限制在有限区间上的几个 Hilbert 型积分不等式及其逆

陈广生
（广西现代职业技术学院计算机工程系，广西河池 547000）

摘要：通过引入两个独立参数 λ1，λ2 和两对共轭指数(p,q)，(r,s),利用权函数方法和实分析技巧，在有限区间（a,b）,(a,∞)，（0，b）(0<a<b<∞)上，分别建立了几个广义核的限制在有限区间上的具有多参数的 Hilbert 型积分不等式,并证明了一些不等式的常数因子是的最佳值.作为应用，建立它们逆向不等式及其等价形式，并考虑一些特殊结果.

关键词： Hilbert 类积分不等式；权函数；Holder 不等式；

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Some Hilbert-type integral inequalities and their revers on the Finite interval

CHEN Guangsheng
(Department of Computer Engineering,Guangxi Modern Vocational Technology College, Hechi, Guangxi 547000)

Abstract: By introducing two independent parameters λ1，λ2 ， and two pairs of conjugate exponents (p,q)，(r,s), using weight function and the method of real analisis. On the finite interval （a,b）,(a,∞)，（0，b）(0<a<b<∞), some Hilbert's type integral inequalities were established respectively with some parameters and general kernel, and constant factors of some inequalities are proved to be the optimum value. As its application, we have built their revers and equivalent form ,and considered some special results.

Key words: Hilbert's type integral inequality；weight function；Holder's inequality.

0 引言

设 $p>1$，$\frac{1}{p}+\frac{1}{q}=1$，$f(x)$，$g(x)\ge 0$，$0<\int_0^\infty f^p(x)dx<\infty$，$0<\int_0^\infty g^q(x)dx<\infty$，则有如下著名的Hardy-Hilbert积分不等式 [1]：

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy<\frac{\pi}{\sin(\pi/p)}[\int_0^\infty f^p(x)dx]^{\frac{1}{p}}[\int_0^\infty g^q(x)dx]^{\frac{1}{q}}, \tag{0.1}$$

这里，常数因子 $\frac{\pi}{\sin(\pi/p)}$ 是最佳值。它在分析学中有重要的应用[2].

最近，文献[3]引入单参量及两共轭指数给出了以下的不等式：

设 $p>1$，$\frac{1}{p}+\frac{1}{q}=1$，$r>1$，$\frac{1}{r}+\frac{1}{s}=1$，$\lambda_1$，$\lambda_2>0$，$f(x)$，$g(x)\ge 0$，使得 $0<\int_a^b x^{p(1-\lambda_1/r)-1}f^p(x)dx<\infty$，$0<\int_a^b x^{q(1-\lambda_2/s)-1}g^q(x)dx<\infty$，则有

作者简介：陈广生（1979－），男，广西北流人，讲师，硕士，从事解析不等式、小波分析和热辐射研究. E-mail: cgswavelets@yahoo.com.cn

$$\int_a^b\int_a^b \frac{f(x)g(y)}{\max\{x^{\lambda_1},y^{\lambda_2}\}}dxdy < C\{\int_a^b x^{p(1-\lambda_1/r)-1} f^p(x)dx\}^{\frac{1}{p}}\{\int_a^b x^{q(1-\lambda_2/s)-1} g^q(x)dx\}^{\frac{1}{q}}, \tag{0.2}$$

这里，$C=\frac{rs}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}[1-(\frac{a}{b})^{\frac{\lambda_2}{rs}}]^{\frac{1}{p}}[1-(\frac{a}{b})^{\frac{\lambda_1}{rs}}]^{\frac{1}{q}}$.

本文的目的是引入两个独立参数λ_1，λ_2，利用权函数方法和实分析技巧，在有限区间上，建立核为一般非齐次函数的Hilbert型积分不等式.作为应用，建立它们的逆向不等式及其等价式，考虑了一些特殊结果，而且所建立的逆向不等式推广了文献[4]的结果.

1 限制在区间(a,b)（$0<a<b<\infty$）上的 Hilbert 型积分不等式

1.1 两个等价不等式

引理 **1.1.1** 设λ_1，$\lambda_2>0$，$k(x^{\lambda_1},y^{\lambda_2})\ge 0$为$(0,\infty)\times(0,\infty)$上的非齐次可测函数，$k(1,u)$为$(0,\infty)\times(0,\infty)$上的-1 齐次函数，$\frac{1}{r}+\frac{1}{s}=1$，$r>1$，$k=\int_0^\infty k(1,u)u^{-\frac{1}{r}}du$为一个正数，且在$(0,1]$上，$k(1,u)$与$k(u,1)>0$ a.e. 于$(0,1]$，$(a,b)\subseteq(0,\infty)$，定义权函数

$$\omega_{\lambda_1,\lambda_2}(s,x,a,b)=\int_a^b k(x^{\lambda_1},y^{\lambda_2})x^{\lambda_1/r}y^{\lambda_2/s-1}dy, \tag{1.1.1}$$

$$\omega_{\lambda_1,\lambda_2}(r,y,a,b)=\int_a^b k(x^{\lambda_1},y^{\lambda_2})y^{\lambda_2/s}x^{\lambda_1/r-1}dx. \tag{1.1.2}$$

则对$x,y\in(a,b)$，$\omega_{\lambda_1,\lambda_2}(s,x,a,b)$，$\omega_{\lambda_1,\lambda_2}(r,y,a,b)$都有正的下界

证明 设$x\in(a,b)$（$0<a<b<\infty$）.在下式，对两积分分别做变换令$u=y^{\lambda_2}/x^{\lambda_1}$及$u=x^{\lambda_1}/y^{\lambda_2}$，则有

$$\begin{aligned}\omega_{\lambda_1,\lambda_2}(s,x,a,b)&=x^{\lambda_1/r}[\int_a^x k(x^{\lambda_1},y^{\lambda_2})y^{\lambda_2/s-1}+\int_x^b k(x^{\lambda_1},y^{\lambda_2})y^{\lambda_2/s-1}]\\&=\frac{1}{\lambda_2}[\int_{a^{\lambda_2}/x^{\lambda_1}}^{x^{\lambda_2}/x^{\lambda_1}} k(1,u)u^{\frac{1}{s}-1}du+\int_{x^{\lambda_1}/b^{\lambda_2}}^{x^{\lambda_1}/x^{\lambda_2}} k(u,1)u^{\frac{1}{r}-1}du].\end{aligned}$$

由条件知，$\omega_{\lambda_1,\lambda_2}(s,x,a,b)$在$x\in[a,b]$是两个不定积分之和，因而绝对连续.易见它在$[a,b]$上连续且恒正，因而有正的最小值.故$\omega_{\lambda_1,\lambda_2}(s,x,a,b)$在在$x\in[a,b]$必有正的下界.另外，还可求得

$$\omega_{\lambda_1,\lambda_2}(s,x,0,b)=\frac{1}{\lambda_2}[\int_0^{x^{\lambda_2}/x^{\lambda_1}} k(1,u)u^{\frac{1}{s}-1}du+\int_{x^{\lambda_1}/b^{\lambda_2}}^{x^{\lambda_1}/x^{\lambda_2}} k(u,1)u^{\frac{1}{r}-1}du]\ge\frac{1}{\lambda_2}\int_0^{x^{\lambda_2}/x^{\lambda_1}} k(1,u)u^{\frac{1}{s}-1}du>0, x\in(0,b),$$

$$\omega_{\lambda_1,\lambda_2}(s,x,a,\infty)=\frac{1}{\lambda_2}[\int_{a^{\lambda_2}/x^{\lambda_1}}^{x^{\lambda_2}/x^{\lambda_1}} k(1,u)u^{\frac{1}{s}-1}du+\int_0^{x^{\lambda_1}/x^{\lambda_2}} k(u,1)u^{\frac{1}{r}-1}du]\ge\frac{1}{\lambda_2}\int_{a^{\lambda_2}/x^{\lambda_1}}^{x^{\lambda_2}/x^{\lambda_1}} k(1,u)u^{\frac{1}{s}-1}du>0, x\in(a,\infty),$$

$$\omega_{\lambda_1,\lambda_2}(s,x,0,\infty)=\frac{k}{\lambda_2}>0,\quad x\in(0,\infty).$$

同理可证明$\omega_{\lambda_1,\lambda_2}(r,y,a,b)$在$(a,b)\subseteq(0,\infty)$有正的下界.证毕.

定理 1.1.1 设λ_1，$\lambda_2>0$，p，$r>1$，$\frac{1}{p}+\frac{1}{q}=1$，$\frac{1}{r}+\frac{1}{s}=1$，$k(x^{\lambda_1},y^{\lambda_2})\ge 0$为$(0,\infty)\times(0,\infty)$上的非齐次可测函数，$k(1,u)$为$(0,\infty)\times(0,\infty)$上的-1 齐次函数，$k=\int_0^\infty k(1,u)u^{-\frac{1}{r}}du$为有限

数，且在$(0,1]$上，$k(1,u)$与$k(u,1)>0$ a.e. 于$(0,1]$，且$a=0$或$b=\infty$，有(a,b)（$\subseteq(0,\infty)$）上的可测函数$\tilde{k}_{\lambda_1,\lambda_2}(x)$，$k_{\lambda_1,\lambda_2}(y)$，适合$0<\tilde{k}_{\lambda_1,\lambda_2}(x),k_{\lambda_1,\lambda_2}(y)\le 1$及

$$\omega_{\lambda_1,\lambda_2}(s,x,a,b)\le\tilde{k}_{\lambda_1,\lambda_2}(x)\frac{k}{\lambda_2}，\quad \omega_{\lambda_1,\lambda_2}(r,y,a,b)\le k_{\lambda_1,\lambda_2}(y)\frac{k}{\lambda_1}，\quad x,y\in(a,b). \tag{1.1.3}$$

若$f(x)$，$g(x)$为(a,b)的非负可测函数，使

$$0<\int_a^b x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx<\infty，\quad 0<\int_a^b x^{q(1-\frac{\lambda_2}{s})-1}g^q(x)dx<\infty \tag{1.1.4}$$

