Differential-difference equations associated with the fractional Lax operators

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Abstract. We study integrable hierarchies associated with spectral problems of the form $P\psi = \lambda Q\psi$ where P, Q are difference operators. The corresponding nonlinear differential-difference equations can be viewed as inhomogeneous generalizations of the Bogoyavlensky type lattices. While the latter turn into the Korteweg–de Vries equation under the continuous limit, the lattices under consideration provide discrete analogs of the Sawada–Kotera and Kaup–Kupershmidt equations. The r-matrix formulation and several simplest explicit solutions are presented.

Keywords: Lax pair, discretization, Bogoyavlensky lattice, Sawada–Kotera equation, Kaup–Kupershmidt equation

MSC: 35Q53, 37K10

1 Introduction

The simplest example studied in this paper is the lattice equation

$$u_{,t} = u^2(u_2u_1 - u_{-1}u_{-2}) - u(u_1 - u_{-1}) \tag{1}$$

where we use the shorthand notations

$$u = u(n,t), \quad u_{,t} = \partial_t(u), \quad u_j = u(n+j,t).$$

For the first time, this equation was derived by Tsujimoto and Hirota [1, eq. (4.12)] as the continuous limit of the reduced discrete BKP hierarchy. Recall that both equations

$$u_{,t'} = u(u_1 - u_{-1}) \qquad \text{and} \qquad u_{,t''} = u^2(u_2u_1 - u_{-1}u_{-2}) \tag{2}$$

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are very well known integrable models: respectively, the Volterra lattice [2, 3] and the modified Narita–Itoh–Bogoyavlensky lattice of the second order [4, 5, 6]. One can easily verify that the flows $\partial_{t'}$ and $\partial_{t''}$ do not commute, that is, these equations belong to the different hierarchies. Hence, one should not expect a priori that their linear combination remains integrable. Nevertheless, this is the case: we will show that equation (1) admits the Lax representation

$$L_{,t} = [A, L]$$

with the operator L equal to a ratio of two difference operators, namely, $L = (T^2 + u)^{-1}(uT^2 + 1)T$ where T denotes the shift operator $u_k \to u_{k+1}$.

Equation (1) can be cast into the Hirota's bilinear form which admits a family of generalizations depending on a pair of integer parameters (l, m). These generalizations were discovered by Hu, Clarkson and Bullough [7, eq. (4)] who searched for bilinear equations admitting N-soliton solutions. One of the goals of our paper is to demonstrate that this family of equations is associated with the fractional Lax operators of the form

$$L = (T^m + u)^{-1}(uT^m + 1)T^l. \tag{3}$$

As usually, any such L is associated with a whole commutative hierarchy of equations corresponding to the sequence of difference operators A of increasing order. We denote this hierarchy $\mathrm{dSK}^{(l,m)}$, since it can be viewed as a discretization of the hierarchy containing the Sawada–Kotera equation [8, 9]

$$U_{,\tau} = U_5 + 5UU_3 + 5U_1U_2 + 5U^2U_1 \tag{SK}$$

where we denote

$$U = U(x, \tau), \quad U_{,\tau} = \partial_\tau(U), \quad U_j = \partial_x^j(U).$$

For instance, equation (1) belongs to $\mathrm{dSK}^{(1,2)}$. The concrete formula of the continuous limit in this example is the following, at $\varepsilon \to 0$:

$$u(n, t) = \frac{1}{3} + \frac{\varepsilon^2}{9} U\Big(x - \frac{4}{9}\varepsilon t, \tau + \frac{2\varepsilon^5}{135} t\Big), \quad x = \varepsilon n \tag{4}$$

and an analogous formula exists for any (l, m). It should be noted that each of equations (2) apart defines a discretization of the Korteweg–de Vries (KdV) equation $U_{,t} = U_3 + 6UU_1$ rather than the SK one. Moreover, it is well known that actually all Bogoyavlensky type lattices serve as discretizations of the KdV equation or its higher symmetries, so that an infinite family of discrete hierarchies correspond to just one continuous. Quite analogously, the whole family of $\mathrm{dSK}^{(l,m)}$ hierarchies serve as discrete analogs of the SK hierarchy. We hope that this observation makes clear the place of these equations in the big picture of integrable systems.

On the other hand, the differential and difference cases are not quite parallel. First, Lax operator for the SK equation

$$L = D^3 + UD = (D - f)(D + f)D$$

is not fractional. Lax operators given by the ratio of differential operators were studied by Krichever [10], however it seems that these examples and (3) are unrelated.

Second, let us consider the problem of discretization for another important example, the Kaup–Kupershmidt equation [11, 12, 13]

$$U_{,\tau} = U_5 + 5UU_3 + \frac{25}{2}U_1U_2 + 5U^2U_1. \tag{KK}$$

Recall that it is associated with the operator

$$L = D^3 + UD + \frac{1}{2}U_{,x} = (D + f)D(D - f)$$

and both SK and KK equations are connected through the Miura substitutions obtained by factorization of Lax operators [14, 15]:

$$U_{\mathrm{SK}} = f_{,x} - f^2, \qquad U_{\mathrm{KK}} = -2f_{,x} - f^2.$$

Despite of this close relation, it was noted that some properties of the SK and KK equations are rather different, see e.g. [16]. It seems that distinctions between the lattice analogs of these equations are even more deep. A discretization of the KK equation is presented in section 4, however, we were able to find just one operator L in this case comparing to infinite family (3) in the SK case, and no discrete analog of Miura type substitution between dSK and dKK is known.

The contents of the paper is the following. Section 2 contains some necessary information on the lattices of Bogoyavlensky type, see also books [17, 18]. Section 3 devoted to discretization of the SK equation contains the main results of the paper. A general construction of the Lax pairs with operator (3) is given in section 3.1. In section 3.3, the r-matrix approach in the difference setting [19, 20, 18] is used to obtain explicit formulas for the operator A and to prove the commutativity of the dSK$^{(l,m)}$ hierarchy. The continuous limit, the bilinear representation, the simplest breather type solutions are presented in sections 3.4, 3.5. Section 4 is devoted to discretization of the KK equation and section 5 contains several examples of coupled lattice equations associated with more general fractional Lax operators.

2 Preliminaries

2.1 Definitions and notations

We consider differential-difference (lattice) equations of the evolutionary form

$$u_{,t} = f(u_m, \dots, u_{-m}), \quad u = u(n,t), \quad u_{,t} = \partial_t(u), \quad u_j = u(n+j,t). \quad (5)$$

Such equations can be viewed as discrete analogs of continuous evolutionary equations like KdV or SK

$$U_{,\tau} = F(U_k, \dots, U), \quad U = U(x,\tau), \quad U_{,\tau} = \partial_\tau(U), \quad U_j = \partial_x^j(U)$$

(the orders m and k may not coincide under the continuous limit). The shift operator $T : u_j \mapsto u_{j+1}$ plays the same role for equations (5) as the total x-derivative $D : U_j \mapsto U_{j+1}$ plays in the continuous case. Differential operators are polynomials with respect to D, with the multiplication defined by the Leibniz rule $DA = D(A) + AD$ and the conjugation defined by the rule $D^\dagger = -D$. In contrast, difference operators are in general Laurent polynomials, that is contain powers of both T and T^{-1}, and the rules for the multiplication and the conjugation are $TA = T(A)T$ and $T^\dagger = T^{-1}$. For short, we will use subscripts also for denoting action of T on operators, $A_j = T^j(A)$.

