Mathematical modeling of magnetostrictive nanowires for sensor application

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Abstract

Magnetostrictive wires of diameter in the nanometer scale have been proposed for application as acoustic sensors [Downey et al., 2008], [Yang et al., 2006]. The sensing mechanism is expected to operate in the bending regime. In this work we derive a variational theory for the bending of magnetostrictive nanowires starting from the full 3-dimensional continuum theory of magnetostriction. We recover a theory which looks like the typical Euler-Bernoulli bending model but includes an extra term contributed by the magnetic part of the energy. The solution of this variational theory for an important, newly developed magnetostricitve alloy called Galfenol (cf. [Clark et al., 2000]) is compared with the result of experiments on actual nanowires (cf. [Downey, 2008]) which shows agreement.

1 Introduction

Magnetostrictive solids are those in which reversible elastic deformations are caused by changes in the magnetization. These materials have a coupling of ferromagnetic energies with elastic energies. Typically magnetostriction is a small effect in the range of 20-200 ppm for commonly occurring ferromagnetic materials like Fe, Co and Ni alloys. In the 1970's giant magnetostrictive alloys like $Tb_{0.3}Dy_{0.7}Fe_2$ were developed. This alloy called Terfenol has high magnetostriction of the order ~ 2000 ppm, but is very brittle, and has low tensile strength of the order ~ 100 MPa. For this reason in most sensor/actuator applications it is used under compressive strain. Recent research by Clark et al. [Clark et al., 2000] has led to the development of a new alloy called Galfenol with formula $Fe_{100-x}Ga_x$ where x ranges from $10\% - 30\%$. These alloys have relatively high magnetostriction ~ 400 ppm and high tensile strengths $\sim 400 MPa$.

In recent years a lot of new experimental techniques have been developed to manufacture ferromagnetic wires of nanometer diameter such as electron-beam lithography, step growth electro-deposition, and template-assisted electro-deposition. A possible application of these nanosize wires is in making acoustic sensors. The inspiration for this application comes from the structure of the human ear. The inner ear has fine cilia like hair whose response to impinging acoustic waves is transmitted through the nervous system to the brain. Such biologically inspired devices have been proposed to detect acoustic, fluid flow and tactile inputs (cf. [Yang et al., 2006]). One possible arrangement of galfenol nanowires is in the form of an array depicted in Fig 1 . Here impinging acoustic waves are expected to change the magnetization of the wire array by inducing bending deformation.

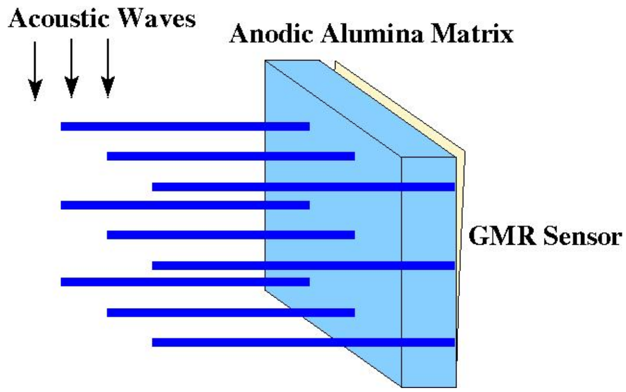

Figure 1: Proposed model device using nanowires of Galfenol

The model of a vibrating string or the bending of a beam are important models in elasticity which are known to approximate the full 3-D behavior of a deformable body in the linear strain regime. Starting in the 80's rigorous mathematical methods based on the theory of Γ-convergence were used to justify these 1-D models as the correct approximation of 3-D elasticity, loosely speaking under asymptotic conditions as the diameter of the 3-D body approaches zero. The basic references for these results are [Acerbi et al., 1991] and [Anzellotti et al., 1994], while reference for Γ -convergence can be found in [Braides, 2002].

Meanwhile in the micromagnetics literature there has been extensive use of Γ-convergence based methods to derive reduced dimension models for ferromagnetic thin films. The earliest results in this direction are [Gioia and James, 1997] and [Carbou, 2001]. Since our nanowires are expected to be used for the proposed sensor application in the bending deformation regime, the main goal of this paper is to combine the ideas of the references cited above from the elasticity and micromagnetics literature to derive similar asymptotic models for magnetostrictive nanowires in bending. The nanowires we are modeling have diameters in the 10-100nm range with lengths in the range 2-5μm. We will show that the bending behavior of a magnetostrictive nanowire resembles the classical Euler-Bernoulli bending with an extra term which come from the magnetic part of the energy.

§ 1.1 gives a brief review of the micromagnetic theory of magnetostriction and defines the classical energy $\mathscr{E}(\widetilde{\boldsymbol{m}},\widetilde{\boldsymbol{u}})$ in the continuum theory of magnetostriction as a function of the magnetization-deformation pair $(\widetilde{\boldsymbol{m}},\widetilde{\boldsymbol{u}})$. The first part of § 2 gives a simple heuristic argument to show the various scales of elastic and magnetic energy relevant to the final result. In § 2.3 we start with the energy $\mathscr{E}(\widetilde{\boldsymbol{m}},\widetilde{\boldsymbol{u}})$ defined on a wire of diameter ε and on rescaling the wire to have unit diameter, recover a new energy $\mathscr{I}^{\varepsilon}(\boldsymbol{m},\boldsymbol{u})$ which equals the energy $\mathscr{E}(\widetilde{\boldsymbol{m}},\widetilde{\boldsymbol{u}})$ per unit wire cross-sectional area and depending on a rescaled magnetization-deformation pair $(\boldsymbol{m},\boldsymbol{u})$ now defined on the wire with unit diameter. Starting with minimizers $(\boldsymbol{m}^{\varepsilon},\boldsymbol{u}^{\varepsilon})$ of the energy $\mathscr{I}^{\varepsilon}(\boldsymbol{m},\boldsymbol{u})$ in § 3 we derive the first variational limit problem which physically represents the magnetoelastic equivalent to the elastic theory of an extensible string. § 4 gives the next order correction to the first variational problem which only involves magnetic terms. § 5 gives the following order variational problem which is the main result of this paper and describes the bending behavior of the magnetostrictive nanowires. Here we show that we can extract a deformation $\boldsymbol{w}^{\varepsilon}$ (cf. (5.14)) from the energy minimizing pair $(\boldsymbol{m}^{\varepsilon},\boldsymbol{u}^{\varepsilon})$ which itself minimizes an energy $\mathscr{I}_2^{o}$ (cf. (5.20)) where $\mathscr{I}_2^{o}$ is an energy which resembles the classical Euler-Bernoulli bending energy with some correction terms coming from the magnetization. The method of proof involves the idea of convergence of minimizers, and we do not use the more abstract Γ-convergence method.

Basic notation: $\alpha,\beta,\gamma,\cdots$ are scalars; $\boldsymbol{a},\boldsymbol{u},\boldsymbol{m},\cdots$ denote vectors in $\mathbb{R}^3$; $\boldsymbol{A},\boldsymbol{B},\boldsymbol{E},\cdots$ are tensors in $\mathbb{R}^{3\times 3}$ and $S^2\subset\mathbb{R}^3$ represents the surface of the unit ball in $\mathbb{R}^3$. Components of any vector $\boldsymbol{m}$ are denoted by either m_1,m_2,m_3 or m_x,m_y,m_y. For any matrix $\boldsymbol{A}$, $\boldsymbol{A}^T$ denotes the transpose of the matrix. We use standard function space notation of $L^2(\Omega,\mathbb{R}^3)$, $H^1(\mathbb{R}^3,\mathbb{R}^3)$, $H_0^1(\Omega,\mathbb{R}^3)$; for details refer [Adams and Fournier, 2009]. By Young's inequality we mean a variation of the classical Young's inequality, $2ab\le\delta^{-1}a^2+\delta b^2$ for $\mathbb{R}\ni\delta>0$.

1.1 Micromagnetics

The initial model for ferromagnetic solids was proposed in [Landau and Lifshitz, 1935] where they also derived the equation for magnetization dynamics. The continuum theory of ferromagnetic materials was developed in the works of Brown [Brown, 1963] which was subsequently expanded to a theory for magnetostriction in [Brown, 1966], where a variational model for magnetostriction with small strain is developed.

Let Ω_ε be a smooth bounded reference configuration in $\mathbb{R}^3$ depending on a parameter ε. In the following sections we fill specify this dependence. Let $\widetilde{\boldsymbol{m}}(\boldsymbol{y})$ be the magnetization vector at a point $\boldsymbol{y}\in\Omega_\varepsilon$. Below the Curie temperature, the magnetization is constrained to have constant euclidean norm i.e.,

$$|\widetilde{\boldsymbol{m}}(\boldsymbol{y})| = m_s \quad a.e. \qquad \boldsymbol{y} \in \Omega_\varepsilon.$$

For a bounded domain, this constraint implies $\widetilde{\boldsymbol{m}} \in L^p(\Omega_\varepsilon, m_s S^2),\ \forall 1 \le p \le \infty$. We extend $\widetilde{\boldsymbol{m}}$ by 0 outside Ω_ε whenever necessary and denote it by $\widetilde{\boldsymbol{m}}\chi_{\Omega_\varepsilon} = \widetilde{\boldsymbol{m}}(\boldsymbol{y})\chi_{\Omega_\varepsilon}(\boldsymbol{y})$ where then we have $\widetilde{\boldsymbol{m}}\chi_{\Omega_\varepsilon} \in L^p(\mathbb{R}^3, \mathbb{R}^3),\ \forall 1 \le p \le \infty$. We denote by $\widetilde{\boldsymbol{u}} \in H^1(\Omega_\varepsilon, \mathbb{R}^3)$ the displacement map. The infinitesimal strain corresponding to $\widetilde{\boldsymbol{u}}(\boldsymbol{y})$ is, $\big(\nabla^y$ is gradient w.r.t. $\boldsymbol{y}\big)$

$$\widetilde{\boldsymbol{E}}[\widetilde{\boldsymbol{u}}](\boldsymbol{y}) = \frac{1}{2}(\nabla^y \widetilde{\boldsymbol{u}}(\boldsymbol{y}) + \nabla^y \widetilde{\boldsymbol{u}}(\boldsymbol{y})^T). \tag{1.1}$$

The interaction of magnetic properties with crystalline structure of magnetic solids generates an interaction energy modeled by a function, $\varphi : m_s S^2 \to [0, \infty)$. This energy has a finite number of wells (say N) along a set of magnetization vectors $\{\widetilde{\boldsymbol{m}}^{(k)}\}$ where the index $k \in \{1, 2, \cdots N\}$ and on which without loss of generality we can set $\varphi(\widetilde{\boldsymbol{m}}^{(k)}) = 0$. The anisotropy energy thus becomes,

$$E_{anis} = \int_{\Omega_\varepsilon} \varphi(\widetilde{\boldsymbol{m}}(\boldsymbol{y}))\, \boldsymbol{dy}.$$

