ON EMBEDDING OF DENDRIFORM ALGEBRAS INTO ROTA—BAXTER ALGEBRAS

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Abstract. Following a recent work by C. Bai, O. Bellier, L. Guo, X. Ni (arXiv:1106.6080) we define what is a dendriform di- or trialgebra in an arbitrary variety Var of binary algebras (associative, commutative, Poisson, etc.). We prove that every dendriform dialgebra in Var can be embedded into a Rota—Baxter algebra of weight zero in the same variety, and every dendriform trialgebra can be embedded into a Rota—Baxter algebra of nonzero weight.

1. Introduction

In 1960 Glen Baxter [7] introduced an identity defining what is now called Rota—Baxter operator in developing works of F. Spitzer [33] in fluctuation theory. By definition, a Rota—Baxter operator R of weight λ on an algebra A is a linear map on A such that

$$R(x)R(y) = R(xR(y) + R(x)y) + \lambda R(xy), \quad x, y \in A,$$

where λ is a scalar from the base field.

Later, commutative associative algebras with such an operator were studied by G.-C. Rota and others [10, 31]. In 1980s, these operators appeared in the context of Lie algebras independently in works A. A. Belavin and V. G. Drinfeld [8] and M. A. Semenov-Tian-Shansky [32] in research of solutions of classical Young—Baxter equation named in the honour of physicists Chen Ning Yang and Rodney Baxter.

For the present time, numerous connections of Rota—Baxter operators with other areas of mathematics are found. The latter include quantum field theory, Young—Baxter equations, operads, Hopf algebras, number theory etc. [2, 13, 14, 16, 24].

The notion of a Leibniz algebra introduced by J.-L. Loday [25] is originated from cohomology theory of Lie algebras; this is a noncommutative analogue of Lie algebras. Associative dialgebras (now often called diassociative algebras) emerged in the paper by J.-L. Loday and T. Pirashvili [26], they play the role of universal enveloping associative algebras for Leibniz algebras. Dendriform dialgebras were defined by J.-L. Loday in 1999 [27] in his study of algebraic K-theory. Moreover, they occur to be Koszul-dual to diassociative algebras. In 2001, J.-L. Loday and V. Ronco [28] introduced a generalization of dialgebras—trialgebras and dual to them dendriform trialgebras.

M. Aguiar in 2000 [1] was the first who noticed a relation between Rota—Baxter algebras and dendriform algebras. He proved that an associative algebra with a Rota—Baxter operator R of weight zero relative to operations $a \prec b = aR(b)$, $a \succ b = R(a)b$ is a dendriform dialgebra. In 2002, K. Ebrahimi-Fard [15] generalized this fact to the case of Rota—Baxter algebras of arbitrary weight and obtained as result both dendriform dialgebra and dendriform trialgebra. In the paper by K. Ebrahimi-Fard and L. Guo [17] in 2007, universal enveloping Rota—Baxter algebras of weight λ for dendriform dialgebras and trialgebras were defined.

The natural question: Whether an arbitrary dendriform di- or trialgebra can be embedded into its universal enveloping Rota—Baxter algebra was solved positively in [17] for free dendriform algebras only. In 2010, Y. Chen and Q. Mo proved that any dendriform dialgebra over a field of characteristic zero can be embedded into an appropriate Rota—Baxter algebra of weight zero [12] using the Gröbner—Shirshov bases technique for Rota—Baxter algebras developed in [9].

To solve the problem for any dendriform dialgebra (or trialgebra) from a Rota—Baxter algebra of arbitrary weight, C. Bai, L. Guo and K. Ni [4] introduced in 2010 a notion of $\mathcal{O}$-operators, a generalization of Rota—Baxter operators and proved that every dendriform di- or trialgebra can be explicitly obtained from an algebra with a $\mathcal{O}$-operator.

In a recent work [5], the results of Aguiar and Ebrahimi-Fard were extended to the case of arbitrary operad of Rota—Baxter algebras and dendriform dialgebras and trialgebras.

In the present work, we solve the following problem. Given a binary operad $\mathcal{P}_{\mathrm{Var}}$ governing a variety Var of Ω-algebras (Ω is a set of binary operations), we define what is a di- or tri-Var-dendriform algebra (following [5]). Then we construct a Rota—Baxter Ω-algebra from the variety Var such that the initial dendriform di- or trialgebra embeds into this Rota—Baxter algebra in the sense of Aguiar and Ebrahimi-Fard (for trialgebras, we demand $\lambda \neq 0$).

The idea of the construction can be easily illustrated as follows. Suppose $(A, \prec, \succ, \cdot)$ is an (associative) dendriform trialgebra. Then the direct sum of two isomorphic copies of A, the space $\hat{A} = A \oplus A'$, endowed with a binary operation

$$a * b = a \prec b + a \succ b + a \cdot b, \quad a * b' = (a \succ b)', \quad a' * b = (a \prec b)', \quad a' * b' = (a \cdot b)'$$

for $a, b \in A$, is an associative algebra. Moreover, the map $R(a') = a$, $R(a) = -a$ is a Rota—Baxter operator of weight 1 on $\hat{A}$. The embedding of A into $\hat{A}$ is given by $a \mapsto a'$, $a \in A$.

In this work, we also introduce and consider some modification of Loday's notion of trialgebras which we will call skew trialgebras (or s-trialgebras, for short). This class of algebras appears from differential and $\mathbb{Z}$-conformal algebras. Associative skew trialgebras turn to be related with a natural noncommutative analogue of Poisson algebras. Dendriform s-trialgebras are Koszul dual to s-trialgebras and they are also connected with Rota—Baxter algebras in the same way as usual dendriform dialgebras and trialgebras.

2. Operads for di- and trialgebras

Our main object of study is the class of dendriform di- or tri-algebras. In this section, we start with objects from the "dual world" in the sense of Koszul duality.

The notion of an operad once introduced in [29] has been reincarnated in the beginning of 2000s. We address the reader to either of perfect expositions of this notion and its applications in universal algebra, e.g., [18, 23, 34].

Throughout this paper, $\Bbbk$ is an arbitrary base field. All operads are assumed to be families of linear spaces, compositions are linear maps, and the actions of symmetric groups are also linear.

By an Ω-algebra we mean a linear space equipped with a family of binary linear operations $\Omega = \{\circ_i \mid i \in I\}$. Denote by $\mathcal{F}$ the free operad governing the variety of all Ω-algebras. For every natural $n > 1$, the space $\mathcal{F}(n)$ can be identified with the space spanned by all binary trees with n leaves labeled by $x_1, \dots, x_n$, where each vertex (which is not a leaf) has a label from Ω.

Let Var be a variety of Ω-algebras defined by a family S of poly-linear identities of any degree (which is greater than one). Denote by $\mathcal{P}_{\mathrm{Var}}$ the binary operad governing the variety Var, i.e., every algebra from Var is a functor from $\mathcal{P}_{\mathrm{Var}}$ to the multi-category Vec of linear spaces with poly-linear maps.

Denote by $\Omega^{(2)}$ and $\Omega^{(3)}$ the sets of binary operations $\{\vdash_i, \dashv_i \mid i \in I\}$ and $\Omega^{(2)} \cup \{\perp_i \mid i \in I\}$, respectively. Similarly, let $\mathcal{F}^{(2)}$ and $\mathcal{F}^{(3)}$ stand for the free operads governing the varieties of all $\Omega^{(2)}$- and $\Omega^{(3)}$-algebras, respectively.

We will need the following important operads.

Example 1. Operad Perm introduced in [11] is governing the variety of Perm-algebras [39, p. 17]. Namely, $\mathrm{Perm}(n) = \Bbbk^n$ with a standard basis $e_i^{(n)}$, $i = 1, \dots, n$. Every $e_i^{(n)}$ can be identified with an associative and commutative poly-linear monomial in $x_1, \dots, x_n$ with one emphasized variable x_i.