A lattice equation

$$u_{,t'} = g(u_k, \dots, u_{-k})$$

is called symmetry of (5) if the compatibility condition $D_{,t}(g) = D_{,t'}(f)$ is fulfilled, that is

$$[f,g]_* := \sum_{s=-m}^{m} \partial_{u_s}(f) T^s(g) - \sum_{s=-k}^{k} \partial_{u_s}(g) T^s(f) = 0. \quad (6)$$

The lattice is called integrable if it admits an infinite sequence of symmetries with the order k greater than any fixed number. The linear space of all symmetries is called hierarchy. A conservation law is a relation of the form

$$D_{,t}(\rho(u_k, \dots, u)) = (T-1)(\sigma(u_{k+m-1}, \dots, u_{-m}))$$

which holds true in virtue of equation (5). The discussion of these notions and applications to the problem of classification of integrable lattice equations can be found in the review article by Yamilov [21].

2.2 Bogoyavlensky lattices

Understanding the structure of $\mathrm{dSK}^{(l,m)}$ hierarchy is not possible without understanding the homogeneous hierarchies of Bogoyavlensky type. A general pattern of (local) equations from $\mathrm{dSK}^{(l,m)}$ is given by the formula

$$u_{,t_k} = F^{(L+KM)} + \dots + F^{(L+M)} + F^{(L)}$$

where $F^{(s)}$ denotes a homogeneous polynomial of degree s with respect to the variables u_j and K, L, M are related somehow with the parameters l, m and the order k of the flow. Moreover, the first and the last terms in the sum always correspond to some (modified) lattices of Bogoyavlensky type belonging to the different hierarchies.

This structure is explained by the following arguments, starting from the Lax representation with the operator L (3). Let us consider the scaling $u \to \delta^{-m}u$, $T \to \delta T$, then it is easy to see that the limit $\delta \to \infty$ sends L to the operator $L' = u_{-m}T^l + T^{l-m}$ and the limit $\delta \to 0$ leads to $L'' = T^{m+l} + u^{-1}T^l$. Each of these operators corresponds to its own hierarchy of homogeneous lattice equations. The total inhomogeneous equation contains both of them together with the intermediate terms which are necessary for preserving commutativity of the flows.

Let us consider the concrete example. One can check that the lattice

$$\begin{aligned} u_{,t'} = u\big(&w_1(w_3 + w_2 + w_1 + w) - w_{-1}(w + w_{-1} + w_{-2} + w_{-3}) \\ &- u_1(w_3 + w_{-1}) + u_{-1}(w_1 + w_{-3})\big), \quad w := u(1 - u_1u_{-1}) \end{aligned} \tag{7}$$

is a higher symmetry of equation (1). Collecting the homogeneous terms yields

$$u_{,t} = F^{(4)} + F^{(2)}, \quad u_{,t'} = G^{(7)} + G^{(5)} + G^{(3)}$$

and the consistency condition of the flows splits to relations

$$\begin{gathered} [F^{(4)}, G^{(7)}]_* = 0, \quad [F^{(4)}, G^{(5)}]_* + [F^{(2)}, G^{(7)}]_* = 0, \\ [F^{(4)}, G^{(3)}]_* + [F^{(2)}, G^{(5)}]_* = 0, \quad [F^{(2)}, G^{(3)}]_* = 0 \end{gathered}$$

where commutator $[,]_*$ is defined by equation (6). As it was already said in Introduction, polynomials $F^{(4)}$ and $F^{(2)}$ correspond to the modified Bogoyavlensky and Volterra lattices. Polynomials $G^{(7)}$ and $G^{(3)}$ correspond to their symmetries and the intermediate polynomial $G^{(5)}$ compensates inconsistency of the hierarchies.

The Bogoyavlensky hierarchy $\mathrm{B}^{(m)}$ is associated with the operator $L = T + uT^{-m}$ and we recall here several basic formulas regarding this case. A detailed theory can be found in the books [6, 18]. More general operators of the form $L = T^l + uT^{-m}$ were considered recently in the paper [22].

The simplest equation from the $\mathrm{B}^{(m)}$ hierarchy reads

$$u_t = u(u_m + \dots + u_1 - u_{-1} - \dots - u_{-m}). \tag{8}$$

This equations and its higher symmetries are associated with the difference spectral problem

$$\psi_1 + u\psi_{-m} = \lambda\psi$$

and admit the Lax representations

$$L_{,t_k} = [A^{(k)}, L], \quad L = T + uT^{-m}, \quad A^{(k)} = \pi_+\big(L^{(m+1)k}\big) \tag{9}$$

where π_+ denotes the projection of any formal series $A = \sum_{j<\infty} a^{(j)}T^j$ onto the linear space of polynomials with respect to T:

$$\pi_+(A) = \sum_{0\le j<\infty} a^{(j)}T^j, \quad \pi_-(A) = \sum_{j<0} a^{(j)}T^j.$$

In particular,

$$A^{(1)} = T^{m+1} + v, \quad v := u_m + \cdots + u$$

and equation (9) at $k = 1$ is equivalent to lattice (8). The check is easy:

$$\begin{aligned} L_{,t} - [A^{(1)}, L] &= u_{,t}T^{-m} - [T^{m+1} + v, T + uT^{-m}] \\ &= u_{,t}T^{-m} - (u_{m+1} - u + v - v_1)T - u(v - v_{-m})T^{-m}, \end{aligned} \tag{10}$$

the terms with T cancel and the rest yields the equation.

In order to prove that equation (9) correctly defines the lattice for any k, we have to check that all powers of T except for T^{-m} vanish in the commutator $[A^{(k)}, L]$. Since L^{m+1} is a Laurent polynomial with respect to T^{m+1}, hence $A^{(k)}$ is a polynomial with respect to T^{m+1}. Therefore the commutator contains only powers of the form $T^{(m+1)j+1}$, $j \ge -1$. On the other hand,

$$[A^{(k)}, L] = -[\pi_-\big(L^{(m+1)k}\big), L],$$

so that the commutator does not contain positive powers of T and only one possible power T^{-m} remains.

It can be proven that equations (9) define a special reduction in the Lax pair with a generic operator $L = T + u^{(0)} + u^{(1)}T^{-1} + \cdots + u^{(m)}T^{-m}$. In this case one can choose operators A in the form $A = \pi_+(L^k)$ with arbitrary k. For instance, the Toda lattice hierarchy appears at $m = 1$. This type of multi-field systems was studied, for instance, in papers [20, 22].