For cubic materials $\varphi(\boldsymbol{m}) = \frac{\Pi_1}{m_s^4}(\widetilde{m}_1^2\widetilde{m}_2^2 + \widetilde{m}_1^2\widetilde{m}_3^2 + \widetilde{m}_2^2\widetilde{m}_3^2) + \frac{\Pi_2}{m_s^6}(\widetilde{m}_1^2\widetilde{m}_2^2\widetilde{m}_3^2)$, which along with the constraint $|\widetilde{\boldsymbol{m}}| = m_s$ gives the following bound,

$$0 \le \int_{\Omega} \varphi(\widetilde{\boldsymbol{m}}(\boldsymbol{y}))\, \boldsymbol{dy} \le K_1 |\Omega_\varepsilon|. \tag{1.2}$$

The exchange energy penalizes variations in the magnetization in a body and thus tends to prefer constant magnetizations. It is modeled as follows,

$$E_{exc} = d \int_{\Omega_\varepsilon} |\nabla^y \widetilde{\boldsymbol{m}}|^2\, \boldsymbol{dy}.$$

Here d is called the exchange constant. Magnetized bodies generate a magnetic self field in all of $\mathbb{R}^3$. This field $\widetilde{\boldsymbol{h}}^\varepsilon_{\widetilde{\boldsymbol{m}}}(\boldsymbol{y})$ is given by the following equation,

$$\begin{aligned}
&\nabla^y \cdot (-\nabla^y \widetilde{\phi}^\varepsilon(\boldsymbol{y}) + 4\pi \widetilde{\boldsymbol{m}}(\boldsymbol{y})) = 0 \qquad \forall \boldsymbol{y} \in \mathbb{R}^3,\\
&\widetilde{\boldsymbol{h}}^\varepsilon_{\widetilde{\boldsymbol{m}}}(\boldsymbol{y}) = -\nabla^y \widetilde{\phi}^\varepsilon(\boldsymbol{y}),\\
&[|\nabla^y \widetilde{\phi}^\varepsilon \cdot \widetilde{\boldsymbol{n}}|] = [| -\widetilde{\boldsymbol{h}}^\varepsilon_{\widetilde{\boldsymbol{m}}} \cdot \widetilde{\boldsymbol{n}}|] = 4\pi\ \widetilde{\boldsymbol{m}} \cdot \widetilde{\boldsymbol{n}} \qquad \text{on } \partial\Omega_\varepsilon.
\end{aligned}$$

$[|\cdot|]$ represents the jump of a quantity across any oriented surface with unit normal $\widetilde{\boldsymbol{n}}$. The demagnetization energy is generated by the interaction of the magnetization with $\widetilde{\boldsymbol{h}}^\varepsilon_{\widetilde{\boldsymbol{m}}}$ and equals

$$E_{demag}(\widetilde{\boldsymbol{m}}) = \frac{1}{8\pi} \int_{\mathbb{R}^3} |\widetilde{\boldsymbol{h}}^\varepsilon_{\widetilde{\boldsymbol{m}}}(\boldsymbol{y})|^2\, \boldsymbol{dy} = -\frac{1}{2} \int_{\Omega_\varepsilon} \widetilde{\boldsymbol{h}}^\varepsilon_{\widetilde{\boldsymbol{m}}}(\boldsymbol{y}) \cdot \widetilde{\boldsymbol{m}}(\boldsymbol{y})\, \boldsymbol{dy}.$$

A standard upper and lower bound for E_{demag} is given by,

$$0 \le E_{demag}(\widetilde{\boldsymbol{m}}) \le \frac{1}{2}\int_{\Omega_\varepsilon} |\widetilde{\boldsymbol{m}}(\boldsymbol{y})|^2 \boldsymbol{dy} = \frac{1}{2}|\Omega_\varepsilon| m_s^2. \tag{1.3}$$

The energy of interaction between an external applied field $\widetilde{\boldsymbol{h}_a} \in L^2(\Omega, \mathbb{R}^3)$ and the magnetization over the body is modeled by the following,

$$E_{app}(\widetilde{\boldsymbol{m}}) = -\int_{\Omega_\varepsilon} \widetilde{\boldsymbol{h}_a}(\boldsymbol{y}) \cdot \widetilde{\boldsymbol{m}}(\boldsymbol{y}) \boldsymbol{dy}.$$

which along with Hölder's inequality gives

$$-K_2 \le E_{app}(\widetilde{\boldsymbol{m}}) \le K_2 \qquad\qquad K_2 = \|\widetilde{\boldsymbol{h}_a}\|_{L^2(\Omega)} \|\widetilde{\boldsymbol{m}}\|_{L^2(\Omega)}. \tag{1.4}$$

The elastic energy for the magnetoelastic solid for small strains is given by,

$$E_{el} = \int_{\Omega_\varepsilon} \frac{1}{2}(\widetilde{\boldsymbol{E}}[\widetilde{\boldsymbol{u}}] - \widetilde{\boldsymbol{E}}_s(\widetilde{\boldsymbol{m}})) : \mathbb{C}[\widetilde{\boldsymbol{E}}[\widetilde{\boldsymbol{u}}] - \widetilde{\boldsymbol{E}}_s(\widetilde{\boldsymbol{m}})] \boldsymbol{dy}$$

where in this paper we write the integrand as,

$$(\widetilde{\boldsymbol{E}}[\widetilde{\boldsymbol{u}}] - \widetilde{\boldsymbol{E}}_s(\widetilde{\boldsymbol{m}})) : \mathbb{C}[\widetilde{\boldsymbol{E}}[\widetilde{\boldsymbol{u}}] - \widetilde{\boldsymbol{E}}_s(\widetilde{\boldsymbol{m}})] = \mathbb{C}[\widetilde{\boldsymbol{E}}[\widetilde{\boldsymbol{u}}] - \widetilde{\boldsymbol{E}}_s(\widetilde{\boldsymbol{m}})]^2.$$

Here $\widetilde{\boldsymbol{E}}_s(\widetilde{\boldsymbol{m}})$ is the spontaneous strain due to magnetization

$$\widetilde{\boldsymbol{m}} \mapsto \widetilde{\boldsymbol{E}}_s(\widetilde{\boldsymbol{m}}) \in \boldsymbol{M}^{3\times 3}_{sym},$$

where $\boldsymbol{M}^{3\times 3}_{sym}$ denotes the set of symmetric matrices of 3×3 dimension. For cubic materials it's form is

$$\widetilde{\boldsymbol{E}}_s(\widetilde{\boldsymbol{m}}) = \frac{3}{2m_s^2}\begin{bmatrix} \lambda_{100}\,\widetilde{m_1}^2 & \lambda_{111}\,\widetilde{m_1}\widetilde{m_2} & \lambda_{111}\,\widetilde{m_1}\widetilde{m_3} \\ \lambda_{111}\,\widetilde{m_1}\widetilde{m_2} & \frac{3}{2}\lambda_{100}\,\widetilde{m_2}^2 & \lambda_{111}\,\widetilde{m_2}\widetilde{m_3} \\ \lambda_{111}\,\widetilde{m_1}\widetilde{m_3} & \lambda_{111}\,\widetilde{m_2}\widetilde{m_3} & \lambda_{100}\,\widetilde{m_3}^2 \end{bmatrix} - \frac{\lambda_{100}}{2} I$$

where I is the Identity matrix in $\mathbb{R}^{3\times 3}$. The form for $\widetilde{\boldsymbol{E}}_s(\widetilde{\boldsymbol{m}})$ and $\mathbb{C}$ being symmetric positive definite 4^{th} order tensor gives for some $\gamma, \Gamma > 0$,

$$|\widetilde{\boldsymbol{E}}_s(\widetilde{\boldsymbol{m}})| \le K_3 \qquad\qquad \text{as } |\widetilde{\boldsymbol{m}}| = m_s \tag{1.5}$$

$$\gamma|M^2| \ge \mathbb{C}[M]^2 \ge \Gamma|M^2|, \qquad\qquad \forall M \in \boldsymbol{M}^{3\times 3}_{sym}. \tag{1.6}$$

In addition to these, energy due to external force acting on the body in the form of body force or surface traction is included in the general energy. However since these terms are lower order in deformation $\widetilde{\boldsymbol{u}}$, they do not affect the final form of the limit problem. For this investigation, we neglect this term to reduce the length of the computation. Thus the full energy functional for magnetostriction is,

$$\begin{aligned}\mathscr{E}^{\varepsilon}(\widetilde{\boldsymbol{m}},\widetilde{\boldsymbol{u}}) &= E_{exc}+E_{anis}+E_{app}+E_{el}+E_{demag}\\ &= \int_{\Omega_\varepsilon}\left\{d|\nabla^{\boldsymbol{y}}\widetilde{\boldsymbol{m}}|^2+\varphi(\widetilde{\boldsymbol{m}})-\widetilde{\boldsymbol{h}_a}\cdot\widetilde{\boldsymbol{m}}+\frac{\mathbb{C}}{2}[\widetilde{\boldsymbol{E}}[\widetilde{\boldsymbol{u}}]-\widetilde{\boldsymbol{E}}_s(\widetilde{\boldsymbol{m}})]^2\right\}\boldsymbol{dy}+\frac{1}{8\pi}\int_{\mathbb{R}^3}|\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widetilde{\boldsymbol{m}}}|^2\boldsymbol{dy}\end{aligned} \tag{1.7}$$

2 Heuristic Scaling of energy

In § 2.1 and § 2.2, we start with a cylindrical domain with radius ε and length 1, and show how an isotropic linear elastic energy and the magnetostatic energy scale with respect to ε. The linear elastic scaling laws have been know for long in the engineering literature, but a rigorous derivation starting from a three-dimensional linear elastic theory is relatively recent. In § 2.3 we rescale the original energy $\mathscr{E}(\widetilde{\boldsymbol{m}},\widetilde{\boldsymbol{u}})$ defined on a domain Ω_ε depending on ε to a new energy $\mathscr{I}^{\varepsilon}$ set on a fixed domain Ω.

