A heat kernel version of Hardy's theorem for the Laguerre hypergroup

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Abstract: The uncertainty principle says that a function and its Fourier transform can't simultaneously decay very rapidly at infinity. A classical version of uncertainty principle, known as Hardy's theorem, was first proved by Hardy on R. The Hardy's theorem has been extended to various settings. More results can be found in the book by Thangavelu and the references therein. Hardy's theorem is well explained in terms of the heat kernel. In view of this point, Thangavelu proved a heat kernel version of Hardy's theorem for the Heisenberg group. Thangavelu's result is remarkable because the heat kernel with respect to the sublaplacian on the Heisenberg group decays as much slower than the heat kernel for Euclidean space when the central variable t is concerned. In this paper, we prove a heat kernel version of Hardy's theorem for the Laguerre hypergroup.

Keywords: Laguerre hypergroup; Uncertainty principle; Hardy's theorem

0 Introduction

The uncertainty principle says that a function and its Fourier transform can't simultaneously decay very rapidly at infinity. A classical version of uncertainty principle, known as Hardy's theorem, was first proved by Hardy [1] on R. We state Hardy's theorem on R as follows.

Theorem 1. Suppose f is a measurable function on R^n and satisfies

$$|f(x)| \le C(1+|x|^2)^k e^{-a|x|^2},$$

$$|\hat{f}(y)| \le C(1+|y|^2)^k e^{-b|y|^2},$$

where a, b >0. Then f=0 whenever $ab > \frac{1}{4}$ and when $ab = \frac{1}{4}$, $f(x) = P(x)e^{-a|x|^2}$, where P is a polynomial of degree $\le 2k$.

The Hardy's theorem has been extended to various settings. More results can be found in the book [2] by Thangavelu and the references therein. We note that the heat kernel h_s on R^n is given by $h_s(x) = (4\pi s)^{-n/2} e^{-|x|^2/(4s)}, \hat{h}_s(y) = e^{-s|y|^2}$.

Thus Hardy's theorem is well explained in terms of the heat kernel. In view of this point, Thangavelu [3] proved a heat kernel version of Hardy's theorem for the Heisenberg group. Thangavelu's result is remarkable because the heat kernel with respect to the sublaplacian on the Heisenberg group decays as $e^{-a|t|}$ much slower than $e^{-a|t|^2}$ when the central variable t is concerned. In this paper we will prove a heat kernel version of Hardy's theorem for the Laguerre hypergroup.

Throughout the paper, we will use C to denote the positive constant, which is not necessarily same at each occurrence.

1 Preliminaries

In this section, we set some notations and collect some basic results about the Laguerre

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hypergroup. For more about the Laguerre hypergroup we refer the reader to [4], [5], [6] and [7].

We also give Hardy's theorem for the Hankel transform, which we will use in the sequel.

Given $\alpha \ge 0$, let $K = [0,\infty) \times R$ equipped with the measure

$$dm_\alpha(x,t) = \frac{1}{\pi\Gamma(\alpha+1)} x^{2\alpha+1} dxdt.$$

We simply write $L^p(K)$ instead of $L^p(k, dm_\alpha)$. For $(x,t) \in K$, the generalized translation operators $T^\alpha_{(x,t)}$ are defined by

$$T^\alpha_{(x,t)} f(y,s) = \frac{1}{2\pi}\int_0^{2\pi} f(\sqrt{x^2+y^2+2xy\cos\theta}, s+t+xy\sin\theta)d\theta \quad \text{for} \quad \alpha = 0 \quad \text{and}$$

$$T^\alpha_{(x,t)} f(y,s) = \frac{\alpha}{2\pi}\int_0^{2\pi}\int_0^1 f(\sqrt{x^2+y^2+2xyr\cos\theta}, s+t+xyr\sin\theta) r(1-r^2)^{\alpha-1} drd\theta$$

for $\alpha > 0$.