3 Discretizations of the Sawada–Kotera equation

3.1 Lax representation

Let us consider the difference spectral problem

$$u\psi_{m+l} + \psi_l = \lambda(\psi_m + u\psi) \tag{11}$$

where m, l are integers. We assume that m, l are positive and coprime, without loss of generality, since the general case can be obtained by refinement of the mesh and/or change of its directions. It is less obvious that the numbers m and l can be exchanged: spectral problem (11) is equivalent to

$$u\varphi_{m+l} + \varphi_m = \mu(\varphi_l + u\varphi)$$

under the change

$$\psi(n) = \varkappa^n \varphi(n), \quad \lambda = -\varkappa^l, \quad \mu = -\varkappa^{-m}. \tag{12}$$

In the operator form, equation (11) reads

$$P\psi = \lambda Q\psi, \quad P = (uT^m + 1)T^l, \quad Q = T^m + u. \tag{13}$$

The isospectral deformations are defined by equation $\psi_{,t} = A\psi$ with some difference operator A. The corresponding Lax equation

$$L_{,t} = [A, L], \quad L = Q^{-1}P \tag{14}$$

can be rewritten as the system

$$P_{,t} = BP - PA, \quad Q_{,t} = BQ - QA \tag{15}$$

where one of equations can be considered just as a definition of B. Let P, Q be as in (13), then this system is equivalent to equations

$$\begin{gathered} u_{,t} = B(T^m + u) - (T^m + u)A, \\ B(T^{2m} - 1) = A_m T^{2m} - A_l + uAT^m - uA_{m+l}T^m. \end{gathered} \tag{16}$$

In order to resolve the latter we make the assumption that operator A is of the form

$$A = F(T^m - T^{-m}) \tag{17}$$

then B is found as the difference operator

$$B = F_m T^m - F_1 T^{-m} + u(F - F_{m+1}) \tag{18}$$

while first equation (16) turns into

$$u_{,t} = T^m F u + uFT^{-m} - uT^m F_l - F_l T^{-m} u + F_m - F_l + u(F - F_{m+l})u. \tag{19}$$

It is clear that the same evolution of the variable u is defined by the conjugated operator $F^\dagger$ and, moreover, all terms T^j, $j \nmid m$ can be thrown away. This means that we can find F as a self-adjoint operator $F = F^\dagger$ which is a Laurent polynomial with respect to the powers T^m:

$$F = f^{(k)}T^{km} + \dots + f^{(1)}T^m + f^{(0)} + T^{-m}f^{(1)} + \dots + T^{-km}f^{(k)}, \quad k \geq 0. \tag{20}$$

Certainly, the coefficients depend on k, l, m, so that it would be more rigorous to write $f^{(j,k,l,m)}$ instead of $f^{(j)}$, but we will consider these numbers fixed at the moment.

Collecting the coefficients at T^{jm}, $j > 0$, yields the relations

$$\begin{aligned} u_{jm} f^{(j-1)}_m - u f^{(j-1)}_{m+l} &= f^{(j)}_l - f^{(j)}_m + uu_{jm}(f^{(j)}_{m+l} - f^{(j)}) \\ &\quad + u_{jm} f^{(j+1)}_l - u f^{(j+1)}, \quad j = 1, \dots, k+1, \end{aligned} \tag{21}$$

where it is assumed for convenience that $f^{(j)}=0$ at $j>k$. The coefficient at T^0 gives an evolutionary equation for u:

$$u_{,t}=2u(f^{(1)}-f^{(1)}_l)+u^2(f^{(0)}-f^{(0)}_{m+l})+f^{(0)}_m-f^{(0)}_l. \tag{22}$$

System of equations (21), (22) defines the k-th flow in the hierarchy $\mathrm{dSK}^{(l,m)}$.

If we are interested in the local evolution only then we require that all $f^{(j)}$ can be recurrently found as functions of an finite set of variables u_i. In this case a certain restriction on the values of k appears and a part of the flows is rejected. Indeed, consider equation (21) at $j=k+1$,

$$u_{(k+1)m}f^{(k)}_m=uf^{(k)}_{m+l}, \tag{23}$$

or

$$(T^l-1)(\log f^{(k)}_m)=(T^{(k+1)m}-1)(\log u).$$

It can be proven that it is solvable with respect to $f^{(k)}$ if and only if $(k+1)m$ is divisible by l and the solution is, up to a constant factor,

$$f^{(k)}=u_{-m}u_{l-m}\cdots u_{(s-1)l-m},\quad (k+1)m=sl. \tag{24}$$

Since l and m are coprimes, hence the local flows may appear only if $k=pl-1$ and $s=mp$. The fact that the rest equations (21) for such k are solvable indeed will be verified later in section 3.3. The case $l=1$ is the only one when there are no restrictions on k and the simplest choice $k=0$ brings in this case to the following family of lattices.

Theorem 1. *For any $m>0$, the simplest equation in the hierarchy dSK$^{(1,m)}$*

$$u_{,t}=u^2(u_m\cdots u_1-u_{-1}\cdots u_{-m})-u(u_{m-1}\cdots u_1-u_{-1}\cdots u_{1-m}) \tag{25}$$

possesses Lax representation (14) with the operators

$$P=uT^{m+1}+T,\quad Q=T^m+u,$$
$$A=f(T^{-m}-T^m),\quad B=f_1T^{-m}-f_mT^m+u(f_{m+1}-f)$$

where $f=u_{-1}\cdots u_{-m}$.

Proof. A direct computation (cf with (10)) proves that both equations (15) with given P,Q,A,B are equivalent to relations

$$u_mf_m=uf_{m+1},\quad u_{,t}=u^2(f_{m+1}-f)-f_m+f_1.$$

The former defines the variable f (up to a constant factor) and the latter is equivalent to lattice (25). □

In particular, equation (25) at $m=2$ coincide with (1) and at $m=1$ it is just the modified Volterra lattice

$$u_{,t}=u^2(u_1-u_{-1}).$$

It should be remarked that gauge equivalence (12) between the spectral problems can be extended on the level of nonlinear equations and the same flow (25) appears also as a member of $\mathrm{dSK}^{(m,1)}$ hierarchy. However, operator (20) is much more complicated in this case: it contains all powers $T^{m-1},T^{m-2},\dots,T^{1-m}$ comparing with just $F=f^{(0)}$ in $\mathrm{dSK}^{(1,m)}$ case.

Computing of higher symmetries quickly becomes involved, because finding of F requires (discrete) integration of rather bulky expressions. For instance, the second flow in the hierarchy $\mathrm{dSK}^{(1,m)}$ is, according to (22), of the form

$$u_{,t'}=2u(f^{(1)}-f^{(1)}_1)+u^2(f^{(0)}-f^{(0)}_{m+1})+f^{(0)}_m-f^{(0)}_1$$

where functions $f^{(1)},f^{(0)}$ are defined by relations

$$u_{2m}f^{(1)}_m=uf^{(1)}_{m+1},\quad u_mf^{(0)}_m-uf^{(0)}_{m+1}=f^{(1)}_1-f^{(1)}_m-uu_m(f^{(1)}-f^{(1)}_{m+1}).$$

This yields, up to integration constants,

$$f^{(1)}=u_{m-1}\cdots u_{-m},\quad f^{(0)}=(w+\dots+w_{-2m+1})u_{-1}\cdots u_{-m},\\ w:=(1-u_{m-1}u_{-1})u_{m-2}\cdots u_0$$

(at $m=2$ equation (7) appears). One can check straightforwardly that the obtained flow commutes with (25) indeed. A general proof and a way to bypass the integration are given below in section 3.3.

