A complete set of multidimensional Bell inequalities

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Abstract. We give a multidimensional generalisation of the complete set of Bell-correlation inequalities given by Werner and Wolf in [20] for the two-dimensional case. Our construction applies for the n parties, two-observables case, where each observable is d-valued. The d^{d^n} inequalities obtained involve homogeneous polynomials. They define the facets of a polytope in a complex vector space of dimension d^n.

1. Introduction

The search for Bell inequalities has been the subject of a lot of work. The problem is the following. Assume that a physical system is made of n subsystems. For each subsystem, a set of m different observables is considered. The outcomes of each of the nm observables belong to a set of cardinality d. The problem is to find inequalities which must be satisfied when a local realistic model is assumed. The first such inequalities were provided by Bell [2] for the case $(n, m, d) = (2, 2, 2)$. It was also shown that Quantum Mechanics violate these inequalities. The CHSH inequalities given in [4] were shown in [6] to be a complete set for the case $(2, 2, 2)$. This means that these inequalities provide necessary and sufficient conditions for the existence of a local realistic model.

The authors of [20] gave a complete set of 2^{2^n} Bell inequalities for dichotomic observables, with arbitrary number of parties (case $(n, 2, 2)$). The structure of these inequalities was further studied in [16], where a recursive method to compute Bell inequalities is also given. The tool for this construction was the Walsh-Hadamard transform of Boolean functions. See also [18] which gives some insight and useful details.

A method to obtain a complete set of dichotomic Bell inequalities was given in [15]. It has notably been used to exhibit a complete set for the case $(2, 3, 2)$.

The multidimensional case has also been considered in numerous references. Reasons to explore beyond the two-dimensional case include that multidimensional entangled quantum states are known to be more resistant to noise, and that they can lead to stronger violations of local realism [10]. Also there are specific uses of the tridimensional case for quantum cryptography [11]. The pioneer work for multiple outcome Bell inequalities was [5], where a family of multidimensional Bell inequalities, that generalize CHSH, was obtained. Moreover, these inequalities have been later proved tight [13].

However, no complete set has been given yet, beyond the two-dimensional case.

Instead of the joint probabilities used by many authors for the multi- or three-dimensional case ([1], [5], [12], [15]), we study the correlations between different observables using correlation functions. In general, if $X_i(\lambda)$ and $Y_j(\lambda)$ are the values obtained by party i for the observable $\hat{X}$ and by party j for the observable $\hat{Y}$, the corresponding correlation is given by

$$\int_\Lambda X_i(\lambda) Y_j(\lambda) \rho(\lambda)\, d\lambda$$

where Λ is the domain of the hidden variables λ and ρ with $\int_\Lambda \rho(\lambda)\, d\lambda = 1$ is a density function. These correlation functions have been widely used for the study of the two-dimensional case, where the outcomes belong to $\{\pm 1\}$. We use the same correlation functions also for the multidimensional case, but the outcomes are now d-th roots of unity in $\mathbb{C}$. This approach has yet been considered for the $d = 3$ case in [8], [11], [12], [19].

We use a geometrical approach. Froissart [7] has apparently been the first to do so, and then the authors of [9] independently. It was shown in [14] that the local-realistic domain is a convex polytope (for joint

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probability distributions). The polytope corresponding to joint probabilities and the one corresponding to correlation functions are strongly related because of the relation $E(X) = 2p(X = 1) - 1$ between expectation values and probabilities, in the case $d = 2$. The polytope we consider belongs to a complex vector space of dimension d^n.

Our inequalities are tight. This means that they define the facets of the polytope. The problem of obtaining all the (tight) inequalities was only solved in the two-dimensional setting ([6], [15], with joint probabilities, and [20] with correlation functions).

Our inequalities involve powers of observables, arranged in homogeneous polynomial expressions. Powers of observables have already been used in [19]. It turns out that the method developed for $(n, 2, 2)$ generalizes pretty well for the multidimensional, two-observables per party case, by means of multidimensional discrete Fourier transform. With this tool, we are able to give a complete set of tight Bell inequalities for the case $(n, 2, d)$.

In this paper, we first presents background about multidimensional Fourier transform (DFT for short). Then we recall some facts about the duality of polytopes in (finite dimensional) Hilbert spaces and study some useful relations between DFT and duality. Then we produce d^{d^n} Bell inequalities which generalize those obtained in [20]. We study the polynomials involved in these inequalities and give some facts about the symmetries observed. Then we prove that our Bell inequalities form a complete set of tight ones. Finally we briefly explore the case $d = 3$.

2. Multidimensional discrete Fourier transform

There are numerous references for the discrete Fourier transform. One of them is [3]. However, we give here all the material we need for our purposes.

Maps from $\mathbb{Z}_d^n$ to the set of d-th roots of 1

The main tool for the classification of dichotomic Bell inequalities is the Walsh-Hadamard transform for Boolean functions. For our generalisation of the dichotomic case, we will use d-valued functions and multidimensional discrete Fourier transform.