Example 2. Operad CommTrias introduced in [36] is governing the variety of associative and commutative trialgebras, see [39, p. 25]. Namely, $\mathrm{CommTrias}(n)$ has a standard basis $e_H^{(n)}$, where $\emptyset \neq H \subseteq \{1, \dots, n\}$. Such an element (corolla) can be identified with a commutative and associative monomial with several emphasized variables x_j, $j \in H$.

The number of observations made, for example, in [37, 11, 22] lead to the following natural definition.

Definition 1. A *di*-Var-*algebra* is a functor from $\mathcal{P}_{\mathrm{Var}} \otimes \mathrm{Perm}$ to Vec, i.e., an $\Omega^{(2)}$-algebra satisfying the following identities:

$$(x_1 \dashv_i x_2) \vdash_j x_3 = (x_1 \vdash_i x_2) \vdash_j x_3, \quad x_1 \dashv_i (x_2 \vdash_j x_3) = x_1 \dashv_i (x_2 \dashv_j x_3), \tag{1}$$

$$f(x_1, \dots, \dot{x}_k, \dots, x_n), \quad f \in S,\ n = \deg f,\ k = 1, \dots, n, \tag{2}$$

where $i, j \in I$, and $f(x_1, \dots, \dot{x}_k, \dots, x_n)$ stands for $\Omega^{(2)}$-identity obtained from f by means of replacing all products $\circ_i$ with either $\dashv_i$ or $\vdash_i$ in such a way that all horizontal dashes point to the selected variable x_k.

Example 3. Let $|\Omega| = 1$. The variety of diassociative algebras (or associative dialgebras) [26] is given by (1) together with

$$\begin{gathered} x_1 \dashv (x_2 \dashv x_3) = (x_1 \dashv x_2) \dashv x_3, \quad x_1 \vdash (x_2 \dashv x_3) = (x_1 \vdash x_2) \dashv x_3, \\ x_1 \vdash (x_2 \vdash x_3) = (x_1 \vdash x_2) \vdash x_3. \end{gathered} \tag{3}$$

Example 4. Consider the class of Poisson algebras ($|\Omega| = 2$), where $\circ_1$ is an associative and commutative product (we will denote $x \circ_1 y$ simply by xy) and $\circ_2$ is a Lie product ($x \circ_2 y = [x, y]$) related with $\circ_1$ by the following identity:

$$[x_1x_2, x_3] = [x_1, x_3]x_2 + x_1[x_2, x_3].$$

Then a di-Poisson algebra is a linear space equipped by four operations $(\cdot * \cdot)$, $[\cdot * \cdot]$, $* \in \{\vdash, \dashv\}$ satisfying (1) and (2). Commutativity of the first product and anticommutativity of the second one allow to reduce these four operations to only two, since (2) implies

$$(x_1 \dashv x_2) = (x_2 \vdash x_1), \quad [x_1 \dashv x_2] = -[x_2 \vdash x_1].$$

With respect to the operations

$$xy := (x \vdash y), \quad [x, y] := [x \vdash y],$$

the identities (1) and (2) are equivalent to the following system:

$$\begin{gathered} x_1(x_2x_3) = (x_1x_2)x_3, \quad (x_1x_2)x_3 = (x_2x_1)x_3, \\ [x_1, [x_2, x_3]] - [x_2, [x_1, x_3]] = [[x_1, x_2], x_3], \\ [x_1x_2, x_3] = x_1[x_2, x_3] + x_2[x_1, x_3], \\ [x_1, x_2x_3] = [x_1, x_2]x_3 + x_2[x_1, x_3]. \end{gathered}$$

In [27], a more general class was introduced (without assuming commutativity of the associative product).

A similar approach works for trialgebras.

Definition 2. A *tri-*Var-*algebra* is a functor from $\mathcal{P}_{\mathrm{Var}} \otimes \mathrm{CommTrias}$ to Vec, i.e., an $\Omega^{(3)}$-algebra satisfying the following identities:

$$\begin{gathered} (x_1 *_i x_2) \vdash_j x_3 = (x_1 \vdash_i x_2) \vdash_j x_3, \quad x_1 \dashv_i (x_2 *_j x_3) = x_1 \dashv_i (x_2 \dashv_j x_3), \\ * \in \{\vdash, \dashv, \perp\}, \end{gathered} \tag{4}$$

$$\begin{gathered} f(x_1, \dots, \dot{x}_{k_1}, \dots, \dot{x}_{k_l}, \dots, x_n), \\ f \in S, \ n = \deg f, \ 1 \le k_1 < \dots < k_l \le n, \ l = 1, \dots, n. \end{gathered}$$

where $i, j \in I$, and $f(x_1, \dots, \dot{x}_{k_1}, \dots, \dot{x}_{k_l}, \dots, x_n)$ is the result of a procedure described below. (It is somewhat similar to the tri-successor procedure from [5]).

Suppose $u = u(x_1, \dots, x_n) \in \mathcal{F}(n)$ is a non-associative Ω-monomial. Fix l indices $1 \le k_1 < \dots < k_l \le n$, and denote the the monomial u with l emphasized variables x_{k_j}, $j = 1, \dots, l$, by u^H, $H = \{k_1, \dots, k_l\}$. Now, identify u^H with an element from $\mathcal{F}(n) \otimes \mathrm{CommTrias}(n)$ in the natural way:

$$u^H \equiv u \otimes e^{(n)}_{k_1,\dots,k_l}.$$

It can be considered as a binary tree from $\mathcal{F}(n)$ with l emphasized leaves.

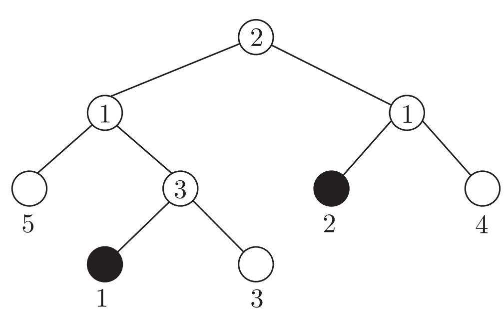

Fig. 1

Example: $u^H = (x_5 \circ_1 (\dot{x}_1 \circ_3 x_3)) \circ_2 (\dot{x}_2 \circ_1 x_4)$, $H = \{1, 2\}$. Emphasized leaves are colored in black, others—in white.

Now the task is to mark all vertices of u^H with appropriate labels from $\Omega^{(3)}$. Define the family of maps

$$\Phi(n) : \mathcal{F}(n) \otimes \mathrm{CommTrias}(n) \to \mathcal{F}^{(3)}(n), \quad n \geq 1,$$

as follows. Given $u \otimes e^{(n)}_{k_1,\dots,k_l} \in \mathcal{F}(n) \otimes \mathrm{CommTrias}(n)$, the structure of the tree u as well as labels of leaves do not change. For $n = 1$, there is nothing to do. If $u = v \circ_i w$ then the set $H = \{k_1, \dots, k_l\}$ of emphasized variables splits into two subsets, $H = H_1 \dot\cup H_2$, where H_1 consists of all k_j such that x_{k_j} appears in v. Assume $\deg v = p$, then $\deg w = n - p$. Set

$$\Phi(n)(u^H) = \begin{cases} \Phi(p)(v^{H_1}) \perp_i \Phi(n-p)(w^{H_2}), & \text{if } H_1, H_2 \neq \emptyset, \\ v^{\vdash} \vdash_i \Phi(n-p)(w^H), & \text{if } H_1 = \emptyset, \\ \Phi(p)(v^H) \dashv_i w^{\dashv}, & \text{if } H_2 = \emptyset, \end{cases}$$

where $v^{\vdash}$ (or $w^{\dashv}$) stands for the tree where each vertex label $\circ_j$ turns into $\vdash_j$ ($\dashv_j$).