Adopting nonlocal variables leads to some extension of the hierarchy. In this case we consider equation (23) as a constraint which defines the variable $f^{(k)}$ for any k. Then we arrive to the following system which generalizes (25) for any l, making the picture more uniform. We will return to this system in section 3.5.

Theorem 2. *For any coprime m,l, the simplest system in the extended $dSK^{(l,m)}$ hierarchy*

$$u_mf_m=uf_{m+l},\quad u_{,t}=u^2(f-f_{m+l})+f_m-f_l \tag{26}$$

possesses Lax representation (14) with operators

$$P=uT^{m+l}+T^l,\quad Q=T^m+u,\\ A=f(T^{-m}-T^m),\quad B=f_lT^{-m}-f_mT^m+u(f_{m+l}-f).$$

$m=2:$	$u_{,t} = u^2(u_2u_1 - u_{-1}u_{-2}) - u(u_1 - u_{-1})$
$u = v_1v$	$v_{,t} = v_1v^3v_{-1}(v_2v_1 - v_{-1}v_{-2}) - v^2(v_1 - v_{-1})$
$m=3:$	$u_{,t} = u^2(u_3u_2u_1 - u_{-1}u_{-2}u_{-3}) - u(u_2u_1 - u_{-1}u_{-2})$
$v = u_1u$	$v_{,t} = v(v_3v_1 + v_2v - vv_{-2} - v_{-1}v_{-3}) - v(v_2 + v_1 - v_{-1} - v_{-2})$
$u = v_2v_1v$	$v_{,t} = v_2v_1^2v^4v_{-1}^2v_{-2}(v_3v_2v_1 - v_{-1}v_{-2}v_{-3}) - v_1v^3v_{-1}(v_2v_1 - v_{-1}v_{-2})$
$m=4:$	$u_{,t} = u^2(u_4u_3u_2u_1 - u_{-1}u_{-2}u_{-3}u_{-4}) - u(u_3u_2u_1 - u_{-1}u_{-2}u_{-3})$
$u = v_2v$	$v_{,t} = v_2v_1v^3v_{-1}v_{-2}(v_4v_3v_2v_1 - v_{-1}v_{-2}v_{-3}v_{-4}) - v_1v^2v_{-1}(v_3v_2v_1 - v_{-1}v_{-2}v_{-3})$

Table 1. Examples of lattices (25) from dSK$^{(1,m)}$ and their modifications

3.2 Modified lattices

Equations under consideration can be rewritten in several ways by use of difference substitutions. The simplest kind of substitution is introducing a potential. Let A be a constant operator, then substitution $u = A(v)$ maps solutions of equation $v_{,t} = f[A(v)]$ into solutions of equation $u_{,t} = A(f[u])$. Table 1 contains several instances of such kind, up to the change $u \to e^u$, $v \to e^v$.

Another kind of substitutions are Miura type transformations. Let φ be a particular solution of spectral problem (11) corresponding to a value $\lambda = \alpha$ of the spectral parameter. Then one readily finds that the ratio $h = \varphi_1/\varphi$ is related with the potential u by formula

$$M^- : \quad u = \frac{\alpha h_{m-1}\cdots h - h_{l-1}\cdots h}{h_{m+l-1}\cdots h - \alpha}.$$

This defines a difference substitution, according to the following statement.

Theorem 3. *Let u satisfies an equation (22) from dSK$^{(l,m)}$, then $h = \varphi_1/\varphi$ also satisfies a lattice equation which can be written as a conservation law*

$$(\log h)_{,t} = (T-1)S[h]. \tag{27}$$

Proof. Since φ is governed by equation $\varphi_{,t} = A\varphi = F(\varphi_m - \varphi_{-m})$, hence

$$(\log h)_{,t} = (T-1)(\log\varphi)_{,t} = (T-1)\Big(\frac{1}{\varphi}F(\varphi_m - \varphi_{-m})\Big).$$

Coefficients of the operator F are functions on the variables h_j, being functions on u_j's. The ratios of the form φ_k/φ can be expressed through h_j as well and therefore an equation of the form (27) holds. $\square$

It is worth noticing that an infinite sequence of conservation laws for the original lattice (22) can be obtained from (27) by use of the classical trick with the inversion of Miura map $u = M^-(h,\alpha)$ as a formal power series with respect to α [23].

Second Miura map is obtained by replacing $h \to 1/h$, $\alpha \to 1/\alpha$ which results in the mapping

$$M^+ : \quad u = \frac{\alpha h_{m+l-1}\cdots h_l - h_{m+l-1}\cdots h_m}{h_{m+l-1}\cdots h - \alpha}.$$

This substitution relates the same equations as M^-, due to invariance of the spectral problem with respect to the change $n \to -n$, $\lambda \to 1/\lambda$. Therefore, the composition $M^-(M^+)^{-1}$ defines a Bäcklund transformation which relates two copies of the $\mathrm{dSK}^{(l,m)}$ hierarchy. Recall that Bäcklund transformation for the continuous SK equation was derived in [24].

A particular example at $l = 2, m = 1$ is given by substitutions

$$M^- : \ u = \frac{(\alpha - h_1)h}{h_2h_1h - \alpha}, \quad M^+ : \ u = \frac{h_2(\alpha - h_1)}{h_2h_1h - \alpha}$$

which map solutions of the modified equation

$$h_{,t} = \frac{h(\alpha - h)}{h_1hh_{-1} - \alpha}\left(\frac{h(\alpha - h_1)(\alpha - h_{-1})(h_2h_1 - h_{-1}h_{-2})}{(h_2h_1h - \alpha)(hh_{-1}h_{-2} - \alpha)} - h_1 + h_{-1}\right)$$

into solutions of (1).

3.3 r-matrix formulation

In this section we prove that:

(i) if the constraint (23) is resolved by formula (24) then the further recurrent relations (21) are solved in the local form as well, so that the (local) hierarchy $\mathrm{dSK}^{(l,m)}$ is correctly defined;

(ii) the flows corresponding to the different k commute.