Let $M_b(K)$ denote the space of bounded Radon measures on K. The convolution on $M_b(K)$ is defined by

$$(\mu * \nu)(f) = \int_{K\times K} T^\alpha_{(x,t)} f(y,s) d\mu(x,t) d\nu(y,s).$$

It is easy to see that $\mu * \nu = \nu * \mu$. If $f, g \in L^1(K)$ and $\mu = fm_\alpha, \nu = gm_\alpha$, then $\mu * \nu = (f * g)m_\alpha$, where $f * g$ is the convolution of functions f and g defined by

$$f * g(x,t) = \int_K T^\alpha_{(x,t)} f(y,s) g(y,-s) dm_\alpha(y,s).$$

$(K, *, i)$ is a hypergroup in the sense of Jewett (cf. [8], [9]), where i denotes the involution defined by i(x,t)=(x,-t). If $\alpha = n-1$ is a nonnegative integer, then the Laguerre hypergroup K can be identified with the hypergroup of radial functions on the Heisenberg group H^n.

The dilations on K are defined by $\delta_r(x,t) = (rx, r^2 t), r > 0$.

It is clear that the dilations are consistent with the structure of hypergroup. Let

$$(\delta_r f)(x,t) = r^{-(2\alpha+4)} f(\frac{x}{r}, \frac{t}{r^2}). \quad \text{Then we have} \quad \|\delta_r f\|_{L^1(K)} = \|f\|_{L^1(K)}.$$

Let us consider the partial differential operator $L = -(\frac{\partial^2}{\partial x^2} + \frac{2\alpha+1}{x}\frac{\partial}{\partial x} + x^2\frac{\partial^2}{\partial t^2})$.

L is positive and symmetric in $L^2(K)$, and is homogeneous of degree 2 with respect to the dilations defined above. When $\alpha = n-1$, L is the radial part of the sublaplacian on the Heisenberg group H^n. We call L the generalized sublaplacian.

Let L^α_m be the Laguerre polynomial of degree m and order α defined in terms of the generating function by $\sum_{m=0}^{\infty} s^m L^\alpha_m(x) = \frac{1}{(1-s)^{(\alpha+1)}} e^{-\frac{xs}{1-s}}$.

Set $\varphi^{(\alpha)}_m(x) = e^{-\frac{x^2}{2}} L^{(\alpha)}_m(x^2)$. (1)

Lemma 1. For any $\lambda \neq 0$, the system $\left\{ \left(\frac{2|\lambda|^{\alpha+1} m!}{\Gamma(m+\alpha+1)} \right)^{1/2} \varphi_m^{(\alpha)}(\sqrt{|\lambda|}x) : m \in N \right\}$ forms an orthonormal basis of the space $L^2([0,\infty), x^{2\alpha+1}dx)$.

For $(\lambda, m) \in R \times N$, we put

$$\psi_{(\lambda,m)}(x,t) = \left(\frac{m!\Gamma(\alpha+1)}{\Gamma(m+\alpha+1)} \right)^{1/2} e^{i\lambda t} \varphi_m^{(\alpha)}(\sqrt{|\lambda|}x).$$

Lemma 2. The functions $\psi_{(\lambda,m)}$ satisfy that

(a) $\left\| \psi_{(\lambda,m)} \right\|_{L^\infty} = \psi_{(\lambda,m)}(0,0) = 1,$

(b) $T^\alpha_{(x,t)} \psi_{(\lambda,m)}(y,s) = \psi_{(\lambda,m)}(x,t) \psi_{(\lambda,m)}(y,s),$

(c) $\mathrm{L}\psi_{(\lambda,m)} = 2|\lambda|(2m+\alpha+1)\psi_{(\lambda,m)}.$

Let $f \in L^1(K)$, the generalized Fourier transform of f is defined by

$$\hat{f}(\lambda,m) = \int_K f(x,t)\, \psi_{(-\lambda,m)}(x,t) dm_\alpha(x,t).$$

We note that

$$\hat{f}(\lambda,m) = \frac{m!}{\rho\Gamma(m+\alpha+1)} \int_0^\infty f^\lambda(x) \varphi_m^{(\alpha)}(\sqrt{|\lambda|}x) x^{2\alpha+1} dx, \tag{2}$$

Where $f^\lambda(x) = \int_{-\infty}^{+\infty} f(x,t) e^{-i\lambda t} dt$ is the Fourier transform of f(x,t) in the t-variable.