One may extend $\Phi(n)$ by linearity, so, if $f(x_1, \dots, x_n) = \sum\limits_{\xi} \alpha_\xi u_\xi \in \mathcal{F}(n)$ then

$$f(x_1, \dots, \dot{x}_{k_1}, \dots, \dot{x}_{k_l}, \dots, x_n) := \sum_{\xi} \alpha_\xi \Phi(n)(u_\xi^H).$$

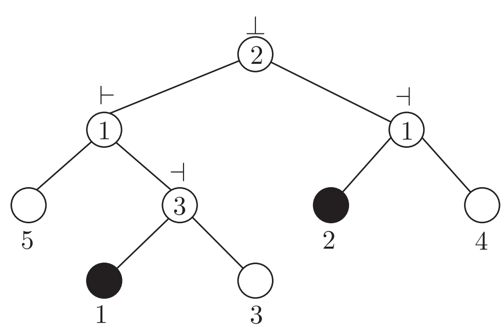

Fig. 2

Example: $\Phi(5)(u^H) = (x_5 \vdash_1 (x_1 \dashv_3 x_3)) \perp_2 (x_2 \dashv_1 x_4)$ for u and H as on Fig. 1. For each vertex which is not a leaf we assign $\perp$ if both left and right branches have emphasized leaves. If only left branch contains an emphasized leaf then we assign $\dashv$ to this vertex and to all vertices of the right branch. Symmetrically, if only right branch contains an emphasized leaf then we assign $\vdash$ to this vertex and to all vertices of the left branch.

Example 5 (Tri-associative algebra)**.** Let Var = As be the variety of associative algebras. It has only one defining identity $(x_1, x_2, x_3) = x_1(x_2x_3) - (x_1x_2)x_3$, that turns into seven identities (5). Indeed, each nonempty subset $H \subseteq \{1, 2, 3\}$ gives rise to an identity of $\Omega^{(3)}$-algebras, $\Omega^{(3)} = \{\vdash, \dashv, \perp\}$. If $|H| = 1$ then these are just the identities of a di-As-algebra (3). For $|H| = 2$, we obtain three identities, e.g., if $H = \{1, 3\}$ then the corresponding identity $(\dot{x}_1, x_2, \dot{x}_3)$ is $x_1 \perp (x_2 \vdash x_3) - (x_1 \dashv x_2) \perp x_3$ If $H = \{1, 2, 3\}$ then we obtain the relation of associativity for $\perp$. Together with (4), these are exactly the defining identities of triassociative algebras [39, p. 23].

Example 6. Let A be an associative algebra. Then the space $A^{\otimes 3}$ with respect to operations

$$a\otimes b\otimes c\vdash a'\otimes b'\otimes c' = abca'\otimes b'\otimes c',\quad a\otimes b\otimes c\dashv a'\otimes b'\otimes c' = a\otimes b\otimes ca'b'c',$$
$$a\otimes b\otimes c\perp a'\otimes b'\otimes c' = a\otimes bca'b'\otimes c'$$

is a triassociative algebra.

The following construction invented in [30] for dialgebras also works for trialgebras.

Let A be a 0-trialgebra, i.e., an $\Omega^{(3)}$-algebra which satisfies (4). Then $A_0 = \mathrm{Span}\,\{a\vdash_i b - a\dashv_i b,\ a\vdash_i b - a\perp_i b \mid a,b\in A,\ i\in I\}$ is an ideal of A. The quotient $\bar A = A/A_0$ carries a natural structure of an Ω-algebra. Consider the formal direct sum $\hat A = \bar A\oplus A$ with (well-defined) operations

$$\bar a\circ_i x = a\vdash_i x,\quad x\circ_i \bar a = x\dashv_i a,\quad \bar a\circ_i \bar b = \overline{a\vdash_i b},\quad x\circ_i y = x\perp_i y, \tag{5}$$

$\bar a,\bar b\in\bar A$, $x,y\in A$.

Proposition 1. *A trialgebra A satisfying* (4) *is a tri-*Var*-algebra if and only if $\hat A$ is an algebra from the variety* Var*.*

Proof. The claim follows from the following observation. If $f(x_1,\dots,x_n)\in\mathcal F(n)$ then the value $f(\bar a_1,\dots,\bar a_n)$ in $\bar A\subset\hat A$ is just the image of $[\Phi(n)(f^H)](a_1,\dots,a_n)$ in $\bar A$ for any subset H; moreover, the value of $f(x_1,\dots,\dot x_{k_1},\dots,\dot x_{k_l},\dots,x_n)$ on $a_1,\dots,a_n\in A$ is equal to $f(\bar a_1,\dots,a_{k_1},\dots,a_{k_l},\dots,\bar a_n)\in\hat A$, i.e., one has to add bars to all non-emphasized variables. □

Assuming $x\perp_i y\equiv 0$ for all $x,y\in A$, $i\in I$, we obtain the construction from [30]. This construction turns to be useful in the study of dialgebras (see, e.g., [20, 38]). For a variety Var, let us denote by $\mathcal D_{\mathrm{Var}}$ and $\mathcal T_{\mathrm{Var}}$ the operads governing di- and tri-Var-algebras, respectively.

The structure of a di-Var-algebra may be recovered from a structure of a Var-pseudo-algebra over an appropriate bialgebra H. Let us recall this notion from [6]. Suppose H is a cocommutative bialgebra with a coproduct Δ and counit ε. We will use the Swedler notation for Δ, e.g., $\Delta(h) = h_{(1)}\otimes h_{(2)}$, $\Delta^2(h) := (\Delta\otimes\mathrm{id})\Delta(h) = (\mathrm{id}\otimes\Delta)\Delta(h) = h_{(1)}\otimes h_{(2)}\otimes h_{(3)}$, $h\in H$. The operation $F\cdot h = F\Delta^{n-1}(h)$, $F\in H^{\otimes n}$, $h\in H$, turns $H^{\otimes n}$ into a right H-module (the outer product of right regular H-modules).

A unital left H-module C gives rise to an operad (also denoted by C) such that

$$C(n) = \{f : C^{\otimes n}\to H^{\otimes n}\otimes_H C \mid f \text{ is } H^{\otimes n}\text{-linear}\}.$$

For example, if $\dim H = 1$ then what we obtain is just a linear space with poly-linear maps. The composition of such maps as well as the action of a symmetric group is defined in [6].

In these terms, if Var is a variety of Ω-algebras defined by a system of poly-linear identities S then a Var-pseudo-algebra structure on an H-module C is a functor from $\mathcal P_{\mathrm{Var}}$ to the operad C. Such a functor is determined by a family of $H^{\otimes 2}$-linear maps $*_i : C\otimes C\to H^{\otimes 2}\otimes_H C$ satisfying the identities (c.f. [21])

$$f^*(x_1,\dots,x_n) = 0,\quad f\in S,\ \deg f = n,$$

where f^* is obtained from f in the following way. Assume a poly-linear Ω-monomial u in the variables $x_1,\dots,x_n$ turns into a word $x_{\sigma(1)}\dots x_{\sigma(n)}$, $\sigma\in S_n$, after removing all brackets and symbols $\circ_i$, $i\in I$. Denote by u^* the same monomial u with all $\circ_i$ replaced with $*_i$. Then u^* can be considered as a map $C^{\otimes n}\to H^{\otimes n}\otimes_H C$, which is not necessary $H^{\otimes n}$-linear. However, $u^{(*)}=(\sigma\otimes_H \mathrm{id})u^*$ is $H^{\otimes n}$-linear. Finally, if $f=\sum\limits_\xi \alpha_\xi u_\xi$, $\alpha_\xi\in\Bbbk$, then

$$f^*(x_1,\dots,x_n)=\sum_\xi \alpha_\xi u_\xi^{(*)}.$$

Example 7 (c.f. [6]). Consider an Ω-algebra A, a cocommutative bialgebra H, and define $C=H\otimes A$. Then C is an pseudo-algebra with respect to the operations

$$(f\otimes a)*_i(h\otimes b)=(f\otimes h)\otimes_H(a\circ_i b),\quad f,h\in H,\ a,b\in A,\ i\in I.$$

Such a pseudo-algebra is denoted by $\mathrm{Cur}\,A$ (current pseudo-algebra). If A belongs to Var then, obviously, $\mathrm{Cur}\,A$ is a Var-pseudo-algebra over H.