There are two equivalent ways to define Boolean functions: it can be a map F from $\{0,1\}^n$ to $\{0,1\}$ (additive convention), or a map f from $\{0,1\}^n$ to $\{1,-1\}$ (multiplicative convention). The equivalence is of course given by $f = (-1)^F$. The multiplicative convention is more comfortable when dealing with Walsh-Hadamard transforms. We also adopt a multiplicative convention, and the considered functions will take their values in the set

$$\mathcal{U} = \{1, \omega, \ldots, \omega^{d-1}\} \qquad \text{where } \omega = \exp(2i\pi/d). \tag{1}$$

We put $\mathbb{Z}_d = \{0, 1, \ldots, d-1\}$ and denote by $\mathbb{Z}_d^n$ the set of n-uples with components in $\mathbb{Z}_d$ ($d, n \in \mathbb{N}^*$). Also, we denote by $\mathcal{F}$ or $\mathcal{F}_{d,n}$ the set of maps from $\mathbb{Z}_d^n$ to $\mathcal{U}$. There are d^{d^n} such functions.

The DFT

Let f be a map from $\mathbb{Z}_d^n$ to the complex field $\mathbb{C}$ (or to $\mathcal{U}$ as a particular case). The (multidimensional) discrete Fourier transform of f is the map $\mathrm{DFT}\, f = \hat{f}$, also from $\mathbb{Z}_d^n$ to $\mathbb{C}$, defined by

$$\hat{f}(r_1, \ldots, r_n) = \sum_{s_1, \ldots, s_n \in \mathbb{Z}_d} \omega^{r_1 s_1 + \cdots + r_n s_n} f(s_1, \ldots, s_n) \tag{2}$$

or, written in compact form, $\hat{f}(r) = \sum_{s \in \mathbb{Z}_d^n} \omega^{r \cdot s} f(s)$ where $r \cdot s = \sum_{i=1}^n r_i s_i$ is the standard scalar product of the n-uples r and s.

We denote as H_d the matrix $(\omega^{ij})_{0 \leqslant i,j \leqslant n-1}$. The n-th tensor power of H_d is the $D \times D$ matrix, with $D = d^n$, given by

$$H_d^{\otimes n} := (\omega^{r \cdot s})_{r,s \in \mathbb{Z}_d^n}.$$

The matrices $H_d^{\otimes n}$ can be built up from blocks using recursion on n:

$$H_d^{\otimes 0} = (1) \qquad \text{and} \qquad H_d^{\otimes n} = (\, \omega^{ij} H_d^{\otimes n-1}\,)_{0 \leqslant i,j \leqslant n-1}\,. \tag{3}$$

These matrices are a generalization of the usual Hadamard matrices which are obtained in the special case $d = 2$ (hence $\omega = -1$).

A map f from $\mathbb{Z}_d^n$ to $\mathbb{C}$ can be identified to the vector of its values $(f(s))_{s\in\mathbb{Z}_d^n}$. The (column) vector of the values of $\hat{f}$ can be obtained applying the matrix $H_d^{\otimes n}$ to the (column) vector of the values of f :

$$\begin{pmatrix} \hat{f}(0,0,\ldots,0) \\ \hat{f}(1,0,\ldots,0) \\ \vdots \\ \hat{f}(d-1,\ldots,d-1) \end{pmatrix} = H_d^{\otimes n} \begin{pmatrix} f(0,0,\ldots,0) \\ f(1,0,\ldots,0) \\ \vdots \\ f(d-1,\ldots,d-1) \end{pmatrix}.$$

Hence, the map DFT: $f \mapsto \hat{f}$ is a linear map from $\mathbb{C}^{d^n}$ to itself.

Inverse DFT

Let also define $H_d^{*\otimes n}$ the matrix $(\omega^{-r\cdot s})_{r,s\in\mathbb{Z}_d^n}$. It can be checked that

$$H_d^{*\otimes n} H_d^{\otimes n} = d^n I.$$

Hence, the inverse transform DFT^{-1} is obtained by

$$f(s_1,\ldots,s_n) = \frac{1}{d^n} \sum_{r_1,\ldots,r_n\in\mathbb{Z}_d} \omega^{-(r_1 s_1+\cdots+r_n s_n)} \hat{f}(r_1,\ldots,s_n)$$

or, in compact form, $f(s) = \frac{1}{d^n}\sum_{r\in\mathbb{Z}_d^n} \omega^{-r\cdot s}\hat{f}(r)$.

In the particular case $d = 2$, the multidimensional discrete Fourier transform is the Walsh-Hadamard transform of Boolean functions:

$$\hat{f}(w) = \sum_{x\in\{0,1\}^n} (-1)^{w\cdot x} f(x)$$

(using the multiplicative convention: $f(x) \in \{1,-1\}$).

Some easy results

Some easy results can be derived from the definition given by Equation (2), between the discrete Fourier transforms of two elements of $\mathcal{F}_{d,n}$ which are related in some way:

2.1. — Proposition. *Put $\hat{f} = \mathrm{DFT}\, f$ and $\hat{g} = \mathrm{DFT}\, g$ where f and g belong to $\mathcal{F}_{d,n}$.*
(a) If $g(s) = f(-s)$ for all $s \in \mathbb{Z}_d^n$, then $\hat{g}(r) = \hat{f}(-r)$ for all $r \in \mathbb{Z}_d^n$.
(b) If $g(s) = f(-s)^*$ for all $s \in \mathbb{Z}_d^n$, then $\hat{g}(r) = \hat{f}(r)^*$ for all $r \in \mathbb{Z}_d^n$ (denotes complex conjugation).*
(c) Let $\delta \in \mathbb{Z}_d^n$. If $g(s) = f(s+\delta)$ for all $s \in \mathbb{Z}_d^n$ (addition in $\mathbb{Z}_d^n$ is assumed component-wise and modulo d), then $\hat{g}(r) = \omega^{-r\cdot\delta}\hat{f}(r)$ for all $r \in \mathbb{Z}_d^n$.
(d) Let $\delta \in \mathbb{Z}_d^n$. If $g(s) = \omega^{\delta\cdot s} f(s)$ for all $s \in \mathbb{Z}_d^n$, then $\hat{g}(r) = \hat{f}(r+\delta)$ for all $r \in \mathbb{Z}_d^n$.
(e) Let σ be a permutation of the set $\{1,\ldots,n\}$. For $s = (s_1,\ldots,s_n) \in \mathbb{Z}_d^n$, we use the shorthand notation $\sigma(s) = (s_{\sigma(1)},\ldots,s_{\sigma(n)})$. If $g(s) = f\big(\sigma(s)\big)$ for all $s \in \mathbb{Z}_d^n$, then $\hat{g}(r) = \hat{f}\big(\sigma(r)\big)$ for all $r \in \mathbb{Z}_d^n$.

PROOF — We show only the last two assertions and leave the first three to the reader. Assume that $g(s) = \omega^{\delta\cdot s} f(s)$ for all $s \in \mathbb{Z}_d^n$. Then

$$\hat{g}(r) = \sum_{s\in\mathbb{Z}_d^n} \omega^{r\cdot s} g(s) = \sum_{s\in\mathbb{Z}_d^n} \omega^{r\cdot s}\omega^{\delta\cdot s} f(s)$$

for all $r \in \mathbb{Z}_d^n$. Hence,

$$\hat{g}(r-\delta) = \sum_{s\in\mathbb{Z}_d^n} \omega^{(r-\delta)\cdot s}\omega^{\delta\cdot s} f(s) = \sum_{s\in\mathbb{Z}_d^n} \omega^{r\cdot s} f(s) = \hat{f}(r).$$

This proves assertion (d). Assume now that $g(s) = f\big(\sigma(s)\big)$ for all $s \in \mathbb{Z}_d^n$. Then, for all $r \in \mathbb{Z}_d^n$,

$$\begin{aligned}\hat{g}(r) &= \sum_{s\in\mathbb{Z}_d^n} \omega^{r\cdot s} g(s) = \sum_{s\in\mathbb{Z}_d^n} \omega^{r\cdot s} f\big(\sigma(s)\big) \\ &= \sum_{s\in\mathbb{Z}_d^n} \omega^{r\cdot\sigma^{-1}(s)} f(s) \qquad \text{because } \sigma \text{ induces a permutation on } \mathbb{Z}_d^n.\end{aligned}$$

Hence

$$\hat{g}\big(\sigma^{-1}(r)\big) = \sum_{s\in\mathbb{Z}_d^n} \omega^{\sigma^{-1}(r)\cdot\sigma^{-1}(s)} f(s) = \sum_{s\in\mathbb{Z}_d^n} \omega^{r\cdot s} f(s) = \hat{f}(r).$$

This proves assertion (e). □

3. Convex hulls

Let $D \in \mathbb{N}$. We denote $\langle\beta,\gamma\rangle = \sum_{i=1}^D \beta_i^*\gamma_i$ the usual Hermitian inner product in $\mathbb{C}^D$. The complex vector space C^D can also be viewed as a vector space over $\mathbb{R}$, with dimension $2D$. Each element $\beta \in \mathbb{C}^D$ can be alternatively written as a D-uple $(\beta_1,\dots,\beta_D)$ of coordinates belonging to $\mathbb{C}$ or as a $2D$-uple $(x_1,y_1,\dots,x_D,y_D)$ of coordinates belonging to $\mathbb{R}$, with the relations $\beta_k = x_k + iy_k$. Recall that the real part of the inner product $\langle\cdot,\cdot\rangle$ is nothing more than the usual scalar product in $\mathbb{R}^{2D}$:

$$\operatorname{Re}\langle\beta,\gamma\rangle = \operatorname{Re}\sum_{k=1}^D \beta_k^*\gamma_k = \sum_{k=1}^D (x_k z_k + y_k t_k) \qquad \begin{array}{l}\text{if } \beta_k = x_k + iy_k \text{ and } \gamma_k = z_k + it_k,\\ \text{with } x_k, y_k, z_k, t_k \in \mathbb{R}.\end{array}$$

Let S be a subset of $\mathbb{C}^D$. The *convex hull* of S is the set

$$\operatorname{Hull} S := \left\{ \sum_k p_k\beta_k \text{ with } \beta_k \in S \text{ and } p_k \in \mathbb{R}_+ \text{ such that } \sum_k p_k = 1 \right\}.$$

The *dual* (or *polar*) of the set S is, by definition, the set

$$T = S^\circ := \{\gamma \in \mathbb{C}^D \mid \operatorname{Re}\langle\beta,\gamma\rangle \leqslant 1, \forall\beta \in S\}. \tag{4}$$

When S is a polytope containing 0, the vertices of the dual T correspond to the facets of S. To be precise, γ is a vertice of T if and only if the hyperplane defined by the equation $\operatorname{Re}\langle\beta,\gamma\rangle = 1$ contains a facet of S.