2.1 Linear Elastic Energy

Let $\Theta = B_\varepsilon(0)\times(0,1)$ be a cylindrical domain of radius ε centered at the origin and length 1 with axis aligned along the x_3 axis. Let Y be the Young's Modulus, $A=\pi\varepsilon^2$ is the cross-sectional area, and $I=\frac{\pi}{4}\varepsilon^4$ be the second moment of area of the cross section. Let (u_1,u_2,u_3) be the displacements in (x_1,x_2,x_3) directions. From the engineering literature on rods and beams we know that the extensional energy of a rod along its axis is given as

$$\int_0^1 Y A\,|\partial_3 u_3|^2 = Y\pi\varepsilon^2\int_0^1|\partial_3 u_3|^2 = O(\varepsilon^2), \tag{2.1}$$

where u_3 is the extension of the rod along its axis. From the Euler-Bernoulli model for a beam bending in the direction of the x_1 axis, the bending energy is

$$\int_0^1 Y I\,|\partial_{33}u_1|^2 = Y\frac{\pi\varepsilon^4}{4}\int_0^1|\partial_{33}u_1|^2 = O(\varepsilon^4). \tag{2.2}$$

The different scaling of the two energies with respect to ε suggests to us that a linear elastic isotropic energy of the form

$$\mathscr{W}^{\varepsilon}(\boldsymbol{u}) = \int_\Theta \mu|\boldsymbol{E}(\boldsymbol{u})|^2+\frac{\lambda}{2}|\operatorname{tr}(\boldsymbol{E}(\boldsymbol{u}))|^2\boldsymbol{dx} \tag{2.3}$$

should factor into terms which are of different orders in powers of ε. Using Γ-convergence this factorization into orders of powers of ε has been proven in [Anzellotti et al., 1994]. They have shown that,

$$\mathscr{W}^{\varepsilon}(\boldsymbol{u}) = \varepsilon^2 \mathscr{W}_1(\widehat{u}_3) + \varepsilon^4 \mathscr{W}_2(\widehat{u}_1, \widehat{u}_2, \widehat{u}_4) + \text{ higher order terms} \tag{2.4}$$

where $(\widehat{u}_1, \widehat{u}_2, \widehat{u}_3)(x_3) = \frac{1}{|B_\varepsilon(0)|} \int_{B_\varepsilon(0)} \boldsymbol{u} \boldsymbol{d\sigma}(x_1, x_2)$ is the averaged cross-sectional displacement and $\widehat{u}_4(x_3) = \frac{2}{\varepsilon^2 |B_\varepsilon(0)|} \int_{B_\varepsilon(0)} (x_2 u_1 - x_1 u_2) \boldsymbol{d\sigma}$ gives the torsional component.

2.2 Magnetostatic energy

For an ellipsoidal body it is well known cf. [Maxwell, 1873] that the demagnetization field for a constant magnetization $\boldsymbol{m}$ is,

$$\boldsymbol{h}_{\boldsymbol{m}} = -4\pi \boldsymbol{D}\boldsymbol{m} \qquad\qquad E_{demag} = 2\pi \left(\text{Volume of body} \right) \times \boldsymbol{D}\boldsymbol{m} \cdot \boldsymbol{m} \tag{2.5}$$

where $\boldsymbol{D} \in \mathbb{R}^{3\times3}$ is called demagnetization tensor. $\boldsymbol{D}$ is independent of position $\boldsymbol{x}$, and has trace 1. For non-ellipsoidal bodies supporting a constant magnetization $\boldsymbol{m}$, it still is true that $\boldsymbol{h}_{\boldsymbol{m}} = -4\pi \boldsymbol{D}\boldsymbol{m}$, however the demagnetization tensor (with trace still 1) now depends on position $\boldsymbol{x}$. The magnetostatic energy is now given by $\mathscr{E}_{demag} = \left(\text{Volume of body} \right) \times \boldsymbol{D}^m \boldsymbol{m} \cdot \boldsymbol{m}$, where $\boldsymbol{D}^m$ (known as the magnetometric demagnetization factor) is the volumetric average of $\boldsymbol{D}$. For our cylindrical domain $\Theta = B_\varepsilon(0) \times (0,1)$, $\boldsymbol{D}^m$ is a diagonal matrix with entries

$$D^m_{33} = \frac{8\varepsilon}{3\pi} - \frac{\varepsilon^2}{2} + O(\varepsilon^4), \qquad\qquad D^m_{11} = D^m_{22} = \frac{1}{2} - \frac{4\varepsilon}{3\pi} + \frac{\varepsilon^2}{4} + O(\varepsilon^4).$$

See [Joseph, 1966] for a simple derivation of this result. The demagnetization energy for a constant magnetization $\boldsymbol{m} = (m_1, m_2, m_3)$ is given by

$$\begin{aligned} E_{demag} &= 2\pi^2 \Big[\varepsilon^2 \frac{m_1^2 + m_2^2}{2} - \varepsilon^3 \frac{4}{3\pi} \Big((m_1^2 + m_2^2) - 2m_3^2 \Big) + \frac{\varepsilon^4}{2} \Big(\frac{m_1^2 + m_2^2}{2} - m_3^2 \Big) \Big] \\ &= \pi^2 (m_1^2 + m_2^2)\, \varepsilon^2 + O(\varepsilon^3) + O(\varepsilon^4). \end{aligned} \tag{2.6}$$

Thus, for a cylindrical domain Θ with constant magnetization we can already see the presence of various orders of scales in the magnetostatic and elastic energy. In the following sections we will see that the magnetostatic and elastic energy are the only terms in the energy which appear beyond $O(\varepsilon^2)$.

2.3 Rescaling

In this subsection we rescale the domain Ω_ε depending on a parameter ε to a fixed domain Ω. The space variable in the original domain Ω_ε is either denoted by $\boldsymbol{y}$ or $\boldsymbol{z}$ and in the rescaled domain by $\boldsymbol{x}$. The gradient operator w.r.t. $\boldsymbol{y}$ and $\boldsymbol{z}$ are denoted $\nabla^{\boldsymbol{y}}$ and $\nabla^{\boldsymbol{z}}$ respectively and gradient w.r.t. $\boldsymbol{x}$ is denoted as just ∇. For any vector $\boldsymbol{v}$ taking values in $\mathbb{R}^3$ we will write $\boldsymbol{v} = (v_1, v_2, v_3) = (\boldsymbol{v}_p, v_3)$ where $p = 1,2$ and $\boldsymbol{v}_p$ represents the planar component of $\boldsymbol{v}$. Correspondingly, the gradient operator may be denoted by $\nabla = (\nabla_p, \partial_3)$.

Let $\Omega_\varepsilon := [\,\boldsymbol{y}_p \in \omega_\varepsilon,\, y_3 \in (0,1)]$ be a domain with cross-section $\omega_\varepsilon \subset \mathbb{R}^2$ being any Lipschitz domain in 2-dimensions. While the results of all the subsequent sections hold for any arbitrary cross-section ω_ε, for the sake of simplicity we set ω_ε to be a ball with radius ε in 2-dimensions. We rescale the domain Ω_ε to $\Omega := [\boldsymbol{x}_p \in \omega,\, x_3 \in (0,1)]$ (ω is now the ball with unit radius in 2 dimensions) by the following one-to-one map

$$x_1 = \frac{y_1}{\varepsilon} \qquad\qquad x_2 = \frac{y_2}{\varepsilon} \qquad\qquad x_3 = y_3 \tag{2.7}$$

and the vector fields $\widetilde{\boldsymbol{m}}(\boldsymbol{y})$, $\widetilde{\boldsymbol{u}}(\boldsymbol{y})$ and $\widetilde{\boldsymbol{h}_a}(\boldsymbol{y})$ using the one-to-one map

$$\boldsymbol{m}(\boldsymbol{x}) = \widetilde{\boldsymbol{m}}(\boldsymbol{y}), \qquad \boldsymbol{u}(\boldsymbol{x}) = \widetilde{\boldsymbol{u}}(\boldsymbol{y}), \qquad \boldsymbol{h}_a(\boldsymbol{x}) = \widetilde{\boldsymbol{h}_a}(\boldsymbol{y}), \qquad \boldsymbol{h}^\varepsilon_{\boldsymbol{m}}(\boldsymbol{x}) = \widetilde{\boldsymbol{h}}^\varepsilon_{\widetilde{\boldsymbol{m}}}(\boldsymbol{y}). \tag{2.8}$$

Note that the map $\boldsymbol{m}(\boldsymbol{x}) = \widetilde{\boldsymbol{m}}(\boldsymbol{y})$ being one-to-one, means that we can invert the rescaled magnetization $\boldsymbol{m}(\boldsymbol{x})$ back to the unscaled magnetization $\widetilde{\boldsymbol{m}}(\boldsymbol{y})$. The field $\boldsymbol{h}^\varepsilon_{\boldsymbol{m}}$ is defined as the field which on unscaling, solves the Maxwell's equation for the unscaled magnetization $\widetilde{\boldsymbol{m}}(\boldsymbol{y})$ on Ω_ε. Thus the ε superscript on $\boldsymbol{h}^\varepsilon_{\boldsymbol{m}}$. The gradient operator $\nabla^{\boldsymbol{y}} = (\nabla^{\boldsymbol{y}}_p, \partial^{\boldsymbol{y}}_3)$ operating on $\widetilde{\boldsymbol{m}}(\boldsymbol{y})$ or $\widetilde{\boldsymbol{u}}(\boldsymbol{y})$ correspondingly scales as,

$$\nabla^{\boldsymbol{y}}_p\, \widetilde{\boldsymbol{m}}(\boldsymbol{y}) = \frac{1}{\varepsilon}\nabla_p\, \boldsymbol{m}(\boldsymbol{x}) \qquad\qquad \partial^{\boldsymbol{y}}_3\, \widetilde{\boldsymbol{m}}(\boldsymbol{y}) = \partial_3\, \boldsymbol{m}(\boldsymbol{x}).$$

We define a new field $\boldsymbol{\kappa}^\varepsilon(\boldsymbol{x})$ through the rescaling of the strain,

$$\begin{aligned}
\widetilde{\boldsymbol{E}}[\widetilde{\boldsymbol{u}}](\boldsymbol{y}) &= \frac{1}{2}[\ \nabla^{\boldsymbol{y}}\widetilde{\boldsymbol{u}}(\boldsymbol{y}) + \nabla^{\boldsymbol{y}}\widetilde{\boldsymbol{u}}(\boldsymbol{y})^T\ ] \\
&= \begin{bmatrix} \frac{1}{\varepsilon}\partial_1 u_1 & \frac{1}{2\varepsilon}(\partial_1 u_2 + \partial_2 u_1) & \frac{1}{2}(\frac{1}{\varepsilon}\partial_1 u_3 + \partial_3 u_1) \\ \frac{1}{2\varepsilon}(\partial_1 u_2 + \partial_2 u_1) & \frac{1}{\varepsilon}\partial_2\, u_2 & \frac{1}{2}(\frac{1}{\varepsilon}\partial_2 u_3 + \partial_3 u_2) \\ \frac{1}{2}(\frac{1}{\varepsilon}\partial_1 u_3 + \partial_3 u_1) & \frac{1}{2}(\frac{1}{\varepsilon}\partial_2 u_3 + \partial_3 u_2) & \partial_3 u_3 \end{bmatrix} \\
&=: \boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}](\boldsymbol{x}).
\end{aligned} \tag{2.9}$$

