A Note on Distributivity of Open Filter Domains *

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Abstract

Filters and open filters are useful utilities to explore structures of domains. In this paper, it is proved that for a continuous distributive semilattice L, $OFilt(L)$ is a distributive lattice iff L is stably continuous. And an example is given to show that in the general case, the distributivity of L cannot imply that of $OFilt(L)$.

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1 Introduction

In domain theory, filters and open filters are useful utilities in the study of order and topological structures, so it is natural to explore how kinds of property are reflected between a ground domain and its corresponding (open) filter domain. There is an interesting problem posed in [6] which is whether the open filters poset $(OFilt(L), \subseteq)$ on a continuous distributive semilattice L is distributive. In this note we present a negative answer to this problem, and prove that stable continuity property of a distributive semilattice is in a very great degree equivalent with distributivity of $OFilt(L)$.

In the following, a poset L is said to be a semilattice if for $\forall x, y \in L$, the infimum $x \wedge y$ exists. L is said to be a dcpo if any directed subset of L has a supremum. For $\forall x \in L$, $\Downarrow x = \{y \in L : y \ll x\}$. If L is a dcpo and for $\forall x \in L$, $\Downarrow x$ is directed and $\vee \Downarrow x = x$, then L is called a continuous domain. $Filt(L)$ and $OFilt(L)$ denote the set of all the filters and that of all the Scott-open filters on L respectively. $\sigma(L)$ is the Scott topology on L, and $Q(L)$ is the poset of all Scott compact upper subsets of L with the reverse inclusion order. For a semilattice L and $A, B \subseteq L$, let $A \wedge_L B = \{a \wedge b : a \in A, b \in B\}$. $\forall x \in L$, let $\uparrow x = \{y \in L : x \leq y\}$ and $\Uparrow x = \{y \in L : x \ll y\}$. The way-below relation $\ll$ on L is said to be multiplicative iff $\forall x, y, z \in L, z \ll x$ and $z \ll y$ always imply $z \ll x \wedge y$. When $\ll$ is multiplicative, L is said to be stably continuous. $\forall x, y \in L$, x and y are consistent iff they have a common upper bound, i.e., $\exists z \in L$, such that $x, y \leq z$.

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Note that for a continuous (algebraic) domain L, $OFilt(L)$ is also a continuous (algebraic) one.

2 Main Results

Definition 1.[6] A semilattice L is said to be distributive if for $\forall a, b, x \in L$, $a \wedge b \leq x$ always implies the existence of elements $c, d \in L$ with $a \leq c, b \leq d$ and $c \wedge d = x$.

It is easy to check that a lattice L is distributive iff it is a distributive semilattice in the sense of the above definition.

Lemma 2. If L is a distributive semilattice, then any two elements in it are consistent. Furthermore, if L is finite, then L must be a lattice.

Proof. For $\forall x, y \in L$, since $x \wedge y \leq x$, there exist some $x^{'}, y^{'} \in L$, such that $x \leq x^{'}, y \leq y^{'}$, and $x^{'} \wedge y^{'} = x$. Then $x, y \leq y^{'}$.

If L is finite, then the upper bound subset of any two elements x, y is non-empty and thus just has its meet as the supremum of x and y. □

Proposition 3. For a distributive semilattice L, we have that

(i) $(Filt(L), \subseteq)$ is a lattice , and for $\forall F_1, F_2 \in Filt(L), F_1 \wedge F_2 = F_1 \cap F_2, F_1 \vee F_2 = F_1 \wedge_L F_2 = \{x_1 \wedge x_2 : x_1 \in F_1, x_2 \in F_2\} = \cup_{x_1 \in F_1, x_2 \in F_2} \uparrow (x_1 \wedge x_2)$.

(ii) $(OFilt(L), \subseteq)$ is a semilattice, and for $\forall F_1, F_2 \in OFilt(L), F_1 \wedge F_2 = F_1 \cap F_2$.