The following result holds (the bipolar Theorem, see [17]):

3.1. — Theorem. *For any subset S of $\mathbb{C}^D$ containing 0, the dual $S^{\circ\circ}$ of the dual of S is the convex hull of S.*

The hull of $\mathcal{U}$ and its dual

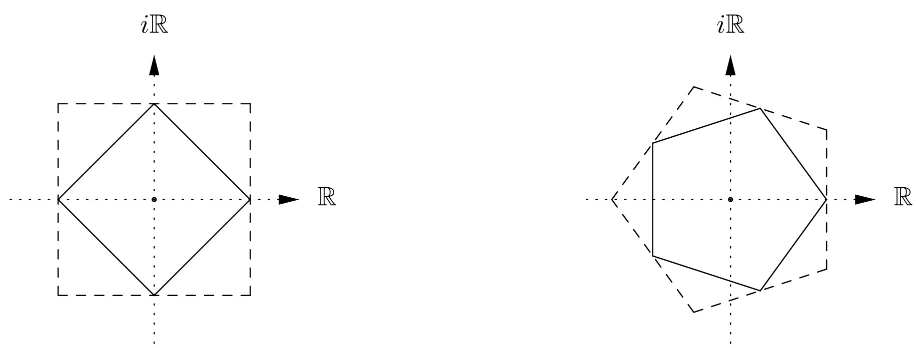
We assume here $d > 2$. The convex hull of the set $\mathcal{U}$ is a regular polygon. The dual of $\mathcal{U}$ is also a regular polygon with d vertices (see Figure 1):

3.2. — Lemma. *The dual $\mathcal{U}^\circ$ of $\mathcal{U}$ (with $d > 2$) is the polygon with vertices set:*

$$\mathcal{V} = \left\{ \frac{1}{\cos(\pi/d)} \exp\left(\frac{2k+1}{d} i\pi\right) \;\middle|\; k = 0,\dots,d-1 \right\}.$$

Proof — For $\beta_k = \exp\left(\frac{2ki\pi}{d}\right) \in \mathcal{U}$ and $\gamma_l = \exp\left(\frac{(2l+1)i\pi}{d}\right)\Big/\cos\left(\frac{\pi}{d}\right) \in \mathcal{V}$ we have

$$\operatorname{Re}\langle\beta_k,\gamma_l\rangle = \frac{\operatorname{Re}\big(\exp(-2ki\pi/d)\exp\big((2l+1)i\pi/d\big)\big)}{\cos(\pi/d)} = \frac{\cos\big((2l+1-2k)\pi/d\big)}{\cos(\pi/d)}.$$



Even example $d = 4$ Odd example $d = 5$

Figure 1. The boundaries of the convex hull of $\mathcal{U}$ (solid) and its dual (dashed)

Thus, $\operatorname{Re}\langle \beta_k, \gamma_l \rangle = 1$ when $k = l$ or when $k = l + 1$ (the vertice γ_l of $\mathcal{U}^\circ$ corresponds to the edge $\delta_l = (\beta_l, \beta_{l+1})$ of $\operatorname{Hull} \mathcal{U}$). For the other values of k, we have $\operatorname{Re}\langle \beta_k, \gamma_l \rangle < 1$ because β_k is in the half-plane delimited by δ_l and containing 0. □

3.3. — Lemma. *Define $\rho = \exp(i\pi/d)$. For each $\beta \in \operatorname{Hull} \mathcal{U}$, the following inequality holds:*

$$\operatorname{Re}(\rho\beta) \leqslant \cos(\pi/d).$$

Proof — From Lemma 3.2, we have $\mathcal{U}^\circ = \frac{\rho}{\cos(\pi/d)} \operatorname{Hull} \mathcal{U}$. Hence $\rho\beta = \cos(\pi/d)\gamma$ for some $\gamma \in \mathcal{U}^\circ$. Thus, $\operatorname{Re}(\rho\beta) = \cos(\pi/d) \operatorname{Re}(\gamma)$. But we have $\operatorname{Re}(\gamma) = \operatorname{Re}\langle 1, \gamma \rangle \leqslant 1$ because $1 \in \mathcal{U}$. The result follows because $\cos(\pi/d) > 0$ (we assumed $d > 2$, note also that case $d = 2$ is trivially true). □

Duality and DFT

As in Section 2, we put $D = d^n$. The map DFT is linear and its matrix $U = H_d^{\circ n}$ (in the canonical basis of $\mathbb{C}^D$) satisfies $U^\dagger U = DI$, where $U^\dagger$ is the conjugate transpose of U. This has some useful consequences.