Substituting the above transformations into $\mathscr{E}^\varepsilon(\widetilde{\boldsymbol{m}}, \widetilde{\boldsymbol{u}})$ (cf. (1.7)) we get

$$\begin{aligned}
\mathscr{E}^\varepsilon(\widetilde{\boldsymbol{m}}, \widetilde{\boldsymbol{u}}) = \varepsilon^2 \int_\Omega \Big[\ \frac{d}{\varepsilon^2}|\nabla_p \boldsymbol{m}|^2 + d\,|\partial_3 \boldsymbol{m}|^2 + \varphi(\boldsymbol{m}) - \boldsymbol{h}_a \cdot \boldsymbol{m} + \frac{\mathbb{C}}{2}[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}] - \boldsymbol{E}_s(\boldsymbol{m})]^2 \Big] \boldsymbol{dx} \\
+ \frac{\varepsilon^2}{8\pi}\int_{\mathbb{R}^3} |\boldsymbol{h}^\varepsilon_{\boldsymbol{m}}(\boldsymbol{x})|^2\, \boldsymbol{dx}.
\end{aligned}$$

Dividing by ε^2 we then define the energy $\mathscr{I}^\varepsilon(\boldsymbol{m},\boldsymbol{u}) = \varepsilon^{-2}\mathscr{E}(\widetilde{\boldsymbol{m}},\widetilde{\boldsymbol{u}})$ giving,

$$\mathscr{I}^\varepsilon(\boldsymbol{m},\boldsymbol{u}) = \int_\Omega \left\{\frac{d}{\varepsilon^2}|\nabla_p \boldsymbol{m}|^2 + d\ |\partial_3 \boldsymbol{m}|^2 + \varphi(\boldsymbol{m}) - \boldsymbol{h}_a \cdot \boldsymbol{m} + \frac{\mathbb{C}}{2}[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}] - \boldsymbol{E}_s(\boldsymbol{m})]^2\right\}\boldsymbol{dx} + \mathscr{E}_d^\varepsilon(\boldsymbol{m}).$$

$\mathscr{E}_d^\varepsilon(\boldsymbol{m})$ is defined and bounded by rescaling the standard demagnetization bound (1.3)

$$\mathscr{E}_d^\varepsilon(\boldsymbol{m}) := \frac{1}{8\pi}\int_{\mathbb{R}^3} |\boldsymbol{h}_{\boldsymbol{m}}^\varepsilon(\boldsymbol{x})|^2 \boldsymbol{dx} \le \frac{1}{2}\int_\Omega |\boldsymbol{m}(\boldsymbol{x})|^2 \boldsymbol{dx} = \frac{1}{2}|\Omega| m_s^2. \tag{2.10}$$

We investigate the asymptotic nature of the problem

$$(\mathscr{P}^\varepsilon)\ \ \inf_{\mathscr{A}_\varepsilon} \mathscr{I}^\varepsilon(\boldsymbol{m},\boldsymbol{u}), \quad \mathscr{A}_\varepsilon = \{(\boldsymbol{m},\boldsymbol{u}) \in H^1(\Omega, m_s S^2) \times H^1_\sharp(\Omega,\mathbb{R}^3)\} \tag{2.11}$$

where $H^1_\sharp(\Omega,\mathbb{R}^3) = \left\{\boldsymbol{u}(\boldsymbol{x}) \in H^1(\Omega,\mathbb{R}^3) \mid \boldsymbol{u}(x_1,x_2,0) = \boldsymbol{0},\ \ \forall (x_1,x_2) \in \omega\right\}$ enforces Dirichlet boundary conditions at the base of a cantilever beam. For the subsequent sections we also use the notation $H^1_\sharp(0,1)$ defined as

$$H^1_\sharp((0,1),\mathbb{R}) = \left\{\ \boldsymbol{u}(x_3) \in H^1((0,1),\mathbb{R}^3) \mid \boldsymbol{u}(x_3 = 0) = \boldsymbol{0}\ \right\}. \tag{2.12}$$

In the next section, we will start with a sequence of minimizers $(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$ of $\mathscr{I}^\varepsilon(\boldsymbol{m},\boldsymbol{u})$ and show that we can extract a subsequence whose limit relates to the minimizers of a simpler lower dimensional problem $\mathscr{I}^o$.

3 First variational limiting problem

Let $(\widehat{\cdot})$ denote the cross-sectional average of any scalar/vector, i.e. for any field $\boldsymbol{a}(\boldsymbol{x})$ set

$$\widehat{\boldsymbol{a}}(x_3) = \fint_\omega \boldsymbol{a}(\boldsymbol{x}_p, x_3)\boldsymbol{dx}_p. \tag{3.1}$$

For ε fixed, let $(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$ be a minimizer of $\mathscr{I}^\varepsilon(\boldsymbol{m},\boldsymbol{u})$. We look at the behavior of $\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon,\boldsymbol{u}^\varepsilon)$ as $\varepsilon \to 0$. For that, we will first show that $\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon,\boldsymbol{u}^\varepsilon)$ is bounded above and below independent of ε. Then we will show that from the sequence $(\boldsymbol{m}^\varepsilon,\boldsymbol{u}^\varepsilon)$, we can extract a subsequence(unrelabeled) such that $(\boldsymbol{m}^\varepsilon, \widehat{u_3}^\varepsilon)$ on the subsequence converges weakly to some $(\boldsymbol{m}^o, v)$ in an appropriate space. This convergence will be improved to strong, and the limit

$(\boldsymbol{m}^o, v)$ will be shown to minimize a new functional $\mathscr{I}^o$ in Theorem 3.1 .

3.1 Boundedness of $\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$

For an upper bound on $\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$ we compare its energy with a test function $(\boldsymbol{m}, \boldsymbol{0})$ with $\boldsymbol{m}$ any constant vector on $m_s S^2$ and $\boldsymbol{u} = 0$ to get,

$$\begin{aligned}\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) \le \mathscr{I}^\varepsilon(\boldsymbol{m}, \boldsymbol{0}) &= \int_\Omega \left[\phi(\boldsymbol{m}) + \frac{1}{2}\mathbb{C}[\boldsymbol{E}_s(\boldsymbol{m})]^2 - \boldsymbol{h}_a \cdot \boldsymbol{m}\right] \boldsymbol{dx} + \mathscr{E}_d^\varepsilon(\boldsymbol{m}) \\ &\le K_4 + \frac{m_s^2}{2}|\Omega|,\end{aligned} \tag{3.2}$$

where the anisotropy, elastic, Zeeman and magnetostatic terms are bounded using (1.2), (1.5), (1.6), (1.4) and (2.10). Positivity of the rescaled exchange, anisotropy, demag and elastic energy and (1.4) gives the lower bound,

$$\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) \ge -\int_\Omega \boldsymbol{h}_a \cdot \boldsymbol{m}\, \boldsymbol{dx} \ge -K_2.$$

Based on (3.1) set the cross-sectional averages of $\boldsymbol{u}^\varepsilon$ and $\boldsymbol{m}^\varepsilon$ as $\widehat{\boldsymbol{u}}^\varepsilon(x_3)$ and $\widehat{\boldsymbol{m}}^\varepsilon(x_3)$.

Proposition 3.1. *Given $\widehat{\boldsymbol{m}}^\varepsilon \in H^1(\Omega)$ and $\widehat{\boldsymbol{u}}^\varepsilon \in H^1_\sharp(\Omega)$, we have the following,*

$$\begin{aligned}\|\partial_3 \widehat{\boldsymbol{u}}^\varepsilon\|^2_{L^2(\Omega)} = |\omega|\ \|\partial_3 \widehat{\boldsymbol{u}}^\varepsilon\|^2_{L^2(0,1)} \le \|\partial_3 \boldsymbol{u}^\varepsilon\|^2_{L^2(\Omega)},\\ \|\partial_3 \widehat{\boldsymbol{m}}^\varepsilon\|^2_{L^2(\Omega)} = |\omega|\ \|\partial_3 \widehat{\boldsymbol{m}}^\varepsilon\|^2_{L^2(0,1)} \le \|\partial_3 \boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega)},\end{aligned}$$

and for $i = \{1,2,3\}$

$$\|\widehat{u_i}^\varepsilon\|^2_{L^2(\Omega)} \le K_5 \|\partial_3 u_i^\varepsilon\|^2_{L^2(\Omega)}.$$

Proof. For $i \in \{1,2,3\}$, using Jensen's inequality

$$\begin{aligned}\int_0^1 |\partial_3 \widehat{u_i}^\varepsilon|^2 dx_3 = \int_0^1 \left|\partial_3\left\{\frac{1}{|\omega|}\int_\omega u_i^\varepsilon \boldsymbol{d\sigma}\right\}\right|^2 dx_3 = \int_0^1 \frac{1}{|\omega|^2}\left|\int_\omega \partial_3 u_i^\varepsilon \boldsymbol{d\sigma}\right|^2 dx_3 \\ \le \int_0^1 \frac{1}{|\omega|}\int_\omega |\partial_3 u_i^\varepsilon|^2 \boldsymbol{d\sigma}\, dx_3 = \frac{1}{|\omega|}\int_\Omega |\partial_3 u_i^\varepsilon|^2 \boldsymbol{dx}.\end{aligned}$$

Integrating over ω and summing over i gives us the first result. Similar calculation with $\boldsymbol{m}^\varepsilon$ replacing $\boldsymbol{u}^\varepsilon$ gives the second result. Noting the Dirichlet Boundary conditions on $\boldsymbol{u}^\varepsilon$ at $x_3 = 0$ we get using 1-D Poincaré inequality over $(0,1)$,

$$\int_0^1 |\widehat{u_i}^\varepsilon|^2 dx_3 \le K_1 \int_0^1 |\partial_3 \widehat{u_i}^\varepsilon|^2 dx_3 \le K_1 \int_\Omega \frac{1}{|\omega|}|\partial_3 u_i^\varepsilon|^2 \boldsymbol{dx}$$

where K_1 is the Poincaré constant. Integrating over ω gives the result. □

3.2 Weak compactness of minimizers $(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$ as $\varepsilon \to 0$

The upper and lower bounds on $\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$ gives,

$$K_5 > \int_\Omega \frac{d}{\varepsilon^2}|\nabla_p \boldsymbol{m}^\varepsilon|^2 + d \mid \partial_3 \boldsymbol{m}^\varepsilon|^2 \geq d \int_\Omega |\nabla \boldsymbol{m}^\varepsilon|^2. \tag{3.3}$$

Thus we have for some unrelabelled subsequence

$$\|\nabla_p \boldsymbol{m}^\varepsilon(\boldsymbol{x})\|^2_{L^2(\Omega)} \leq \frac{K_5}{d}\varepsilon^2, \qquad \boldsymbol{m}^\varepsilon \to \boldsymbol{m}^o \text{ in } L^2(\Omega), \qquad \nabla \boldsymbol{m}^\varepsilon \rightharpoonup \nabla \boldsymbol{m}^o \text{ in } L^2(\Omega). \tag{3.4}$$

