bob aborts the protocol.
4. If the answer is YES, Bob performs another projective measurement on the post-measurement state using the codeword subspace projector Π_{c_k}.
5. If the answer is YES, then Bob declares to have received the kth message and stops the protocol.
6. If the answer is NO, then Bob increments k and goes back to Step 2 if $k < |\mathcal{M}|$. If $k = |\mathcal{M}|$, Bob declares that an error has occurred and aborts the protocol.

As derived in Ref. [18], the following POVM $\{\Lambda_m\}_{m\in\mathcal{M}}$ corresponds to the above sequential decoding scheme:

$$\Lambda_m \equiv \bar{Q}_{c_1} \cdots \bar{Q}_{c_{m-1}} \bar{\Pi}_{c_m} \bar{Q}_{c_{m-1}} \cdots \bar{Q}_{c_1}, \tag{A.1}$$

where for any operator Θ, we define $\bar{\Theta}$ as

$$\bar{\Theta} \equiv \Pi\Theta\Pi, \tag{A.2}$$

and

$$Q_x \equiv I - \Pi_x. \tag{A.3}$$

We analyze the performance of this sequential decoding scheme by computing a lower bound on the expectation of the average success probability, where the expectation is with respect to all possible codes:

$$\mathbb{E}_{\mathcal{C}}\left\{\bar{p}_{\text{succ}}\left(\mathcal{C}\right)\right\} = \sum_{c_1,\ldots,c_{|\mathcal{M}|}} p_X\left(c_1\right)\cdots p_X\left(c_{|\mathcal{M}|}\right)\frac{1}{|\mathcal{M}|}\sum_{m=1}^{|\mathcal{M}|}\text{Tr}\left\{\Pi_{c_m}\bar{Q}_{c_{m-1}}\cdots\bar{Q}_{c_1}\rho_{c_m}\bar{Q}_{c_1}\cdots\bar{Q}_{c_{m-1}}\right\} \tag{A.4}$$

$$= \frac{1}{|\mathcal{M}|}\sum_{l=0}^{|\mathcal{M}|-1}\sum_{x,c_1,\ldots,c_l} p_X\left(x\right)p_X\left(c_1\right)\cdots p_X\left(c_l\right)\text{Tr}\left\{\Pi_x\bar{Q}_{c_l}\cdots\bar{Q}_{c_1}\rho_x\bar{Q}_{c_1}\cdots\bar{Q}_{c_l}\right\} \tag{A.5}$$

$$= \frac{1}{|\mathcal{M}|}\sum_{l=0}^{|\mathcal{M}|-1}\sum_{x,c_1,\ldots,c_l} p_X\left(x\right)p_X\left(c_1\right)\cdots p_X\left(c_l\right)\sum_{y}\sum_{y'\in\mathcal{T}_x}\lambda_{x,y}\left|\left\langle\psi_{x,y'}\right|\bar{Q}_{c_1}\cdots\bar{Q}_{c_l}\left|\psi_{x,y}\right\rangle\right|^2. \tag{A.6}$$

We obtain the last equality by writing out the spectral decomposition of Π_x and ρ_x:

$$\rho_x = \sum_y \lambda_{x,y}\left|\psi_{x,y}\right\rangle\left\langle\psi_{x,y}\right|, \tag{A.7}$$

$$\Pi_x = \sum_{y\in\mathcal{T}_x}\left|\psi_{x,y}\right\rangle\left\langle\psi_{x,y}\right|. \tag{A.8}$$

We note that ρ_x and Π_x commute by assumption and therefore share common eigenstates. We use $\mathcal{T}_x$ to index a subset of the eigenstates of ρ_x.