Proof. For arbitrary two (open) filters F_1 and F_2 of L, let $x_1 \in F_1$, $x_2 \in F_2$. By lemma 2, $\exists z \in L$ satisfying $x_1, x_2 \leq z$. Then $z \in F_1 \cap F_2$, and thus $F_1 \cap F_2 \neq \Phi$. For $\forall x, y \in F_1 \cap F_2, x \wedge y \in F_1 \cap F_2$. Hence $F_1 \cap F_2$ is the least (open) filter contained by both F_1 and F_2.

$F_1 \vee F_2 = \cup_{x_1 \in F_1, x_2 \in F_2} \uparrow (x_1 \wedge x_2)$ apparently holds, and so we only need to verify the equality $F_1 \vee F_2 = F_1 \wedge_L F_2$. Firstly, for $\forall x \in F_1, y \in F_2, z \in L$ with $x \wedge y \leq z$, by the distributivity of L, $\exists x^{'} \geq x, y^{'} \geq y$ satisfying $x^{'} \wedge y^{'} = z$. Then $z \in F_1 \wedge F_2$, and $F_1 \wedge_L F_2$ is an upper subset. Secondly, from the above we have seen that $\forall x \in F_1, \exists y \in F_2$ satisfying $x \leq y$, and vice versa, so $F_1 \wedge_L F_2$ contains F_1, F_2 and is the least filter containing both F_1 and F_2. □

Example. $OFilt(L)$ need not be a lattice for a semilattice L. In fact, let $L = \{a_1, a_2, \ldots, a_n, \ldots\} \cup \{a, b, c, \top\}$, where $a_1 \leq a_2 \leq \ldots \leq a_n \leq \ldots \leq a \leq b \leq \top$ and $a \leq c \leq \top$. Then L is a continuous distributive complete lattice, while the open filters $F_1 = \uparrow b$ and $F_2 = \uparrow c$ have no least upper bound in $OFilt(L)$. □

The following proposition shows that the distributivity of L can determine that of $Filt(L)$.

Proposition 4.[6] If L is a distributive semilattice, so is $Filt(L)$.

Proof. Since the notion of distributivity in Definition 1 coincides with that defined in classical lattice theory, we just need to prove for $\forall F_1, F_2, F_3 \in Filt(L)$, it holds that $F_1 \wedge (F_2 \vee F_3) = (F_1 \wedge F_2) \vee (F_1 \wedge F_3)$. Indeed, l.h.s.$= F_1 \cap (F_2 \vee F_3) = F_1 \cap (\bigcup_{x \in F_2, y \in F_3} \uparrow (x \wedge y)) = \bigcup_{x \in F_2, y \in F_3} (F_1 \cap \uparrow (x \wedge y)) = \bigcup_{x \in F_2, y \in F_3} ((F_1 \cap \uparrow x) \vee (F_1 \cap \uparrow y)) = \bigcup_{x \in F_2, y \in F_3} \bigcup_{z \in F_1 \cap \uparrow x, z' \in F_1 \cap \uparrow y} \uparrow (z \wedge z^{'}) = \bigcup_{z \in F_1 \cap F_2, z' \in F_1 \cap F_3} \uparrow (z \wedge z^{'}) = (F_1 \cap F_2) \vee (F_1 \cap F_3) = (F_1 \wedge F_2) \vee (F_1 \wedge F_3)$=r.h.s..□

However the following example shows that even for a completely distributive algebraic lattice, the open filter domain need not be distributive.