Let $d\gamma_\alpha$ be the positive measure defined on $R \times N$ by

$$\int_{R\times N} g(\lambda,m) d\gamma_\alpha(\lambda,m) = \sum_{m=0}^\infty \frac{\Gamma(m+\alpha+1)}{m!\Gamma(\alpha+1)} \int_R g(\lambda,m) |\lambda|^{\alpha+1} d\lambda.$$

Write $L^p(\hat{K})$ instead of $L^p(R\times N, d\gamma_\alpha)$. We have the following

Plancherel formula $\|f\|_{L^2(K)} = \left\| \hat{f} \right\|_{L^2(\hat{K})}, f \in L^1(K) \cap L^2(K).$

We also have the inverse formula of the generalized Fourier transform.

$$f(x,t) = \int_{R\times N} \hat{f}(\lambda,m) \psi_{(\lambda,m)}(x,t) d\gamma_\alpha(\lambda,m).$$

Let $\{H^s : s > 0\} = \{e^{-sL} : s > 0\}$ be the heat semigroup generated by L.

There is an unique smooth function $h((x,t),s) = h_s(x,t)$ on $K \times [0,\infty)$ such that $H^s f(x,t) = f * h_s(x,t)$. h_s is called the heat kernel associated to L.

By the definition of the generalized Fourier transform and Lemma 2, it is easy to know that

$$\widehat{\delta_r f}(\lambda,m) = \hat{f}(r^2\lambda,m),\ \widehat{Lf}(\lambda,m) = 2|\lambda|(2m+\alpha+1)\hat{f}(\lambda,m),$$

$$\widehat{f*g}(\lambda,m) = \hat{f}(\lambda,m)\hat{g}(\lambda,m).$$

Therefore

$$\hat{h}_s(\lambda,m) = e^{-2|\lambda|(2m+\alpha+1)s},\ h_{s_1} * h_{s_2}(\lambda,m) = h_{s_1+s_2},$$

$$h_s(x,t) = s^{-(\alpha+2)} h_1(\frac{x}{\sqrt{s}}, \frac{t}{s}).$$

Although the heat kernel $h_s(x,t)$ is not explicitly known, we do have the explicit expression of $h_s^\lambda(x)$, from which the estimate for $h_s(x,t)$ is obtained (cf. [4]).

Lemma 3. $h_s^\lambda(x) = 2\pi(\frac{\lambda}{2\sinh(2\lambda s)})^{\alpha+1} e^{-\frac{1}{2}\lambda \coth(2\lambda s) x^2}$.

Lemma 4. There exists A>0 such that

$$0 < h_s(x,t) \le s^{-(\alpha+2)} e^{-\frac{A}{s}(|x|^2+|t|)}.$$

Now we turn to the Hankel transform. For $z \in C$, the Bessel function of first kind and order α is defined by

$$J_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{-\alpha-2k} z^{\alpha+2k}}{\Gamma(k+1)\Gamma(k+\alpha+1)} = \frac{2^{-\alpha} z^\alpha}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_{-1}^{1} e^{izs}(1-s^2)^{\alpha-\frac{1}{2}} ds.$$

In this paper we only concern about the case $\alpha \ge 0$ although $\mathrm{J}_\alpha(z)$ are well defined for all $\alpha \in C$.

We refer the reader to Watson's book [10] for the reference about the Bessel function. Here we point out that

$$\frac{d}{dz}(z^{-\alpha} J_\alpha(z)) = -z^{-\alpha} J_{\alpha+1}(z), \qquad (3)$$

$$|J_\alpha(it)| \le C t^{-\frac{1}{2}} e^t, t > 0. \qquad (4)$$

Then the Hankel transform of order α of $\mathrm{f} \in \mathrm{L}^1([0,\infty))$ is defined by

$$(\mathrm{H}_\alpha f)(y) = \int_0^\infty f(x) \frac{\mathrm{J}_\alpha(xy)}{(xy)^\alpha} x^{2\alpha+1} dx.$$

The functions $\varphi_m(x)$ defined by (1) are the eigenfunctions of the Hankel transform, i.e.,

$$(\mathrm{H}_\alpha \varphi_m)(x) = (-1)^m \varphi_m(x) \qquad (5)$$

(cf. [11], P. 42). It follows that $\mathrm{H}_\alpha^{-1} = H_\alpha$. Also we have

$$(\mathrm{H}_\alpha f(r\cdot))(y) = r^{-2\alpha-2} (\mathrm{H}_\alpha f)(\frac{y}{r}), r > 0. \qquad (6)$$

We state Hardy's theorem for the Hankel transform as follows.