The following lower bound applies to the rightmost term in (A.6):

$$\sum_y\sum_{y'\in\mathcal{T}_x}\lambda_{x,y}\left|\left\langle\psi_{x,y'}\right|\bar{Q}_{c_1}\cdots\bar{Q}_{c_l}\left|\psi_{x,y}\right\rangle\right|^2 \geq \sum_{y\in\mathcal{T}_x}\lambda_{x,y}\left|\left\langle\psi_{x,y}\right|\bar{Q}_{c_1}\cdots\bar{Q}_{c_l}\left|\psi_{x,y}\right\rangle\right|^2 \tag{A.9}$$

$$= \sum_{y\in\mathcal{T}_x}\lambda_{x,y}\left|\left\langle\psi_{x,y}\right|\bar{Q}_{c_1}\cdots\bar{Q}_{c_l}\left|\psi_{x,y}\right\rangle\right|^2\sum_y\lambda_{x,y} \tag{A.10}$$

$$\geq \left|\sum_{y\in\mathcal{T}_x}\lambda_{x,y}\left\langle\psi_{x,y}\right|\bar{Q}_{c_1}\cdots\bar{Q}_{c_l}\left|\psi_{x,y}\right\rangle\right|^2 \tag{A.11}$$

$$= \left|\text{Tr}\left\{\Pi_x\rho_x\Pi_x\bar{Q}_{c_1}\cdots\bar{Q}_{c_l}\right\}\right|^2. \tag{A.12}$$

The first inequality follows by eliminating some positive terms from the summation. The second inequality follows by applying the Cauchy-Schwarz inequality. The last equality follows by exploiting the assumed commutative relation between Π_x and ρ_x. Therefore, the following lower bound applies to the expectation of the average success probability:

$$\mathbb{E}_{\mathcal{C}}\left\{\bar{p}_{\text{succ}}\left(\mathcal{C}\right)\right\} \geq \frac{1}{|\mathcal{M}|}\sum_{l=0}^{|\mathcal{M}|-1}\sum_{x,c_1,\ldots,c_l} p_X\left(x\right)p_X\left(c_1\right)\cdots p_X\left(c_l\right)\left|\text{Tr}\left\{\Pi_x\rho_x\Pi_x\bar{Q}_{c_1}\cdots\bar{Q}_{c_l}\right\}\right|^2. \tag{A.13}$$

We again apply the Cauchy-Schwarz inequality to the inner summation:

$$\sum_{x,c_1,\ldots,c_l} p_X\left(x\right)p_X\left(c_1\right)\cdots p_X\left(c_l\right)\left|\text{Tr}\left\{\Pi_x\rho_x\Pi_x\bar{Q}_{c_1}\cdots\bar{Q}_{c_l}\right\}\right|^2 \tag{A.14}$$

$$\geq \left|\sum_{x,c_1,\ldots,c_l} p_X\left(x\right)p_X\left(c_1\right)\cdots p_X\left(c_l\right)\text{Tr}\left\{\Pi_x\rho_x\Pi_x\bar{Q}_{c_1}\cdots\bar{Q}_{c_l}\right\}\right|^2 \tag{A.15}$$

$$= \left|\text{Tr}\left\{\left(\sum_x p_X\left(x\right)\Pi_x\rho_x\Pi_x\right)\left(\sum_{c_1}p_X\left(c_1\right)\bar{Q}_{c_1}\right)\cdots\left(\sum_{c_l}p_X\left(c_l\right)\bar{Q}_{c_l}\right)\right\}\right|^2 \tag{A.16}$$

$$= \left|\text{Tr}\left\{W_1\mathcal{Q}^l\right\}\right|^2, \tag{A.17}$$

where we define

$$W_q \equiv \sum_x p_X(x)\, \Pi_x \rho_x^q \Pi_x, \tag{A.18}$$

$$\mathcal{Q} \equiv \sum_x p_X(x)\, \bar{Q}_x, \tag{A.19}$$

and it is understood that $\mathcal{Q}^0 = \Pi$ (an abuse of notation explained further below). Therefore, we obtain the following lower bound on the expectation of the average success probability:

$$\mathbb{E}_{\mathcal{C}}\left\{\bar{p}_{\text{succ}}(\mathcal{C})\right\} \geq \frac{1}{|\mathcal{M}|}\sum_{l=0}^{|\mathcal{M}|-1}\left|\operatorname{Tr}\left\{W_1\mathcal{Q}^l\right\}\right|^2. \tag{A.20}$$