3.4. — Lemma. *Assume that $\beta, \gamma \in \mathbb{C}^D$, and put $\hat{\beta} = \operatorname{DFT} \beta$ and $\hat{\gamma} = \operatorname{DFT} \gamma$. We have $\langle \hat{\beta}, \hat{\gamma} \rangle = D \langle \beta, \gamma \rangle$.*

Proof — If we identify β and γ with the column vectors of their coordinates in the canonical basis we can write:

$$\langle \hat{\beta}, \hat{\gamma} \rangle = \hat{\beta}^\dagger \hat{\gamma} = (U\beta)^\dagger U\gamma = \beta^\dagger U^\dagger U \gamma = D \beta^\dagger \gamma = D\langle \beta, \gamma \rangle$$

as claimed. □

3.5. — Proposition. *Let Γ be a polytope in $\mathbb{C}^D$ containing 0, and denote by $\hat{\Gamma}$ its image (which is also a polytope, by linearity of DFT) under the map DFT. We have the following relations between their duals:*

$$\widehat{\Gamma^\circ} = D\,\hat{\Gamma}^\circ.$$

Proof — For $\beta \in \mathbb{C}^D$, we have $\beta \in \Gamma^\circ$ if and only if $\langle \beta, \gamma \rangle \leqslant 1$ for all $\gamma \in \Gamma$. From Lemma 3.4, this is equivalent to $\langle \hat{\beta}, \hat{\gamma} \rangle \leqslant D$ for all $\gamma \in \Gamma$. This condition can be written $\langle \frac{1}{D}\hat{\beta}, \hat{\gamma} \rangle \leqslant 1$, or therefore $\frac{1}{D}\hat{\beta} \in \hat{\Gamma}^\circ$. Finally, it is equivalent to $\hat{\beta} \in D\,\hat{\Gamma}^\circ$. □

4. Homogeneous Bell inequalities

Le n be the number of parties. For each party, we consider two observables, denoted by $\hat{A}_i$ and $\hat{B}_i$ (for $1 \leqslant i \leqslant n$). The outcomes of each measure are assumed to belong to the set $\mathcal{U}$ defined in (1), with $d \geqslant 2$.

Recall, from the identity

$$1 - X^d = (1 - X)(1 + X + X^2 + \cdots + X^{d-1}),$$

that the roots of the polynomial $1 + X + \cdots + X^{d-1}$ are the elements of $\mathcal{U} \setminus \{1\}$. Recall also that $\sum_{u\in\mathcal{U}} u^k$ evaluates to d when k is a multiple of d but is zero otherwise. If $a_i, b_i \in \mathcal{U}$, there exists an integer $r_i \in \mathbb{Z}_d$ such that $a_i/b_i = \omega^{r_i}$. Let also $s_i \in \mathbb{Z}_d$. Then

$$\begin{aligned} a_i^{d-1} + \omega^{s_i} a_i^{d-2} b_i + \cdots + \omega^{(d-1)s_i} b_i^{d-1} &= a_i^{d-1}(1 + \omega^{s_i - r_i} + \cdots + \omega^{(d-1)(s_i - r_i)}) \\ &= \begin{cases} a_i^{d-1} d & \text{if } r_i = s_i, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Let now f be any map from $\mathbb{Z}_d^n$ to $\mathcal{U}$. We have

$$\sum_{s\in\mathbb{Z}_d^n} f(s) \prod_{i=1}^{n} (a_i^{d-1} + \omega^{s_i} a_i^{d-2} b_i + \cdots + \omega^{r_i s_i} a_i^{d-1-r_i} b_i^{r_i} + \cdots + \omega^{(d-1)s_i} b_i^{d-1}) = u d^n \tag{5}$$

where $u \in \mathcal{U}$, because in this sum, exactly one term is non-zero (the one corresponding to $s_i = r_i$ for each i).

If we expand the products in (5), we get

$$\begin{aligned} u d^n &= \sum_{s\in\mathbb{Z}_d^n} f(s) \sum_{r\in\mathbb{Z}_d^n} \prod_{i=1}^{n} \omega^{s_i r_i}\, a_i^{d-1-r_i} b_i^{r_i} \\ &= \sum_{s\in\mathbb{Z}_d^n} f(s) \sum_{r\in\mathbb{Z}_d^n} \omega^{r\cdot s}\, a^r \qquad \text{where } a^r := \prod_{i=1}^{n} a_i^{d-1-r_i} b_i^{r_i} \\ &= \sum_{r\in\mathbb{Z}_d^n} \hat{f}(r)\, a^r \qquad \text{where } \hat{f} = \mathrm{DFT}\, f. \end{aligned}$$

Now, if the a_i and b_i are random variables we can write, about expected values:

$$\sum_{r\in\mathbb{Z}_d^n} \hat{f}(r)\, E(a^r) \in d^n \ \mathrm{Hull}\, \mathcal{U}.$$

From Lemma 3.3, we obtain:

$$\mathrm{Re}\left(\rho \sum_{r\in\mathbb{Z}_d^n} \hat{f}(r)\, E(a^r)\right) \leqslant d^n \cos(\pi/d).$$

When $d > 2$, this also can be written:

$$\mathrm{Re}\left(\frac{\rho}{d^n \cos(\pi/d)} \sum_{r\in\mathbb{Z}_d^n} \hat{f}(r) E(a^r)\right) \leqslant 1 \qquad \text{for } f \in \mathcal{F}_{d,n}. \tag{6}$$

We call these relations homogeneous Bell inequalities. There are d^D of them.

5. Homogeneous Bell polynomials

We now study the polynomials in $2n$ variables A_i and B_i (for $1 \leqslant i \leqslant n$) which are involved in the homogeneous Bell inequalities. Some Bell polynomials where defined in [20] for $d = 2$. As a generalisation to the multidimensional case, we define the homogeneous Bell polynomials to be

$$\mathcal{P}_f = \sum_{r \in \mathbb{Z}_d^n} \hat{f}(r) A^r \qquad \text{where } A^r := \prod_{i=1}^{n} A_i^{d-1-r_i} B_i^{r_i}$$

where f is any map from $\mathbb{Z}_d^n$ to $\mathcal{U}$. Let us denote $\mathcal{H}_{d,n}$ the set of these polynomials. Each element of $\mathcal{H}_{d,n}$ is a homogeneous polynomial of degree $n(d-1)$.