则有如下等价不等式：

$$I_{\lambda_1,\lambda_2}(a,b):=\int_a^b\int_a^b k(x^{\lambda_1},y^{\lambda_2})f(x)g(y)dxdy<\frac{k}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}$$
$$\times\{\int_a^b\tilde{k}_{\lambda_1,\lambda_2}(x)x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx\}^{\frac{1}{p}}\{\int_a^b k_{\lambda_1,\lambda_2}(y)y^{q(1-\frac{\lambda_2}{s})-1}g^q(y)dy\}^{\frac{1}{q}}, \tag{1.1.5}$$

$$J_{\lambda_1,\lambda_2}(a,b)=\int_a^b\frac{y^{\frac{p\lambda_2}{s}-1}}{k_{\lambda_1,\lambda_2}^{p-1}(y)}[\int_a^b k(x^{\lambda_1},y^{\lambda_2})f(x)dx]^p dy<[\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}]^p\int_a^b\tilde{k}_{\lambda_1,\lambda_2}(x)x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx. \tag{1.1.6}$$

证明 由条件及引理1.1.1，$\tilde{k}_{\lambda_1,\lambda_2}(x)$，$k_{\lambda_1,\lambda_2}(y)$在$(a,b)$有正的上界及下界$m>0$，因而有如下与式（1.1.4）等价的不等式：

$$0<\int_a^b\tilde{k}_{\lambda_1,\lambda_2}(x)x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx<\infty，\quad 0<\int_a^b k_{\lambda_1,\lambda_2}(y)x^{q(1-\frac{\lambda_2}{s})-1}g^q(x)dx<\infty$$

由Hölder 不等式[4]和引理1.1.1得

$$\int_a^b\int_a^b k(x^{\lambda_1},y^{\lambda_2})f(x)g(y)dxdy=\int_a^b\int_a^b k(x^{\lambda_1},y^{\lambda_2})f(x)g(y)\frac{y^{(\lambda_2/s-1)/p}}{x^{(\lambda_1/r-1)/q}}\frac{x^{(\lambda_1/r-1)/q}}{y^{(\lambda_2/s-1)/p}}dxdy$$
$$\le\{\int_a^b\omega_{\lambda_1,\lambda_2}(s,x,a,b)x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx\}^{\frac{1}{p}}\{\int_a^b\omega_{\lambda_1,\lambda_2}(r,y,a,b)y^{q(1-\frac{\lambda_2}{s})-1}g^q(y)dy\}^{\frac{1}{q}}, \tag{1.1.7}$$

下面证明式（1.1.7）中间取严格不等号，若不然，必存在不全为0的常数A，B使得$Af^p(x)y^{\lambda_2/s-1}x^{(p-1)(1-\lambda_1/r)}=Bg^q(y)x^{\lambda_1/r-1}y^{(q-1)(1-\lambda_2/s)}$ a.e 于$(a,b)\times(a,b)$. 即有 $Ax^{p(1-\frac{\lambda_1}{r})}f^p(x)=By^{q(1-\frac{\lambda_2}{s})}g^q(y)$ a.e 于$(a,b)\times(a,b)$. 于是有常数C，使$Axx^{p(1-\frac{\lambda_1}{r})-1}f^p(x)=C$ a.e 于(a,b).不妨设$A\neq 0$，则可得$x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)=\frac{C}{A}x^{-1}$ a.e 于(a,b)，无论C是否为0，积分的结果必与 $0<\int_a^b x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx<\infty$相矛盾于是式（1.1.5）成立.

设在(a,b)上，$[f(x)]_n=f(x)$，$f(x)<n$；$[f(x)]_n=n$，$f(x)\ge n$（$n\in\mathrm{N}$）.必有$[a_n,b_n]\subset(a,b)$，$\lim_{n\to\infty}[a_n,b_n]=(a,b)$及$n_0$，使$0<\int_{a_n}^{b_n}\tilde{k}_{\lambda_1,\lambda_2}(x)x^{p(1-\frac{\lambda_1}{r})-1}[f(x)]_n^p dx<\infty$（$n\ge n_0$）.令$g_n(y)=\frac{y^{\frac{p\lambda_2}{s}-1}}{k_{\lambda_1,\lambda_2}^{p-1}(y)}[\int_{a_n}^{b_n}k(x^{\lambda_1},y^{\lambda_2})[f(x)]_n dx]^{p-1}$，$y\in(a_n,b_n)$，$n\ge n_0$.

必存在$M>0$，使得$[f(x)]_n\le n\le Mx^{\frac{\lambda_1}{r}-1}$，$x\in[a_n,b_n]$及

$$0<\int_{a_n}^{b_n}k_{\lambda_1,\lambda_2}(y)y^{q(1-\frac{\lambda_2}{s})-1}g_n^q(y)dy=\int_{a_n}^{b_n}\frac{y^{\frac{p\lambda_2}{s}-1}}{k_{\lambda_1,\lambda_2}^{p-1}(y)}[\int_{a_n}^{b_n}k(x^{\lambda_1},y^{\lambda_2})[f(x)]_n dx]^p dy$$
$$\le\frac{M^p}{m^{p-1}}\int_{a_n}^{b_n}y^{\frac{p\lambda_2}{s}-1}[\int_0^\infty k(x^{\lambda_1},y^{\lambda_2})x^{\frac{\lambda_1}{r}-1}dx]^p dy=M^p\frac{1}{m^{p-1}}(\frac{k}{\lambda_1})^p\int_{a_n}^{b_n}y^{-1}dy=(\frac{Mk}{\lambda_1})^p\frac{1}{m^{p-1}}\ln\frac{b_n}{a_n}<\infty. \tag{1.1.8}$$

故当$n \ge n_0$时，应用式（1.1.5），当$x, y \in (a,b) \setminus [a_n, b_n]$，$g_n(y) = [f(x)]_n = 0$，因此有

$$0 < \int_{a_n}^{b_n} k_{\lambda_1,\lambda_2}(y) y^{q(1-\frac{\lambda_2}{s})-1} g_n^q(y) dy = \int_{a_n}^{b_n} \frac{y^{\frac{p\lambda_2}{s}-1}}{k_{\lambda_1,\lambda_2}^{p-1}(y)} [\int_{a_n}^{b_n} k(x^{\lambda_1}, y^{\lambda_2}) [f(x)]_n dx]^p dy$$

$$= \int_{a_n}^{b_n} \int_{a_n}^{b_n} k(x^{\lambda_1}, y^{\lambda_2}) [f(x)]_n g_n(y) dx dy \le \frac{k}{\lambda_1^{1/q} \lambda_2^{1/p}} [\int_{a_n}^{b_n} \tilde{k}_{\lambda_1,\lambda_2}(x) x^{p(1-\frac{\lambda_1}{r})-1} [f(x)]_n^p dx]^{\frac{1}{p}} \quad (1.1.9)$$

$$\times [\int_{a_n}^{b_n} k_{\lambda_1,\lambda_2}(y) y^{q(1-\frac{\lambda_2}{s})-1} g_n^q(y) dy]^{\frac{1}{q}} < \infty,$$

$$0 < \int_{a_n}^{b_n} k_{\lambda_1,\lambda_2}(y) y^{q(1-\frac{\lambda_2}{s})-1} g_n^q(y) dy < [\frac{k}{\lambda_1^{1/q} \lambda_2^{1/p}}]^p \int_a^b \tilde{k}_{\lambda_1,\lambda_2}(x) x^{p(1-\frac{\lambda_1}{r})-1} f^p(x) dx < \infty. \quad (1.1.10)$$

因而$0 < \int_a^b k_{\lambda_1,\lambda_2}(y) y^{q(1-\frac{\lambda_2}{s})-1} g_\infty^q(y) dy < \infty$，亦有$0 < \int_a^b y^{q(1-\frac{\lambda_2}{s})-1} g_\infty^q(y) dy < \infty$.故当$n \to \infty$时，由式（1.1.5），式（1.1.9），（1.1.10）取严格不等号；故式（1.1.6）成立.

反之，设式（1.1.6）成立，由Holder不等式，有

$$I_{\lambda_1,\lambda_2}(a,b) = \int_a^b (\frac{y^{\frac{\lambda_2}{s}-\frac{1}{p}}}{k_{\lambda_1,\lambda_2}^{1/q}(y)} \int_a^b k(x^{\lambda_1}, y^{\lambda_2}) f(x) dx)(k_{\lambda_1,\lambda_2}^{1/q}(y) y^{\frac{1}{p}-\frac{\lambda_2}{s}} g(y)) dy \quad (1.1.11)$$

$$\le J_{\lambda_1,\lambda_2}^{1/p} \{\int_a^b k_{\lambda_1,\lambda_2}(y) y^{q(1-\frac{\lambda_2}{s})-1} g^q(y) dy\}^{\frac{1}{q}},$$

由式（1.1.6），因$0 < \int_a^b k_{\lambda_1,\lambda_2}(y) x^{q(1-\frac{\lambda_2}{s})-1} g^q(x) dx < \infty$得式（1.1.5），因此式（1.1.6）和式（1.1.5）等价.

1.1.2 两个引理

引理 1.1.2 设λ_1，$\lambda_2 > 0$，$k(x^{\lambda_1}, y^{\lambda_2}) \ge 0$为$(0,\infty) \times (0,\infty)$上的非齐次可测函数，$k(u,1)$为$(0,1]$上的递减连续可微函数，$0 < \sigma_{\lambda_1,\lambda_2} < \min\{\lambda_1, \lambda_2\}$使函数

$$h_{\lambda_1,\lambda_2}(y) := y^{-\sigma_{\lambda_1,\lambda_2}} \int_0^y k(u,1) u^{\sigma_{\lambda_1,\lambda_2}-1} du \quad (y \in (0,1])$$

为有限值，则有$h_{\lambda_1,\lambda_2}(y) \ge h_{\lambda_1,\lambda_2}(1) = \int_0^1 k(u,1) u^{\sigma_{\lambda_1,\lambda_2}-1} du$（$y \in (0,1]$）.