In order to proceed, we note that

$$\begin{aligned}\mathcal{Q} &= \sum_x p_X(x)\, \bar{Q}_x && \text{(A.21)}\\ &= \Pi\left(\sum_x p_X(x)\,(I - \Pi_x)\right)\Pi && \text{(A.22)}\\ &\leq I, && \text{(A.23)}\end{aligned}$$

and therefore

$$\operatorname{Tr}\left\{W_1\mathcal{Q}^l\right\} = \operatorname{Tr}\left\{W_1\mathcal{Q}^{\frac{l-1}{2}}\mathcal{Q}\mathcal{Q}^{\frac{l-1}{2}}\right\} \leq \operatorname{Tr}\left\{W_1\mathcal{Q}^{l-1}\right\}. \tag{A.24}$$

Given this observation, we can further lower bound the success probability by taking the smallest term of the summation from (A.20):

$$\begin{aligned}\mathbb{E}_{\mathcal{C}}\left\{\bar{p}_{\text{succ}}(\mathcal{C})\right\} &\geq \left|\operatorname{Tr}\left\{W_1\mathcal{Q}^{|\mathcal{M}|-1}\right\}\right|^2 && \text{(A.25)}\\ &= \left|\operatorname{Tr}\left\{W_1\left(\bar{I} - \bar{W}_0\right)^{|\mathcal{M}|-1}\right\}\right|^2 && \text{(A.26)}\\ &= \left|\operatorname{Tr}\left\{\sum_{z=0}^{|\mathcal{M}|-1}\binom{|\mathcal{M}|-1}{z}(-1)^z\, W_1\bar{I}^{|\mathcal{M}|-z}\bar{W}_0^z\right\}\right|^2 && \text{(A.27)}\\ &= \left|\sum_{z=0}^{|\mathcal{M}|-1}\binom{|\mathcal{M}|-1}{z}(-1)^z \operatorname{Tr}\left\{W_1\Pi\bar{W}_0^z\right\}\right|^2 && \text{(A.28)}\end{aligned}$$

Here, we define a function f_z as

$$f_z \equiv \operatorname{Tr}\left\{W_1\Pi\bar{W}_0^z\right\}, \tag{A.29}$$

where z is a nonnegative integer. We abused the notation of $\bar{W}_0^z$ here, which does *not* mean to raise the eigenvalues of $\bar{W}_0$ to the power z in its spectral decomposition, but rather

$$\bar{W}_0^z = \prod_{i=1}^{z}\bar{W}_0, \tag{A.30}$$

as it arises from the binomial expansion. We note that $f_0 = \text{Tr}\{W_1 \Pi\}$ and the function f_z is always positive. Thus, the above expression is equal to the following one:

$$\left| \sum_{z=0}^{|\mathcal{M}|-1} \binom{|\mathcal{M}|-1}{z} (-1)^z f_z \right|^2 \tag{A.31}$$

$$= \left| f_0 + \sum_{z=1}^{|\mathcal{M}|-1} \binom{|\mathcal{M}|-1}{z} (-1)^z f_z \right|^2 \tag{A.32}$$

$$= |A|^2 , \tag{A.33}$$

with

$$A \equiv f_0 + \sum_{z=1}^{|\mathcal{M}|-1} \binom{|\mathcal{M}|-1}{z} (-1)^z f_z. \tag{A.34}$$

We then have

$$A \geq 2 f_0 - \sum_{z=0}^{|\mathcal{M}|-1} \binom{|\mathcal{M}|-1}{z} f_z. \tag{A.35}$$