In achieving this goal the r-matrix approach is an indispensable tool, see e.g. [19, 20, 18]. Let us consider the Lie algebra of the formal Laurent series with respect to the powers T^m of the shift operator:

$$\mathfrak{g}^{(m)} = \Big\{\sum_{j<\infty} g^{(j)}T^{jm}\Big\}$$

with the commutator $[A, B] = AB - BA$. It is easy to see that any element

$$G = g^{(k+1)}T^{(k+1)m} + g^{(k)}T^{km} + g^{(k-1)}T^{(k-1)m} + \dots$$

of this Lie algebra admits an unique decomposition of the form

$$G = F(T^m - T^{-m}) + H \tag{28}$$

where $F = F^\dagger$ is a self-conjugated difference operator and H is a formal series which contains only nonpositive powers of T^m. Each of the linear spaces

$$\mathfrak{g}_+^{(m)} = \big\{F(T^m - T^{-m})|\ F = F^\dagger\big\}, \quad \mathfrak{g}_-^{(m)} = \Big\{\sum_{j\le 0} h^{(j)} T^{jm}\Big\}$$

constitutes a Lie algebra: for $\mathfrak{g}_-^{(m)}$ this is obvious and for $\mathfrak{g}_+^{(m)}$ we have

$$[F(T^m - T^{-m}), F'(T^m - T^{-m})] = (P + P^\dagger)(T^m - T^{-m})$$

where $P = F(T^m - T^{-m})F'$.

Thus, formula (28) is the decomposition (in the vector space sense)

$$\mathfrak{g}^{(m)} = \mathfrak{g}_+^{(m)} \oplus \mathfrak{g}_-^{(m)}$$

of the Lie algebra into the direct sum of two Lie subalgebras. This decomposition defines the projections $\pi_\pm$ on the $\mathfrak{g}_\pm^{(m)}$ component and the r-matrix $r = \frac{1}{2}(\pi_+ - \pi_-)$. Now we can formulate the following theorem about Lax equations (13), (14) with fractional L operator.

Theorem 4. *Let l, m be coprime, $P = (uT^m + 1)T^l$, $Q = T^m + u$ and let $L = Q^{-1}P$ be expanded as a formal Laurent series. Then the flows*

$$L_{,t_p} = [\pi_+(L^{pm}), L] \tag{29}$$

are correctly defined for all $p = 1, 2, \dots,$ coincide with the $dSK^{(l,m)}$ flows introduced by equations (21), (22) and commute with each other.

Proof. After expanding, L takes the form

$$\begin{aligned} L &= (1 - u_{-m}T^{-m} + (u_{-m}T^{-m})^2 - \dots)(u_{-m} + T^{-m})T^l \\ &= u_{-m}T^l + (1 - u_{-m}u_{-2m})T^{l-m} + \dots . \end{aligned}$$

Differentiating this series turns (29) into an infinite system of equations for a single variable u, and the correctness means that all these equations must coincide. To prove this, we compare representation (29) with Lax equation (14) in fractional form.

Notice that L itself does not belong to the Lie algebra $\mathfrak{g}^{(m)}$, but its power $G = L^{pm}$ does, so that the projection $A = \pi_+(G) = F(T^m - T^{-m})$ makes sense. We denote the order of operator F as $k = pl - 1$, in agreement with

(20) and (24). The coefficients of F are uniquely computed from coefficients of G accordingly to the recurrent relations

$$f^{(k+2)} = f^{(k+1)} = 0, \quad f^{(j)} = g^{(j+1)} + f^{(j+2)}, \quad j = k, k-1, \dots, 0$$

so that all coefficients are local functions of u_j (in particular, $f^{(k)}$ is given by (24)). Moreover, the order of (29) right hand side is equal to l, because $[\pi_+(G), L] = -[\pi_-(G), L]$. This proves that F provides a solution of the recurrent relations (21) as well (which is unique up to integration constants). Indeed, these relations were derived from the condition that terms with $T^{(k+1)m}, \dots, T^m$ in equation (19) cancel which is equivalent to cancellation of the powers $T^{(k+1)m+l}, \dots, T^{m+l}$ in the original Lax equation (14). Thus, flow (29) coincides with a flow from $\mathrm{dSK}^{(l,m)}$ which is, therefore, local. On the other hand, this proves correctness of (29), since the whole infinite set of equations turns out to be equivalent to the single equation (22).

The proof of the commutativity is standard. Let $G' = L^{p'm}$ and $A' = \pi_+(G')$ then

$$(L_{,t_p})_{,t_{p'}} - (L_{,t_{p'}})_{,t_p} = [A_{t_{p'}} - A'_{t_p} + [A, A'], L],$$

so it is sufficient to prove that

$$A_{t_{p'}} - A'_{t_p} + [A, A'] = 0.$$

Since $A_{t_{p'}} = \pi_+([A', G])$ and $[G, G'] = 0$, this is equivalent to

$$\begin{aligned} &\pi_+\big([A', G] - [A, G'] + [A, A']\big) \\ &\quad = \pi_+\big([G' - \pi_-(G'), G] - [G - \pi_-(G), G'] + [G - \pi_-(G), G' - \pi_-(G')]\big) \\ &\quad = \pi_+\big([\pi_-(G), \pi_-(G')]\big) = 0 \end{aligned}$$

as required. □

3.4 Continuous limit

Here we compute the continuous limit for the basic flow of the extended hierarchy $\mathrm{dSK}^{(l,m)}$ defined by equation (26). There is a certain technical difficulty in the prolongation of the continuous limit on the variable f which is not local at $l \neq 1$. In order to solve the constraint, this variable should be considered as a series with respect to the small parameter. Up to this complication the continuous limit is very similar to example (4) from Introduction. We postulate that, at $\varepsilon \to 0$, the variables u, f are of the form

$$\begin{aligned} u(n,t) &= a + ab\varepsilon^2 U(x + c\varepsilon t, \tau + d\varepsilon^2 t), \\ f(n,t) &= 1 + \sum_{s=2}^{\infty} \varepsilon^s Y_s(x + c\varepsilon t, \tau + d\varepsilon^2 t), \quad x = \varepsilon n \end{aligned} \tag{30}$$

with undetermined coefficients a, b, c, d. Functions Y_s are expressed through the function U and its partial derivatives with respect to x after substituting into first equation (26) and taking the Taylor expansion about $\varepsilon = 0$ (clearly, one can neglect the dependence on t here). We find, omitting the unessential integration constants:

$$
\begin{aligned}
Y_2 &= \frac{mb}{l}U,\\
Y_3 &= -\frac{m(m+l)b}{2l}U_1,\\
Y_4 &= \frac{m(m+l)(2m+l)b}{12l}U_2 + \frac{m(m-l)b^2}{2l^2}U^2,\\
Y_5 &= -\frac{m^2(m+l)^2b}{24l}U_3 - \frac{m(m^2-l^2)b^2}{2l^2}UU_1,\\
Y_6 &= \frac{m(m+l)(2m+l)(3m^2+3ml-l^2)b}{720l}U_4 + \frac{m(m^2-l^2)(3m+2l)b^2}{24l^2}U_1^2\\
&\quad + \frac{m(m^2-l^2)(2m+l)b^2}{12l^2}UU_2 + \frac{m(m-l)(m-2l)b^3}{6l^3}U^3.
\end{aligned}
$$

This is enough, since we need only terms up to ε^7 when substituting into second equation (26). The coefficients a, c are found from the requirement that the low order terms vanish while the coefficients b, d are responsible for the scaling of U and t and can be chosen arbitrarily. Finally, we come to the following statement.