证明　由条件，有$k'(u,1) \le 0$，$y \in (0,1)$.因而

$$h'_{\lambda_1,\lambda_2}(y) := -\sigma_{\lambda_1,\lambda_2} y^{-\sigma_{\lambda_1,\lambda_2}-1} \int_0^y k(u,1) u^{\sigma_{\lambda_1,\lambda_2}-1} du + y^{-\sigma_{\lambda_1,\lambda_2}} k(y,1) y^{\sigma_{\lambda_1,\lambda_2}-1}$$

$$= -y^{-\sigma_{\lambda_1,\lambda_2}-1} k(y,1) y^{\sigma_{\lambda_1,\lambda_2}} + y^{-\sigma_{\lambda_1,\lambda_2}-1} \int_0^y u^{\sigma_{\lambda_1,\lambda_2}} dk(u,1) + k(y,1) y^{-1}$$

$$= y^{-\sigma_{\lambda_1,\lambda_2}-1} \int_0^y u^{\sigma_{\lambda_1,\lambda_2}} k'(u,1) du \le 0, \quad y \in (0,1).$$

因$h_{\lambda_1,\lambda_2}(y)$在$y=1$左连续，故$h_{\lambda_1,\lambda_2}(y) \ge h_{\lambda_1,\lambda_2}(1)$（$y \in (0,1]$）.证毕.

引理1.1.3 设λ_1，$\lambda_2 > 0$，$k(x^{\lambda_1}, y^{\lambda_2}) \ge 0$为$(0,\infty) \times (0,\infty)$上的非齐次可测函数，$k(u,1)$为$(0,1]$上的递减连续可微函数，$0 < \sigma_{\lambda_1,\lambda_2} < \min\{\lambda_1, \lambda_2\}$使函数

$$\tilde{h}_{\lambda_1,\lambda_2}(y) := y^{-\sigma_{\lambda_1,\lambda_2}} \int_0^y k(1,u) u^{\sigma_{\lambda_1,\lambda_2}-1} du \quad (y \in (0,1])$$

为有限值，则有$\tilde{h}_{\lambda_1,\lambda_2}(y) \ge \tilde{h}_{\lambda_1,\lambda_2}(1) = \int_0^1 k(1,u) u^{\sigma_{\lambda_1,\lambda_2}-1} du$（$y \in (0,1]$）.

证明：类似于引理1.1.2可证引理成立.

1.2 若干定理及推论

在定理1.1.1中，取$\tilde{k}_{\lambda_1,\lambda_2}(x)=k_{\lambda_1,\lambda_2}(y)=1$，有

定理 1.2.1 设$0<a<b<\infty$，在定理 1.1.1 的条件下，若式（1.1.3）中至少有一式几乎处处严格取不等号，则有

$$\int_a^b\int_a^b k(x^{\lambda_1},y^{\lambda_2})f(x)g(y)dxdy<\frac{k}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}\{\int_a^b x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx\}^{\frac{1}{p}}\{\int_a^b y^{q(1-\frac{\lambda_2}{s})-1}g^q(y)dy\}^{\frac{1}{q}},\qquad (1.2.1)$$

$$\int_a^b y^{\frac{p\lambda_2}{s}-1}[\int_a^b k(x^{\lambda_1},y^{\lambda_2})f(x)dx]^p dy<[\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}]^p\int_a^b x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx.\qquad (1.2.2)$$

下面在附加条件的情况下建立式（1.2.1）、式（1.2.2）的若干加强式.

若在$(0,1]$上$k(1,u)$和$k(u,1)$有下界$l>0$，在下式前两个积分中作变换$u=\frac{y^{\lambda_2}}{x^{\lambda_1}}$，在最后积分中作变换$u=\frac{x^{\lambda_1}}{y^{\lambda_2}}$，有

$$\omega_{\lambda_1,\lambda_2}(s,x,a,b)=\int_a^b k(x^{\lambda_1},y^{\lambda_2})x^{\lambda_1/r}y^{\lambda_2/s-1}dy$$
$$=\frac{1}{\lambda_2}[\int_0^\infty k(1,u)u^{\frac{1}{s}-1}du-(\int_0^{a^{\lambda_2}/x^{\lambda_1}}k(1,u)u^{\frac{1}{s}-1}du+\int_0^{x^{\lambda_1}/b^{\lambda_2}}k(u,1)u^{\frac{1}{r}-1}du)]$$
$$\le\frac{1}{\lambda_2}[k-l(\int_0^{a^{\lambda_2}/x^{\lambda_1}}u^{\frac{1}{s}-1}du+\int_0^{x^{\lambda_1}/b^{\lambda_2}}u^{\frac{1}{r}-1}du)]=\frac{k}{\lambda_2}[1-\frac{l}{k}(\int_0^{a^{\lambda_2}/x^{\lambda_1}}u^{\frac{1}{s}-1}du+\int_0^{x^{\lambda_1}/b^{\lambda_2}}u^{\frac{1}{r}-1}du)]$$
$$=\frac{k}{\lambda_2}[1-\frac{l}{k}(s(a^{\lambda_2}/x^{\lambda_1})^{\frac{1}{s}}+r(x^{\lambda_1}/b^{\lambda_2})^{\frac{1}{r}})]=\frac{k}{\lambda_2}[1-\frac{lrs}{k}(\frac{1}{r}(a^{\lambda_2}/x^{\lambda_1})^{\frac{1}{s}}+\frac{1}{s}(x^{\lambda_1}/b^{\lambda_2})^{\frac{1}{r}})].$$

由广义的算术-几何平均值不等式[4]，有

$$\frac{1}{r}(\frac{a}{x})^{\frac{\lambda}{s}}+\frac{1}{s}(\frac{x}{b})^{\frac{\lambda}{r}}\ge[(\frac{a}{x})^{\frac{\lambda}{s}}]^{\frac{1}{r}}[(\frac{x}{b})^{\frac{\lambda}{r}}]^{\frac{1}{s}}=(\frac{a}{b})^{\frac{\lambda}{rs}}$$

因而有

$$\omega_{\lambda_1,\lambda_2}(s,x,a,b)\le\frac{k}{\lambda_2}[1-\frac{lrs}{k}(\frac{1}{r}(a^{\lambda_2}/x^{\lambda_1})^{\frac{1}{s}}+\frac{1}{s}(x^{\lambda_1}/b^{\lambda_2})^{\frac{1}{r}})]\le\frac{k}{\lambda_2}[1-\frac{lrs}{k}(\frac{a}{b})^{\frac{\lambda_2}{rs}})].$$

同理，有类似于上式的如下不等式：

$$\omega_{\lambda_1,\lambda_2}(r,y,a,b)\le\frac{k}{\lambda_1}[1-\frac{lrs}{k}(\frac{1}{s}(\frac{a^{\lambda_1}}{y^{\lambda_2}})^{\frac{1}{r}}+\frac{1}{r}(\frac{y^{\lambda_2}}{b^{\lambda_1}})^{\frac{1}{s}})]\le\frac{k}{\lambda_1}[1-\frac{lrs}{k}(\frac{a}{b})^{\frac{\lambda_1}{rs}})].$$

于是，由注1.1.2有

推论1.2.1 在定理1.2.1的条件下，若在$(0,1]$上$k(1,u)$和$k(u,1)$有下界$l>0$，且至少有一函数不会几乎处处等于l，则如下等价不等式：

$$I_{\lambda_1,\lambda_2}(a,b):=\int_a^b\int_a^b k(x^{\lambda_1},y^{\lambda_2})f(x)g(y)dxdy<\frac{k}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}$$
$$\times\{\int_a^b[1-\frac{lrs}{k}(\frac{1}{r}(a^{\lambda_2}/x^{\lambda_1})^{\frac{1}{s}}+\frac{1}{s}(x^{\lambda_1}/b^{\lambda_2})^{\frac{1}{r}})]x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx\}^{\frac{1}{p}}\qquad (1.2.3)$$
$$\{\int_a^b[1-\frac{lrs}{k}(\frac{1}{s}(\frac{a^{\lambda_1}}{y^{\lambda_2}})^{\frac{1}{r}}+\frac{1}{r}(\frac{y^{\lambda_2}}{b^{\lambda_1}})^{\frac{1}{s}})]y^{q(1-\frac{\lambda_2}{s})-1}g^q(y)dy\}^{\frac{1}{q}},$$

$$J_{\lambda_1,\lambda_2}(a,b)=\int_a^b \frac{y^{\frac{p\lambda_2}{s}-1}}{[1-\frac{lrs}{k}(\frac{1}{s}(\frac{a^{\lambda_1}}{y^{\lambda_2}})^{\frac{1}{r}}+\frac{1}{r}(\frac{y^{\lambda_2}}{b^{\lambda_1}})^{\frac{1}{s}})]^{p-1}}[\int_a^b k(x^{\lambda_1},y^{\lambda_2})f(x)dx]^p dy$$
$$<[\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}]^p\int_a^b[1-\frac{lrs}{k}(\frac{1}{r}(a^{\lambda_2}/x^{\lambda_1})^{\frac{1}{s}}+\frac{1}{s}(x^{\lambda_1}/b^{\lambda_2})^{\frac{1}{r}})]x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx. \quad (1.2.4)$$

特别地，（不必要求至少有一函数不会几乎处处等于 l 的条件），还有如下等价不等式：

$$I_{\lambda_1,\lambda_2}(a,b):=\int_a^b\int_a^b k(x^{\lambda_1},y^{\lambda_2})f(x)g(y)dxdy$$
$$<\frac{1}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}[k-lrs(\frac{a}{b})^{\frac{\lambda_2}{rs}})]^{1/p}[k-lrs(\frac{a}{b})^{\frac{\lambda_1}{rs}})]^{1/q}\{\int_a^b x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx\}^{\frac{1}{p}}\{\int_a^b y^{q(1-\frac{\lambda_2}{s})-1}g^q(y)dy\}^{\frac{1}{q}}, \quad (1.2.5)$$

$$J_{\lambda_1,\lambda_2}(a,b)=\int_a^b y^{\frac{p\lambda_2}{s}-1}[\int_a^b k(x^{\lambda_1},y^{\lambda_2})f(x)dx]^p dy$$
$$<\{\frac{1}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}[k-lrs(\frac{a}{b})^{\frac{\lambda_2}{rs}})]^{1/p}[k-lrs(\frac{a}{b})^{\frac{\lambda_1}{rs}})]^{1/q}\}^p\int_a^b x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx. \quad (1.2.6)$$