The function f_z satisfies the following two properties:

$$f_0 \geq 1 - 2\epsilon, \tag{A.36}$$

$$f_z \leq \left(\frac{d}{D}\right)^z f_0. \tag{A.37}$$

We now prove this. First we show that f_0 is ϵ-close to one:

$$f_0 = \text{Tr}\{W_1 \Pi\} \tag{A.38}$$

$$= \sum_x p_X(x) \, \text{Tr}\{\Pi_x \rho_x \Pi_x \Pi\} \tag{A.39}$$

$$= \sum_x p_X(x) \, \text{Tr}\{\Pi_x \rho_x \Pi\} \tag{A.40}$$

$$= \sum_x p_X(x) \, \text{Tr}\{(I - (I - \Pi_x)) \rho_x \Pi\} \tag{A.41}$$

$$= \sum_x p_X(x) \, \text{Tr}\{\rho_x \Pi\} - \sum_x p_X(x) \, \text{Tr}\{(I - \Pi_x) \rho_x \Pi\} \tag{A.42}$$

$$\geq \sum_x p_X(x) \, \text{Tr}\{\rho_x \Pi\} - \sum_x p_X(x) \, \text{Tr}\{(I - \Pi_x) \rho_x\} \tag{A.43}$$

$$\geq \sum_x p_X(x) \, \text{Tr}\{\rho_x \Pi\} - \epsilon$$

$$\geq 1 - 2\epsilon \tag{A.44}$$

The third equality follows by the commutative relation between Π_x and ρ_x. The second inequality follows from the condition (1.4). Now we will upper bound the function f_z in terms of f_{z-1}, and we show that we can upper bound f_z in terms of f_{z-1}, and as a result, in terms of f_0.

$$
\begin{aligned}
f_z &= \operatorname{Tr}\left\{W_1 \bar{W}_0^z\right\} && \text{(A.45)}\\
&= \operatorname{Tr}\left\{\sqrt{W_1}\bar{W}_0^{\frac{z-1}{2}}\bar{W}_0\bar{W}_0^{\frac{z-1}{2}}\sqrt{W_1}\right\} && \text{(A.46)}\\
&= \operatorname{Tr}\left\{\sqrt{W_1}\bar{W}_0^{\frac{z-1}{2}}\Pi\left(\sum_x p_X(x)\,\Pi_x\right)\Pi\bar{W}_0^{\frac{z-1}{2}}\sqrt{W_1}\right\} && \text{(A.47)}\\
&\leq d\cdot\operatorname{Tr}\left\{\sqrt{W_1}\bar{W}_0^{\frac{z-1}{2}}\Pi\left(\sum_x p_X(x)\,\Pi_x\rho_x\Pi_x\right)\Pi\bar{W}_0^{\frac{z-1}{2}}\sqrt{W_1}\right\} && \text{(A.48)}\\
&\leq d\cdot\operatorname{Tr}\left\{\sqrt{W_1}\bar{W}_0^{\frac{z-1}{2}}\Pi\rho\Pi\bar{W}_0^{\frac{z-1}{2}}\sqrt{W_1}\right\} && \text{(A.49)}\\
&\leq \frac{d}{D}\operatorname{Tr}\left\{\sqrt{W_1}\bar{W}_0^{\frac{z-1}{2}}\Pi\bar{W}_0^{\frac{z-1}{2}}\sqrt{W_1}\right\} && \text{(A.50)}\\
&\leq \frac{d}{D}\operatorname{Tr}\left\{W_1\bar{W}_0^{z-1}\right\} && \text{(A.51)}\\
&= \frac{d}{D}f_{z-1} && \text{(A.52)}\\
\Rightarrow f_z &\geq \left(\frac{d}{D}\right)^z f_0 && \text{(A.53)}
\end{aligned}
$$

In this derivation, we used the conditions (1.5) and (1.6) and the fact that W_1 and $\bar{W}_0$ are positive.