By the weak convergence of $\nabla \boldsymbol{m}^\varepsilon(\boldsymbol{x})$ to $\nabla \boldsymbol{m}(\boldsymbol{x})$ and the lower semi-continuity of norm operator $\|(\cdot)\|_{L^2(\Omega)}$ w.r.t. weak convergence we have,

$$\begin{aligned} &\|\nabla_p \boldsymbol{m}^o(\boldsymbol{x})\|_{L^2(\Omega)} \leq \liminf_{\varepsilon\to 0}\|\nabla_p \boldsymbol{m}^\varepsilon(\boldsymbol{x})\|_{L^2(\Omega)} \leq \liminf_{\varepsilon \to 0}\sqrt{\frac{K_5}{d}}\varepsilon = 0. \\ &\lim_{\varepsilon\to 0} \boldsymbol{m}^\varepsilon(\boldsymbol{x}) = \boldsymbol{m}^o(\boldsymbol{x}) = \boldsymbol{m}^o(x_3) \qquad \text{in } L^2(\Omega). \end{aligned} \tag{3.5}$$

The strong convergence of $\boldsymbol{m}^\varepsilon$ to $\boldsymbol{m}^o$ in $L^2(\Omega)$ also gives pointwise convergence $a.e.$ for a (unrelabelled) subsequence. The cross-sectional average of this subsequence $\widehat{\boldsymbol{m}}^\varepsilon(x_3) = \fint_\omega \boldsymbol{m}^\varepsilon(\boldsymbol{x})\boldsymbol{dx}_p$ converges pointwise $a.e.$ to $\fint_\omega \boldsymbol{m}^o(x_3)\boldsymbol{dx}_p = \boldsymbol{m}^o(x_3)$. Since from Jensen's inequality $|\widehat{\boldsymbol{m}}^\varepsilon(x_3)| \leq |\boldsymbol{m}^\varepsilon| = m_s$, we get using dominated convergence theorem for this unrelabeled subsequence

$$\lim_{\varepsilon\to 0} \widehat{\boldsymbol{m}}^\varepsilon(x_3) = \boldsymbol{m}^o(x_3) \qquad \text{in } L^2((0,1)). \tag{3.6}$$

Also integrating the above across the cross-section Using Young's inequality, positive definiteness of $\mathbb{C}$ in (1.5), and (1.6) we have

$$\begin{aligned} \int_\Omega |\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^\varepsilon]|^2 &= \int_\Omega |\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^\varepsilon] - \boldsymbol{E}_s(\boldsymbol{m}^\varepsilon) + \boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)|^2 \leq 2\int_\Omega |\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^\varepsilon] - \boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)|^2 + |\boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)|^2 \\ &\leq \frac{2}{\gamma}\mathbb{C}\int_\Omega |\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^\varepsilon] - \boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)|^2 + 2\int_\Omega |\boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)|^2 \leq K_6 < \infty. \end{aligned}$$

Combining this with Proposition 3.1 we have

$$\int_\Omega |\widehat{u_3}^\varepsilon|^2 dx_3 \leq K_1 \int_\Omega |\partial_3 \widehat{u_3}^\varepsilon|^2 dx_3 \leq K_1 \int_\Omega |\partial_3 u_3^\varepsilon|^2 \leq K_1 \int_\Omega |\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^\varepsilon]|^2 < K_6.$$

Thus $\|\widehat{u_3}^\varepsilon\|_{H^1(0,1)} \leq \infty$ and due to Dirichlet conditions on $\boldsymbol{u}^\varepsilon$ we get, $\widehat{u_3}^\varepsilon \in H^1_\sharp(0,1)$ with $H^1_\sharp(0,1) = \{v \in H^1(0,1) : v(0) = 0\}$. For an unrelabeled subsequence,

$$\widehat{u_3}^\varepsilon(x_3) \to v^o(x_3) \text{ in } L^2(0,1), \qquad\qquad \partial_3\widehat{u_3}^\varepsilon(x_3) \rightharpoonup \partial_3 v^o(x_3) \text{ in } L^2(0,1) \tag{3.7}$$

Already from the fact that $\boldsymbol{m}^o$ and v^o depends only on the x_3 space variable the 1-D nature of the limit problem becomes evident. Equation (A.20) gives

$$\frac{1}{8\pi}\int_{\mathbb{R}^3}|\boldsymbol{h}^\varepsilon_{\boldsymbol{m}}(\boldsymbol{x})|^2\boldsymbol{dx}-J_0(\widehat{\boldsymbol{m}}^\varepsilon)=\frac{1}{8\pi}\int_{\mathbb{R}^3}|\boldsymbol{h}^\varepsilon_{\boldsymbol{m}}(\boldsymbol{x})|^2\boldsymbol{dx}-\pi|\omega|\int_0^1\left|\widehat{\boldsymbol{m}}^\varepsilon_p(x_3)\right|dx_3=O(\varepsilon).$$

which implies on using (3.6) nd taking $\lim_{\varepsilon\to 0}$,

$$\lim_{\varepsilon\to 0}\frac{1}{8\pi}\int_{\mathbb{R}^3}|\boldsymbol{h}^\varepsilon_{\boldsymbol{m}}(\boldsymbol{x})|^2\boldsymbol{dx}=\lim_{\varepsilon\to 0}\pi|\omega|\int_0^1\left|\widehat{\boldsymbol{m}}^\varepsilon_p(x_3)\right|dx_3=\pi|\omega|\int_0^1|\boldsymbol{m}^o_p(x_3)|^2dx_3. \tag{3.8}$$

3.3 Strong compactness of $(\boldsymbol{m}^\varepsilon,\boldsymbol{u}^\varepsilon)$ and variational problem

$$\text{Set } f_0(s):=\min\left[\mathbb{C}[\boldsymbol{E}]:\boldsymbol{E}\ ;\ \boldsymbol{E}\in\boldsymbol{M}^{3\times3}_{sym}\text{ and }E_{33}=s\right]. \tag{3.9}$$

Note that f_0 defined above in (3.9) can be evaluated as

$$f_0(s)=(c_{11}|s|^2-2\sigma c_{12})|s|^2:=Y|s|^2 \tag{3.10}$$

with $\sigma=\dfrac{c_{12}}{c_{11}+c_{12}}$ being Poisson's ratio and $Y=(c_{11}-2\sigma c_{12})$ is the Young's modulus. We then state the main result of this section.

Theorem 3.1. *There exists a subsequence* $(\boldsymbol{m}^\varepsilon,\boldsymbol{u}^\varepsilon)$ *not relabeled such that* $\boldsymbol{m}^\varepsilon\to\boldsymbol{m}^o$, $\widehat{u_3}^\varepsilon\to v^o$ *strongly in* $H^1(\Omega,\mathbb{R}^3)\times H^1_\sharp(\Omega,\mathbb{R})$ *and* $(\boldsymbol{m}^o,v^o)$ *minimizes* $\mathscr{I}^o(\boldsymbol{m},v)$ *in* $\mathscr{A}_o=\{(\boldsymbol{m}(x_3),v(x_3))\in H^1((0,1),\mathbb{R}^3)\times H^1_\sharp((0,1),\mathbb{R})\}$ *where* $\mathscr{I}^o(\boldsymbol{m},v)$ *is defined as*

$$\mathscr{I}^o(\boldsymbol{m},v)=\int_0^1 d|\partial_3\boldsymbol{m}|^2+\varphi(\boldsymbol{m})+\pi|\boldsymbol{m}_p|^2+\frac{1}{2}f_0(\partial_3 v-E_{s_{33}}(\boldsymbol{m}))-\boldsymbol{h}_a\cdot\boldsymbol{m}. \tag{3.11}$$

Proof. Recall $\boldsymbol{h}^\varepsilon_{\boldsymbol{m}^o}$ be the rescaled field of $\widetilde{\boldsymbol{h}}^\varepsilon_{\widetilde{\boldsymbol{m}}^o}$, where $\widetilde{\boldsymbol{h}}^\varepsilon_{\widetilde{\boldsymbol{m}}^o}$ solves Maxwell's equation for $\widetilde{\boldsymbol{m}}^o$ on Ω_ε, where $\widetilde{\boldsymbol{m}}^o$ is defined from $\boldsymbol{m}^o$ through the equation (2.8). Using triangle inequality, $\left|\|\boldsymbol{h}^\varepsilon_{\boldsymbol{m}^\varepsilon}\|_{L^2(\mathbb{R}^3)}-\|\boldsymbol{h}^\varepsilon_{\boldsymbol{m}^o}\|_{L^2(\mathbb{R}^3)}\right|\le\|\boldsymbol{h}^\varepsilon_{\boldsymbol{m}^\varepsilon}-\boldsymbol{h}^\varepsilon_{\boldsymbol{m}^o}\|_{L^2(\mathbb{R}^3)}$. Recall $\mathscr{E}^\varepsilon_d(\boldsymbol{m})=\frac{1}{8\pi}\|\boldsymbol{h}^\varepsilon_{\boldsymbol{m}}\|^2_{L^2(\mathbb{R}^3)}$. Then

$$\begin{aligned}\left|\mathscr{E}^\varepsilon_d(\boldsymbol{m}^\varepsilon)-\mathscr{E}^\varepsilon_d(\boldsymbol{m}^o)\right|&=\left|\|\boldsymbol{h}^\varepsilon_{\boldsymbol{m}^\varepsilon}\|_{L^2(\mathbb{R}^3)}-\|\boldsymbol{h}^\varepsilon_{\boldsymbol{m}^o}\|_{L^2(\mathbb{R}^3)}\right|\ \left|\|\boldsymbol{h}^\varepsilon_{\boldsymbol{m}}\|_{L^2(\mathbb{R}^3)}+\|\boldsymbol{h}^o_{\boldsymbol{m}}\|_{L^2(\mathbb{R}^3)}\right|\\&\le\|\boldsymbol{h}^\varepsilon_{\boldsymbol{m}}-\boldsymbol{h}^o_{\boldsymbol{m}}\|_{L^2(\mathbb{R}^3)}\left|\|\boldsymbol{h}^\varepsilon_{\boldsymbol{m}}\|_{L^2(\mathbb{R}^3)}+\|\boldsymbol{h}^o_{\boldsymbol{m}}\|_{L^2(\mathbb{R}^3)}\right|\\&\le\frac{1}{\sqrt{2}}\|\boldsymbol{m}^\varepsilon-\boldsymbol{m}^o\|_{L^2(\Omega)}\frac{2}{\sqrt{2}}|\Omega|^2m_s^2=K\|\boldsymbol{m}^\varepsilon-\boldsymbol{m}^o\|_{L^2(\Omega)}\end{aligned} \tag{3.12}$$

where we have used the rescaled demag bound (2.10), $\|\boldsymbol{h}^\varepsilon_{\boldsymbol{m}}-\boldsymbol{h}^o_{\boldsymbol{m}}\|_{L^2(\mathbb{R}^3)}\le\frac{1}{\sqrt{2}}\|\boldsymbol{m}^\varepsilon-\boldsymbol{m}^o\|_{L^2(\Omega)}$ and similarly for the other terms. Using (3.4) to get $\boldsymbol{m}^\varepsilon(\boldsymbol{x})\to\boldsymbol{m}^o(x_3)$ strongly in $L^2(\Omega)$

$$\lim_{\varepsilon\to 0}\left|\mathscr{E}^\varepsilon_d(\boldsymbol{m}^\varepsilon)-\mathscr{E}^\varepsilon_d(\boldsymbol{m}^o)\right|\le\lim_{\varepsilon\to 0}K\|\boldsymbol{m}^\varepsilon-\boldsymbol{m}^o\|_{L^2(\Omega)}=0. \tag{3.13}$$