若 $k(1,u)$， $k(u,1)$ 在 $(0,1]$ 上是递减的连续可微函数，由引理1.1.2和1.1.3得

$$(\frac{a^{\lambda_2}}{x^{\lambda_1}})^{-\frac{1}{s}}\int_0^{\frac{a^{\lambda_2}}{x^{\lambda_1}}}k(1,u)u^{\frac{1}{s}-1}du\ge\int_0^1 k(1,u)u^{\frac{1}{s}-1}du=\tilde{\theta}(s)$$

$$(\frac{x^{\lambda_1}}{b^{\lambda_2}})^{-\frac{1}{s}}\int_0^{\frac{x^{\lambda_1}}{b^{\lambda_2}}}k(u,1)u^{\frac{1}{r}-1}du\ge\int_0^1 k(1,u)u^{\frac{1}{r}-1}du=\theta(r),\ a<x<b.$$

故，可得

$$\omega_{\lambda_1,\lambda_2}(s,x,a,b)\le\frac{k}{\lambda_2}-\frac{1}{\lambda_2}[\tilde{\theta}(s)(\tfrac{a^{\lambda_2}}{x^{\lambda_1}})^{\frac{1}{s}}+\theta(r)(\tfrac{x^{\lambda_1}}{b^{\lambda_2}})^{\frac{1}{r}}],\ a<x<b.$$

同理，可得

$$\omega_{\lambda_1,\lambda_2}(r,y,a,b)\le\frac{k}{\lambda_1}-\frac{1}{\lambda_1}[\theta(r)(\tfrac{a^{\lambda_1}}{y^{\lambda_2}})^{\frac{1}{r}}+\tilde{\theta}(s)(\tfrac{y^{\lambda_2}}{b^{\lambda_1}})^{\frac{1}{s}}],\ a<y<b.$$

推论 1.2.2 设 $0<a<b<\infty$，在定理1.1.1的条件下，若 $k(1,u)$， $k(u,1)$ 在 $(0,1]$ 上是递减的连续可微函数，且至少有一函数在子区间严格递减，则有如下等价不等式：

$$I_{\lambda_1,\lambda_2}(a,b):=\int_a^b\int_a^b k(x^{\lambda_1},y^{\lambda_2})f(x)g(y)dxdy<\frac{k}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}$$
$$\times\{\int_a^b[1-\frac{1}{k}(\tilde{\theta}(s)(\tfrac{a^{\lambda_2}}{x^{\lambda_1}})^{\frac{1}{s}}+\theta(r)(\tfrac{x^{\lambda_1}}{b^{\lambda_2}})^{\frac{1}{r}})]x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx\}^{\frac{1}{p}} \quad (1.2.7)$$
$$\{\int_a^b[1-\frac{1}{k}(\theta(r)(\tfrac{a^{\lambda_1}}{y^{\lambda_2}})^{\frac{1}{r}}+\tilde{\theta}(s)(\tfrac{y^{\lambda_2}}{b^{\lambda_1}})^{\frac{1}{s}})]y^{q(1-\frac{\lambda_2}{s})-1}g^q(y)dy\}^{\frac{1}{q}},$$

$$J_{\lambda_1,\lambda_2}(a,b)=\int_a^b\frac{y^{\frac{p\lambda_2}{s}-1}}{[1-\frac{1}{k}(\theta(r)(\tfrac{a^{\lambda_1}}{y^{\lambda_2}})^{\frac{1}{r}}+\tilde{\theta}(s)(\tfrac{y^{\lambda_2}}{b^{\lambda_1}})^{\frac{1}{s}})]^{p-1}}[\int_a^b k(x^{\lambda_1},y^{\lambda_2})f(x)dx]^p dy$$
$$<[\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}]^p\int_a^b[1-\frac{1}{k}(\tilde{\theta}(s)(\tfrac{a^{\lambda_2}}{x^{\lambda_1}})^{\frac{1}{s}}+\theta(r)(\tfrac{x^{\lambda_1}}{b^{\lambda_2}})^{\frac{1}{r}})]x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx. \quad (1.2.8)$$

这里， $\tilde{\theta}(s)=\int_0^1 k(1,u)u^{\frac{1}{s}-1}du$， $\theta(r)=\int_0^1 k(1,u)u^{\frac{1}{r}-1}du$.特别地，当 $r=s=2$ 时，若在 $(0,1]$ 上， $k(1,u)=k(u,1)$，还有以下的等价不等式：

$$I_{\lambda_1,\lambda_2}(a,b):=\int_a^b\int_a^b k(x^{\lambda_1},y^{\lambda_2})f(x)g(y)dxdy<\frac{k}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}$$
$$\times\{\int_a^b[1-\frac{1}{2}(\frac{a^{\lambda_2}}{x^{\lambda_1}})^{\frac{1}{2}}-\frac{1}{2}(\frac{x^{\lambda_1}}{b^{\lambda_2}})^{\frac{1}{2}}]x^{p(1-\frac{\lambda_1}{2})-1}f^p(x)dx\}^{\frac{1}{p}} \tag{1.2.9}$$
$$\{\int_a^b[1-\frac{1}{2}(\frac{a^{\lambda_1}}{y^{\lambda_2}})^{\frac{1}{2}}-\frac{1}{2}(\frac{y^{\lambda_2}}{b^{\lambda_1}})^{\frac{1}{2}}]y^{q(1-\frac{\lambda_2}{2})-1}g^q(y)dy\}^{\frac{1}{q}},$$

$$J_{\lambda_1,\lambda_2}(a,b)=\int_a^b\frac{y^{\frac{p\lambda_2}{2}-1}}{[1-\frac{1}{2}(\frac{a^{\lambda_1}}{y^{\lambda_2}})^{\frac{1}{2}}-\frac{1}{2}(\frac{y^{\lambda_2}}{b^{\lambda_1}})^{\frac{1}{2}}]^{p-1}}[\int_a^b k(x^{\lambda_1},y^{\lambda_2})f(x)dx]^p dy$$
$$<[\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}]^p\int_a^b[1-\frac{1}{2}(\frac{a^{\lambda_2}}{x^{\lambda_1}})^{\frac{1}{2}}-\frac{1}{2}(\frac{x^{\lambda_1}}{b^{\lambda_2}})^{\frac{1}{2}}]x^{p(1-\frac{\lambda_1}{2})-1}f^p(x)dx. \tag{1.2.10}$$

算术-几何平均值不等式[5]

$$\frac{1}{2}(\frac{a}{x})^{\frac{\lambda}{2}}+\frac{1}{s}(\frac{x}{b})^{\frac{\lambda}{2}}\ge[(\frac{a}{x})^{\frac{\lambda}{2}}]^{\frac{1}{2}}[(\frac{x}{b})^{\frac{\lambda}{2}}]^{\frac{1}{2}}=(\frac{a}{b})^{\frac{\lambda}{4}}.$$

由式（1.2.9）和(1.2.10)，有

推论 1.2.3 设$0<a<b<\infty$，在定理1.1.1的条件下，若$k(1,u)=k(u,1)$在$(0,1]$上是递减的连续可微函数，$k=\int_0^\infty k(1,u)u^{-\frac{1}{2}}du$ 则有如下等价不等式：

$$I_{\lambda_1,\lambda_2}(a,b):=\int_a^b\int_a^b k(x^{\lambda_1},y^{\lambda_2})f(x)g(y)dxdy<\frac{k}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}\times$$
$$[1-(\tfrac{a}{b})^{\frac{\lambda_2}{4}}]^{1/p}[1-(\tfrac{a}{b})^{\frac{\lambda_1}{4}}]^{1/q}\{\int_a^b x^{p(1-\frac{\lambda_1}{2})-1}f^p(x)dx\}^{\frac{1}{p}}\{\int_a^b y^{q(1-\frac{\lambda_2}{2})-1}g^q(y)dy\}^{\frac{1}{q}}, \tag{1.2.11}$$

$$J_{\lambda_1,\lambda_2}(a,b)=\int_a^b y^{\frac{p\lambda_2}{2}-1}[\int_a^b k(x^{\lambda_1},y^{\lambda_2})f(x)dx]^p dy$$
$$<[\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}[1-(\tfrac{a}{b})^{\frac{\lambda_2}{4}}]^{1/p}[1-(\tfrac{a}{b})^{\frac{\lambda_1}{4}}]^{1/q}]^p\int_a^b x^{p(1-\frac{\lambda_1}{2})-1}f^p(x)dx. \tag{1.2.12}$$

注1.2.1 （1）由式（1.2.5）及（1.2.6）知，式（1.2.1）及（1.2.2）的常数因子不是最佳的；（2）当$r=s=2$时，式（1.2.5）弱于（1.2.11）.