As in [17] where the case $d = 2$ is handled, we give a recursive construction of the homogeneous Bell polynomials. This construction is a direct consequence of Equation (3). If $\mathcal{P}_0, \dots, \mathcal{P}_{d-1}$ are homogeneous Bell polynomials in the $2(n-1)$ the variables A_i, B_i with $1 \leqslant i \leqslant n-1$, then we get a homogeneous Bell polynomial in $2n$ variables by the d-ary operation $\bowtie$:

$$\mathcal{P}_0 \bowtie \cdots \bowtie \mathcal{P}_{d-1} := \sum_{r_n=0}^{d-1} \left(\sum_{t=0}^{d-1} \omega^{r_n t} \mathcal{P}_t \right) A_n^{d-1-r_n} B_n^{r_n}.$$

Conversely, every element of the set $\mathcal{H}_{d,n}$ can be obtained this way.

For example, with $d = 2$, the polynomials obtained are ± 1 for $n = 0$, $\pm 2A_1$ and $\pm 2B_1$ for $n = 1$, and

$$\begin{aligned}
&\pm 4A_1A_2, && \pm 2(-A_1A_2 + A_1B_2 + B_1A_2 + B_1B_2),\\
&\pm 4A_1B_2, && \pm 2(A_1A_2 - A_1B_2 + B_1A_2 + B_1B_2),\\
&\pm 4B_1A_2, && \pm 2(A_1A_2 + A_1B_2 - B_1A_2 + B_1B_2),\\
&\pm 4B_1B_2, && \pm 2(A_1A_2 + A_1B_2 + B_1A_2 - B_1B_2),
\end{aligned}$$

for $n = 2$ (we recognize the polynomials involved in the CHSH inequalities). Examples for $d = 3$ will be given in Section 7.

Symmetries

The set $\mathcal{H}_{d,n}$ of homogeneous Bell polynomials has some symmetries we briefly discuss now. They are consequences of Proposition 2.1.

a. If the maps f and $g \in \mathbb{Z}_d^n$ are the same, up to the order of their arguments:

$$g(s_1, \dots, s_n) = f(s_{\sigma(1)}, \dots, s_{\sigma(n)}) \qquad \text{for all } s \in \mathbb{Z}_d^n,$$

for some permutation σ, then the polynomial $\mathcal{P}_g$ can be obtained from $\mathcal{P}_f$ by changing each variable A_i (resp. B_i) to $A_{\sigma(i)}$ (resp. $B_{\sigma(i)}$). This symmetry corresponds to the fact that the n subsystems are indistinguishable.

b. If, for some i_0,

$$g(s) = \omega^{-s_{i_0}} f(s) \qquad \text{for all } s \in \mathbb{Z}_d^n,$$

then, from Proposition 2.1, we have $\hat{g}(r) = \hat{f}(r - \delta)$ for all $r \in \mathbb{Z}_d^n$, where $\delta = (0, \dots, 0, 1, 0, \dots, 0)$ has its only non-null component at index i_0. Hence, we obtain

$$\mathcal{P}_g = \sum_{r \in \mathbb{Z}_d^n} \hat{g}(r) A^r = \sum_{r \in \mathbb{Z}_d^n} \hat{f}(r - \delta) A^r = \sum_{r \in \mathbb{Z}_d^n} \hat{f}(r) A^{r+\delta}.$$

This shows that we obtain $\mathcal{P}_g$ from $\mathcal{P}_f$ by the circular monomial substitution

$$A_{i_0}^{d-1} \longrightarrow A_{i_0}^{d-2} B_{i_0} \longrightarrow \cdots \longrightarrow A_{i_0} B_{i_0}^{d-2} \longrightarrow B_{i_0}^{d-1} \longrightarrow A_{i_0}^{d-1}.$$

Also, the set $\mathcal{H}_{d,n}$ is invariant, under the swap operation $A_{i_0} \leftrightarrow B_{i_0}$ (this can be algebraically checked with the help of Proposition 2.1(a)). Hence, for each i_0, the set $\mathcal{H}_{d,n}$ is invariant under the action of the dihedral group of order $2d$ over the monomials made of the variables A_{i_0} and B_{i_0}.

c. Of course, the set $\mathcal{H}_{d,n}$ is also invariant under multiplication by ω, and by complex conjugation (Proposition 2.1(b) can be used to check this latter fact).

6. The classical domain

We now show that the homogeneous Bell inequalities obtained in Section 4 are tight and completely characterize a local realistic model, for $n \in \mathbb{N}^*$ parties, $m = 2$ observables for each site, and d-outcomes measurements with $d > 2$.

The values a_i and b_i, when a local realistic model is applied, of these $2n$ observables are assumed to belong to the set $\mathcal{U}$. We consider the monomials

$$A^s = \prod_{i=1}^{n} A_i^{d-1-s_i} B_i^{s_i}$$

which appear in homogeneous Bell polynomials. There are $D = d^n$ of them. For each experiment, the data set of the values obtained for these monomials form a vector $\xi = (a^s)_{s \in \mathbb{Z}_d^n}$ in $\mathbb{C}^D$. Our aim is to show that the domain accessible to the expected values of ξ is the polytope defined by the inequalities (6).