Comparing energy of $\mathscr{I}^\varepsilon$ at its minimizer $(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$ with the test function $(\boldsymbol{m}^o, \boldsymbol{u}^\varepsilon)$ we get $\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) \le \mathscr{I}^\varepsilon(\boldsymbol{m}^o, \boldsymbol{u}^\varepsilon)$ which gives

$$\int_\Omega \Big[\frac{d}{\varepsilon^2}|\nabla_p \boldsymbol{m}^\varepsilon|^2 + d\,|\partial_3 \boldsymbol{m}^\varepsilon|^2 + \varphi(\boldsymbol{m}^\varepsilon) - \boldsymbol{h}_a \cdot \boldsymbol{m}^\varepsilon + \frac{\mathbb{C}}{2}[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^\varepsilon] - \boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)]^2 \Big] \boldsymbol{dx} + \mathscr{E}_d^\varepsilon(\boldsymbol{m}^\varepsilon)$$
$$\le \int_\Omega \Big[d\,|\partial_3 \boldsymbol{m}^o|^2 + \varphi(\boldsymbol{m}^o) - \boldsymbol{h}_a \cdot \boldsymbol{m}^o + \frac{\mathbb{C}}{2}[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^\varepsilon] - \boldsymbol{E}_s(\boldsymbol{m}^o)]^2 \Big] + \mathscr{E}_d^\varepsilon(\boldsymbol{m}^o).$$

Taking lim-sup of both sides w.r.t. ε, canceling common terms, using equation (3.13) and noting that $\boldsymbol{m}^\varepsilon(\boldsymbol{x}) \to \boldsymbol{m}^o(x_3)$ strongly in $L^2(\Omega)$, we can simplify the above equation to get

$$\limsup_{\varepsilon\to 0} \int_\Omega \Big[\frac{d}{\varepsilon^2}|\nabla_p \boldsymbol{m}^\varepsilon|^2 + d\,|\partial_3 \boldsymbol{m}^\varepsilon|^2 \Big] \le \int_\Omega d\,|\partial_3 \boldsymbol{m}^o|^2 \boldsymbol{dx}.$$

But weak convergence of $\nabla \boldsymbol{m}^\varepsilon$ in (3.4) implies $\liminf_{\varepsilon\to 0} \int_\Omega |\partial_3 \boldsymbol{m}^o|^2 \le \int_\Omega |\partial_3 \boldsymbol{m}^\varepsilon|^2$ which combined with above $\limsup$ gives the strong convergence,

$$\partial_3 \boldsymbol{m}^\varepsilon \to \partial_3 \boldsymbol{m}^o \text{ in } L^2(\Omega) \qquad \frac{1}{\varepsilon}\nabla_p \boldsymbol{m}^\varepsilon \to \boldsymbol{0} \text{ in } L^2(\Omega). \tag{3.14}$$

Now we show strong convergence of the elastic terms. Set $s^\varepsilon(\boldsymbol{x}) := \partial_3 u_3^\varepsilon(\boldsymbol{x}) - E_{s_{33}}(\boldsymbol{m}^o)$. Then from (3.1), $\widehat{s}^{\,\varepsilon}(x_3) = [\partial_3 \widehat{u_3^\varepsilon - E_{s_{33}}}(\boldsymbol{m}^o)](x_3) = \widehat{\partial_3 u_3^\varepsilon}(x_3) - E_{s_{33}}(\boldsymbol{m}^o) = \partial_3 \widehat{u_3}^\varepsilon(x_3) - E_{s_{33}}(\boldsymbol{m}^o)$ where we have used the fact that $\boldsymbol{m}^o = \boldsymbol{m}^o(x_3)$. Then using definition of $f_0(s) = Y|s|^2$ from (3.9) and Jensen's inequality we get

$$\int_\Omega f_0(\partial_3 \widehat{u_3}^\varepsilon - E_{s_{33}}(\boldsymbol{m}^o)) \boldsymbol{dx} = \int_0^1 \int_\omega Y|\widehat{s}^{\,\varepsilon}|^2 \boldsymbol{dx} \le \int_0^1 Y \Big[\int_\omega |s^\varepsilon|^2 \boldsymbol{d\sigma} \Big] dx_3 = \int_\Omega f_0(s^\varepsilon(\boldsymbol{x})) \boldsymbol{dx}$$
$$\le \int_\Omega \mathbb{C}[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^\varepsilon] - \boldsymbol{E}_s(\boldsymbol{m}^o)]^2 \boldsymbol{dx}. \tag{3.15}$$

To improve the convergence in (3.7) we need to compare the energy $\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$ with some test function based on $\boldsymbol{m}^o$ and v^o. But the lack of regularity of $v \in H^1(0,1)$ requires a mollification procedure. Let $v^h(x_3) \in \mathscr{D}(0,1)$ and $v^h(x_3) \to v^o(x_3)$ in $H^1(0,1)$ as $h \to 0$. Set $s^h(x_3) := (\partial_3 v^h(x_3) - E_{s_{33}}(\boldsymbol{m}^o))$. Note $\lim_{h\to 0} s^h = (\partial_3 v^o(x_3) - E_{s_{33}}(\boldsymbol{m}^o))$. Define

$$\boldsymbol{v}_h^\varepsilon(\boldsymbol{x}) := \begin{bmatrix} \varepsilon E_{s_{11}}(\boldsymbol{m}^o)x_1 + 2E_{s_{13}}(\boldsymbol{m}^o)x_3 + \varepsilon E_{s_{12}}(\boldsymbol{m}^o)x_2 - \varepsilon\sigma\, s^h(x_3)x_1 \\ \varepsilon E_{s_{22}}(\boldsymbol{m}^o)x_2 + 2E_{s_{23}}(\boldsymbol{m}^o)x_3 + \varepsilon E_{s_{12}}(\boldsymbol{m}^o)x_1 - \varepsilon\sigma\, s^h(x_3)x_2 \\ v^h(x_3) \end{bmatrix}.$$

For $\boldsymbol{v}_h^\varepsilon$ defined above, $\boldsymbol{\kappa}^\varepsilon[\boldsymbol{v}_h^\varepsilon] - \boldsymbol{E}_s(\boldsymbol{m}^\varepsilon) = \boldsymbol{E}_s(\boldsymbol{m}^o) - \boldsymbol{E}_s(\boldsymbol{m}^\varepsilon) + \boldsymbol{\alpha}^h(x_3)$ where

$$\boldsymbol{\alpha}^h(x_3) = \begin{bmatrix} -\sigma s^h(x_3) & 0 & -\varepsilon\sigma\partial_3 s^h(x_3)x_1 \\ 0 & -\sigma s^h(x_3) & -\varepsilon\sigma\partial_3 s^h(x_3)x_2 \\ -\varepsilon\sigma\partial_3 s^h(x_3)x_1 & -\varepsilon\sigma\partial_3 s^h(x_3)x_2 & s^h(x_3) \end{bmatrix},$$

and a straight forward computation gives

$$\int_\Omega \mathbb{C}[\boldsymbol{\alpha}^h]^2 \boldsymbol{dx} = \int_\Omega f_0(s^h)\boldsymbol{dx} + \varepsilon^2 \int_\Omega c_{44}\,|\,\sigma\,\partial_3 s^h\,|^2(x_1^2 + x_2^2)\boldsymbol{dx}. \tag{3.16}$$

Then comparing energy of the test function $(\boldsymbol{m}^\varepsilon, \boldsymbol{v}_h^\varepsilon)$ with $\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon$ gives

$$\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) \le \mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{v}_h^\varepsilon).$$

First we cancel like terms on both sides in the above equation. Than fixing h and taking limsup of both sides w.r.t. ε, using definition of f_0 from (3.9) and strong convergence in (3.14), (3.15) and (3.16) we get,

$$\begin{aligned}
\limsup_{\varepsilon\to 0} \int_\Omega \frac{\mathbb{C}}{2}[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^\varepsilon] - \boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)]^2 \boldsymbol{dx} &\le \limsup_{\varepsilon\to 0} \int_\Omega \frac{\mathbb{C}}{2}[\boldsymbol{\kappa}^\varepsilon(\boldsymbol{v}_h^\varepsilon) - \boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)]^2 \boldsymbol{dx} \\
&\le \limsup_{\varepsilon\to 0} \int_\Omega \frac{\mathbb{C}}{2}[\boldsymbol{\alpha}^h + \boldsymbol{E}_s(\boldsymbol{m}^o) - \boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)]^2 \boldsymbol{dx} \\
&= \limsup_{\varepsilon\to 0} \int_\Omega \frac{\mathbb{C}}{2}[\boldsymbol{\alpha}^h]^2 \boldsymbol{dx} = \int_\Omega \frac{1}{2} f_0(s^h)\boldsymbol{dx}.
\end{aligned}$$

Now taking $\lim_{h\to 0}$ of L.H.S. and noting that $\lim_{h\to 0} s^h = (\partial_3 v^o(x_3) - E_{s_{33}}(\boldsymbol{m}^o))$ gives

$$\limsup_{\varepsilon\to 0} \int_\Omega \frac{\mathbb{C}}{2}[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^\varepsilon] - \boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)]^2 \boldsymbol{dx} \le \int_\Omega \frac{1}{2} f_0(\partial_3 v^o(x_3) - E_{s_{33}}(\boldsymbol{m}^o))\boldsymbol{dx}. \tag{3.17}$$

Taking $\limsup_{\varepsilon\to 0}$ of (3.15) and combining with equation (3.17) above gives

$$\limsup_{\varepsilon\to 0} \int_\Omega f_0(\partial_3 \widehat{u_3}^\varepsilon(x_3) - E_{s_{33}}(\boldsymbol{m}^o))\boldsymbol{dx} \le \int_\Omega \frac{1}{2} f_0(\partial_3 v^o(x_3) - E_{s_{33}}(\boldsymbol{m}^o))\boldsymbol{dx}. \tag{3.18}$$