2 限制在区间(a,∞)（$a>0$）上的 Hilbert 型积分不等式

定理2.1.1 设$a>0$，$b=\infty$，则在定理1.1.1的条件下，有如下等价不等式：

$$\int_a^\infty\int_a^\infty k(x^{\lambda_1},y^{\lambda_2})f(x)g(y)dxdy<\frac{k}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}$$
$$\times\{\int_a^\infty\tilde{k}_{\lambda_1,\lambda_2}(x)x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx\}^{\frac{1}{p}}\{\int_a^\infty k_{\lambda_1,\lambda_2}(y)y^{q(1-\frac{\lambda_2}{s})-1}g^q(y)dy\}^{\frac{1}{q}}, \tag{2..1.1}$$

$$\int_a^\infty\frac{y^{\frac{p\lambda_2}{s}-1}}{k_{\lambda_1,\lambda_2}^{p-1}(y)}[\int_a^\infty k(x^{\lambda_1},y^{\lambda_2})f(x)dx]^p dy<[\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}]^p\int_a^\infty\tilde{k}_{\lambda_1,\lambda_2}(x)x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx. \tag{2.1.2}$$

这里，常数因子$\frac{k}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}$，$[\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}]^p$均为最佳值.特别地，

$$\int_a^\infty\int_a^\infty k(x^{\lambda_1},y^{\lambda_2})f(x)g(y)dxdy<\frac{k}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}\{\int_a^\infty x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx\}^{\frac{1}{p}}\{\int_a^\infty y^{q(1-\frac{\lambda_2}{s})-1}g^q(y)dy\}^{\frac{1}{q}}, \tag{2.1.3}$$

$$\int_a^\infty y^{\frac{p\lambda_2}{s}-1}[\int_a^\infty k(x^{\lambda_1},y^{\lambda_2})f(x)dx]^p dy<[\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}]^p\int_a^\infty x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx. \qquad (2.1.4)$$

证明 类似于定理1.1.1，可证明定理2.1.1成立，用反证法可证明定理中的常数因子为最佳值.

推论2.1.1 在定理2.1.1的条件下，若在$(0,1]$上$k(1,u)$和$k(u,1)$有下界$l>0$，则如下等价不等式：

$$\int_a^\infty\int_a^\infty k(x^{\lambda_1},y^{\lambda_2})f(x)g(y)dxdy<\frac{k}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}$$
$$\times\{\int_a^\infty[1-\frac{ls}{k}(a^{\lambda_2}/x^{\lambda_1})^{\frac{1}{s}}]x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx\}^{\frac{1}{p}}\{\int_a^\infty[1-\frac{lr}{k}(\frac{a^{\lambda_1}}{y^{\lambda_2}})^{\frac{1}{r}}]y^{q(1-\frac{\lambda_2}{s})-1}g^q(y)dy\}^{\frac{1}{q}}, \qquad (2.1.5)$$

$$\int_a^\infty\frac{y^{\frac{p\lambda_2}{s}-1}}{[1-\frac{lr}{k}(\frac{a^{\lambda_1}}{y^{\lambda_2}})^{\frac{1}{r}}]^{p-1}}[\int_a^\infty k(x^{\lambda_1},y^{\lambda_2})f(x)dx]^p dy$$
$$<[\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}]^p\int_a^\infty[1-\frac{ls}{k}(a^{\lambda_2}/x^{\lambda_1})^{\frac{1}{s}}]x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx. \qquad (2.1.6)$$

推论2.1.2 在定理2.1.1的条件下，若在$(0,1]$上$k(1,u)$和$k(u,1)$是递减的连续可微函数，则如下等价不等式：

$$\int_a^\infty\int_a^\infty k(x^{\lambda_1},y^{\lambda_2})f(x)g(y)dxdy<\frac{k}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}$$
$$\times\{\int_a^\infty[1-\frac{\tilde{\theta}(s)}{k}(\frac{a^{\lambda_2}}{x^{\lambda_1}})^{\frac{1}{s}}]x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx\}^{\frac{1}{p}}\{\int_a^\infty[1-\frac{\theta(r)}{k}(\frac{a^{\lambda_1}}{y^{\lambda_2}})^{\frac{1}{r}}]y^{q(1-\frac{\lambda_2}{s})-1}g^q(y)dy\}^{\frac{1}{q}}, \qquad (2.1.7)$$

$$\int_a^b\frac{y^{\frac{p\lambda_2}{s}-1}}{[1-\frac{\theta(r)}{k}(\frac{a^{\lambda_1}}{y^{\lambda_2}})^{\frac{1}{r}}]^{p-1}}[\int_a^b k(x^{\lambda_1},y^{\lambda_2})f(x)dx]^p dy$$
$$<[\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}]^p\int_a^b[1-\frac{\tilde{\theta}(s)}{k}(\frac{a^{\lambda_2}}{x^{\lambda_1}})^{\frac{1}{s}}]x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx. \qquad (2.1.8)$$

3 限制在区间$(0,b)$（$b>0$）上的 Hilbert 型积分不等式

定理2.1.1 设$a=0$，$0<b<\infty$，则在定理1.1.1的条件下，有如下等价不等式：

$$\int_0^b\int_0^b k(x^{\lambda_1},y^{\lambda_2})f(x)g(y)dxdy<\frac{k}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}$$
$$\times\{\int_0^b\tilde{k}_{\lambda_1,\lambda_2}(x)x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx\}^{\frac{1}{p}}\{\int_0^b k_{\lambda_1,\lambda_2}(y)y^{q(1-\frac{\lambda_2}{s})-1}g^q(y)dy\}^{\frac{1}{q}}, \qquad (3..1.1)$$

$$\int_0^b\frac{y^{\frac{p\lambda_2}{s}-1}}{k_{\lambda_1,\lambda_2}^{p-1}(y)}[\int_0^b k(x^{\lambda_1},y^{\lambda_2})f(x)dx]^p dy<[\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}]^p\int_0^b\tilde{k}_{\lambda_1,\lambda_2}(x)x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx. \qquad (3.1.2)$$

这里，常数因子$\frac{k}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}$，$[\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}]^p$均为最佳值.特别地，

$$\int_0^b\int_0^b k(x^{\lambda_1},y^{\lambda_2})f(x)g(y)dxdy<\frac{k}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}\{\int_0^b x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx\}^{\frac{1}{p}}\{\int_0^b y^{q(1-\frac{\lambda_2}{s})-1}g^q(y)dy\}^{\frac{1}{q}}, \qquad (3.1.3)$$

$$\int_0^b y^{\frac{p\lambda_2}{s}-1}[\int_0^b k(x^{\lambda_1},y^{\lambda_2})f(x)dx]^p dy<[\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}]^p\int_0^b x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx. \quad (3.1.4)$$

推论3.1.1 在定理3.1.1的条件下，若在$(0,1]$上$k(1,u)$和$k(u,1)$有下界$l>0$，则如下等价不等式：

$$\int_a^\infty\int_a^\infty k(x^{\lambda_1},y^{\lambda_2})f(x)g(y)dxdy<\frac{k}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}$$
$$\times\{\int_a^\infty[1-\frac{lr}{k}(x^{\lambda_1}/b^{\lambda_2})^{\frac{1}{r}}]x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx\}^{\frac{1}{p}} \quad (3.1.5)$$
$$\{\int_a^\infty[1-\frac{ls}{k}(\frac{y^{\lambda_2}}{b^{\lambda_1}})^{\frac{1}{s}})]y^{q(1-\frac{\lambda_2}{s})-1}g^q(y)dy\}^{\frac{1}{q}},$$

$$\int_0^b\frac{y^{\frac{p\lambda_2}{s}-1}}{[1-\frac{ls}{k}(\frac{y^{\lambda_2}}{b^{\lambda_1}})^{\frac{1}{s}})]^{p-1}}[\int_0^b k(x^{\lambda_1},y^{\lambda_2})f(x)dx]^p dy \quad (3.1.6)$$
$$<[\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}]^p\int_0^b[1-\frac{lr}{k}(x^{\lambda_1}/b^{\lambda_2})^{\frac{1}{r}}]x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx.$$

推论3.1.2 在定理3.1.1的条件下，若在$(0,1]$上$k(1,u)$和$k(u,1)$是递减的连续可微函数，则如下等价不等式：

$$\int_a^b\int_a^b k(x^{\lambda_1},y^{\lambda_2})f(x)g(y)dxdy<\frac{k}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}$$
$$\times\{\int_a^b[1-\frac{\theta(r)}{k}(\frac{x^{\lambda_1}}{b^{\lambda_2}})^{\frac{1}{r}}]x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx\}^{\frac{1}{p}} \quad (3.1.7)$$
$$\{\int_a^b[1-\frac{\tilde{\theta}(s)}{k}(\frac{y^{\lambda_2}}{b^{\lambda_1}})^{\frac{1}{s}}]y^{q(1-\frac{\lambda_2}{s})-1}g^q(y)dy\}^{\frac{1}{q}},$$

$$\int_0^b\frac{y^{\frac{p\lambda_2}{s}-1}}{[1-\frac{\tilde{\theta}(s)}{k}(\frac{y^{\lambda_2}}{b^{\lambda_1}})^{\frac{1}{s}})]^{p-1}}[\int_0^b k(x^{\lambda_1},y^{\lambda_2})f(x)dx]^p dy<[\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}]^p\int_0^b[1-\frac{\theta(r)}{k}(\frac{x^{\lambda_1}}{b^{\lambda_2}})^{\frac{1}{r}})]x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx. \quad (3.1.8)$$

这里，$\tilde{\theta}(s)=\int_0^1 k(1,u)u^{\frac{1}{s}-1}du$，$\theta(r)=\int_0^1 k(1,u)u^{\frac{1}{r}-1}du$.特别地，当$r=s=2$时，若在$(0,1]$上，$k(1,u)=k(u,1)$，还有以下的等价不等式：

$$\int_0^b\int_0^b k(x^{\lambda_1},y^{\lambda_2})f(x)g(y)dxdy<\frac{k}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}$$
$$\times\{\int_0^b[1-\frac{1}{2}(\frac{x^{\lambda_1}}{b^{\lambda_2}})^{\frac{1}{2}}]x^{p(1-\frac{\lambda_1}{2})-1}f^p(x)dx\}^{\frac{1}{p}}\{\int_0^b[1-\frac{1}{2}(\frac{y^{\lambda_2}}{b^{\lambda_1}})^{\frac{1}{2}}]y^{q(1-\frac{\lambda_2}{2})-1}g^q(y)dy\}^{\frac{1}{q}}, \quad (3.1.9)$$

$$\int_0^b\frac{y^{\frac{p\lambda_2}{2}-1}}{[1-\frac{1}{2}(\frac{y^{\lambda_2}}{b^{\lambda_1}})^{\frac{1}{2}}]^{p-1}}[\int_0^b k(x^{\lambda_1},y^{\lambda_2})f(x)dx]^p dy<[\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}]^p\int_0^b[1-\frac{1}{2}(\frac{x^{\lambda_1}}{b^{\lambda_2}})^{\frac{1}{2}}]x^{p(1-\frac{\lambda_1}{2})-1}f^p(x)dx. \quad (3.1.10)$$