The polytope Ω

Put

$$\xi_r = (\omega^{r \cdot s})_{s \in \mathbb{Z}_d^n} \in \mathbb{C}^D \qquad \text{for each } r \in \mathbb{Z}_d^n.$$

The $d^{n+1} = dD$ vectors $u\xi_r$, for $u \in \mathcal{U}$ and $r \in \mathbb{Z}_d^n$ are all distinct. In a local realistic model, each experimental data set assigns a value

$$\prod_{i=1}^{n} a_i^{d-1-s_i} b_i^{s_i} = \prod_{i=1}^{n} a_i^{d-1} \prod_{i=1}^{n} \omega^{r_i s_i} = \prod_{i=1}^{n} a_i^{d-1} \omega^{r \cdot s}$$

to each monomial A^s where $\omega^{r_i} = b_i / a_i$ (for $1 \leqslant i \leqslant n$). Thus, the vector ξ obtained from experimental data is one of the vectors $u\xi_r$, where $u = \prod_{i=1}^{n} a_i^{d-1}$, and $r = (r_i)_{1 \leqslant i \leqslant n}$ with the r_i just defined.

Conversely, it is possible to design classical experiments which assign independently any value in $\mathcal{U}$ to the $2n$ variables and which assign any $u\xi_r$ to the data set vector ξ. Then, if the values assigned to the variables follow some probability distributions, expected values for the vectors ξ obtained, are convex combinations of the $u\xi_r$. Hence the classically accessible region for ξ is the convex hull of the $u\xi_r$, which will be denoted by Ω as it was in [20] for the case $d = 2$. The domain Ω is a polytope in $\mathbb{C}^D$ and has dD vertices. Notice that Ω has a d-order symmetry: $\omega\Omega = \Omega$.

The polytope $\Pi = \mathrm{DFT}^{-1}\,\Omega$

We can find all the inequalities defining the facets of the polytope Ω. They will be the d^D homogeneous Bell inequalities (6) we obtained in Section 4.

Let $(\pi_s)_{s \in \mathbb{Z}_d^n}$ be the canonical basis of the complex vector space $\mathbb{C}^D$. The discrete Fourier transform maps the π_s to the ξ_s. We consider the following polytope:

$$\Pi := \mathrm{Hull}\{u\pi_s \mid u \in \mathcal{U}, s \in \mathbb{Z}_d^n\}.$$

Then Ω is $\hat{\Pi}$, the image of Π under DFT. To find the facets of Ω, we have to study its dual. But from Proposition 3.5,

$$\Omega^\circ = \hat{\Pi}^\circ = \frac{1}{d^n}\,\widehat{\Pi^\circ}. \tag{7}$$

Let's first study Π°.

6.1. — Proposition. *The vertices of the polytope Π° are the $\beta = (\beta_1, \ldots, \beta_s)$ such that*

$$\beta_s = \frac{\rho}{\cos(\pi/d)} f(s) \qquad \text{where } f \text{ is any element of } \mathcal{F}_{d,n}.$$

Proof — By definition,

$$\Pi^\circ = \{\beta \in \mathbb{C}^D \mid \mathrm{Re}\langle \beta, u\pi_s \rangle \leqslant 1, \forall u \in \mathcal{U}, s \in \mathbb{Z}_d^n\}.$$

Using the d-order symmetry of $\mathcal{U}^\circ$, and using $\langle \beta, u\pi_s\rangle = u\langle \beta, \pi_s\rangle$, we can write

$$\Pi^\circ = \{\beta \in \mathbb{C}^D \mid \langle \beta, \pi_s\rangle \in \mathcal{U}^\circ, \forall s \in \mathbb{Z}_d^n\}.$$

We are interested with the extremal points of Π°. These are obtained when $\langle \beta, \pi_s\rangle$ are in a corner of $\mathcal{U}^\circ$ (see Lemma 3.2):

$$\langle \beta, \pi_s\rangle \in \mathcal{V} = \frac{\rho}{\cos(\pi/d)}\mathcal{U}.$$

Hence, there exists $f \in \mathcal{F}_{d,n}$ such that:

$$\beta_s^* = \langle \beta, \pi_s\rangle = \frac{\rho}{\cos(\pi/d)} f(s) \qquad \text{for all } s \in \mathbb{Z}_d^n.$$

But Π° is symmetric under complex conjugation. Hence we can change β_s^* for β_s. □

The dual of Ω

6.2. — Theorem. *The vertices of the polytope Ω° are given by*

$$\frac{\rho}{d^n \cos(\pi/d)} \big(\hat{f}(r)\big)_{r\in\mathbb{Z}_d^n} \qquad \text{for } f \in \mathcal{F}_{d,n}.$$

Proof — The result follows from Equation (7) and Proposition 6.1. □

To end this section, note that the inequalities (6) can be written

$$\mathrm{Re}\langle \beta_f, \xi\rangle \leqslant 1 \qquad \text{with} \quad \beta_f^* = \frac{\rho}{d^n \cos(\pi/d)} \big(\hat{f}(r)\big)_{r\in\mathbb{Z}_d^n} \quad \text{and} \quad \xi = \big(E(a^r)\big)_{r\in\mathbb{Z}_d^n}.$$

Hence the theorem just obtained shows that our homogeneous Bell inequalities define the facets of the polytope Ω. Thus they form a complete set of tight Bell inequalities.