Example. Let L_1 be an algebraic domain denoted by the following graph,

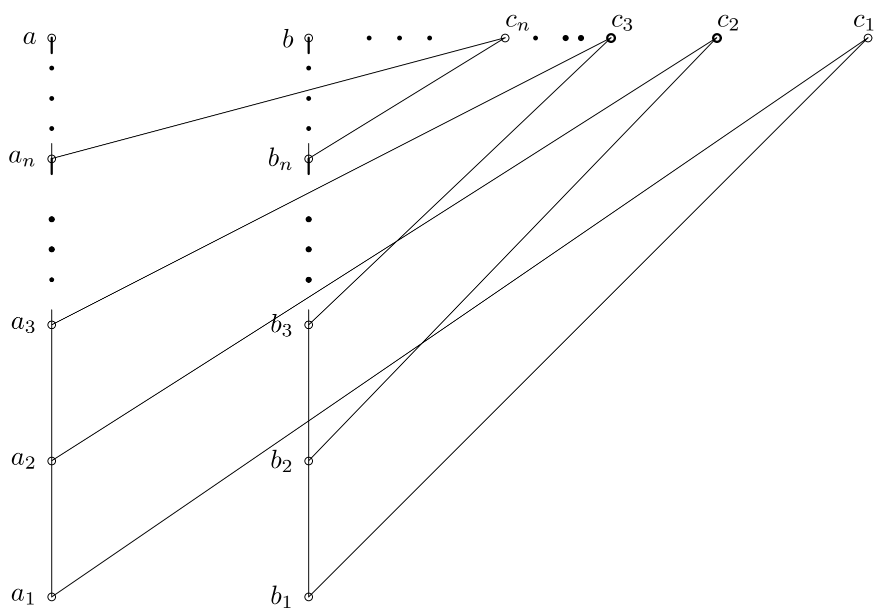

Figure 1:

and $L = (\sigma(L_1), \subseteq)$ be the lattice of Scott topology on L_1. Since L_1 is algebraic, L is a completely distributive algebraic lattice. But $OFilt(L) = OFilt(\sigma(L_1))$ is not distributive.

In fact, assume that $OFilt(L) = OFilt(\sigma(L_1))$ is distributive. By Hofmann-Mislove Theorem, $OFilt(\sigma(L_1)) \cong Q(L_1)$, and so $Q(L_1)$ must be distributive. Thus for $\forall K, K_1, K_2 \in Q(L_1)$ with $K \subseteq K_1 \cup K_2$, there exist $K_1^{'}, K_2^{'} \in Q(L_1)$ such that $K_1^{'} \subseteq K_1, K_2^{'} \subseteq K_2$, and $K = K_1^{'} \cup K_2^{'}$. Now let $K_1 = \{c_1, c_2, c_3, \ldots, c_n, \ldots\} \cup \{a\}$, $K_2 = \{b\}$, and $K = \{c_1, c_2, c_3, \ldots, c_n, \ldots\} \cup \{b\}$. Then K, K_1 and K_2 clearly belong to $Q(L_1)$. If there exist $K_1^{'}, K_2^{'} \in Q(L_1)$ such that $K_1^{'} \subseteq K_1, K_2^{'} \subseteq K_2$ and $K = K_1^{'} \cup K_2^{'}$, then $K_2^{'}$ must be $\{b\}$. It follows that $K_1^{'} = \{c_1, c_2, c_3, \ldots, c_n, \ldots\}$. But $K_1^{'}$ is apparently non-compact, a contradiction. $\square$

Although $OFilt(L)$ is not distributive for a general continuous distributive semilattice L, it is always distributive when L is a finite distributive semilattice.

Proposition 5. $OFilt(L)$ on the finite distributive semilattice L is a distributive lattice.

Proof. Since L is a finite distributive semilattice, by Lemma 2, we know that L is a lattice, and every filter in it must be of the principal form $\uparrow x$. Thus for any two open filters $F_1, F_2, F_3 \in OFilt(L)$, there exist $x_1, x_2, x_3 \in L$ such that $F_i = \uparrow x_i$ for $i = 1, 2, 3$. It follows that $F_1 \vee F_2 = \uparrow (x_1 \wedge x_2)$, and $F_1 \wedge (F_2 \vee F_3) = \uparrow (x_1 \vee (x_2 \wedge x_3)) = (F_1 \wedge F_2) \vee (F_1 \wedge F_3)$. $\square$

Next we consider those continuous distributive semilattices L such that $OFilt(L)$ is distributive.

Proposition 6. For a continuous distributive semilattice L, $\wedge : (L, \sigma(L)) \times (L, \sigma(L)) \longrightarrow (L, \sigma(L)), (x, y) \longmapsto x \wedge y$ is an open map iff $OFilt(L)$ is a distributive lattice and for $\forall F_1, F_2 \in OFilt(L), F_1 \vee F_2 = F_1 \wedge_L F_2$.