4 限制在区间(a,b)（$0<a<b<\infty$）上的逆向 Hilbert 型积分不等式

4.1 限制在区间(a,b)（$0<a<b<\infty$）上的逆向 Hilbert 型积分不等式

定理 4.1.1 设λ_1，$\lambda_2>0$，$0<p<1$，$r>1$，$\frac{1}{p}+\frac{1}{q}=1$，$\frac{1}{r}+\frac{1}{s}=1$，$k(x^{\lambda_1},y^{\lambda_2})\geq 0$为$(0,\infty)\times(0,\infty)$上的非齐次可测函数，$k(1,u)$为$(0,\infty)\times(0,\infty)$上的-1 齐次函数，

$k=\int_0^\infty k(1,u)u^{-\frac{1}{r}}du$ 为有限数，且在 $(0,1]$ 上，$k(1,u)$ 与 $k(u,1)>0$ a.e. 于 $(0,1]$，且 $a=0$ 或 $b=\infty$，有 (a,b)（$\subseteq(0,\infty)$）上的可测函数 $\tilde{\mu}_{\lambda_1,\lambda_2}(x)$，$k_{\lambda_1,\lambda_2}(y)$，适合 $0<\tilde{\mu}_{\lambda_1,\lambda_2}(x),k_{\lambda_1,\lambda_2}(y)\le 1$ 及 $m_{\lambda_1,\lambda_2}(r)>0$，成立不等式

$$m_{\lambda_1,\lambda_2}(r)\le\tilde{\mu}_{\lambda_1,\lambda_2}(x)\frac{k}{\lambda_2}\le\omega_{\lambda_1,\lambda_2}(s,x,a,b),\quad x\in(a,b) \tag{4.1.3}$$

$$\omega_{\lambda_1,\lambda_2}(r,y,a,b)\le k_{\lambda_1,\lambda_2}(y)\frac{k}{\lambda_1},\qquad y\in(a,b). \tag{4.1.4}$$

若 $f(x)$，$g(x)$ 为 (a,b) 的非负可测函数，使

$$0<\int_a^b x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx<\infty,\quad 0<\int_a^b x^{q(1-\frac{\lambda_2}{s})-1}g^q(x)dx<\infty \tag{4.1.5}$$

则有如下等价不等式：

$$\begin{aligned}I_{\lambda_1,\lambda_2}(a,b):=\int_a^b\int_a^b k(x^{\lambda_1},y^{\lambda_2})f(x)g(y)dxdy>\frac{k}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}\\ \times\{\int_a^b\tilde{\mu}_{\lambda_1,\lambda_2}(x)x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx\}^{\frac{1}{p}}\{\int_a^b k_{\lambda_1,\lambda_2}(y)y^{q(1-\frac{\lambda_2}{s})-1}g^q(y)dy\}^{\frac{1}{q}}\end{aligned} \tag{4.1.6}$$

$$J_{\lambda_1,\lambda_2}(a,b)=\int_a^b\frac{y^{\frac{p\lambda_2}{s}-1}}{k_{\lambda_1,\lambda_2}^{p-1}(y)}[\int_a^b k(x^{\lambda_1},y^{\lambda_2})f(x)dx]^p dy>[\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}]^p\int_a^b\tilde{\mu}_{\lambda_1,\lambda_2}(x)x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx \tag{4.1.7}$$

证明 由条件及引理1.1.1，$\tilde{\mu}_{\lambda_1,\lambda_2}(x)$，$k_{\lambda_1,\lambda_2}(y)$ 在 (a,b) 有正的上界及下界 $m_{\lambda_1,\lambda_2}(r)>0$，因而有如下与式（1.1.5）等价的不等式：

$$0<\int_a^b\tilde{\mu}_{\lambda_1,\lambda_2}(x)x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx<\infty,\quad 0<\int_a^b k_{\lambda_1,\lambda_2}(y)x^{q(1-\frac{\lambda_2}{s})-1}g^q(x)dx<\infty$$

由逆向 Hölder 不等式[5]和引理1.1.1得

$$\begin{aligned}&\int_a^b\int_a^b k(x^{\lambda_1},y^{\lambda_2})f(x)g(y)dxdy=\int_a^b\int_a^b k(x^{\lambda_1},y^{\lambda_2})f(x)g(y)\frac{y^{(\lambda_2/s-1)/p}}{x^{(\lambda_1/r-1)/q}}\frac{x^{(\lambda_1/r-1)/q}}{y^{(\lambda_2/s-1)/p}}dxdy\\ &\ge\{\int_a^b\int_a^b k(x^{\lambda_1},y^{\lambda_2})f^p(x)x^{\lambda_1/r}y^{\lambda_2/s-1}x^{p(1-\frac{\lambda_1}{r})-1}dxdy\}^{\frac{1}{p}}\{\int_a^b\int_a^b k(x^{\lambda_1},y^{\lambda_2})g^q(y)x^{\lambda_1/r-1}y^{\lambda_2/s}y^{q(1-\frac{\lambda_2}{s})-1}dxdy\}^{\frac{1}{q}}\\ &=\{\int_a^b\omega_{\lambda_1,\lambda_2}(s,x,a,b)x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx\}^{\frac{1}{p}}\{\int_a^b\omega_{\lambda_1,\lambda_2}(r,y,a,b)y^{q(1-\frac{\lambda_2}{s})-1}g^q(y)dy\}^{\frac{1}{q}}\end{aligned} \tag{4.1.8}$$

下面证明式（4.1.8）中间取严格不等号，若不然，必存在不全为0的常数 A，B 使得 $Af^p(x)y^{\lambda_2/s-1}x^{(p-1)(1-\lambda_1/r)}=Bg^q(y)x^{\lambda_1/r-1}y^{(q-1)(1-\lambda_2/s)}$ a.e 于 $(a,b)\times(a,b)$．即有 $Ax^{p(1-\frac{\lambda_1}{r})}f^p(x)=By^{q(1-\frac{\lambda_2}{s})}g^q(y)$ a.e 于 $(a,b)\times(a,b)$.于是有常数 C，使 $Axx^{p(1-\frac{\lambda_1}{r})-1}f^p(x)=C$ a.e 于 (a,b).不妨设 $A\ne 0$，则可得 $x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)=\frac{C}{A}x^{-1}$ a.e 于 (a,b)，无论 C 是否为0，积分的结果必与 $0<\int_a^b x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx<\infty$ 相矛盾于是式（4.1.6）成立.

令 $g(y)=\frac{y^{\frac{p\lambda_2}{s}-1}}{k_{\lambda_1,\lambda_2}^{p-1}(y)}[\int_a^b k(x^{\lambda_1},y^{\lambda_2})f(x)dx]^{p-1}$，$y\in[a,b]$.

显然，$J_{\lambda_1,\lambda_2}(a,b)>0$．若 $J_{\lambda_1,\lambda_2}(a,b)=\infty$，则式（4.1.7）自然成立；若 $0<J_{\lambda_1,\lambda_2}(a,b)=\int_a^b k_{\lambda_1,\lambda_2}(y)x^{q(1-\frac{\lambda_2}{s})-1}g^q(x)dx<\infty$，则有 $0<\int_a^b x^{q(1-\frac{\lambda_2}{s})-1}g^q(x)dx<\infty$，应用式（4.1.6），有

$$\infty > \int_a^b k_{\lambda_1,\lambda_2}(y) y^{q(1-\frac{\lambda_2}{s})-1} g^q(y)dy = J_{\lambda_1,\lambda_2}(a,b) = I_{\lambda_1,\lambda_2}(a,b)$$
$$> \frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}[\int_a^b \tilde{\mu}_{\lambda_1,\lambda_2}(x) x^{p(1-\frac{\lambda_1}{r})-1} f^p(x)dx]^{\frac{1}{p}}[\int_a^b k_{\lambda_1,\lambda_2}(y) y^{q(1-\frac{\lambda_2}{s})-1} g^q(y)dy]^{\frac{1}{q}} > 0$$

$$J_{\lambda_1,\lambda_2}(a,b) = \int_a^b k_{\lambda_1,\lambda_2}(y) y^{q(1-\frac{\lambda_2}{s})-1} g^q(y)dy = [\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}]^p \int_a^b \tilde{\mu}_{\lambda_1,\lambda_2}(x) x^{p(1-\frac{\lambda_1}{r})-1} f^p(x)dx$$

故有式（4.1.7）.反之，设式（4.1.7）成立.配方并有逆向的Holder不等式，有

$$I_{\lambda_1,\lambda_2}(a,b) = \int_a^b (\frac{y^{\frac{\lambda_2}{s}-\frac{1}{p}}}{k_{\lambda_1,\lambda_2}^{1/q}(y)} \int_a^b k(x^{\lambda_1}, y^{\lambda_2}) f(x)dx)(k_{\lambda_1,\lambda_2}^{1/q}(y) y^{\frac{1}{p}-\frac{\lambda_2}{s}} g(y))dy \ge J_{\lambda_1,\lambda_2}^{1/p} \{\int_a^b k_{\lambda_1,\lambda_2}(y) y^{q(1-\frac{\lambda_2}{s})-1} g^q(y)dy\}^{\frac{1}{q}}$$

再由式（4.1.7），因$0 < \int_a^b k_{\lambda_1,\lambda_2}(y) y^{q(1-\frac{\lambda_2}{s})-1} g^q(y)dy < \infty$，有式（4.1.6）.故式（4.1.6）和（1.1.7）等价.

注4.1.2 当$0 < a < b < \infty$时，易见在定理4.1.1的条件下，等价式（4.1.5）和（4.1.6）可能取到等号；若式（4.1.3）中一式几乎处处取严格不等号，则仍有等价的严格不等式（4.1.5）和（4.1.6）.