7. The case $d = 3$

We illustrate our results with the first multidimensional case: $d = 3$ (sometimes called trichotomic). Note that the factor $1/\cos(\pi/d)$ in Equation (6) is maximal in this case, and this might lead to higher violations with non-classical models.

DFT

Here, $\omega = \exp(2i\pi/3)$ and

$$H_3^{\otimes 1} = H_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \qquad H_3^{\otimes 2} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & 1 & \omega & \omega^2 & 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega & 1 & \omega^2 & \omega & 1 & \omega^2 & \omega \\ 1 & 1 & 1 & \omega & \omega & \omega & \omega^2 & \omega^2 & \omega^2 \\ 1 & \omega & \omega^2 & \omega & \omega^2 & 1 & \omega^2 & 1 & \omega \\ 1 & \omega^2 & \omega & \omega & 1 & \omega^2 & \omega^2 & \omega & 1 \\ 1 & 1 & 1 & \omega^2 & \omega^2 & \omega^2 & \omega & \omega & \omega \\ 1 & \omega & \omega^2 & \omega^2 & 1 & \omega & \omega & \omega^2 & 1 \\ 1 & \omega^2 & \omega & \omega^2 & \omega & 1 & \omega & 1 & \omega^2 \end{pmatrix}.$$

The hull of $\mathcal{U}$ and its dual

The hull of $\mathcal{U}$ is the triangle with vertices 1, ω, ω^2 and its edges are defined by the three inequalities

$$x + \sqrt{3}y \leqslant 1, \qquad -2x \leqslant 1, \qquad x - \sqrt{3}y \leqslant 1.$$

Hence, the dual $\mathcal{U}^\circ$ has vertices $1 + i\sqrt{3}, -2, 1 - i\sqrt{3}$, which are obtained from the vertices of $\mathrm{Hull}\,\mathcal{U}$ by multiplication by $\exp(i\pi/3)/\cos(\pi/3) = -2\omega^2$.

Bell polynomials

We did some computations, with the help of the `Magma` computer algebra system. For the (virtual) case $n = 0$, the trichotomic Bell polynomials are the constant polynomials 1, ω and ω^2. For $n = 1$, there are yet 27 homogeneous trichotomic Bell polynomials which can be given by the expression

$$u\big(3M + (v-1)(A^2 + AB + B^2)\big)$$

where $u, v \in \{1, \omega, \omega^2\}$ and $M \in \{A^2, AB, B^2\}$.

For $n = 2$, there are 19683 homogeneous trichotomic Bell polynomials. Among them, 18792 are irreducible polynomials. The number of elements in $\mathcal{H}_{3,2}$ with only real coefficients is 81 (the aim of this criterion here is just to reduce the list size). We can list them, up to the symmetries discussed in Section 5, as there remain only 4 ones:

$$\begin{gathered}
9A_1^2A_2^2 \\
3(A_1^2A_2^2 - A_1^2B_2^2 + 2A_1B_1A_2B_2 + A_1B_1B_2^2 - B_1^2A_2^2 + B_1^2A_2B_2) \\
3(-A_1^2A_2B_2 + A_1^2B_2^2 + A_1B_1A_2^2 + 2A_1B_1B_2^2 - B_1^2A_2^2 + B_1^2A_2B_2) \\
3(2A_1^2A_2^2 - A_1^2A_2B_2 - A_1^2B_2^2 + A_1B_1A_2^2 + A_1B_1A_2B_2 + A_1B_1B_2^2).
\end{gathered}$$

We found also that there are 243 elements in $\mathcal{H}_{3,2}$ up to these symmetries.

Bell inequalities

The factor $\frac{\rho}{d^n \cos(\pi/d)}$ with appear in Inequalities (6) is in this case $-2\omega^2/3^n$. By changing f to ωf, we can remove the ω^2 to obtain the following homogeneous trichotomic Bell inequalities:

$$-\operatorname{Re}\left(\frac{2}{3^n}\sum_{r\in\mathbb{Z}_3^n}\hat{f}(r)E(a^r)\right) \leqslant 1 \qquad \text{for each } f \in \mathcal{F}_{3,n}. \tag{8}$$

8. Conclusion

In this paper, we defined homogeneous Bell inequalities and we showed that they correspond to the boundaries of the domain accessible with local-realistic models, for the general multipartite and multidimensional case with two observables per party. We studied homogeneous Bell polynomials and their symmetries. It turns out that the classical domain is the image under DFT of a polytope obtained from the canonical basis, and we used this fact to compute its dual. With this, we were able to show that the homogeneous Bell inequalities form a complete set.

The complex valued correlation function we used is a natural mathematical generalisation of the two-dimensional one. Fu in [8] argued that it has also a physical meaning, at least in Quantum Mechanics. It was a crucial and fruitful ingredient in the present work, and this raises interrogations about the precise extent of this physical meaning.

We have said nothing about violations. The $d = 3$ case was only slightly explored and has not been fully compared with previous results (for example the inequality obtained in [12]). This work is planned for a near future.

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