一个多参数的 Hilbert 型积分不等式

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摘要：通过引入两个参数，利用实分析技巧和权函数方法研究双参数型 Hilbert 不等式，得到了一个多参数的 Hilbert 型积分不等式及其等价式，证明了它们的常数因子是最佳值，且在结果中选取合适的参数，得到了一系列形式简洁的 Hilbert 型积分不等式.

关键词： Hilbert 类积分不等式；权函数；Holder 不等式；最佳常数因子
中图分类号： O178

A Hilbert-type integral inequality with several parameters

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Abstract: A Hilbert's inequality of bi-parameter type was investigated by introducing two parameters and ,using the method of real analisis and weight function.A Hilbert-type integral inequality with several parameters and its equivalent form were obtained,and their constant factors were proved to be the optimum value. A series of Hilbert's type integral inequalities with more simple forms were obtained via selecteting the appropriate parameters satifying the conditions selected in the results.

Key words: Hilbert's integral inequality; weight function; Ho1der's inequality;the best constant factor

0 引言

设 $p>1$，$\frac{1}{p}+\frac{1}{q}=1$，$f(x)$，$g(x)\geq 0$，$0<\int_0^\infty f^p(x)dx<\infty$，$0<\int_0^\infty g^q(x)dx<\infty$，则有[1]：

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy<\frac{\pi}{\sin(\pi/p)}[\int_0^\infty f^p(x)dx]^{\frac{1}{p}}[\int_0^\infty g^q(x)dx]^{\frac{1}{q}} \quad (1)$$

$$\int_0^\infty[\int_0^\infty \frac{f(x)}{x+y}dx]^p dy<[\frac{\pi}{\sin(\pi/p)}]^p\int_0^\infty f^p(x)dx \quad (2)$$

这里，常数因子$\frac{\pi}{\sin(\pi/p)}$和$[\frac{\pi}{\sin(\pi/p)}]^p$都是最佳值[1-2]；式（1）称为 Hardy-Hilbert 积分不等式，在分析学中有重要作用[3]，式（2）是式（1）的等价形式.

文献[4-5]通过引入双参数λ_1，λ_2得到了式（1）和式（2）的双参数推广式：

设 $p>1$，$\frac{1}{p}+\frac{1}{q}=1$，$\lambda_1$，$\lambda_2>0$，$f(x)\geq 0$，$g(x)\geq 0$，

$0<\int_0^\infty x^{(p-1)((1-\lambda_1)}f^p(x)dx<\infty$，$0<\int_0^\infty x^{(q-1)((1-\lambda_2)}g^q(x)dx<\infty$，则

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$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x^{\lambda_1}+y^{\lambda_2}}dxdy < \frac{\pi}{\lambda_1^{1/q}\lambda_2^{1/p}\sin(\pi/p)} \times[\int_0^\infty x^{(p-1)((1-\lambda_1)} f^p(x)dx]^{\frac{1}{p}}[\int_0^\infty x^{(q-1)((1-\lambda_2)} g^q(x)dx]^{\frac{1}{q}} \quad (3)$$

其等价式为

$$\int_0^\infty y^{\lambda_2-1}[\int_0^\infty \frac{f(x)}{x^{\lambda_1}+y^{\lambda_2}}dx]^p dy < [\frac{\pi}{\lambda_1^{1/q}\lambda_2^{1/p}\sin(\pi/p)}]^p[\int_0^\infty x^{(p-1)(1-\lambda_1)} f^p(x)dx] \quad (4)$$

这里，常数因子$\frac{\pi}{\lambda_1^{1/q}\lambda_2^{1/p}\sin(\pi/p)}$及$[\frac{\pi}{\lambda_1^{1/q}\lambda_2^{1/p}\sin(\pi/p)}]^p$都是最佳值.

本文利用权函数和实分析方法得到一个多参数核为$\frac{1}{A\min\{x^{\lambda_1},y^{\lambda_2}\}+B\max\{x^{\lambda_1},y^{\lambda_2}\}}$的积分不等式及其等价式，并证明了它们的常数因子为最佳值.