But from (3.7) we have $\partial_3 \widehat{u_3}^\varepsilon(x_3) - E_{s_{33}}(\boldsymbol{m}^o) \rightharpoonup \partial_3 v^o(x_3) - E_{s_{33}}(\boldsymbol{m}^o)$ in $L^2((0,1))$ which gives

$$\int_\Omega \frac{1}{2} f_0(\partial_3 v^o(x_3) - E_{s_{33}}(\boldsymbol{m}^o))\boldsymbol{dx} \le \liminf_{\varepsilon\to 0} \int_\Omega f_0(\partial_3 \widehat{u}^\varepsilon(x_3) - E_{s_{33}}(\boldsymbol{m}^o))\boldsymbol{dx}. \tag{3.19}$$

Thus (3.18) and (3.19) together gives the strong convergence,

$$\lim_{\varepsilon\to 0} \widehat{u_3}^\varepsilon \to v^o \text{ in } H^1(0,1), \tag{3.20a}$$

$$\lim_{\varepsilon\to 0} \int_\Omega \frac{\mathbb{C}}{2}[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^\varepsilon] - \boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)]^2 \boldsymbol{dx} \to \int_\Omega \frac{1}{2} f_0(\partial_3 v^o - E_{s_{33}}(\boldsymbol{m}^o))\boldsymbol{dx}. \tag{3.20b}$$

Finally its easy to see that the strong convergence from (3.14) and (3.20) together gives,

$$\lim_{\varepsilon\to 0} \mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) = |\omega| \mathscr{I}^o(\boldsymbol{m}^o, v^o). \qquad \square$$

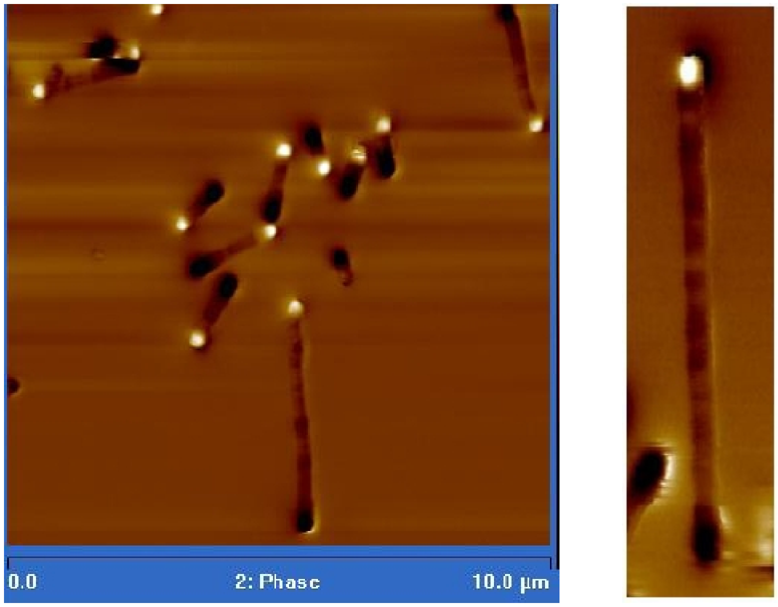

Figure 2: Left: MFM scan for several Galfenol nanowires, Right: Magnified scan of single nanowire, scale of the wires shown in bottom of left figure (Scans courtesy of Downey [Downey, 2008]).

3.4 Minimization of limit problem

The minimization of $\mathscr{I}^o(\boldsymbol{m}, v)$ is a substantially simpler problem than the original one. One can see that if the applied field is a constant over the domain, the terms $\varphi(\boldsymbol{m}) + \pi|\boldsymbol{m}_p|^2 - \boldsymbol{h}_a \cdot \boldsymbol{m}$ behaves like an "effective anisotropy". If this is minimized over constant vector $\boldsymbol{m} \in m_s S^2$ to give $\boldsymbol{m}^o$, then its easy to see that $(\boldsymbol{m}^o, E_{s_{33}}(\boldsymbol{m}^o)x_3)$ minimizes $\mathscr{I}^o(\boldsymbol{m}, v)$.

For a large class of ferromagnetic materials, the largest energy in the "effective anisotropy" for small applied fields is the demagnetization term $\pi|\boldsymbol{m}_p|^2$ which finds its minimum if $\boldsymbol{m}^o$ is an axial magnetization $(0, 0, m_s)$. In particular for our nanowires of Galfenol this is true. Experimentally produced nanowires of Galfenol of 30-100 nanometer diameter shows strong alignment of magnetization along the axis in the absence of applied fields and needs large applied fields in transverse direction to alter this state. Experimental verification of these results for Galfenol wires can be seen from Magnetic Force Microscopy (MFM) scans in Figures 2 taken from [Downey, 2008].

These scans are done for wires with 100 nanometer diameter and <110> crystallographic orientation with no applied field. For cubic anisotropy, <110> is a local minimum of the anisotropy energy and gives zero magnetostatic energy contribution making it a global minimum of the "effective anisotropy". The uniformity of the scan along the wire length depicts a uniform state of magnetization and the bright and dark spots at the two ends are interpreted to be the field lines due to an axial magnetization producing net positive and negative poles at the ends.

For the following sections we assume that field $\boldsymbol{h}_a$ is fixed. This assumption simplifies the presentation in the following sections without effecting the main presentation of the asymptotic limiting problem. Based on aforementioned remarks let us set

$$P_0 = \int_\Omega \varphi(\boldsymbol{m}^o) + \pi|\boldsymbol{m}_p^o|^2 - \boldsymbol{h}_a \cdot \boldsymbol{m}^o \tag{3.21}$$

where $\boldsymbol{m}^o$ minimizes $\varphi(\boldsymbol{m}) + \pi|\boldsymbol{m}_p|^2 - \boldsymbol{h}_a \cdot \boldsymbol{m}$ in $m_s S^2$. Then $(\boldsymbol{m}^o, E_{s_{33}}(\boldsymbol{m}^o)x_3)$ minimizes $\mathscr{I}^o$. Setting

$$\boldsymbol{u}^o(\boldsymbol{x}) := \begin{bmatrix} \varepsilon E_{s_{11}}(\boldsymbol{m}^o)x_1 + 2E_{s_{13}}(\boldsymbol{m}^o)x_3 + \varepsilon E_{s_{12}}(\boldsymbol{m}^o)x_2 \\ \varepsilon E_{s_{22}}(\boldsymbol{m}^o)x_2 + 2E_{s_{23}}(\boldsymbol{m}^o)x_3 + \varepsilon E_{s_{12}}(\boldsymbol{m}^o)x_1 \\ E_{s_{33}}(\boldsymbol{m}^o)x_3 \end{bmatrix}.$$

Then it is easy to check using (2.6) that $\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^o] = \boldsymbol{E}_s(\boldsymbol{m}^o)$ and

$$\int_\Omega \frac{\mathbb{C}}{2}[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^o] - \boldsymbol{E}_s(\boldsymbol{m}^o)]^2 \, \boldsymbol{dx} = \int_\Omega \frac{1}{2} f_0(\partial_3 v^o - E_{s_{33}}(\boldsymbol{m}^o)) \, \boldsymbol{dx} = 0 \tag{3.22a}$$

$$\mathscr{I}^\varepsilon(\boldsymbol{m}^o, \boldsymbol{u}^o) = P_0 + \mathscr{E}_d^\varepsilon(\boldsymbol{m}^o) - \pi|\boldsymbol{m}_p^o|^2 = P_0 - \big(\varepsilon\frac{8\pi}{3} - \varepsilon^2\frac{\pi^2}{2}\big)\big[\,|\boldsymbol{m}_p^o|^2 - 2|m_3^0|^2\big] \tag{3.22b}$$

$$|\omega| \inf_{\mathscr{A}_o} \mathscr{I}^o(\boldsymbol{m}, v) = P_0. \tag{3.22c}$$

4 Second order variational limit problem

§ 3 gives a rigorous derivation of the first order variational approximation $\mathscr{I}^o(\boldsymbol{m}, v)$ in the sense that for a sequence of minimizers $(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$ of $\mathscr{I}^\varepsilon(\boldsymbol{m}, \boldsymbol{u})$, $\lim_{\varepsilon\to 0} \mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) = |\omega|\mathscr{I}^o(\boldsymbol{m}^o, v^o) + o(\varepsilon)$ with $(\boldsymbol{m}^o, v^o)$ minimizing $\mathscr{I}^o(\boldsymbol{m}, v)$ in an appropriate space. Correctors to this approximation come up as higher order theories which involve an expansion of the $o(\varepsilon)$ term. These higher terms can be understood as an asymptotic Γ- series of variational problems in the sense of [Anzellotti and Baldo, 1993].

With this in mind we define $\mathscr{I}_1^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) := \varepsilon^{-1}(\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) - P_0)$. We look at the limit minimization problem corresponding to $\mathscr{I}_1^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$. For this we first show that $\mathscr{I}_1^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$ is bounded above and below independently of ε so that its limit $\varepsilon \to 0$ makes sense. We then show that a limit exists as $\varepsilon \to 0$ for the quantity $\mathscr{I}_1^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$. Note that

$$\begin{aligned}\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon,\boldsymbol{u}^\varepsilon) - P_0 &= \Big[\int_\Omega \frac{d}{\varepsilon^2}|\nabla_p \boldsymbol{m}^\varepsilon|^2\Big] + \Big[\mathscr{E}_d^\varepsilon(\boldsymbol{m}^\varepsilon) - \int_\Omega \pi|\boldsymbol{m}_p^\varepsilon|^2\Big] + \Big[\int_\Omega \Big\{ d\,|\partial_3 \boldsymbol{m}^\varepsilon|^2 \\ &\qquad + \varphi(\boldsymbol{m}^\varepsilon) - \boldsymbol{h}_a\cdot\boldsymbol{m}^\varepsilon + \pi|\boldsymbol{m}_p^\varepsilon|^2 + \frac{\mathbb{C}}{2}[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^\varepsilon] - \boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)]^2 - P_0\Big\}\Big] \\ &= \mathfrak{A}^\varepsilon + \mathfrak{B}^\varepsilon + \mathfrak{C}^\varepsilon \end{aligned} \tag{4.1}$$

where $\mathfrak{A}^\varepsilon, \mathfrak{B}^\varepsilon$ and $\mathfrak{C}^\varepsilon$ are the terms in the big square brackets.

4.1 Bounds for $\mathscr{I}_1^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$

Since $(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$ minimizes $\mathscr{I}^\varepsilon(\boldsymbol{m}, \boldsymbol{u})$, we have $\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) \le \mathscr{I}^\varepsilon(\boldsymbol{m}^o, \boldsymbol{u}^o)$ which on using (3.22b) along with the definition of $\mathscr{I}_1^\varepsilon(\boldsymbol{m}, \boldsymbol{u})$ gives us the inequality

$$\mathscr{I}_1^\varepsilon(\boldsymbol{m}^\varepsilon,\boldsymbol{u}^\varepsilon) \le \mathscr{I}_1^\varepsilon(\boldsymbol{m}^o,\boldsymbol{u}^o) = \frac{1}{\varepsilon}\big(-\varepsilon\frac{8\pi}{3} + \varepsilon^2\frac{\pi^2}{2}\big)\big[\,|\boldsymbol{m}_p^o|^2 - 2|m_3^0|^2\big] \le K_9. \tag{4.2}$$

The lower bound for $\mathscr{I}_1^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$ requires the following technical condition. See Result 8.2 in [Bhattacharya and James, 1999] and Definition 5.2 from [Le Dret and Meunier, 2005] to see other contexts where such a condition is necessary to get lower bound estimates.