Given a pseudo-algebra C with operations $*_i$, $i\in I$, one may define operations $\vdash_i$, $\dashv_i$ on the same space C as follows: if $a*_i b=\sum\limits_\xi (h_\xi\otimes f_\xi)\otimes_H d_\xi$ then

$$a\vdash_i b=\sum_\xi \varepsilon(h_\xi)f_\xi d_\xi,\quad a\dashv_i b=\sum_\xi h_\xi\varepsilon(f_\xi)d_\xi. \tag{6}$$

Proposition 2. *Let C be a* Var*-pseudo-algebra. Then $C^{(0)}$ is a di-*Var*-algebra.*

Proof. It is enough to check that (1) and (2) hold on $C^{(0)}$. Indeed, if $a*_i b=\sum\limits_\xi (h_\xi\otimes f_\xi)\otimes_H d_\xi$, $d_\xi *_j c=\sum\limits_\eta (h'_\eta\otimes f'_\eta)\otimes_H e_\eta$ then

$$(a\vdash_i b)*_j c=\sum_\eta\left(\sum_\xi \varepsilon(h_\xi)f_\xi d_\xi\right)*_j c=\sum_{\eta,\xi}(\varepsilon(h_\xi)f_\xi h'_\eta\otimes f'_\eta)\otimes_H e_\eta.$$

Hence,

$$(a\vdash_i b)\vdash_j c=\sum_{\eta,\xi}\varepsilon(h_\xi f_\xi h'_\eta)f'_\eta e_\eta.$$

On the other hand,

$$(a\dashv_i b)*_j c=\sum_\eta\left(\sum_\xi h_\xi\varepsilon(f_\xi)d_\xi\right)*_j c=\sum_{\eta,\xi}(h_\xi\varepsilon(f_\xi)h'_\eta\otimes f'_\eta)\otimes_H e_\eta,$$

so $(a\vdash_i b)\vdash_j c=(a\dashv_i b)\vdash_j c$ for all $a,b,c\in C$. The second identity in (1) can be proved in the same way.

Consider a poly-linear identity $f\in S$. This is straightforward to check (c.f. [22]) that if

$$f^{(*)}(a_1,\dots,a_n)=\sum_\xi (h_{1\xi}\otimes\dots\otimes h_{n\xi})\otimes_H c_\xi$$

then $f(a_1,\dots,\dot a_k,\dots,a_n)=\sum\limits_\xi h_{1\xi}\dots\varepsilon(h_{k\xi})\dots h_{n\xi}c_\xi$ in $C^{(0)}$. It is clear that if $f^{(*)}$ vanishes in C then $C^{(0)}$ satisfies the identity (2). $\square$

In particular, if B is a Var-algebra then $(\mathrm{Cur}\,B)^{(0)}$ is a di-Var-algebra.

Proposition 3. *If H contains a nonzero element T such that $\varepsilon(T)=0$ then every di-*Var*-algebra A embeds into $(\mathrm{Cur}\,\hat A)^{(0)}$.*

Proof. Recall that $\hat{A} = \bar{A} \oplus A$, $\mathrm{Cur}\,\hat{A} = H \otimes \hat{A}$. Define

$$\iota : A \to H \otimes \hat{A}, \quad \iota(a) = 1 \otimes \bar{a} + T \otimes a. \tag{7}$$

This map is obviously injective, and

$$\iota(a) *_i \iota(b) = (1\otimes 1)\otimes_H (1\otimes \overline{a \vdash_i b}) + (T\otimes 1)\otimes_H (1\otimes a \dashv_i b) + (1\otimes T)\otimes_H (1\otimes a \vdash_i b).$$

Since $\overline{a \vdash_i b} = \overline{a \dashv_i b}$ in $\hat{A}$, we have

$$\iota(a) \vdash_i \iota(b) = 1 \otimes \overline{a \vdash_i b} + T \otimes a \vdash_i b = \iota(a \vdash_i b),$$
$$\iota(a) \dashv_i \iota(b) = 1 \otimes \overline{a \dashv_i b} + T \otimes a \dashv_i b = \iota(a \dashv_i b).$$

□

3. Dendriform di- and trialgebras

The operad Dend of associative dendriform algebras is known to be Koszul dual (see [18] for details of Koszul duality) to the operad Dias $= \mathcal{D}_{\mathrm{As}}$ of diassociative algebras. Since Dias $\simeq$ As $\otimes$ Perm and it was noticed in [37] that for Perm (as well as for CommTrias) the Hadamard product $\otimes$ coincides with the Manin white product $\circ$, we have Dend $:= (\mathrm{As} \otimes \mathrm{Perm})^! = \mathrm{As} \bullet \mathrm{PreLie}$, where $\mathrm{As}^! = \mathrm{As}$, PreLie is the operad of pre-Lie algebras which is Koszul dual to Perm, $\bullet$ stands for the Manin black product of operads [18].

In general, for a binary operad $\mathcal{P}$ the *successor procedure* described in [5] gives rise to what is natural to call defining identities of di- or tri-$\mathcal{P}$-dendriform algebras. In addition, if $\mathcal{P}$ is quadratic then these $\mathcal{P}$-dendriform algebras are dual to the corresponding di- or tri-$\mathcal{P}^!$-algebras. In this case, obviously, $(\mathcal{P}^! \otimes \mathrm{Perm})^! = P \bullet \mathrm{PreLie}$ for dialgebras, and $(\mathcal{P}^! \otimes \mathrm{CommTrias})^! = \mathcal{P}^! \bullet \mathrm{PostLie}$, where $\mathrm{PreLie} = \mathrm{Perm}^!$ $\mathrm{PostLie} = \mathrm{CommTrias}^!$. This is closely related with Proposition 4 below.

In terms of identities, we do not need $\mathcal{P}$ to be quadratic (in fact, it is easy to generalize the successor procedure even for algebras with n-ary operations, $n \geq 2$).

Usually, the operations in a dendriform di- or trialgebra are denoted by $\prec$, $\succ$, and $\cdot$. We will use $\dashv$, $\vdash$, and $\perp$ instead.

Suppose Var is a variety of Ω-algebras defined by a family S of poly-linear identities, as above.

Definition 3. A *tri-*Var*-dendriform algebra* is an $\Omega^{(3)}$-algebra satisfying the identities

$$f^*(x_1, \dots, \dot{x}_{k_1}, \dots, \dot{x}_{k_l}, \dots, x_n), \quad f \in S,\ n = \deg f,\ 1 \leq k_1 < \dots < k_l \leq n, \tag{8}$$

for all $l = 1, \dots, n$, where $f^*(x_1, \dots, \dot{x}_{k_1}, \dots, \dot{x}_{k_l}, \dots, x_n)$ is obtained from f by means of the following procedure.

If u is a non-associative monomial, $u = vw$, $H = \{k_1, \dots, k_l\}$, $\deg v = p$, $H = H_1 \dot\cup H_2$ as above, then

$$\Phi^*(n)(u^H) = \begin{cases} \Phi^*(p)(v^{H_1}) \perp_i \Phi^*(n-p)(w^{H_2}), & \text{if } H_1, H_2 \neq \emptyset, \\ v^* \vdash_i \Phi^*(n-p)(w^H), & \text{if } H_1 = \emptyset, \\ \Phi^*(p)(v^H) \dashv_i w^*, & \text{if } H_2 = \emptyset, \end{cases}$$

where v^* stands for the linear combination of trees obtained when we replace each label $\circ_j$ in v with $\vdash_j + \dashv_j + \perp_j$).