1 主要结果

引理 1 设λ_1，$\lambda_2>0$，$A\ge 0$，$B>0$，定义权函数

$$\omega_{\lambda_1,\lambda_2}(A,B,x)=\int_0^\infty \frac{x^{\lambda_1/2}y^{\lambda_2/2-1}}{A\min\{x^{\lambda_1},y^{\lambda_2}\}+B\max\{x^{\lambda_1},y^{\lambda_2}\}}dy, \quad x\in(0,\infty)$$

$$\omega_{\lambda_1,\lambda_2}(A,B,y)=\int_0^\infty \frac{x^{\lambda_1/2-1}y^{\lambda_2/2}}{A\min\{x^{\lambda_1},y^{\lambda_2}\}+B\max\{x^{\lambda_1},y^{\lambda_2}\}}dx, \quad y\in(0,\infty).$$

则有

$$\omega_{\lambda_1,\lambda_2}(A,B,x)=\frac{1}{\lambda_2}C(A,B)\text{，}\quad \omega_{\lambda_1,\lambda_2}(A,B,y)=\frac{1}{\lambda_1}C(A,B)$$

其中
$$C(A,B)=\begin{cases}\dfrac{4}{\sqrt{AB}}\arctan\sqrt{\dfrac{A}{B}} & A>0,B>0\\ \dfrac{4}{B} & A=0,B>0\end{cases}$$

证明：令$u=\frac{y^{\lambda_2}}{x^{\lambda_1}}$

（1） 当$A>0$，$B>0$时，有

$$\omega_{\lambda_1,\lambda_2}(A,B,x)=\int_0^\infty \frac{x^{\lambda_1/2}y^{\lambda_2/2-1}}{A\min\{x^{\lambda_1},y^{\lambda_2}\}+B\max\{x^{\lambda_1},y^{\lambda_2}\}}dy$$
$$=\frac{1}{\lambda_2}\int_0^\infty \frac{u^{-1/2}}{A\min\{1,u\}+B\max\{1,u\}}du=\frac{4}{\lambda_2\sqrt{AB}}\arctan\sqrt{\frac{A}{B}}$$

（2） 当$A=0$，$B>0$时，有

$$\omega_{\lambda_1,\lambda_2}(A,B,x)=\int_0^\infty \frac{x^{\lambda_1/2}y^{\lambda_2/2-1}}{B\max\{x^{\lambda_1},y^{\lambda_2}\}}dy=\frac{1}{\lambda_2}\int_0^\infty \frac{u^{-1/2}}{B\max\{1,u\}}du=\frac{4}{\lambda_2 B}$$

所以有$\omega_{\lambda_1,\lambda_2}(A,B,x)=\frac{1}{\lambda_2}C(A,B)$.同理可证：$\omega_{\lambda_1,\lambda_2}(A,B,y)=\frac{1}{\lambda_1}C(A,B)$.

引理 2 设 $p>1$，$\frac{1}{p}+\frac{1}{q}=1$，$\lambda_1$，$\lambda_2>0$，$A\ge 0$，$B>0$，$0<\varepsilon<\frac{p}{2}$，则有

$$\int_1^\infty\int_1^\infty \frac{x^{[p(\lambda_1/2-1)-\lambda_1\varepsilon]/p}y^{[q(\lambda_2/2-1)-\lambda_2\varepsilon]/q}}{A\min\{x^{\lambda_1},y^{\lambda_2}\}+B\max\{x^{\lambda_1},y^{\lambda_2}\}}dxdy \ge \frac{1}{\lambda_1\lambda_2\varepsilon}[C(A,B)+o(1)]-O(1),\ (\varepsilon\to 0^+) \tag{5}$$

证明：令 $u=\frac{x^{\lambda_1}}{y^{\lambda_2}}$，则有

（1）当 $A>0$，$B>0$ 时

$$\int_1^\infty\int_1^\infty \frac{x^{[p(\lambda_1/2-1)-\lambda_1\varepsilon]/p}y^{[q(\lambda_2/2-1)-\lambda_2\varepsilon]/q}}{A\min\{x^{\lambda_1},y^{\lambda_2}\}+B\max\{x^{\lambda_1},y^{\lambda_2}\}}dxdy=\frac{1}{\lambda_1}\int_1^\infty y^{-1-\lambda_2\varepsilon}\int_{y^{-\lambda_2}}^\infty \frac{u^{-1/2-\varepsilon/p}}{A\min\{1,u\}+B\max\{1,u\}}dudy$$

$$=\frac{1}{\lambda_1\lambda_2\varepsilon}\int_0^\infty \frac{u^{-1/2-\varepsilon/p}}{A\min\{1,u\}+B\max\{1,u\}}du-\frac{1}{\lambda_1}\int_1^\infty y^{-1-\lambda_2\varepsilon}\int_0^{y^{-\lambda_2}} \frac{u^{-1/2-\varepsilon/p}}{A\min\{1,u\}+B\max\{1,u\}}dudy$$