4.2 若干定理及推论

引理 4.2.1 设$0 < a < b < \infty$。若在$(0,1]$上$k(1,u)$和$k(u,1)$有下界$l > 0$，定义常数

$$d_{\lambda_1,\lambda_2}(a,b) = \sup_{a<x<b}[\int_0^{a^{\lambda_2}/x^{\lambda_1}} k(1,u)u^{\frac{1}{s}-1}du + \int_{b^{\lambda_2}/x^{\lambda_1}}^{\infty} k(1,u)u^{\frac{1}{s}-1}du]$$

$$c_{\lambda_1,\lambda_2}(a,b) = \sup_{a<x<b}[\frac{1}{r}(a^{\lambda_2}/x^{\lambda_1})^{\frac{1}{s}} + \frac{1}{s}(x^{\lambda_1}/b^{\lambda_2})^{\frac{1}{r}}] \qquad c'_{\lambda_1,\lambda_2}(a,b) = \sup_{a<x<b}[\frac{1}{s}(\frac{a^{\lambda_1}}{y^{\lambda_2}})^{\frac{1}{r}} + \frac{1}{r}(\frac{y^{\lambda_2}}{b^{\lambda_1}})^{\frac{1}{s}})]$$

则如下权函数的不等式：

$$0 < m_{\lambda_1,\lambda_2} := \frac{rsl}{\lambda_2}[1-c_{\lambda_1,\lambda_2}(a,b)] \le \frac{1}{\lambda_2}[k-d_{\lambda_1,\lambda_2}(a,b)] \le \omega_{\lambda_1,\lambda_2}(s,x,a,b) \le \frac{k}{\lambda_2} - \frac{lrs}{\lambda_2}(\tfrac{a}{b})^{\frac{\lambda_2}{rs}} \le \frac{k}{\lambda_2}, x \in (a,b) \qquad (4.2.1)$$

$$0 < m'_{\lambda_1,\lambda_2} := \frac{rsl}{\lambda_1}[1-c'_{\lambda_1,\lambda_2}(a,b)] \le \omega_{\lambda_1,\lambda_2}(r,y,a,b) \le \frac{k}{\lambda_1} - \frac{lrs}{\lambda_1}(\tfrac{a}{b})^{\frac{\lambda_1}{rs}} \le \frac{k}{\lambda_1}, y \in (a,b) \qquad (4.2.2)$$

证明 因在$(0,1]$上，$k(u,1)$有下界$l > 0$，故在$(1,\infty)$，$k(1,u) = u^{-1}k(\frac{1}{u},1) \ge lu^{-1}$，在式（4.1.1）中，固定$x \in (a,b)$，作变换$u = \frac{y^{\lambda_2}}{x^{\lambda_1}}$，由条件及平均不等式[5]，有

$$\omega_{\lambda_1,\lambda_2}(s,x,a,b) = \frac{1}{\lambda_2}\int_{a^{\lambda_2}/x^{\lambda_1}}^{b^{\lambda_2}/x^{\lambda_1}} k(1,u)u^{\frac{1}{s}-1}du \ge \frac{1}{\lambda_2}[k-d_{\lambda_1,\lambda_2}(a,b)] = \frac{1}{\lambda_2}\inf_{a<x<b}[\int_{a^{\lambda_2}/x^{\lambda_1}}^{1} k(1,u)u^{\frac{1}{s}-1}du + \int_1^{b^{\lambda_2}/x^{\lambda_1}} k(1,u)u^{\frac{1}{s}-1}du]$$
$$\ge \frac{l}{\lambda_2}\inf_{a<x<b}[\int_{a^{\lambda_2}/x^{\lambda_1}}^{1} u^{\frac{1}{s}-1}du + \int_1^{b^{\lambda_2}/x^{\lambda_1}} u^{\frac{1}{s}-2}du] = \frac{rsl}{\lambda_2}[1-c_{\lambda_1,\lambda_2}(a,b)] > 0$$

$$\omega_{\lambda_1,\lambda_2}(s,x,a,b) = \frac{k}{\lambda_2} - \frac{1}{\lambda_2}[\int_0^{a^{\lambda_2}/x^{\lambda_1}} k(1,u)u^{\frac{1}{s}-1}du + \int_{b^{\lambda_2}/x^{\lambda_1}}^{\infty} k(1,u)u^{\frac{1}{s}-1}du]$$
$$\le \frac{k}{\lambda_2} - \frac{l}{\lambda_2}[\int_0^{a^{\lambda_2}/x^{\lambda_1}} u^{\frac{1}{s}-1}du + \int_{b^{\lambda_2}/x^{\lambda_1}}^{\infty} u^{\frac{1}{s}-2}du] = \frac{k}{\lambda_2} - \frac{lrs}{\lambda_2}[\frac{1}{r}(a^{\lambda_2}/x^{\lambda_1})^{\frac{1}{s}} + \frac{1}{s}(x^{\lambda_1}/b^{\lambda_2})^{\frac{1}{r}}]$$
$$\le \frac{k}{\lambda_2} - \frac{lrs}{\lambda_2}(\tfrac{a}{b})^{\frac{\lambda_2}{rs}} \le \frac{k}{\lambda_2}$$

故式（4.2.1）成立，同理可证明式（4.1.2）成立.证毕.

在定理4.1.1中，$\tilde{\mu}_{\lambda_1,\lambda_2}(x) = 1 - \frac{d_{\lambda_1,\lambda_2}(a,b)}{k}$，$k_{\lambda_1,\lambda_2}(y) = 1 - \frac{lrs}{k}(\frac{a}{b})^{\frac{\lambda_1}{rs}}$，有

定理 4.2.1 设$0 < a < b < \infty$，在定理 4.1.1 的条件下，若在$(0,1]$上$k(1,u)$和$k(u,1)$有

下界$l>0$，则有如下等价不等式：

$$\int_a^b\int_a^b k(x^{\lambda_1},y^{\lambda_2})f(x)g(y)dxdy$$
$$>\frac{1}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}[k-d_{\lambda_1,\lambda_2}(a,b)]^{\frac{1}{p}}[k-lrs(\tfrac{a}{b})^{\frac{\lambda_1}{rs}}]^{\frac{1}{q}}\{\int_a^b x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx\}^{\frac{1}{p}}\{\int_a^b y^{q(1-\frac{\lambda_2}{s})-1}g^q(y)dy\}^{\frac{1}{q}} \quad (4.2.1)$$

$$\int_a^b y^{\frac{p\lambda_2}{s}-1}[\int_a^b k(x^{\lambda_1},y^{\lambda_2})f(x)dx]^p dy>[\frac{1}{\lambda_1^{1/q}\lambda_2^{1/p}}]^p[k-lrs(\tfrac{a}{b})^{\frac{\lambda_1}{rs}}]^{p-1}[k-d_{\lambda_1,\lambda_2}(a,b)]\int_a^b x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx \quad (4.2.2)$$

5 限制在区间(a,∞)（$a>0$）上的逆向 Hilbert 型积分不等式

定理 5.1.1 设 $a>0$，$b=\infty$，则在定理 1.1.1 的条件下，若有 $0\le\eta<\frac{1}{s}$，使 $0<\theta_{\lambda_1,\lambda_2}(s,x,a)=O\left((\frac{a^{\lambda_2}}{x^{\lambda_1}})^{\frac{1}{s}-\eta}\right)\le\tilde{l}<1$，（$x\to\infty$）及

$\tilde{\mu}_{\lambda_1,\lambda_2}(x)=1-\theta_{\lambda_1,\lambda_2}(s,x,a)\le\frac{\lambda_2\omega_{\lambda_1,\lambda_2}(s,x,a,\infty)}{k}$，$x\in(a,\infty)$.

有如下等价不等式：

$$\int_a^\infty\int_a^\infty k(x^{\lambda_1},y^{\lambda_2})f(x)g(y)dxdy>\frac{k}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}$$
$$\times\{\int_a^\infty[1-\theta_{\lambda_1,\lambda_2}(s,x,a)]x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx\}^{\frac{1}{p}}\{\int_a^\infty y^{q(1-\frac{\lambda_2}{s})-1}g^q(y)dy\}^{\frac{1}{q}} \quad (5..1.1)$$

$$\int_a^\infty y^{\frac{p\lambda_2}{s}-1}[\int_a^\infty k(x^{\lambda_1},y^{\lambda_2})f(x)dx]^p dy>[\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}]^p\int_a^\infty[1-\theta_{\lambda_1,\lambda_2}(s,x,a)]x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx \quad (5.1.2)$$

这里，常数因子$\frac{k}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}$，$[\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}]^p$ 均为最佳值.

证明 这里只证明常数因子$\frac{k}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}$，$[\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}]^p$ 均为最佳值..