Now, extend $\Phi^*(n)$ by linearity and set

$$f^*(x_1,\dots,\dot{x}_{k_1},\dots,\dot{x}_{k_l},\dots,x_n) := \sum_\xi \alpha_\xi \Phi^*(n)(u_\xi^H)$$

for $f(x_1,\dots,x_n)=\sum\limits_\xi \alpha_\xi u_\xi \in \mathcal{F}(n)$, $\alpha_\xi\in\Bbbk$, $H=\{k_1,\dots,k_l\}$.

To get the definition of a di-Var-dendriform algebra, it is enough to set $x\perp y=0$ and consider $|H|=1$ only.

Denote by $\operatorname{dend}\mathcal{D}_{\mathrm{Var}}$ and $\operatorname{dend}\mathcal{T}_{\mathrm{Var}}$ the operads governing di- and tri-Var-dendriform algebras.

Proposition 4. *Suppose $|\Omega|<\infty$. If $\mathcal{P}_{\mathrm{Var}}$ is a binary quadratic operad with $\mathcal{P}_{\mathrm{Var}}(1)=\Bbbk$ then $(\mathcal{D}_{\mathrm{Var}})^! = \operatorname{dend}\mathcal{D}_{\mathrm{Var}^!}$ and $(\mathcal{T}_{\mathrm{Var}})^! = \operatorname{dend}\mathcal{T}_{\mathrm{Var}^!}$, where $\mathrm{Var}^!$ stands for the class of algebras governed by the Koszul dual operad $\mathcal{P}^!_{\mathrm{Var}}$.*

Proof. We consider trialgebra case in details since it covers the dialgebra case.

Suppose $\mathcal{P}_{\mathrm{Var}}=\mathcal{P}(E,R)$ is a binary quadratic operad, i.e., a quotient operad of $\mathcal{F}$, $\mathcal{F}(2)=E$, with respect to the operad ideal generated by S_3-submodule $R\subseteq\mathcal{F}(3)$, see [18] for details.

The space E is spanned by $\mu_i : x_1\otimes x_2\mapsto x_1\circ_i x_2$ and $\mu_i^{(12)} : x_1\otimes x_2\mapsto x_2\circ_i x_1$, $i\in I$. Without loss of generality, we may assume that μ_i, $i\in I$, are linearly independent and

$$\mu_k^{(12)} = \sum_{i\in I}\alpha_{ik}\mu_i,\quad k\in I'\subseteq I,\ \alpha_{ik}\in\Bbbk,$$

are the only defining identities of Var of degree two, $|I'|=d\ge 0$ (if $\operatorname{char}\Bbbk\ne 2$, these are just commutativity and anti-commutativity). Denote by $N=2|I|-d$ the dimension of E.

The space $\mathcal{F}(3)$ can be naturally identified with the induced S_3-module $\Bbbk S_3\otimes_{\Bbbk S_2}(E\otimes E)$, where $E\otimes E$ is considered as an S_2-module via $(\mu\otimes\nu)^{(12)}=\mu\otimes\nu^{(12)}$, $\mu,\nu\in E$. Namely, the basis of $\mathcal{F}(3)$ consists of expressions

$$\sigma\otimes_{\Bbbk S_2}(\mu\otimes\nu),\quad \sigma\in\{e,(13),(23)\},$$

μ and ν range over a chosen basis of E. Therefore, $\dim\mathcal{F}(3)=3N^2$.

In terms of monomials (or binary trees), for example, $e\otimes_{\Bbbk S_2}(\mu_i\otimes\mu_j)$ corresponds to $(x_1\circ_j x_2)\circ_i x_3$, $e\otimes_{\Bbbk S_2}(\mu_i^{(12)}\otimes\mu_j)$ to $x_3\circ_i(x_1\circ_j x_2)$. A permutation $\sigma\in S_3$ in the first tensor factor permutes variables, e.g., $(13)\otimes_{\Bbbk S_2}(\mu_i^{(12)}\otimes\mu_j^{(12)})$ corresponds to $x_1\circ_i(x_2\circ_j x_3)$.

Recall that $E^\vee$ denotes the dual space to E considered as an S_2-module with respect to sgn-twisted action: $\langle\nu^{(12)},\mu\rangle=-\langle\nu,\mu^{(12)}\rangle$, $\nu\in E^\vee$, $\mu\in E$. If $\mathcal{F}^\vee$ is the free binary operad generated by $E^\vee$ then $(\mathcal{F}(3))^\vee\simeq\mathcal{F}^\vee(3)=\Bbbk S_3\otimes_{\Bbbk S_2}(E^\vee\otimes E^\vee)$.

The Koszul-dual operad $\mathcal{P}^!_{\mathrm{Var}}$ is the quotient of $\mathcal{F}^\vee$ by the operad ideal generated by $R^\perp\subset\mathcal{F}^\vee(3)$, the orthogonal space to R.

By the definition, the operad $\mathcal{T}_{\mathrm{Var}}$ governing the variety of tri-Var-algebras is equal to $\mathcal{P}(E^{(3)},R^{(3)})$, where the initial data $E^{(3)}$, $R^{(3)}$ are defined as follows. The space $E^{(3)}$ is spanned by μ_i^*, $(\mu_i^*)^{(12)}$, $i\in I$, $*\in\{\vdash,\dashv,\perp\}$, with respect to the relations

$$(\mu_k^{\vdash})^{(12)}=\sum_{i\in I}\alpha_{ik}\mu_i^{\dashv},\quad (\mu_k^{\dashv})^{(12)}=\sum_{i\in I}\alpha_{ik}\mu_i^{\vdash}$$

$$(\mu_k^{\perp})^{(12)}=\sum_{i\in I}\alpha_{ik}\mu_i^{\perp},\quad k\in I'.$$

The S_3-module $R^{(3)}$ is generated by the defining identities of tri-Var-algebras, i.e.,

$$R^{(3)} = \{\Phi(3)(f^H) \mid f \in R, \emptyset \neq H \subseteq \{1,2,3\}\} \oplus O^{(3)},$$

and $O^{(3)}$ is the S_3-submodule of $\mathcal{F}^{(3)}$ generated by

$$\begin{gathered} \mu_j^{\vdash} \otimes \mu_i^{\dashv} - \mu_j^{\vdash} \otimes \mu_i^{\vdash}, \quad \mu_j^{\vdash} \otimes \mu_i^{\perp} - \mu_j^{\vdash} \otimes \mu_i^{\vdash}, \\ (\mu_i^{\dashv})^{(12)} \otimes \mu_j^{\vdash} - (\mu_i^{\dashv})^{(12)} \otimes \mu_j^{\dashv}, \quad (\mu_i^{\dashv})^{(12)} \otimes \mu_j^{\perp} - (\mu_i^{\dashv})^{(12)} \otimes \mu_j^{\perp}, \\ i,j \in I. \end{gathered} \tag{9}$$

This is easy to calculate that $\dim E^{(3)} = 3N$, $\dim \mathcal{F}^{(3)}(3) = 27N^2$, $\dim O^{(3)} = 6N^2$, so $\dim R^{(3)} = 6N^2 + 7\dim R$. Denote by $O_+^{(3)}$ the S_3-submodule of $\mathcal{F}^{(3)}$ generated by the first summands of all relations from (9).

Suppose $f \in \mathcal{F}(3)$, $g \in \mathcal{F}^\vee(3)$, and let $H_1, H_2 \subseteq \{1,2,3\}$ be nonempty subsets. It follows from the definition of $\Phi(3)$ that $\langle \Phi(3)(f^{H_1}), \Phi(3)(g^{H_2})\rangle = 0$ if $H_1 \neq H_2$. For $H_1 = H_2 = H$, orthogonality of f and g implies $\langle \Phi(3)(f^H), \Phi(3)(g^H)\rangle = 0$ as well. Moreover, for every $f \in \mathcal{F}(3)$ we have $\langle \Phi(3)(f^H), O_+^{(3)}\rangle = 0$ since neither of terms from $O_+^{(3)}$ appears in images of $\Phi(3)$.