Therefore, using the above two inequalities, we get that

$$
\begin{aligned}
A &\geq 2f_0 - f_0\sum_{z=0}^{|\mathcal{M}|-1}\binom{|\mathcal{M}|-1}{z}\left(\frac{d}{D}\right)^z && \text{(A.54)}\\
&= f_0\left(2-\left(1+\frac{d}{D}\right)^{|\mathcal{M}|-1}\right) && \text{(A.55)}\\
&\geq (1-2\epsilon)\left(2-e^{\frac{d}{D}|\mathcal{M}|}\right). && \text{(A.56)}
\end{aligned}
$$

The last inequality follows from the fact that $1+x\leq e^x$ for all x, and our analysis completes with the observation that

$$\mathbb{E}_{\mathcal{C}}\left\{\bar{p}_{\text{succ}}(\mathcal{C})\right\} \geq |A|^2 \geq \left|(1-2\epsilon)\left(2-e^{\frac{d}{D}|\mathcal{M}|}\right)\right|^2, \tag{A.57}$$

as long as $2-\exp\{d\,|\mathcal{M}|\,/D\}$ is positive. ■

References

[1] Christoph Adami and Nicolas J. Cerf. von Neumann capacity of noisy quantum channels. *Physical Review A*, 56(5):3470–3483, November 1997.

[2] Charles H. Bennett, Gilles Brassard, Claude Crépeau, Richard Jozsa, Asher Peres, and William K. Wootters. Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels. *Physical Review Letters*, 70(13):1895–1899, March 1993.

[3] Charles H. Bennett, Peter W. Shor, John A. Smolin, and Ashish V. Thapliyal. Entanglement-assisted classical capacity of noisy quantum channels. *Physical Review Letters*, 83(15):3081–3084, October 1999.

[4] Charles H. Bennett, Peter W. Shor, John A. Smolin, and Ashish V. Thapliyal. Entanglement-assisted capacity of a quantum channel and the reverse Shannon theorem. *IEEE Transactions on Information Theory*, 48:2637–2655, 2002.

[5] Charles H. Bennett and Stephen J. Wiesner. Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states. *Physical Review Letters*, 69(20):2881–2884, November 1992.

[6] Garry Bowen. Quantum feedback channels. *IEEE Transactions in Information Theory*, 50(10):2429–2434, October 2004. arXiv:quant-ph/0209076.

[7] Thomas M. Cover and Joy A. Thomas. *Elements of Information Theory*. Wiley-Interscience, 1991.

[8] Imre Csiszár and János Körner. *Information Theory: Coding Theorems for Discrete Memoryless Systems*. Cambridge University Press, second edition, August 2011.

[9] Nilanjana Datta and Renato Renner. Smooth entropies and the quantum information spectrum. *IEEE Transactions on Information Theory*, 55(6):2807–2815, June 2009. arXiv:0801.0282.

[10] Igor Devetak, Aram W. Harrow, and Andreas Winter. A resource framework for quantum Shannon theory. *IEEE Transactions on Information Theory*, 54(10):4587–4618, October 2008.

[11] Nicolas Dutil. *Multiparty quantum protocols for assisted entanglement distillation*. PhD thesis, McGill University, May 2011. arXiv:1105.4657.

[12] Jens Eisert and Michael M. Wolf. *Quantum Information with Continous Variables of Atoms and Light*, chapter Gaussian quantum channels, pages 23–42. Imperial College Press, London, 2007. arXiv:quant-ph/0505151.

[13] Abbas El Gamal and Young-Han Kim. Lecture notes on network information theory. January 2010. arXiv:1001.3404.

[14] Omar Fawzi, Patrick Hayden, Ivan Savov, Pranab Sen, and Mark M. Wilde. Classical communication over a quantum interference channel. February 2011. arXiv:1102.2624.

[15] Amiel Feinstein. A new basic theorem of information theory. *IEEE Transactions on Information Theory*, 4(4):2–22, 1954.

[16] Christopher Gerry and Peter Knight. *Introductory Quantum Optics*. Cambridge University Press, November 2004.

[17] Vittorio Giovannetti, Saikat Guha, Seth Lloyd, Lorenzo Maccone, Jeffrey H. Shapiro, and Horace P. Yuen. Classical capacity of the lossy bosonic channel: The exact solution. *Physical Review Letters*, 92(2):027902, January 2004.