设n，$n_0\in\mathrm{N}$，$n\ge n_0>\max\left\{\frac{r}{p},\frac{s}{p}\right\}$，定义 $f_n(x)$，$g_n(x)$

$$f_n(x)=g_n(x)=0\quad x\in(0,a)\quad,$$
$$f_n(x)=x^{\frac{\lambda_1}{r}-1-\frac{\lambda_1}{np}},\quad g_n(x)=x^{\frac{\lambda_2}{s}-1-\frac{\lambda_2}{nq}}\quad x\in[a,\infty),$$

若式(5.1.1)的常数因子不是最佳值，则存在正数$K\ge\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}$，使式(5.1.1)的常数因子$\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}$换上$K$仍成立.特别地，有

$$I_n=\int_a^\infty\int_a^\infty k(x^{\lambda_1},y^{\lambda_2})f_n(x)g_n(y)dxdy$$
$$\ge K\{\int_a^\infty[1-\theta_{\lambda_1,\lambda_2}(s,x,a)]x^{p(1-\frac{\lambda_1}{r})-1}f_n^p(x)dx\}^{\frac{1}{p}}\{\int_a^\infty y^{q(1-\frac{\lambda_2}{s})-1}g_n^q(y)dy\}^{\frac{1}{q}} \quad (5.1.3)$$
$$=K\{\int_a^\infty x^{-1-\frac{\lambda_1}{n}}dx-\int_a^\infty O\left((\frac{a^{\lambda_2}}{x^{\lambda_1}})^{\frac{1}{s}-\eta}\right)x^{-1-\frac{\lambda_1}{n}}dx\}^{\frac{1}{p}}\{\int_a^\infty y^{-1-\frac{\lambda_2}{n}}dy\}^{\frac{1}{q}}=K\frac{n}{\lambda_1^{1/p}\lambda_2^{1/q}a^{\lambda_1/np+\lambda_2/nq}}[1-\frac{O(n)}{n}]^{1/p}$$

固定y，做变换$u=\frac{x^{\lambda_1}}{y^{\lambda_2}}$

$$I_n\le\int_a^\infty y^{\frac{\lambda_2}{s}-1-\frac{\lambda_2}{nq}}[\int_0^\infty k(x^{\lambda_1},y^{\lambda_2})x^{\frac{\lambda_1}{r}-1-\frac{\lambda_1}{np}}dx]dy=\frac{n}{\lambda_1\lambda_2a^{\lambda_2/n}}\int_0^\infty k(1,u)u^{\frac{1}{s}-1+\frac{1}{np}}du \quad (5.1.4)$$

结合式（5.1.3），可得

$$\frac{1}{\lambda_1^{1/q}\lambda_2^{1/p}a^{\lambda_2/np-\lambda_1/np}}\int_0^\infty k(1,u)u^{\frac{1}{s}-1+\frac{1}{np}}du\ge K[1-\frac{O(n)}{n}]^{1/p}$$

令 $n\to\infty$，由 Lebesgue 控制收敛定理[5]，可得

$\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}\ge K$.故式（5.1.1）的常数因子 $\frac{k}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}$ 最佳值.若式（5.1.2）的常数因子不是最佳的，则由式（5.1.2）易得出式（5.1.1）的常数因子也不是最佳的矛盾.

推论 5.1.1 设 $a>0$ ， $b=+\infty$ ， 在定理 4.1.1 的条件下， 若 $0<\rho(s)<1$ ， 使 $\tilde{\mu}_{\lambda_1,\lambda_2}(x)=1-\rho(s)(\frac{a^{\lambda_2}}{x^{\lambda_1}})^{\frac{1}{s}}\le\frac{\lambda_2\omega_{\lambda_1,\lambda_2}(s,x,a,\infty)}{k}$， $x\in(a,\infty)$.

则如下等价不等式：

$$\int_a^\infty\int_a^\infty k(x^{\lambda_1},y^{\lambda_2})f(x)g(y)dxdy>\frac{k}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}\times\{\int_a^\infty[1-\rho(s)(a^{\lambda_2}/x^{\lambda_1})^{\frac{1}{s}}]x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx\}^{\frac{1}{p}}\{\int_a^\infty y^{q(1-\frac{\lambda_2}{s})-1}g^q(y)dy\}^{\frac{1}{q}} \tag{5.1.5}$$

$$\int_a^\infty y^{\frac{p\lambda_2}{s}-1}[\int_a^\infty k(x^{\lambda_1},y^{\lambda_2})f(x)dx]^p dy>[\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}]^p\int_a^\infty[1-\rho(s)(a^{\lambda_2}/x^{\lambda_1})^{\frac{1}{s}}]x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx \tag{5.1.6}$$

这里，常数因子 $\frac{k}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}$， $[\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}]^p$ 均为最佳值.

6 限制在区间 $(0,b)$ （$b>0$）上的 Hilbert 型积分不等式

定理6.1.1 设 $a=0$， $0<b<\infty$， 则则在定理4.1.1的条件下，若有 $0\le\eta<\frac{1}{s}$， 使 $0<\theta_{\lambda_1,\lambda_2}(r,x,b)=O\left((\frac{x^{\lambda_1}}{b^{\lambda_2}})^{\frac{1}{r}-\eta}\right)\le\tilde{l}<1$， （$x\to0^+$）及 $\tilde{\mu}_{\lambda_1,\lambda_2}(x)=1-\theta_{\lambda_1,\lambda_2}(r,x,b)\le\frac{\lambda_2\omega_{\lambda_1,\lambda_2}(s,x,0,b)}{k}$， $x\in(0,b)$

有如下等价不等式：

$$\int_0^b\int_0^b k(x^{\lambda_1},y^{\lambda_2})f(x)g(y)dxdy<\frac{k}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}\times\{\int_0^b[1-\theta_{\lambda_1,\lambda_2}(r,x,b)]x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx\}^{\frac{1}{p}}\{\int_0^b y^{q(1-\frac{\lambda_2}{s})-1}g^q(y)dy\}^{\frac{1}{q}} \tag{6..1.1}$$

$$\int_0^b\frac{y^{\frac{p\lambda_2}{s}-1}}{k_{\lambda_1,\lambda_2}^{p-1}(y)}[\int_0^b k(x^{\lambda_1},y^{\lambda_2})f(x)dx]^p dy<[\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}]^p\int_0^b[1-\theta_{\lambda_1,\lambda_2}(r,x,b)]x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx \tag{6.1.2}$$

这里，常数因子 $\frac{k}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}$， $[\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}]^p$ 均为最佳值.

证明 这里只证明常数因子 $\frac{k}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}$， $[\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}]^p$ 均为最佳值..

设 n， $n_0\in\mathrm{N}$， $n\ge n_0>\max\left\{\frac{r}{p},\frac{s}{p}\right\}$，定义 $f_n(x)$， $g_n(x)$

$$f_n(x)=x^{\frac{\lambda_1}{r}-1-\frac{\lambda_1}{np}},\quad g_n(x)=x^{\frac{\lambda_2}{s}-1-\frac{\lambda_2}{nq}}\quad x\in(0,b),$$

若式（6.1.1）的常数因子不是最佳值，则存在正数 $K\ge\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}$，使式（6.1.1）的常数因子

$\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}$ 换上 K 仍成立.特别地，有

$$I_n=\int_0^b\int_0^b k(x^{\lambda_1},y^{\lambda_2})f_n(x)g_n(y)dxdy$$

$$\geq K\{\int_0^b[1-\theta_{\lambda_1,\lambda_2}(r,x,b)]x^{p(1-\frac{\lambda_1}{r})-1}f_n^p(x)dx\}^{\frac{1}{p}}\{\int_0^b y^{q(1-\frac{\lambda_2}{s})-1}g_n^q(y)dy\}^{\frac{1}{q}} \quad (6.1.3)$$

$$=K\{\int_0^b x^{-1-\frac{\lambda_1}{n}}dx-\int_0^b O\left((\tfrac{x^{\lambda_1}}{b^{\lambda_2}})^{\frac{1}{r}-\eta}\right)x^{-1-\frac{\lambda_1}{n}}dx\}^{\frac{1}{p}}\{\int_0^b y^{-1-\frac{\lambda_2}{n}}dy\}^{\frac{1}{q}}=K\frac{n}{\lambda_1^{1/p}\lambda_2^{1/q}b^{\lambda_1/np+\lambda_2/nq}}[1-\frac{O(n)}{n}]^{1/p}$$

固定 y，做变换 $u=\frac{x^{\lambda_1}}{y^{\lambda_2}}$

$$I_n\leq\int_0^b y^{\frac{\lambda_2}{s}-1-\frac{\lambda_2}{nq}}[\int_0^\infty k(x^{\lambda_1},y^{\lambda_2})x^{\frac{\lambda_1}{r}-1-\frac{\lambda_1}{np}}dx]dy=\frac{n}{\lambda_1\lambda_2 b^{\lambda_2/n}}\int_0^\infty k(1,u)u^{\frac{1}{s}-1+\frac{1}{np}}du \quad (6.1.4)$$

结合式（6.1.3），可得

$$\frac{1}{\lambda_1^{1/q}\lambda_2^{1/p}b^{\lambda_2/np-\lambda_1/np}}\int_0^\infty k(1,u)u^{\frac{1}{s}-1+\frac{1}{np}}du\geq K[1-\frac{O(n)}{n}]^{1/p}$$

令 $n\to\infty$，由 Lebesgue 控制收敛定理，可得

$\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}\geq K$.故式（6.1.1）的常数因子 $\frac{k}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}$ 最佳值.若式（6.1.2）的常数因子不是最佳的，则由式（6.1.2）易得出式（6.1.1）的常数因子也不是最佳的矛盾.

推论 6.1.1 设 $a>0$，$b=+\infty$，在定理 4.1.1 的条件下，若 $0<\rho(r)<1$，使 $\tilde{\mu}_{\lambda_1,\lambda_2}(x)=1-\rho(r)(\frac{x^{\lambda_1}}{b^{\lambda_2}})^{\frac{1}{r}}\leq\frac{\lambda_2\omega_{\lambda_1,\lambda_2}(s,x,0,b)}{k}$，$x\in(0,b)$.

则如下等价不等式：

$$\int_a^\infty\int_a^\infty k(x^{\lambda_1},y^{\lambda_2})f(x)g(y)dxdy>\frac{k}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}$$
$$\times\{\int_a^\infty[1-\rho(r)(\tfrac{x^{\lambda_1}}{b^{\lambda_2}})^{\frac{1}{r}}]x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx\}^{\frac{1}{p}}\{\int_a^\infty y^{q(1-\frac{\lambda_2}{s})-1}g^q(y)dy\}^{\frac{1}{q}} \quad (6.1.5)$$

$$\int_0^b y^{\frac{p\lambda_2}{s}-1}[\int_0^b k(x^{\lambda_1},y^{\lambda_2})f(x)dx]^p dy>[\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}]^p\int_0^b[1-\rho(r)(\tfrac{x^{\lambda_1}}{b^{\lambda_2}})^{\frac{1}{r}}]x^{p(1-\frac{\lambda_1}{r})-1}f^p(x)dx \quad (6.1.6)$$

这里，常数因子 $\frac{k}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}$，$[\frac{k}{\lambda_1^{1/q}\lambda_2^{1/p}}]^p$ 均为最佳值.

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