$$>\frac{1}{\lambda_1\lambda_2\varepsilon}[\frac{2}{\sqrt{AB}}\arctan\sqrt{\frac{A}{B}}+o(1)_1+\frac{2}{\sqrt{AB}}\arctan\sqrt{\frac{A}{B}}-o(1)_2]-\frac{1}{\lambda_1}\int_1^\infty y^{-1}\int_0^{y^{-\lambda_2}}\frac{u^{-1/2-\varepsilon/p}}{B}dudy$$

$$=\frac{1}{\lambda_1\lambda_2\varepsilon}[C(A,B)+o(1)]-O(1)$$

（2）当 $A=0$，$B>0$ 时

$$\int_1^\infty\int_1^\infty \frac{x^{[p(\lambda_1/2-1)-\lambda_1\varepsilon]/p}y^{[q(\lambda_2/2-1)-\lambda_2\varepsilon]/q}}{A\min\{x^{\lambda_1},y^{\lambda_2}\}+B\max\{x^{\lambda_1},y^{\lambda_2}\}}dxdy=\frac{1}{\lambda_1}\int_1^\infty y^{-1-\lambda_2\varepsilon}\int_{y^{-\lambda_2}}^\infty \frac{u^{-1/2-\varepsilon/p}}{A\min\{1,u\}+B\max\{1,u\}}dudy$$

$$=\frac{1}{\lambda_1\lambda_2\varepsilon}\int_0^\infty \frac{u^{-1/2-\varepsilon/p}}{A\min\{1,u\}+B\max\{1,u\}}du-\frac{1}{\lambda_1}\int_1^\infty y^{-1-\lambda_2\varepsilon}\int_0^{y^{-\lambda_2}} \frac{u^{-1/2-\varepsilon/p}}{A\min\{1,u\}+B\max\{1,u\}}dudy$$

$$=\frac{1}{\lambda_1\lambda_2\varepsilon}[\int_0^1\frac{u^{-1/2-\varepsilon/p}}{B}du+\int_1^\infty\frac{u^{-1/2-\varepsilon/p}}{Bu}du]-\frac{1}{\lambda_1}\int_1^\infty y^{-1-\lambda_2\varepsilon}\int_0^{y^{-\lambda_2}}\frac{u^{-1/2-\varepsilon/p}}{B}dudy$$

$$=\frac{1}{\lambda_1\lambda_2\varepsilon}[\frac{2}{B(1-2\varepsilon/p)}+\frac{2}{B(1+2\varepsilon/p)}]-\frac{1}{\lambda_1\lambda_2}\frac{2}{B(1+2\varepsilon/q)}$$

$$=\frac{1}{\lambda_1\lambda_2\varepsilon}[C(A,B)+o(1)]-\frac{1}{\lambda_1\lambda_2}\frac{2}{B(1+2\varepsilon/q)}=\frac{1}{\lambda_1\lambda_2\varepsilon}[C(A,B)+o(1)]-O(1)$$

综上，故式（5）成立.

定理 1 设 $p>1$，$\frac{1}{p}+\frac{1}{q}=1$，$\lambda_1$，$\lambda_2>0$，$A\ge 0$，$B>0$，$f(x)$，$g(x)\ge 0$，使得 $0<\int_0^\infty x^{p(1-\lambda_1/2)-1}f^p(x)dx<\infty$，$0<\int_0^\infty x^{q(1-\lambda_2/2)-1}g^q(x)dx<\infty$，则有

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{A\min\{x^{\lambda_1},y^{\lambda_2}\}+B\max\{x^{\lambda_1},y^{\lambda_2}\}}dxdy<\frac{C(A,B)}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}} \times\{\int_0^\infty x^{p(1-\lambda_1/2)-1}f^p(x)dx\}^{\frac{1}{p}}\{\int_0^\infty x^{q(1-\lambda_2/2)-1}g^q(x)dx\}^{\frac{1}{q}} \tag{6}$$

这里，常数因子 $\frac{C(A,B)}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}$ 为最佳值.