Theorem 5. *Continuous limit (30) with the values of parameters*

$$a = \frac{m-l}{m+l}, \quad b = \frac{ml}{6}, \quad c = 2m, \quad d = \frac{m^3(l^2-m^2)}{180}$$

sends systems (26) into the Sawada–Kotera equation

$$U_{,\tau} = U_5 + 5UU_3 + 5U_1U_2 + 5U^2U_1.$$

The higher flows of the SK hierarchy can be derived analogously from suitable linear combinations of the $\mathrm{dSK}^{(l,m)}$ flows. However, the general formulas become rather complicated and we restrict ourselves by the following concrete example corresponding to the local hierarchy $\mathrm{dSK}^{(1,2)}$. Let $u_{,t} = 88u_{,t_1} + 27u_{,t_2}$ where the flows ∂_{t_1} and ∂_{t_2} are defined by equations (1) and (7) respectively, then the formula

$$u(n,t) = \frac{1}{3} + \frac{\varepsilon^2}{9}U\Big(x - \frac{200}{9}\varepsilon t, \tau - \frac{16\varepsilon^7}{189}t\Big), \quad x = \varepsilon n$$

defines the continuous limit to the 7-th order symmetry of SK equation

$$
\begin{aligned}
U_{,\tau} = U_7 + 7UU_5 + 14U_1U_4 + 21U_2U_3 + 14U^2U_3&\\
+ 42UU_1U_2 + 7U_1^3 + \frac{28}{3}U^3U_1&.
\end{aligned}
$$

It is well known that there are gaps in the sequence of orders k of equations from the SK hierarchy, namely, the restrictions $k \nmid 2, 3$ are fulfilled, so that the next higher symmetry is of 11-th order. The natural question appears, how this agrees with relations (20)–(22) or (29) which show that in the discrete case there are no gaps multiple 3. It turns out that their appearance is an artefact of the continuous limit. A straightforward computation shows that if we consider a linear combination with the next $\mathrm{dSK}^{(1,2)}$ flow $u_{,t} = u_{,t_1} + \alpha u_{,t_2} + \beta u_{,t_3}$ and set

$$u(n,t) = a + b\varepsilon^2 U(x + c\varepsilon t, \tau + d\varepsilon^9 t), \quad x = \varepsilon n$$

then all parameters are uniquely determined by the condition of vanishing the terms up to ε^{10}, however then the coefficients at ε^{11} cancel automatically and only the trivial flow $U_{,\tau} = 0$ appears.

3.5 Bilinear equations

The constraint (23) can be solved by introducing additional variables and this leads to a convenient representation of the basic system (26) of the extended $\mathrm{dSK}^{(l,m)}$ hierarchy. Let

$$u = \frac{v_l}{v}, \quad f = \frac{v}{v_{-m}}$$

then first equation (26) is satisfied identically and the second one is equivalent to

$$(T^l - 1)\frac{v_{,t}}{v} = (T^m - 1)\Big(\frac{v}{v_{l-m}} - \frac{v_l}{v_{-m}}\Big).$$

Further substitutions

$$v = \frac{w_m}{w} \quad \Rightarrow \quad u = \frac{w_{m+l}w}{w_m w_l}, \quad f = \frac{w_m w_{-m}}{w^2}$$

bring to the bilinear equation

$$w_{l,t}w - w_l w_{,t} = w_m w_{l-m} - w_{-m}w_{l+m}. \tag{31}$$

For the first time, it appeared in paper [7], in a slightly more general form

$$w_{l,t}w - w_l w_{,t} = w_m w_{l-m} - \alpha w_{-m}w_{l+m} + \beta w w_l$$

which is reduced to (31) by the point change $\tilde{w}(n,t) = e^{\beta t}\alpha^{n^2} w(n,t)$. In particular, it was proven in [7] that this equation admits N-soliton solutions. Here, we consider in more details a specification of 2-soliton formula which leads to the breather solution.

The substitution of the 2-soliton Ansatz

$$w(n,t) = 1 + e_1 + e_2 + A_{12}e_1e_2, \quad e_i = q_i^n \exp(-\omega_i t + \delta_i) \tag{32}$$

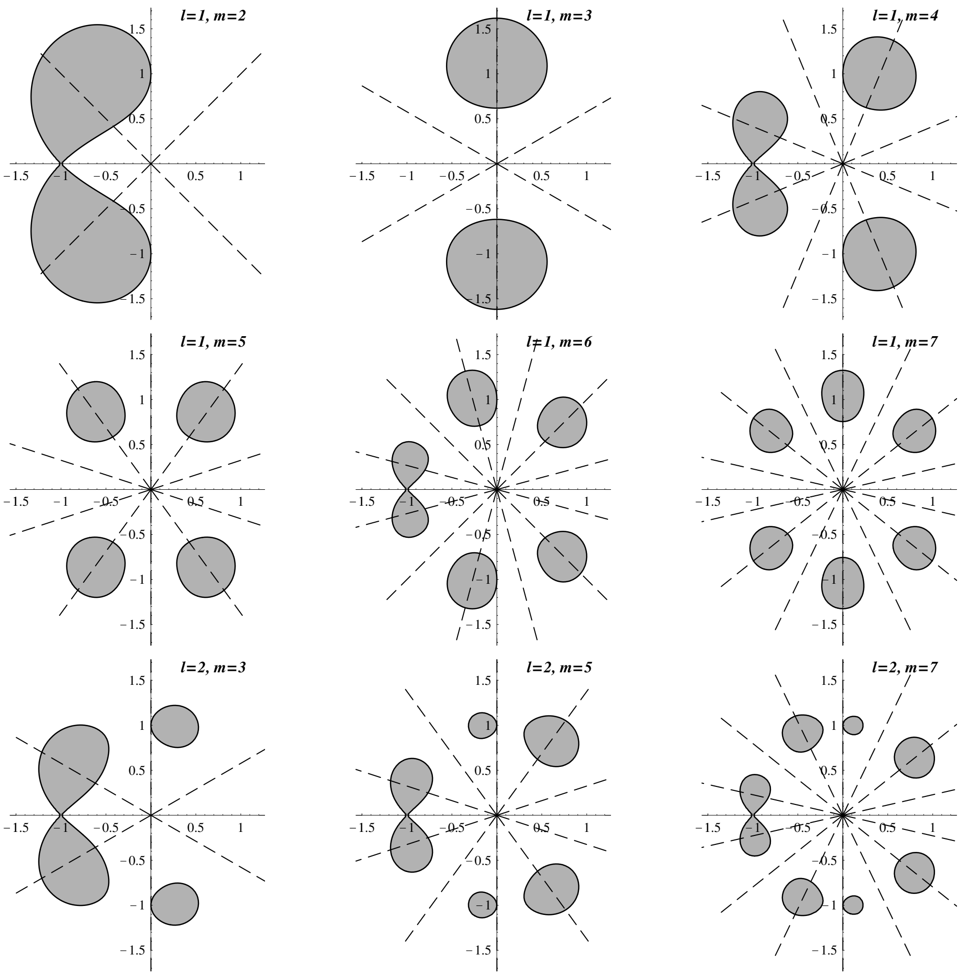

Figure 1. The values of $q = \rho e^{\mathrm{i}\varphi}$ inside the bounded domains in $\mathbb{C}$ correspond to the regular potentials $u(n,t)$. The values along the dashed lines correspond to the potentials periodic in t.