Proof. $\Rightarrow$: Since $\wedge$ is open, for open filters F_1 and F_2 in L, $F_1 \wedge_L F_2$ is open. $F_1 \wedge_L F_2$ is clearly the least filter containing F_1 and F_2. Hence $F_1 \wedge_L F_2$ is just the least upper bound of F_1 and F_2 in $OFilt(L)$. Now the verification for the equation $F_1 \wedge (F_2 \vee F_3) = (F_1 \wedge F_2) \vee (F_1 \wedge F_3)$ is entirely like that in Proposition 4.

$\Leftarrow$: Since L is a continuous domain, $OFilt(L)$ is a basis of $\sigma(L)$. And for $\forall F_1, F_2 \in OFilt(L), F_1 \wedge_L F_2 = F_1 \vee F_2 \in OFilt(L)$. Thus $\wedge$ is an open map. □

Proposition 7. Let L be a continuous distributive semilattice. Then L is stably continuous iff $OFilt(L)$ is a distributive lattice and for $\forall F_1, F_2 \in OFilt(L), F_1 \vee F_2 = F_1 \wedge_L F_2$.

In particular, if L is an arithmetic distributive semilattice, then $OFilt(L)$ is an arithmetic distributive lattice.

Proof. $\Rightarrow$: Let L be stably continuous and $F_1, F_2 \in OFilt(L)$. For $\forall F \in OFilt(L), F = \bigcup_{x \in F} \twoheaduparrow x = \bigcup_{x \in F} \uparrow x$. From Proposition 3, we know that $OFilt(L)$ is a semilattice and $F_1 \wedge_L F_2$ is a filter. $F_1 \wedge_L F_2 = \bigcup_{z \in F_1 \wedge_L F_2} \uparrow z = \bigcup_{x \in F_1, y \in F_2} \uparrow (x \wedge y)$. Since L is distributive and stably continuous, $\bigcup_{x \in F_1, y \in F_2} \uparrow (x \wedge y) = \bigcup_{x \in F_1, y \in F_2} \twoheaduparrow (x \wedge y)$ is open, so $F_1 \vee F_2$ belongs to $OFilt(L)$ and is just the supremum of F_1 and F_2 in $OFilt(L)$. Thus $OFilt(L)$ is a lattice. The verification of distributivity of $OFilt(L)$ is trivial.

$\Leftarrow$: For $\forall a, x, y \in L$ with $a \ll x$ and $a \ll y$, we prove that $a \ll x \wedge y$. In fact, since L is continuous and $x \in \twoheaduparrow a$, $y \in \twoheaduparrow a$, there exist two open filters F_1 and F_2 such that $x \in F_1, y \in F_2$ and $F_1, F_2 \subseteq \twoheaduparrow a \subseteq \uparrow a$. Thus $F_1 \vee F_2 = F_1 \wedge_L F_2 \subseteq \uparrow a$. Since $F_1 \wedge_L F_2$ is Scott open and $x \wedge y$ is in $F_1 \wedge_L F_2$, there exists some $z \in F_1 \wedge_L F_2$ such that $z \ll x \wedge y$. Note that $z \geq a$, and so $a \ll x \wedge y$. □

Remark. Since finite distributive semilattices are always $\ll$-multiplicative, by Proposition 7, we again find out the distributivity of $OFilt(L)$.

Combining Proposition 6 and proposition 7, we obtain the following theorem.

Theorem 8. Let L be a continuous distributive semilattice, then the following are equivalent:

(i) L is stably continuous;

(ii) The map

$$\wedge : (L, \sigma(L)) \times (L, \sigma(L)) \longrightarrow (L, \sigma(L)), (x, y) \longmapsto x \wedge y$$

is open;

(iii) $OFilt(L)$ is a distributive lattice, and for $\forall F_1, F_2 \in OFilt(L)$, $F_1 \vee F_2 = F_1 \wedge_L F_2$. □

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