证明 由带权的 Holder 不等式和引理 1 得

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{A\min\{x^{\lambda_1},y^{\lambda_2}\}+B\max\{x^{\lambda_1},y^{\lambda_2}\}}dxdy$$

$$\le\{\int_0^\infty\int_0^\infty \frac{f^p(x)x^{\lambda_1/2}y^{\lambda_2/2-1}x^{p(1-\lambda_1/2)-1}}{A\min\{x^{\lambda_1},y^{\lambda_2}\}+B\max\{x^{\lambda_1},y^{\lambda_2}\}}dxdy\}^{\frac{1}{p}}$$

$$\times\{\int_0^\infty\int_0^\infty \frac{g^q(y)x^{\lambda_1/2-1}y^{\lambda_2/2}y^{q(1-\lambda_2/2)-1}}{A\min\{x^{\lambda_1},y^{\lambda_2}\}+B\max\{x^{\lambda_1},y^{\lambda_2}\}}dxdy\}^{\frac{1}{q}}$$

$$=\{\int_0^\infty \omega_{\lambda_1,\lambda_2}(A,B,x)x^{p(1-\lambda_1/2)-1}f^p(x)dx\}^{\frac{1}{p}}\{\int_0^\infty \omega_{\lambda_1,\lambda_2}(A,B,y)y^{q(1-\lambda_2/2)-1}g^q(y)dy\}^{\frac{1}{q}}$$

$$=\frac{C(A,B)}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}\{\int_0^\infty x^{p(1-\lambda_1/2)-1}f^p(x)dx\}^{\frac{1}{p}}\{\int_0^\infty y^{q(1-\lambda_2/2)-1}g^q(y)dy\}^{\frac{1}{q}}$$

下面应用 holder 不等式证明上式中间取严格不等号，若不然，必存在不全为 0 的常数 a，b 使得

$af^p(x)y^{\lambda_2/2-1}x^{(p-1)(1-\lambda_1/2)}=bg^q(y)x^{\lambda_1/2-1}y^{(q-1)(1-\lambda_2/2)}$ a.e 于 $(0,\infty)\times(0,\infty)$. 即有 $axf^p(x)x^{p(1-\lambda_1/2)-1}=byg^q(y)y^{q(1-\lambda_2/2)-1}$ a.e 于 $(0,\infty)\times(0,\infty)$.，于是有常数 C，使 $axf^p(x)x^{p(1-\lambda_1/2)-1}=C$ a.e 于 $(0,\infty)$.不妨设 $a\neq0$，则可得等式 $x^{p(1-\lambda_1/2)-1}f^p(x)=\frac{C}{a}x^{-1}$ a.e 于 $(0,\infty)$，无论 C 是否为 0，积分的结果必与 $0<\int_0^\infty x^{p(1-\lambda_1/2)-1}f^p(x)dx<\infty$ 相矛盾.于是式（6）成立.

设 ε 为任意小的正数，定义函数 $f_\varepsilon(x)$，$g_\varepsilon(x)$ 使

$$f_\varepsilon(x)=0,\quad x\in(0,1)\ ;\quad f_\varepsilon(x)=x^{[p(\lambda_1/2-1)-\lambda_1\varepsilon]/p},\quad x\in[1,\infty)$$

$$g_\varepsilon(x)=0,\quad x\in(0,1)\ ;\quad g_\varepsilon(x)=x^{[q(\lambda_2/2-1)-\lambda_2\varepsilon]/q},\quad x\in[1,\infty)$$

若式（6）的常数因子不是最佳值，则存在正数 $K<\frac{C(A,B)}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}$，使式（6）的常数因子 $\frac{C(A,B)}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}$ 换上 K 仍成立.特别地，由式（5），有

$$\frac{1}{\lambda_1\lambda_2}[C(A,B)+o(1)]-\varepsilon O(1)<\varepsilon\int_1^\infty\int_1^\infty \frac{x^{[p(\lambda_1/2-1)-\lambda_1\varepsilon]/p}y^{[q(\lambda_2/2-1)-\lambda_2\varepsilon]/q}}{A\min\{x^{\lambda_1},y^{\lambda_2}\}+B\max\{x^{\lambda_1},y^{\lambda_2}\}}dxdy$$

$$<\varepsilon K[\int_0^\infty x^{p(1-\lambda_1/2)-1}f_\varepsilon^p(x)dx]^{\frac{1}{p}}[\int_0^\infty x^{q(1-\lambda_2/2)-1}g_\varepsilon^q(x)dx]^{\frac{1}{q}}=\varepsilon K\frac{1}{\lambda_1^{1/p}\lambda_2^{1/q}\varepsilon}=\frac{K}{\lambda_1^{1/p}\lambda_2^{1/q}}$$

令 $\varepsilon\to0^+$，有 $K\ge\frac{C(A,B)}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}$.这与假设 $K<\frac{C(A,B)}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}$ 矛盾，故常数因子 $\frac{C(A,B)}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}$ 为最佳值.