This is now easy to see that if $g \in R^\perp \subseteq \mathcal{F}^\vee(3)$ then $\langle f, \Phi^*(3)(g^H)\rangle = 0$ for every $f \in R^{(3)}$. Hence,

$$(R^\perp)^{(3*)} := \{\Phi^*(3)(g^H) \mid g \in R^\perp, \emptyset \neq H \subseteq \{1,2,3\}\} \subseteq (R^{(3)})^\perp.$$

On the other hand, $\dim R^\perp = 3N^2 - \dim R$, so $\dim (R^\perp)^{(3*)} = 21N^2 - 7\dim R$. Therefore, $\dim (R^\perp)^{(3*)} + \dim R^{(3)} = 27N^2$ and $(R^\perp)^{(3*)} = (R^{(3)})^\perp$. It remains to recall that, by definition, $\mathrm{dend}\mathcal{T}_{\mathrm{Var}} = \mathcal{P}(E^{(3)}, (R^\perp)^{(3*)})$. □

4. Embedding into Rota—Baxter algebras

Suppose B is an Ω-algebra. A linear map $R : B \to B$ is called a Rota—Baxter operator of weight $\lambda \in \Bbbk$ if

$$R(x) \circ_i R(y) = R(x \circ_i R(y) + R(x) \circ_i y + \lambda x \circ_i y) \tag{10}$$

for all $x, y \in B$, $i \in I$.

Let A be an $\Omega^{(3)}$-algebra. Consider the isomorphic copy A' of the underlying linear space A (assume $a \in A$ is in the one-to-one correspondence with $a' \in A'$), and define the following Ω-algebra structure on the space $\hat{A} = A \oplus A'$:

$$\begin{gathered} a \circ_i b = a \vdash_i b + a \dashv_i b + a \perp_i b, \quad a \circ_i b' = (a \vdash_i b)', \\ a' \circ_i b = (a \dashv_i b)', \quad a' \circ_i b' = (a \perp_i b)', \end{gathered} \tag{11}$$

for $a, b \in A$, $i \in I$.

Lemma 1. *Given a scalar $\lambda \in \Bbbk$, the linear map $R : \hat{A} \to \hat{A}$ defined by $R(a') = \lambda a$, $R(a) = -\lambda a$ ($a \in A$) is a Rota—Baxter operator of weight λ on the Ω-algebra $\hat{A}$.*

Proof. It is enough to check the relation (10). Straightforward computation shows

$$\begin{aligned} R(a + b') \circ_i R(x + y') &= \lambda^2(-a + b) \circ_i (-x + y) \\ &= \lambda^2(a \vdash_i x + a \dashv_i x + a \perp_i x - a \vdash_i y - a \dashv_i y - a \perp_i y \\ &\quad - b \vdash_i x - b \dashv_i x - b \perp_i x + b \vdash_i y + b \dashv_i y + b \perp_i y). \end{aligned}$$

On the other hand,

$$\begin{aligned}
&R((a+b')\circ_i R(x+y') + R(a+b')\circ_i (x+y') + \lambda(a+b')\circ(x+y')) \\
&= \lambda R((a+b')\circ_i(-x+y) + (-a+b)\circ_i(x+y') + (a+b')\circ(x+y')) \\
&= \lambda R(-a\vdash_i x - a\dashv_i x - a\perp_i x + a\vdash_i y + a\dashv_i y + a\perp_i y \\
&- (b\dashv_i x)' + (b\dashv_i y)' - a\vdash_i x - a\dashv_i x - a\perp_i x + b\vdash_i x + b\dashv_i x + b\perp_i x \\
&- (a\vdash_i y)' + (b\vdash_i y)' + a\vdash_i x + a\dashv_i x + a\perp_i x + (a\vdash_i y)' + (b\dashv_i x)' + (b\perp_i y)') \\
&= \lambda^2(-a\vdash_i y - a\dashv_i y - a\perp_i y + b\dashv_i y + a\vdash_i x + a\dashv_i x \\
&\quad + a\perp_i x - b\vdash_i x - b\dashv_i x - b\perp_i x + b\vdash_i y + b\perp_i y).
\end{aligned}$$

□

Lemma 2. *Let A be a di-*Var*-dendriform algebra. Then the map $R:\hat{A}\to\hat{A}$ defined by $R(a')=a$, $R(a)=0$ is a Rota—Baxter operator of weight $\lambda=0$ on $\hat{A}$.*

The proof is completely analogous to the previous one.

The following statement is well-known (c.f. [1, 15, 35]), but we will state its proof for readers' convenience.

Proposition 5. *Let B be an Ω-algebra with a Rota—Baxter operator R of weight $\lambda\neq 0$. Assume B belongs to* Var*. Then the same linear space B considered as $\Omega^{(3)}$-algebra with respect to the operations*

$$x\vdash_i y = \frac{1}{\lambda}R(x)\circ_i y,\quad x\dashv_i y = \frac{1}{\lambda}x\circ_i R(y),\quad x\perp_i y = x\circ_i y. \tag{12}$$

*is a tri-*Var*-dendriform algebra.*

Proof. Let $u=u(x_1,\dots,x_n)\in\mathcal{F}(n)$ be a poly-linear Ω-monomial. The claim follows from the following relation in B:

$$u^*(x_1,\dots,\dot{x}_{k_1},\dots,\dot{x}_{k_l},\dots,x_n) = \frac{1}{\lambda^{n-l}}u(R(x_1),\dots,x_{k_1},\dots,x_{k_l},\dots,R(x_n)), \tag{13}$$

i.e., in order to get a value of an $\Omega^{(3)}$-monomial in $\hat{A}$ we have to replace every non-emphasized variable x_i ($i\notin H=\{k_1,\dots,k_l\}$) with $\frac{1}{\lambda}R(x_i)$.

Relation (13) is clear for $n=1,2$. In order to apply induction on n, we have to start with the case when $H=\emptyset$. Recall that $u^*(x_1,\dots,x_n)$ stands for the expression obtained from u by means of replacing each $\circ_i$ with $\vdash_i + \dashv_i + \perp_i$. Then

$$R(u^*(x_1,\dots,x_n)) = \frac{1}{\lambda^{n-1}}u(R(x_1),\dots,R(x_n)),\quad n\geq 2, \tag{14}$$

in $\hat{A}$. Indeed, for $n=2$ we have exactly the Rota—Baxter relation. If $u=v\circ_i w$, $v=v(x_1,\dots,x_p)$, $w=w(x_{p+1},\dots,x_n)$, then, by induction,

$$\begin{aligned}
R(u^*) &= R(v^*\vdash_i w^* + v^*\dashv_i w^* + v^*\perp_i w^*) \\
&= \frac{1}{\lambda}R(R(v^*)\circ_i w^* + v^*\circ_i R(w^*) + \lambda v^*\perp_i w^*) = \frac{1}{\lambda}R(v^*)\circ_i R(w^*) \\
&= \frac{1}{\lambda}\frac{1}{\lambda^{p-1}}v(R(x_1),\dots,R(x_p))\circ_i \frac{1}{\lambda^{n-p-1}}w(R(x_{p+1}),\dots,R(x_n)) \\
&= \frac{1}{\lambda^{n-1}}u(R(x_1),\dots,R(x_n)).
\end{aligned}$$

Now, let us finish proving (13). If $u = v \circ_i w$, $\deg v = p$, $H = H_1 \dot\cup H_2$ then there are three cases: (a) $H_1, H_2 \neq \emptyset$; (b) $H_1 = \emptyset$; (c) $H_2 = \emptyset$.

In the case (a), $u^*(x_1, \dots, \dot x_{k_1}, \dots, \dot x_{k_l}, \dots, x_n) = \Phi^*(n)(u^H) = \Phi^*(p)(v^{H_1}) \perp_i \Phi^*(n)(w^{H_2})$, and it remains to apply inductive assumption and the definition of $\perp_i$ from (12).