[18] Vittorio Giovannetti, Seth Lloyd, and Lorenzo Maccone. Achieving the Holevo bound via sequential measurements. December 2010. arXiv:1012.0386.

[19] Vittorio Giovannetti, Seth Lloyd, Lorenzo Maccone, and Peter W. Shor. Broadband channel capacities. *Physical Review A*, 68(6):062323, December 2003.

[20] Vittorio Giovannetti, Seth Lloyd, Lorenzo Maccone, and Peter W. Shor. Entanglement assisted capacity of the broadband lossy channel. *Physical Review Letters*, 91(4):047901, July 2003.

[21] Saikat Guha. *Multiple-User Quantum Information Theory for Optical Communication Channels*. PhD thesis, Massachusetts Institute of Technology, June 2008.

[22] Te Sun Han and Kingo Kobayashi. A new achievable rate region for the interference channel. *IEEE Transactions on Information Theory*, 27(1):49–60, January 1981.

[23] A. W. Harrow. Coherent communication of classical messages. *Physical Review Letters*, 92:097902, 2004.

[24] Masahito Hayashi and Hiroshi Nagaoka. General formulas for capacity of classical-quantum channels. *IEEE Transactions on Information Theory*, 49(7):1753–1768, 2003.

[25] Alexander S. Holevo. On entanglement assisted classical capacity. *Journal of Mathematical Physics*, 43(9):4326–4333, 2002.

[26] Alexander S. Holevo and Reinhard F. Werner. Evaluating capacities of bosonic Gaussian channels. *Physical Review A*, 63(3):032312, February 2001.

[27] M. Horodecki, Jonathan Oppenheim, and Andreas Winter. Partial quantum information. *Nature*, 436:673–676, 2005.

[28] Min-Hsiu Hsieh, Igor Devetak, and Andreas Winter. Entanglement-assisted capacity of quantum multiple-access channels. *IEEE Transactions on Information Theory*, 54(7):3078–3090, 2008.

[29] Min-Hsiu Hsieh and Mark M. Wilde. Entanglement-assisted communication of classical and quantum information. *IEEE Transactions on Information Theory*, 56(9):4682–4704, September 2010.

[30] Min-Hsiu Hsieh and Mark M. Wilde. Trading classical communication, quantum communication, and entanglement in quantum Shannon theory. *IEEE Transactions on Information Theory*, 56(9):4705–4730, September 2010.

[31] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2000.

[32] Tomohiro Ogawa and Hiroshi Nagaoka. Making good codes for classical-quantum channel coding via quantum hypothesis testing. *IEEE Transactions on Information Theory*, 53(6):2261–2266, June 2007.

[33] Pranab Sen. Private communication, July 2011.

[34] Pranab Sen. Sequential decoding for some channels with classical input and quantum output, July 2011.

[35] Claude E. Shannon. A mathematical theory of communication. *Bell System Technical Journal*, 27:379–423, 1948.

[36] Christian Weedbrook, Stefano Pirandola, Raul Garcia-Patron, Nicolas J. Cerf, Timothy C. Ralph, Jeffrey H. Shapiro, and Seth Lloyd. Gaussian quantum information. 2011. In preparation.

[37] Mark M. Wilde. *From Classical to Quantum Shannon Theory*. June 2011. arXiv:1106.1445.

[38] Andreas Winter. Coding theorem and strong converse for quantum channels. *IEEE Transactions on Information Theory*, 45(7):2481–2485, 1999.

[39] Andreas Winter. The capacity of the quantum multiple access channel. *IEEE Transactions on Information Theory*, 47:3059–3065, 2001.

[40] Jon Yard, Patrick Hayden, and Igor Devetak. Capacity theorems for quantum multiple-access channels: Classical-quantum and quantum-quantum capacity regions. *IEEE Transactions on Information Theory*, 54(7):3091–3113, 2008.

[41] Brent J. Yen and Jeffrey H. Shapiro. Multiple-access bosonic communications. *Physical Review A*, 72(6):062312, December 2005.