定理 2 设 $p>1$，$\frac{1}{p}+\frac{1}{q}=1$，$\lambda_1$，$\lambda_2>0$，$A\ge0$，$B>0$，$f\ge0$，使得

$0<\int_0^\infty x^{p(1-\lambda_1/2)-1}f^p(x)dx<\infty$.则有

$$\int_0^\infty y^{p\lambda_2/2-1}[\int_0^\infty \frac{f(x)}{A\min\{x^{\lambda_1},y^{\lambda_2}\}+B\max\{x^{\lambda_1},y^{\lambda_2}\}}dx]^p dy < [\frac{C(A,B)}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}]^p\int_0^\infty x^{p(1-\lambda_1/2)-1}f^p(x)dx \quad (7)$$

这里，常数因子$[\frac{C(A,B)}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}]^p$为最佳值；且与定理 1 等价.

证明　设定$[f(x)]_n=\min\{n,f(x)\}$.因$0<\int_0^\infty x^{p(1-\lambda_1/2)-1}f^p(x)dx<\infty$，故存在$n_0\in\mathrm{N}$，使得当$n\ge n_0$时，有$0<\int_{1/n}^n x^{p(1-\lambda_1/2)-1}[f(x)]_n^p dx<\infty$，定义函数

$g_n(y)=y^{p\lambda_2/2-1}[\int_{1/n}^n \frac{[f(x)]_n}{A\min\{x^{\lambda_1},y^{\lambda_2}\}+B\max\{x^{\lambda_1},y^{\lambda_2}\}}dx]^{p-1}$，$y\in(0,\infty)$，则由（6）式，有

$$\begin{aligned}0&<\int_{1/n}^n y^{q(1-\lambda_2/2)-1}g_n^q(y)dy\\&=\int_{1/n}^n y^{p\lambda_2/2-1}[\int_{1/n}^n \frac{[f(x)]_n}{A\min\{x^{\lambda_1},y^{\lambda_2}\}+B\max\{x^{\lambda_1},y^{\lambda_2}\}}dx]^p dy\\&=\int_{1/n}^n\int_{1/n}^n \frac{[f(x)]_n g_n(y)}{A\min\{x^{\lambda_1},y^{\lambda_2}\}+B\max\{x^{\lambda_1},y^{\lambda_2}\}}dxdy\\&<\frac{C(A,B)}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}[\int_{1/n}^n x^{p(1-\lambda_1/2)-1}[f(x)]_n^p dx]^{\frac{1}{p}}[\int_{1/n}^n y^{q(1-\lambda_2/2)-1}g_n^q(y)dy]^{\frac{1}{q}}\end{aligned} \quad (8)$$

因此有

$$0<\int_{1/n}^n y^{q(1-\lambda_2/2)-1}g_n^q(y)dy<[\frac{C(A,B)}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}]^p\int_0^\infty x^{p(1-\lambda_1/2)-1}f^p(x)dx<\infty \quad (9)$$

当$n\to\infty$时，应用式（6），式（8），（9）取严格不等号；故式（7）成立.

上面由（6）式证明了（7）式.为证等价性，可先设式（7）成立，由 Holder 不等式，有

$$\begin{aligned}&\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{A\min\{x^{\lambda_1},y^{\lambda_2}\}+B\max\{x^{\lambda_1},y^{\lambda_2}\}}dxdy\\&=\int_0^\infty y^{[q(\lambda_2/2-1)+1]/q}[\int_0^\infty \frac{f(x)}{A\min\{x^{\lambda_1},y^{\lambda_2}\}+B\max\{x^{\lambda_1},y^{\lambda_2}\}}dx]y^{[q(1-\lambda_2/2)-1]/q}g(y)dy\\&\le\{\int_0^\infty y^{p\lambda_2/2-1}[\int_0^\infty \frac{f(x)}{A\min\{x^{\lambda_1},y^{\lambda_2}\}+B\max\{x^{\lambda_1},y^{\lambda_2}\}}dx]^p dy\}^{\frac{1}{p}}\{\int_0^\infty y^{q(1-\lambda_2/2)-1}g^q(y)dy\}^{\frac{1}{q}}\\&<\frac{C(A,B)}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}\{\int_0^\infty x^{p(1-\lambda_1/2)-1}f^p(x)dx\}^{\frac{1}{p}}\{\int_0^\infty x^{q(1-\lambda_2/2)-1}g^q(x)dx\}^{\frac{1}{q}}\end{aligned}$$

此不等式即为式（6），因此式（6）和式（7）等价.