In the case (b), $\Phi^*(n)(u^H) = v^* \vdash_i \Phi^*(n-p)(w^H)$, so for any $a_1, \dots, a_n \in B$ we can apply (14) to get

$$[\Phi^*(n)(u^H)](a_1, \dots, a_n) = \frac{1}{\lambda} R(v^*(a_1, \dots, a_p)) \circ_i [\Phi^*(n-p)(w^H)](a_{p+1}, \dots, a_n)$$
$$= \frac{1}{\lambda^p} v(R(a_1), \dots, R(a_p)) \circ_i \frac{1}{\lambda^{n-p-l}} w(R(a_{p+1}, \dots, a_{k_1}, \dots, a_{k_l}, \dots, R(a_n))$$
$$= \frac{1}{\lambda^{n-l}} u(R(a_1, \dots, a_{k_1}, \dots, a_{k_l}, \dots, R(a_n)).$$

The case (c) is completely analogous. □

Remark 1. *Following* [15]*, the dendriform operations on a Rota—Baxter algebra should be defined as*

$$x \vdash_i y = R(x) \circ_i y, \quad x \dashv_i y = x \circ_i R(y), \quad x \perp_i y = \lambda x \circ_i y. \qquad (15)$$

It is easy to see that for $\lambda \neq 0$ one should just re-scale these binary operations to get (12)*.*

Proposition 6 ([1, 35])**.** *Let B be an Ω-algebra with a Rota—Baxter operator R of weight $\lambda = 0$. Assume B belongs to* Var*. Then the same linear space B considered as $\Omega^{(2)}$-algebra with respect to $x \vdash_i y = R(x) \circ_i y$, $x \dashv_i y = x \circ_i R(y)$ is a di-*Var*-dendriform algebra.*

The proof is analogous to the proof of the previous statement.

Theorem 1. *The following statements are equivalent:*

(1) *A is a tri-*Var*-dendriform algebra;*
(2) *$\hat A$ belongs to* Var*.*

Proof. (1) Assume A is a tri-Var-dendriform algebra, and let S be the set of defining identities of Var. We have to check that every $f \in S$ holds on $\hat A$.

First, let us compute a monomial in $\hat A = A \oplus A'$ when all its arguments belong to the first summand.

Lemma 3. *Suppose $u = u(x_1, \dots, x_n) \in \mathcal{F}(n)$ is a poly-linear Ω-monomial of degree n. Then in the Ω-algebra $\hat A$ we have*

$$u(a_1, \dots, a_n) = \sum_H \Phi^*(n)(u^H)(a_1, \dots, a_n), \quad a_i \in A, \qquad (16)$$

where H ranges over all nonempty subsets of $\{1, \dots, n\}$.

Proof. By the definition of multiplication in $\hat A$, $u(a_1, \dots, a_n) = u^*(a_1, \dots, a_n)$, where u^* means the same as in the definition of $\Phi^*(n)$. In particular, for $n = 1, 2$ the statement is clear. Proceed by induction on $n = \deg u$. Assume $u = v \circ_i w$, and, without loss of generality, $v = v(x_1, \dots, x_p)$, $w = w(x_{p+1}, \dots, x_n)$. Then

$$u(a_1, \dots, a_n) = v^*(a_1, \dots, a_p) \vdash_i \left(\sum_{H_2} \Phi^*(n-p)(w^{H_2})(a_{p+1}, \dots, a_n)\right) + \left(\sum_{H_1} \Phi^*(p)(v^{H_1})(a_1, \dots, a_p)\right) \perp_i \left(\sum_{H_2} \Phi^*(n-p)(w^{H_2})(a_{p+1}, \dots, a_n)\right) + \left(\sum_{H_1} \Phi^*(p)(v^{H_1})(a_1, \dots, a_p)\right) \dashv_i w^*(a_{p+1}, \dots, a_n), \quad (17)$$

where H_1 and H_2 range over all nonempty subsets of $\{1, \dots, p\}$ and $\{p+1, \dots, n\}$, respectively. It is easy to see that the overall sum is exactly the right-hand side of (16): The first (second, third) group of summands in (17) corresponds to $H = H_2 \subseteq \{p+1, \dots, n\}$, ($H = H_1 \cup H_2$, $H = H_1 \subseteq \{1, \dots, p\}$, respectively). □

Next, assume that $l > 0$ arguments belong to A'.

Lemma 4. *Suppose $u = u(x_1, \dots, x_n) \in \mathcal{F}(n)$ is a poly-linear Ω-monomial of degree n, $H = \{k_1, \dots, k_l\}$ is a nonempty subset of $\{1, \dots, n\}$. Then in the Ω-algebra $\hat{A}$ we have*

$$u(a_1, \dots, a'_{k_1}, \dots, a'_{k_l}, \dots, a_n) = \big(\Phi^*(n)(u^H)(a_1, \dots, a_n)\big)'. \quad (18)$$

Proof. For $n = 1, 2$ the statement is clear. If $u = v \circ_i w$ for some $i \in I$ as above then we have to consider three natural cases: (a) $H \subseteq \{1, \dots, p\}$; (b) $H \subseteq \{p+1, \dots, n\}$; (c) variables with indices from H appear in both v and w.

In the case (a), the inductive assumption implies

$$u(a_1, \dots, a'_{k_1}, \dots, a'_{k_l}, \dots, a_n) = v(a_1, \dots, a'_{k_1}, \dots, a'_{k_l}, \dots, a_p) \dashv_i w^*(a_{p+1}, \dots, a_n) = \big(\Phi^*(p)(v^H)(a_1, \dots, a_p) \dashv_i w^*(a_{p+1}, \dots, a_n)\big)',$$

and it remains to recall the definition of $\Phi^*(n)$. Case (b) is analogous.

In the case (c), $H = H_1 \dot\cup H_2$ as above and

$$u(a_1, \dots, a'_{k_1}, \dots, a'_{k_l}, \dots, a_n) = \Phi^*(p)(v^{H_1})(a_1, \dots, a_p) \perp_i \Phi^*(n-p)(w^{H_2})(a_{p+1}, \dots, a_n)$$

that proves the claim. □

Finally, suppose $f \in S$ is a poly-linear identity of degree n. Then $\Phi^*(n)(f^H)$ is an identity on the $\Omega^{(3)}$-algebra A, so Lemmas 3 and 4 imply f to hold on $\hat{A}$.

(2) The map $\iota : A \to \hat{A}$, $\iota(a) = a'$, is an embedding of the $\Omega^{(3)}$-algebra A into $\hat{A}$ equipped with operations (12). By Proposition 5, $\hat{A}$ is a tri-Var-dendriform algebra, therefore, so is A. □

Remark 2. *Since $\lambda \neq 0$, we may conclude that every tri-*Var*-dendriform algebra A embeds into a Rota—Baxter algebra $B \in$* Var *of weight λ in the sense of Aguiar* [1] *(see* (15)*): It is sufficient to re-scale the product on B.*

If $\lambda = 0$ then the simple reduction of Theorem 1 by means of Lemma 2 leads to

Theorem 2. *Suppose A is an $\Omega^{(2)}$-algebra, and let $\hat{A}$ stands for an Ω-algebra defined by* (11) *with $x \perp_i y \equiv 0$. Then the following statements are equivalent:*

(1) *A is a di-*Var*-dendriform algebra;*
(2) *$\hat{A}$ belongs to* Var*.*

Remark 3. *It is interesting to note that A is a simple di-*Var*-dendriform algebra if and only if $\hat{A}$ is a simple Rota—Baxter algebra.*

The standard reasoning allows to conclude the following.