Proposition 1. Suppose $\mathrm{f} \in \mathrm{L}^1([0,\infty))$ and satisfies

$$|f(x)| \le C(1+|x|^2)^k e^{-a|x|^2},$$

$$|(H_\alpha f)(y)| \le C(1+|y|^2)^k e^{-b|y|^2},$$

where a, b >0. Then f=0 whenever $ab > \frac{1}{4}$ and when $ab = \frac{1}{4}$, $f(x) = P(x) e^{-a|x|^2}$, where P is a polynomial of degree $\le 2k$.

Proposition 1 is a special case of Hardy's theorem for the Ch\'ebli-Trim\`eche transform (cf.

[12]).

2 Hardy's theorem in terms of heat kernel

First, we prove the following theorem.

Theorem 1. Suppose f is a measurable function on K and satisfies

$$|f(x,t)|\le C(1+|x|^2)^k h_a(x,t),$$

$$|\hat{f}(\lambda,m)|\le C(1+2|\lambda|(2m+\alpha+1))^k e^{-2|\lambda|(2m+\alpha+1)b},$$

where a, b are positive constants. Then f(x, t)=0 whenever a<b.

Proof: By Lemma 4,

$$|f(x,t)|\le C(1+|x|^2)^k e^{-\frac{A}{a}(x^2+|t|)}.$$

For any $x\ge 0$, $f^{\lambda}(x)$ can be extended to a holomorphic function of λ in the strip $\left\{\lambda\in C:|\operatorname{Im}\lambda|<\frac{A}{a}\right\}$. We choose $\delta>0$ such that $a(e^{2b\delta}+e^{-2b\delta})<2b.$ We will prove that $f^{\lambda}(x)=0$ for $0<\lambda<\delta$. This means that $f^{\lambda}(x)=0$ for all $\lambda\in R$. Therefore f(x,t)=0.

It is clear that

$$|f^{\lambda}(x)|\le\int_R|f(x,t)|\,dt\le C(1+x^2)^k e^{-\frac{1}{4a}x^2}. \tag{7}$$

By Lemma 1 and (2),

$$f^{\lambda}(x)=2\pi|\lambda|^{\alpha+1}\sum_{m=0}^{\infty}\hat{f}(\lambda,m)\varphi_m^{(\alpha)}(\sqrt{|\lambda|}x).$$

Take the Hankel transform on both sides of the above equality, by (5), we get

$$(H_\alpha f^{\lambda})(x)=2\pi|\lambda|^{\alpha+1}\sum_{m=0}^{\infty}\hat{f}(\lambda,m)(-1)^{\alpha}\varphi_m^{(\alpha)}(|\lambda|^{-1/2}x).$$

Hence we have

$$|(H_\alpha f^{\lambda})(x)|\le C\sum_{m=0}^{\infty}(1+2|\lambda|(2m+\alpha+1))^k e^{-2|\lambda|(2m+\alpha+1)b}|\varphi_m^{(\alpha)}(|\lambda|^{-1/2}x)|.$$

For $0\le j\le k,\lambda\ne 0$, let us consider

$$g_j(x)=\sum_{m=0}^{\infty}(2|\lambda|(2m+\alpha+1))^j e^{-2|\lambda|(2m+\alpha+1)b}|\varphi_m^{(\alpha)}(|\lambda|^{-1/2}x)|.$$

We have

$$g_j(x)\le C|\lambda|^{-\frac{j_0}{2}}\Big(\sum_{m=0}^{\infty}\frac{m!}{\Gamma(m+\alpha+1)}(2|\lambda|(2m+\alpha+1))^{2j+j_0}e^{-4|\lambda|(2m+\alpha+1)b}(\varphi_m^{(\alpha)}(|\lambda|^{-1/2}x))^2\Big)^{\frac{1}{2}},$$

Where $j_0>\alpha+1$. Let

$$F(t)=\sum_{m=0}^{\infty}\frac{m!}{\Gamma(m+\alpha+1)}e^{-2|\lambda|(2m+\alpha+1)t}(\varphi_m^{(\alpha)}(|\lambda|^{-1/2}x))^2.$$

Then we have

$$g_j(x)\le C|\lambda|^{-\frac{j_0}{2}}|F^{(2j+j_0)}(2b)|^{\frac{1}{2}}.$$

Applying the identity (cf.[13])

$$\sum_{m=0}^{\infty}\frac{m!}{\Gamma(m+\alpha+1)}(L_m^{\alpha}(s))^2 r^m=(1-r)^{-1}(-s^2r)^{-\frac{\alpha}{2}}e^{-\frac{2sr}{1-r}}J_\alpha(\frac{2is\sqrt{r}}{1-r}).$$