into (31) gives us the dispersion relation and the phase shift:

$$\omega_i = q_i^m - q_i^{-m}, \quad A_{ij} = \frac{(q_i^l - q_j^l)(q_i^m - q_j^m)}{(1 - q_i^l q_j^l)(1 - q_i^m q_j^m)}. \tag{33}$$

The direct check proves that then the 3-soliton Ansatz

$$w = 1 + e_1 + e_2 + e_3 + A_{12}e_1e_2 + A_{13}e_1e_3 + A_{23}e_2e_3 + A_{12}A_{13}A_{23}e_1e_2e_3$$

satisfies (31) automatically. It is interesting to compare these formulas with their counterparts for the continuous SK equation [8, 9, 25, 26]

$$e_i = \exp(\varkappa_i x - \omega_i t + \delta_i), \quad \omega_i = \varkappa_i^5, \quad A_{ij} = \frac{(\varkappa_i - \varkappa_j)^2(\varkappa_i^2 - \varkappa_i\varkappa_j + \varkappa_j^2)}{(\varkappa_i + \varkappa_j)^2(\varkappa_i^2 + \varkappa_i\varkappa_j + \varkappa_j^2)}.$$

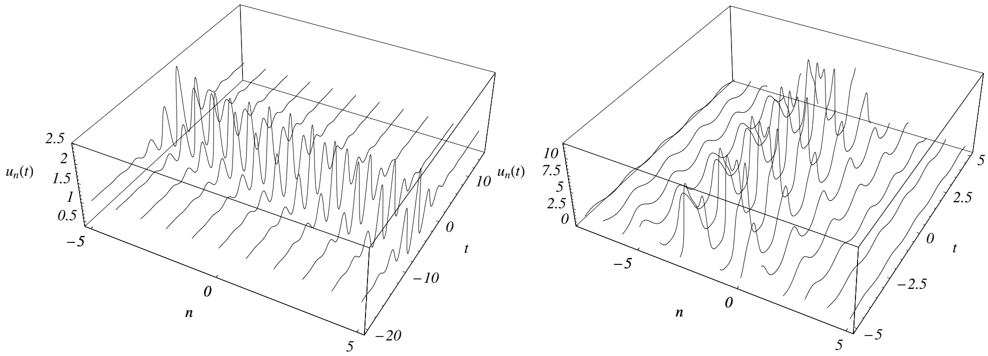

Figure 2. A moving and a stable breathers. The values of parameters: $\rho = 1.2$, $\varphi = 2\pi/3$ (left); $\rho = 1.6$, $\varphi = 3\pi/4$ (right); in both cases $l = 1$, $m = 2$, $\alpha = \beta = 0$.

Formula (32) allows us to obtain the breather-type solutions as well, if we choose

$$q_1 = \rho e^{\mathrm{i}\varphi}, \quad q_2 = \rho e^{-\mathrm{i}\varphi}, \quad \delta_1 = \alpha + \mathrm{i}\beta, \quad \delta_2 = \alpha - \mathrm{i}\beta.$$

The regularity of the potential $u(n,t)$ is achieved under certain restrictions on the value of q. In order to show this, rewrite relations (33) as follows:

$$\omega = \mu + \mathrm{i}\nu, \quad \mu = (\rho^m - \rho^{-m})\cos m\varphi, \quad \nu = (\rho^m + \rho^{-m})\sin m\varphi,$$
$$A_{12} = -\frac{4\rho^{m+l}\sin l\varphi \sin m\varphi}{(1-\rho^{2l})(1-\rho^{2m})},$$

then a simple algebra brings (32) to the form

$$w = 1 + 2z\cos y + A_{12}z^2, \quad y = \varphi n - \nu t + \beta, \quad z = \rho^n e^{\alpha - \mu t}.$$

In particular, if $\varphi = \frac{2k+1}{2m}\pi$ then $\mu = 0$ and solution u is periodic in t. The necessary and sufficient condition for u to be regular is that the function w does not vanish at any n, t. In the generic case the variables y, z are independent and then this is equivalent to the condition that the trinomial $1 + 2z + A_{12}z^2$ does not vanish at real z, that is

$$(\rho^l - \rho^{-l})(\rho^m - \rho^{-m}) + 4\sin l\varphi \sin m\varphi < 0.$$

Thus, we see that already two-phase solutions in these models exhibit a nontrivial zone structure of the spectrum. The corresponding domains in the plane $q = \rho e^{\mathrm{i}\varphi}$ are shown on fig. 1, and the examples of solutions $u(n,t)$ are shown on fig. 2.

4 A discrete analog of the Kaup–Kupershmidt equation

The Kaup–Kupershmidt equation

$$U_{,\tau} = U_5 + 5UU_3 + \frac{25}{2}U_1U_2 + 5U^2U_1$$

is associated with the spectral problem $L\psi = \lambda\psi$ where L is the skew-symmetric ordinary differential operator of third order

$$L = D^3 + UD + \frac{1}{2}U_{,x} = (D-f)D(D+f), \quad U = 2f_{,x} - f^2.$$

When we find a discrete analog, a difficulty is that a symmetric or skew-symmetric difference operator can be of even order only. A way to overcome this is to consider a 6th order difference problem, but on the odd nodes of the lattice only, so that effectively it is of 3rd order with respect to the double shift T^2 (however, the coefficients may depend on the variables associated with the even nodes as well). Let us consider the spectral problem

$$u_{-3}\psi_{-3} + \psi_{-1} = \lambda(\psi_1 + u\psi_3) \tag{34}$$

or, in the operator form, denoting $K = uT^3 + T$:

$$K^\dagger\psi = \lambda K\psi.$$

The Lax equation for the operator $L = K^{-1}K^\dagger$ can be written in the form of system (15). It admits the reduction $B = -A^\dagger$ which yields the equation

$$K_{,t} + A^\dagger K + KA = 0. \tag{35}$$

The operator A is found as a Laurent polynomial with respect to the even powers of T,

$$A = a^{(k)}T^{2k} + \dots + a^{(-k)}T^{-2k},$$

and a direct analysis of equation (35) at $k = 1, 2$ proves the following statement.