Corollary 1 (c.f. [12])**.** *Every di-*Var*-dendriform algebra embeds into its universal enveloping Rota—Baxter* Var*-algebra of weight $\lambda = 0$.*

Corollary 2. *Every tri-*Var*-dendriform algebra embeds into its universal enveloping Rota—Baxter* Var*-algebra of weight $\lambda \neq 0$.*

Remark 4. *All results of this section remain valid for algebras over a commutative ring K if we replace the condition $\lambda \neq 0$ with $\lambda \in K^*$, where K^* is the set of invertible elements of K.*

5. Skew trialgebras and Rota—Baxter algebras

Consider a slightly modified analogue of trialgebras which we shortly call s-trialgebras.

Definition 4. A *s-tri-*Var*-algebra* is an $\Omega^{(3)}$-algebra satisfying the identities (1), (5).

In other words, we exclude the identities $x_1 \dashv_i (x_2 \perp_j x_3) = x_1 \dashv_i (x_2 \dashv_j x_3)$, $(x_1 \perp_i x_2) \vdash_j x_3 = (x_1 \vdash_i x_2) \vdash_j x_3$ from the definition of a tri-Var-algebra.

For any $\Omega^{(3)}$-algebra A satisfying the identities (1) we can also construct (as in the dialgebra case) the Ω-algebra $\hat{A} = \bar{A} \oplus A$ as follows (similar to (5)): $\bar{A} = A/\mathrm{Span}\,\{a \vdash_i b - a \dashv_i b \mid a, b \in A,\ i \in I\}$, $\bar{a} \circ_i \bar{b} = \overline{a \vdash_i b}$, $\bar{a} \circ_i b = a \vdash_i b$, $a \circ_i \bar{b} = a \dashv_i b$, $a \circ_i b = a \perp_i b$. An analogue of Proposition 1 holds for this construction, i.e., it gives an equivalent definition of a s-tri-Var-algebra.

It turns out that s-tri-Var-algebras are closely related with Γ-conformal algebras introduced in [19]. These systems appeared as "discrete analogues" of conformal algebras defined over a group Γ. From the general point of view, these are pseudo-algebras over the group algebra $H = \Bbbk\Gamma$ considered as a Hopf algebra with respect to canonical coproduct $\Delta(\gamma) = \gamma \otimes \gamma$ and counit $\varepsilon(\gamma) = 1$, $\gamma \in \Gamma$.

We consider the case when $\Gamma = \langle \mathbb{Z}, + \rangle$, $H = \Bbbk[t, t^{-1}]$. If C is a pseudo-algebra over H, i.e., a $\mathbb{Z}$-conformal algebra with operations $*_i$, $i \in I$, then for every $a, b \in C$ their pseudo-product $a *_i b \in H^{\otimes 2} \otimes_H C$ can be presented as

$$a *_i b = \sum_{n \in \mathbb{Z}} (t^{-n} \otimes 1) \otimes_H c_n,$$

where almost all c_n are zero. It is convenient to denote c_n by $a_{(n)}b$ [19]. These operations provide an equivalent definition of a $\mathbb{Z}$-conformal algebra: This is a linear space with bilinear operations $\{(\cdot_{(n)}\cdot) \mid n \in \mathbb{Z}\}$ and with a linear invertible mapping t such that the following axioms are satisfied:

(Z1) $a_{(n)}b = 0$ for almost all $n \in \mathbb{Z}$;
(Z2) $ta_{(n)}b = a_{(n+1)}b$;
(Z3) $t(a_{(n)}b) = ta_{(n)}tb$.

A $\mathbb{Z}$-conformal algebra C is associative if $a_{(n)}(b_{(m)}c) = (a_{(n-m)}b)_{(m)}c$ for all $n, m \in \mathbb{Z}$, $a, b, c \in C$.

Proposition 2 implies that every di-Var-algebra can be embedded into a current $\mathbb{Z}$-conformal algebra over an algebra from Var (one may consider, e.g., $T = 1 - t$). For s-trialgebras, a similar statement holds.

Example 8. Let C be an associative $\mathbb{Z}$-conformal algebra. Then, with respect to the operations $\dashv, \vdash$ from (6) and $a \perp b = a_{(0)}b$, the vector space C is a s-tri-associative algebra. Let us denote it also by $C^{(0)}$.

There is an interesting question: Whether a trialgebra or s-trialgebra A can be embedded into $C^{(0)}$ for some $\mathbb{Z}$-conformal algebra C. We have a positive answer for Loday trialgebras and only for $\operatorname{char}\Bbbk = 2$. Then the mapping ϕ from (7) realizes this embedding of A into $\operatorname{Cur}\hat{A}$.

Example 9. A vector space A endowed with two binary operations $\vdash$, $\perp$ belongs to the variety sCommTrias (skew commutative tri-associative algebras) if both operations are associative, $\perp$ is commutative and they also satisfy the following identities:

$$x_1 \vdash (x_2 \perp x_3) = (x_1 \vdash x_2) \perp x_3, \quad (x_1 \vdash x_2) \vdash x_3 = (x_2 \vdash x_1) \vdash x_3.$$

This is easy to derive from the definition that free sCommTrias$[X]$ algebra is nothing but Perm$\langle$Comm$[X]\rangle$, its linear basis consists of words

$$u_1 \vdash u_1 \vdash \ldots \vdash u_k \vdash u_0, \quad u_1 \leqslant \ldots \leqslant u_k,$$

where u_i are basic monomials of the polynomial algebra Comm$[X]$ with respect to the operation $\perp$ and some linear ordering $\leqslant$.

Example 10. Let $\langle A, \cdot\rangle$ be an associative algebra with a derivation d such that $d^2 = 0$. Defining $a \vdash b = d(a)b$, $a \dashv b = ad(b)$ we obtain s-tri-associative algebra $(A, \vdash, \dashv, \cdot)$.

Example 11. An associative s-trialgebra A with respect to the operations $[x, y] = x \dashv y - x \vdash y$ and $x \cdot y = x \perp y$ turns into a dialgebra analogue of a Poisson algebra: The operation $[\cdot,\cdot]$ satisfies the Leibniz identity and $\cdot$ is associative. Moreover, the Poisson identity holds:

$$[xy, z] = x[y, z] + [x, z]y.$$

In [28], the same operations $[\cdot,\cdot]$ and $\cdot$ were considered for ordinary triassociative algebra (in the sense of Definition 2). The noncommutative analogue of a Poisson algebra obtained in this way satisfies one more identity $[x, yz - zy] = [x, [y, z]]$ which does not appear in the case of s-tri-associative algebras.

Let us define a class of "skew" dendriform algebras associated with a variety Var of algebras.

Definition 5. A *s-tri-*Var*-dendriform algebra* is an $\Omega^{(3)}$-algebra satisfying

$$(x_1 \perp_i x_2) \vdash_j x_3 = 0, \quad x_1 \dashv_i (x_2 \perp_j x_3) = 0, \quad i, j \in I,$$

and the analogues of identities (8) with the following difference: to define v^* one should replace $\circ_j$ with $\dashv_j + \vdash_j$.

As above, the class of s-tri-Var-dendriform algebras is Koszul dual to the class of s-tri-Var$^!$-algebras.

We can prove the statement about an embedding of s-trialgebras into corresponding Rota—Baxter algebras.

Theorem 3. *For every s-tri-*Var*-dendriform algebra A there exists an algebra $\hat{A} \in$* Var *with a Rota—Baxter operator R of weight zero and an injective map $\iota : A \to \hat{A}$ such that $\iota(a \vdash_i b) = R(\iota(a)) \circ_i \iota(b)$, $\iota(a \dashv_i b) = \iota(a) \circ_i R(\iota(b))$, and $\iota(a \perp_i b) = \iota(a) \circ_i \iota(b)$.*

Proof. To prove the statement, define Eilenberg construction for A as $\hat{A} = A \oplus A'$ by (11), but also with one difference: $a \circ_i b = a \dashv_i b + a \vdash_i b$. This is a Rota—Baxter algebra with an operator R from Lemma 2. Other steps of the proof of Theorem 1 remain the same. □

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