We get

$$F(t)=(\sinh(2|\lambda|t))^{-\alpha-1}e^{-\frac{\coth(2\lambda t)}{\lambda}x^2}\frac{J_\alpha(\frac{ix^2}{\lambda\sinh(2\lambda t)})}{(\frac{ix^2}{\lambda\sinh(2\lambda t)})^{\alpha}}.$$

Differentiating F(t) and making use of (3) and (4), we obtain the estimate

$$|F^{(2j+j_0)}(2b)|\le C_\lambda(1+x^2)^{2l}e^{-\frac{\tanh(2\lambda b)}{2\lambda}x^2}.$$

Here and below we denote by C_λ a positive number which depends only on λ. So

$$g_j(x)\le C_\lambda(1+x^2)^l e^{-\frac{\tanh(2\lambda b)}{2\lambda}x^2}.$$

It follows that there exists k_0 such that

$$|(H_\alpha f^\lambda)(x)|\le C_\lambda(1+x^2)^{k_0}e^{-\frac{\tanh(2\lambda b)}{2\lambda}x^2}. \qquad (8)$$

When $0<\lambda<\delta,\ \ a<\frac{2b}{e^{2b\lambda}+e^{-2b\lambda}}<\frac{\tanh(2\lambda b)}{2\lambda}$.

By (7), (8) and Proposition 1, we get $f^\lambda(x)=0$ for $0<\lambda<\delta$. This proves Theorem 2. ■

Now we deal with the case a=b.

Theorem 3. Suppose $f\in L^1(K)$ and satisfies

$$|f^\lambda(x)|\le C(1+|x|^2)^k h_a^\lambda(x),$$

$$|\hat{f}(\lambda,m)|\le C(1+2|\lambda|(2m+\alpha+1))^k e^{-2|\lambda|(2m+\alpha+1)a},$$

where a is a positive constant. Then $f(x,t)=q*_t h_a(x,t)$, where q(x,t) is a distribution with

$q^\lambda(x)$ are polynomials of degree $\le 2k$ in the variable x and $*_t$ stands for the usual convolution in the variable t. In particular, when k=0, q(x,t)=q(t) is a distribution on R and its Fourier transform is bounded.

Proof: Because

$$|\hat{f}(\lambda,m)|\le C(1+2|\lambda|(2m+\alpha+1))^k e^{-2|\lambda|(2m+\alpha+1)a},$$

we have, in view of (8),

$$|(H_\alpha f^\lambda)(x)|\le C_\lambda(1+x^2)^{k_0}e^{-\frac{\tanh(2\lambda b)}{2\lambda}x^2}. \qquad (9)$$

By Lemma 3,

$$|f^\lambda(x)|\le C_\lambda(1+|x|^2)^k e^{-\frac{1}{2}\lambda\coth(2a\lambda)x^2}. \qquad (10)$$

From (9), (10) and Proposition 1,

$$f^{\lambda}(x) = P^{\lambda}(x)e^{-\frac{1}{2}\lambda\coth(2a\lambda)x^2},$$

where $P^{\lambda}(x)$ is polynomials of degree $\leq 2k$ in the variable x. Then $f^{\lambda}(x) = q^{\lambda}(x)h_a^{\lambda}(x)$,

where $q^{\lambda}(x) = \frac{1}{2\pi}(\frac{\lambda}{2\sinh(2a\lambda)})^{-\alpha-1}P^{\lambda}(x)$. Theorem 3 is proved. ■

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拉盖尔超群上热核形式的哈代定理

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摘要：不确定性原理是指一个函数和它的傅立叶变换不能同时在无穷远处下降地太快。关于不确定性原理的一个经典的结果是哈代定理，最早由哈代在直线上给出了证明。哈代定理被推广到很多项式，首先是被推广到 n 维欧式空间。后来，逐渐被推广到非交换情形。关于这一方面的结果可以参考 Thangavelu 的著作。欧式空间上的哈代定理可以很好地用热核来解释，这是由于这一原因，Thangavelu 在海森堡群上证明了一种热核形式的哈代定理。这一结果是一个非常重要的结论，因为海森堡群上的热核下降速度比欧式空间要慢很多。在本文中，我们将证明一种拉盖尔超群上热核形式的哈代定理。

关键词：拉盖尔超群；不确定性原理；哈代定理

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