Theorem 6. *Equation (35) with $K = uT^3 + T$ is equivalent to the nonlocal lattice equation*

$$u_{,t_1} = u(f_2u_2 - f_1u_1 + f_{-1}u_{-1} - f_{-2}u_{-2}) + f_1 - f_{-1}, \quad f_3u = f_{-1}u_2$$

under the choice

$$A = -fT^2 + f_{-2}u_{-2} - f_{-1}u_{-1} + f_{-3}T^{-2};$$

and it is equivalent to the local lattice equation

$$u_{,t_2} = u(v_3 - v_2 + v_1 - v_{-1} + v_{-2} - v_{-3} - u_2 + u_{-2}), \quad v := u_1uu_{-1} \tag{36}$$

under the choice

$$A = u_1 T^4 - u_{-4} T^{-4} + (1 - u_{-1} u_{-2})(T^2 - T^{-2})$$
$$+ u_{-1} - u_{-2} - v + v_{-1} - v_{-2} + v_{-3}.$$

It is worth noticing that, alternatively, one can use the following pair of operators (cf with the gauge equivalence (12)):

$$\tilde{K} = uT^3 + T^{-1}, \quad \tilde{A} = -u_1 u_{-1} T^4 + u_{-2} u_{-4} T^{-4} - v + v_{-1} - v_{-2} + v_{-3}. \quad (37)$$

The continuous limit to the KK equation is of the same general form as before, namely, for the flow (36) it reads

$$u(n, t_2) = \frac{1}{3} + \frac{4}{9}\varepsilon^2 U\Big(x - \frac{8}{9}\varepsilon t_2, \tau + \frac{64\varepsilon^5}{135} t_2\Big), \quad x = \varepsilon n.$$

5 Examples related to generic operators

Recall that, according to [20], the Bogoyavlensky type lattices can be viewed as reductions of more general multi-field models associated with the spectral problems $L\psi = \lambda\psi$ for generic difference operators $L = u^{(m)} T^m + u^{(m-1)} T^{m-1} + \cdots + u^{(1-l)} T^{1-l} + u^{(-l)} T^{-l}$. Here m, l are any positive integers, and one can adopt the normalization $u^{(m)} = 1$ or $u^{(l)} = 1$ without loss of generality. A part of the flows from the corresponding hierarchy is consistent with the constraints $u^{(m-1)} = \cdots = u^{(1-l)} = 0$ and this reduction brings to the Bogoyavlensky lattices. A detailed study of some other reductions can be found in [22].

The lattices introduced in the previous sections are related with the spectral problems $P\psi = \lambda Q\psi$ where operators P, Q are binomial. It is natural to expect that these lattices also define reductions for some multi-field equations related with more general operators P, Q. The study of such models is beyond the scope of the present paper and we restrict ourselves by three typical examples.

Example 1. First, let us consider the Lax equations $P_{,t} = BP - PA$, $Q_{,t} = BQ - QA$ for the binomial operators P, Q with different potentials:

$$P = uT^3 + T, \quad Q = T^2 + v.$$

If $v = u$ then operators A, B are given by formulas (18), (20) with a self-adjoint operator F which contains only even powers of T^2. In the general case two sets of operators A, B appear, containing positive or negative powers of T^2. The simplest operators and corresponding flows are the following:

$$
\begin{aligned}
&A^- = v_{-2}v_{-1}T^{-2} + f_{-3} + f_{-2}, \quad B^- = v_{-1}vT^{-2} + f_{-1} + f,\\
&\qquad u_{,t^-} = u(f_{-1} - f_1),\\
&\qquad v_{,t^-} = v(f + f_{-1} - f_{-2} - f_{-3} - v_1 + v_{-1}), \quad f := uv_1v_2;\\
&A^+ = u_{-2}u_{-1}T^2 + g_{-1} + g, \quad B^+ = uu_1T^2 + g + g_1,\\
&\qquad u_{,t^+} = u(g + g_1 - g_2 - g_3 - u_{-1} + u_1),\\
&\qquad v_{,t^+} = v(g_1 - g_{-1}), \qquad g := u_{-2}u_{-1}v.
\end{aligned}
$$

The flows ∂_{t^-} and ∂_{t^+} commute, and the flow $\partial_{,t} = \partial_{t^-} - \partial_{t^+}$ admits the reduction $v = u$ which brings to the dSK equation (1). It should be noted that the same flows can be obtained starting from the gauge equivalent operators $P = uT^3 + T^2$, $Q = T + v$.

Example 2. Now let us consider trinomial operators

$$P = uT^3 + pT^2 + T, \quad Q = T^2 + qT + v.$$

In this case operators A, B contain the odd powers of T as well. The simplest operators and the corresponding flows are of the form

$$
\begin{aligned}
A^- = v_{-1}T^{-1} + v_{-1}p_{-2}, \quad & B^- = vT^{-1} + v_1p,\\
u_{,x^-} &= u(u_{-1}q - u_1q_2 - p + p_1),\\
p_{,x^-} &= p(u_{-1}q - uq_1) + u - u_{-1},\\
v_{,x^-} &= v(u_{-1}q - u_{-2}q_{-1}),\\
q_{,x^-} &= uv_1 - u_{-2}v;
\end{aligned}
$$

$$
\begin{aligned}
A^+ = u_{-2}T + u_{-2}q_{-1}, \quad & B^+ = uT + u_{-1}q,\\
u_{,x^+} &= u(v_1p - v_2p_1),\\
p_{,x^+} &= u_{-1}v - uv_2,\\
v_{,x^+} &= v(v_1p - v_{-1}p_{-2} + q_{-1} - q),\\
q_{,x^+} &= q(v_1p - vp_{-1}) + v - v_1.
\end{aligned}
$$

Example 3. Let us consider the following generalization of the spectral problem (37):

$$K^\dagger\psi = \lambda K\psi, \quad K = uT^3 + v_{-1}T + T^{-1}.$$

The isospectral deformations are defined by the operators $A = a^{(k)}T^{2k} + a^{(k-1)}T^{2k-2} + \cdots + a^{(-k)}T^{-2k}$. The simplest case $k = 1$ results in

$$A = u_{-1}T^2 - u_{-2}T^{-2} + u_{-1}v_{-1} - u_{-2}v_{-2}$$

and equation $K_{,t} + A^\dagger K + KA = 0$ is equivalent to the lattice

$$u_{,t} = -u(u_2 v_2 - u_1 v_1 + u_{-1} v_{-1} - u_{-2} v_{-2} - v_1 + v_{-1}),$$
$$v_{,t} = -v(u_1 v_1 - u_{-1} v_{-1}) + u_2 u_1 - u_{-1} u_{-2} + u_1 - u_{-1}.$$

The higher symmetry corresponding to $k = 2$ is too bulky and we do not write it down, however one can check that it admits the reduction $v = 0$ to the dKK equation (36). In contrast, the flow ∂_t itself does not admit this reduction.

6 Conclusion

In this article we introduced a family of integrable lattice hierarchies associated with fractional Lax operators. In particular, these hierarchies contain equations found earlier in [1, 7] by use of the Hirota bilinear formalism. We proved that these equations serve as semi-discrete analogs of SK and KK equations. An important question which remains open is about the Hamiltonian structure of the presented equations. As usually, the existence of Lax representation allows to obtain a set of conserved quantities which presumably are Hamiltonians, and moreover, the applicability of r-matrix approach suggests that some more or less standard Poisson bracket should exist. However, no explicit answer is found yet. Another intriguing question is about possible relations with the models introduced in [27, 28] within the theory of the lattice W algebras.

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