若式（7）的常数因子不是最佳值，则由式（7）易知式（6）的常数因子也不是最佳的. 这与定理 1 的结论矛盾，说明式（7）的常数因子$[\frac{C(A,B)}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}}]^p$是最佳值.

下面利用定理 1 和定理 2 给出一些推论.

在式（6），（7）中取$\lambda_1=\lambda_2=A=B=1$，有

推论 1 设$p>1$，$\frac{1}{p}+\frac{1}{q}=1$，$f(x)$，$g(x)\geq 0$，使得

$0<\int_0^\infty x^{p/2-1}f^p(x)dx<\infty$，$0<\int_0^\infty x^{q/2-1}g^q(x)dx<\infty$，则有下列等价式

$$\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{x+y}dxdy<\pi\{\int_0^\infty x^{p/2-1}f^p(x)dx\}^{\frac{1}{p}}\{\int_0^\infty x^{q/2-1}g^q(x)dx\}^{\frac{1}{q}} \tag{10}$$

$$\int_0^\infty y^{p/2-1}[\int_0^\infty\frac{f(x)}{x+y}dx]^p dy<\pi^p\int_0^\infty x^{p/2-1}f^p(x)dx \tag{11}$$

这里，常数因子π，π^p为最佳值.

在式（6），（7）中取$\lambda_1=\lambda_2=1/2$，$A=B=1$，有

推论 2 设$p>1$，$\frac{1}{p}+\frac{1}{q}=1$，$f(x)$，$g(x)\geq 0$，使得

$0<\int_0^\infty x^{3p/4-1}f^p(x)dx<\infty$，$0<\int_0^\infty x^{3q/4-1}g^q(x)dx<\infty$，则有下列等价式

$$\begin{aligned}&\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{\min\{\sqrt{x},\sqrt{y}\}+\max\{\sqrt{x},\sqrt{y}\}}dxdy\\&<2\pi\{\int_0^\infty x^{3p/4-1}f^p(x)dx\}^{\frac{1}{p}}\{\int_0^\infty x^{3q/4-1}g^q(x)dx\}^{\frac{1}{q}}\end{aligned} \tag{12}$$

$$\begin{aligned}&\int_0^\infty y^{p/4-1}[\int_0^\infty\frac{f(x)}{\min\{\sqrt{x},\sqrt{y}\}+\max\{\sqrt{x},\sqrt{y}\}}dx]^p dy\\&<(2\pi)^p\int_0^\infty x^{3p/4-1}f^p(x)dx\end{aligned} \tag{13}$$

在式（6），（7）中取$\lambda_1=\lambda_2=\lambda$，$A=0$，$B=1$有

推论 3 设$p>1$，$\frac{1}{p}+\frac{1}{q}=1$，$f(x)$，$g(x)\geq 0$，使得

$0<\int_0^\infty x^{p(1-\lambda/2)-1}f^p(x)dx<\infty$，$0<\int_0^\infty x^{q(1-\lambda/2)-1}g^q(x)dx<\infty$，则有下列等价式

$$\begin{aligned}&\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{\max\{x^\lambda,y^\lambda\}}dxdy<\frac{4}{\lambda}\{\int_0^\infty x^{p(1-\lambda/2)-1}f^p(x)dx\}^{\frac{1}{p}}\\&\times\{\int_0^\infty x^{q(1-\lambda/2)-1}g^q(x)dx\}^{\frac{1}{q}}\end{aligned} \tag{14}$$

$$\int_0^\infty y^{p\lambda/2-1}[\int_0^\infty \frac{f(x)}{\max\{x^\lambda, y^\lambda\}}dx]^p dy < [\frac{4}{\lambda}]^p \int_0^\infty x^{p(1-\lambda/2)-1} f^p(x)dx \tag{15}$$

这里，常数因子$\frac{4}{\lambda}$和$[\frac{4}{\lambda}]^p$为最佳值.特别地当$\lambda=1$时，有

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}}dxdy < 4\{\int_0^\infty x^{p/2-1} f^p(x)dx\}^{\frac{1}{p}}\{\int_0^\infty x^{q/2-1} g^q(x)dx\}^{\frac{1}{q}} \tag{16}$$

$$\int_0^\infty y^{p/2-1}[\int_0^\infty \frac{f(x)}{\max\{x,y\}}dx]^p dy < 4^p \int_0^\infty x^{p/2-1} f^p(x)dx \tag{17}$$

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