Definition 4.1. *We say that a minimizer $(\boldsymbol{m}^o, v^o)$ of $\mathscr{I}^o(\boldsymbol{m}, v)$ (cf. Eqn. (3.11)), satisfies the strong second variation condition if for any $(\boldsymbol{m}(x_3), v(x_3)) \in \mathscr{A}_o$ there exists $\Lambda > 0$ such that,*

$$\begin{aligned}\mathscr{I}^o(\boldsymbol{m},v) - \mathscr{I}^o(\boldsymbol{m}^o,v^o) = \mathscr{I}^o(\boldsymbol{m},v) - P_0 \ge \Lambda\int_0^1 \Big\{\,|\partial_3\boldsymbol{m}(x_3) - \partial_3\boldsymbol{m}^o(x_3)|^2 + |\boldsymbol{m}-\boldsymbol{m}^o|^2 \\ + |\partial_3 v(x_3) - \partial_3 v^o(x_3)|^2\Big\} dx_3. \end{aligned} \tag{4.3}$$

provided $\|\boldsymbol{m} - \boldsymbol{m}^o\|_{L^2(0,1)} < K_{13}\varepsilon$ and $\|v - v^o\|_{L^2(0,1)} < K_{14}\varepsilon$ for some $\varepsilon > 0$ sufficiently small and K_{13}, K_{14} arbitrary constants independent of ε.

Set

$$\boldsymbol{M}^\varepsilon := \boldsymbol{m}^\varepsilon - \boldsymbol{m}^o. \tag{4.4}$$

Using the hypothesis that $\mathscr{I}^o(\boldsymbol{m}, v^o)$ satisfies strong second variation condition let us show the following Lemma,

Lemma 4.1.

$$\mathfrak{C}^\varepsilon \ge \Lambda\Big(\,\|\partial_3\boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)} + \|\boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)}\,\Big).$$

Proof. For fixed $(x,y)\in\omega$ we define $\mathfrak{M}^\varepsilon(x_3) := \boldsymbol{m}^\varepsilon(x_1,x_2,x_3)$, $\mathfrak{V}^\varepsilon(x_3) := \boldsymbol{u}^\varepsilon(x_1,x_2,x_3)$. As $(\boldsymbol{m}^o, v)$ minimizes $\mathscr{I}^o$ using (3.9) and the strong second variation condition we have

$$\int_0^1 d|\partial_3\mathfrak{M}^\varepsilon|^2 + \varphi(\mathfrak{M}^\varepsilon) - \boldsymbol{h}_a\cdot\mathfrak{M}^\varepsilon + \pi|\mathfrak{M}_p^\varepsilon|^2 + \frac{\mathbb{C}}{2}[\boldsymbol{\kappa}^\varepsilon[\mathfrak{V}^\varepsilon] - \boldsymbol{E}_s(\mathfrak{M}^\varepsilon)]^2 - \mathscr{I}^o(\boldsymbol{m}^o(x_3), v(x_3))$$
$$\geq \mathscr{I}^o(\mathfrak{M}^\varepsilon(x_3), \mathfrak{V}_3^\varepsilon(x_3)) - \mathscr{I}^o(\boldsymbol{m}^o(x_3), v(x_3)) \geq \Lambda\int_0^1 \left|\partial_3\big(\mathfrak{M}^\varepsilon - \boldsymbol{m}^o\big)\right|^2 + \left|\mathfrak{M}^\varepsilon - \boldsymbol{m}^o\right|^2 dx_3.$$

Integrating over $\boldsymbol{x}_p = (x_1,x_2)\in\omega$ we get

$$\mathfrak{C}^\varepsilon \geq \int_\omega \Big[\Lambda\int_0^1 \left|\partial_3\big(\mathfrak{M}^\varepsilon - \boldsymbol{m}^o\big)\right|^2 + \left|\mathfrak{M}^\varepsilon - \boldsymbol{m}^o\right|^2 dx_3\Big]\boldsymbol{dx}_p = \Lambda\left\|\boldsymbol{m}^\varepsilon - \boldsymbol{m}^o\right\|^2_{H^1(\Omega)}$$

and we get our result by noting that we defined $\boldsymbol{M}^\varepsilon = \boldsymbol{m}^\varepsilon - \boldsymbol{m}^o$. Since $\boldsymbol{m}^o$ is a constant we also get,

$$\mathfrak{C}^\varepsilon \geq \Lambda\Big(\left\|\partial_3\boldsymbol{m}^\varepsilon\right\|^2_{L^2(\Omega)} + \left\|\boldsymbol{M}^\varepsilon\right\|^2_{L^2(\Omega)}\Big). \tag{4.5}$$

□

We now use this Lemma to get a lower bound. Jensen's inequality gives $|\widehat{\boldsymbol{m}}_p^\varepsilon| \leq |\boldsymbol{m}_p^\varepsilon|$ which gives $\pi\int_\Omega|\widehat{\boldsymbol{m}}_p^\varepsilon|^2\boldsymbol{dx} \leq \pi\int_\Omega|\boldsymbol{m}_p^\varepsilon|^2\boldsymbol{dx}$. Using Young's inequality we have

$$\mathfrak{C}^\varepsilon - D_9\varepsilon^{3/4}\left\|\partial_3\boldsymbol{m}^\varepsilon\right\|_{L^2(\Omega)} \geq \Lambda\left\|\partial_3\boldsymbol{m}^\varepsilon\right\|^2_{L^2(\Omega)} - \frac{\Lambda}{2}\left\|\partial_3\boldsymbol{m}^\varepsilon\right\|^2_{L^2(\Omega)} - \frac{D_9^2}{2\Lambda}\varepsilon\sqrt{\varepsilon} \geq -\frac{D_9^2}{2\Lambda}\varepsilon\sqrt{\varepsilon}.$$

Combining this with equation (A.21) in the Appendix

$$\begin{aligned}\mathfrak{C}^\varepsilon + \mathfrak{B}^\varepsilon &= \mathfrak{C}^\varepsilon + \mathscr{E}_\varepsilon^d(\boldsymbol{m}^\varepsilon) - \pi\int_\Omega|\boldsymbol{m}_p^\varepsilon|^2\boldsymbol{dx} \geq \mathscr{E}_\varepsilon^d(\boldsymbol{m}^\varepsilon) - \pi\int_\Omega|\widehat{\boldsymbol{m}}_p^\varepsilon|^2\\
&\geq \Lambda\left\|\partial_3\boldsymbol{m}^\varepsilon\right\|^2_{L^2(\Omega)} + J_{211}^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon) - 2J_0^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon) + J_{22}^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon) + J_{24}^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon)\\
&\quad - D_9\sqrt{\varepsilon}\left\|\partial_3\boldsymbol{m}^\varepsilon\right\|_{L^2(\Omega)} - \varepsilon\big[D_{11}\|\partial_3\boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega)} + D_{12}\|\partial_3\boldsymbol{m}^\varepsilon\|_{L^2(\Omega)}\big] - D_{10}\varepsilon^2\\
&\geq J_{211}^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon) - 2J_0^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon) + J_{22}^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon) + J_{24}^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon) - \frac{D_9^2}{2\Lambda}\varepsilon\sqrt{\varepsilon} - D_{10}\varepsilon^2\\
&\quad - \varepsilon\big[D_{11}\|\partial_3\boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega)} + D_{12}\|\partial_3\boldsymbol{m}^\varepsilon\|_{L^2(\Omega)}\big].\end{aligned} \tag{4.6}$$

Noting that all terms in equation (4.6) are $O(\varepsilon)$ or higher, and $\mathfrak{A}^\varepsilon \geq 0$, we get a lower bound on $\mathscr{I}_1^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$ as

$$\mathscr{I}_1^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) \geq \frac{1}{\varepsilon}\Big(\mathfrak{B}^\varepsilon + \mathfrak{C}^\varepsilon\Big) \geq -K_9. \tag{4.7}$$

4.2 Convergence of $\mathscr{I}_1^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$

Theorem 4.1. *We have the following convergence,*

$$\lim_{\varepsilon\to 0}\mathscr{I}_1^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) = \frac{16\pi}{3}|m_3^o|^2 - \frac{8\pi}{3}|\boldsymbol{m}_p^o|^2. \tag{4.8}$$

Proof. Dividing equation (4.6) by ε and using the positivity of $\mathfrak{A}^\varepsilon$ gives

$$\mathscr{I}_1^\varepsilon(\boldsymbol{m}^\varepsilon,\boldsymbol{u}^\varepsilon) \geq \frac{1}{\varepsilon}\big(\mathfrak{B}^\varepsilon+\mathfrak{C}^\varepsilon\big) \geq \frac{1}{\varepsilon}\Big(J_{211}^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon)-2J_0^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon)+J_{22}^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon)+J_{24}^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon)\Big)-\frac{D_9^2}{2\Lambda}\sqrt{\varepsilon}$$
$$-D_{10}\varepsilon-\Big(D_{11}\|\partial_3\boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega)}+D_{12}\|\partial_3\boldsymbol{m}^\varepsilon\|_{L^2(\Omega)}\Big). \tag{4.9}$$

Using the strong convergence $\widehat{\boldsymbol{m}}^\varepsilon(x_3)\to\boldsymbol{m}^o(x_3)$ in $L^2(0,1)$ from (3.6) and $\partial_3\boldsymbol{m}^\varepsilon\to\partial_3\boldsymbol{m}^o=0$ in $L^2(\Omega)$ from (3.14), we take $\liminf_{\varepsilon\to0}$ above to get,

$$\liminf_{\varepsilon\to0}\mathscr{I}_1^\varepsilon(\boldsymbol{m}^\varepsilon,\boldsymbol{u}^\varepsilon)\geq\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\Big(J_{211}^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon)-2J_0^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon)+J_{22}^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon)+J_{24}^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon)\Big)$$
$$=\frac{16\pi}{3}|m_3^o|^2-\frac{8\pi}{3}|\boldsymbol{m}_p^o|^2, \tag{4.10}$$

utilizing Proposition A.6 . To get the reverse inequality we take $\limsup$ of the equation (4.2),

$$\limsup_{\varepsilon\to0}\mathscr{I}_1(\boldsymbol{m}^\varepsilon,\boldsymbol{u}^\varepsilon)\leq\limsup_{\varepsilon\to0}\frac{1}{\varepsilon}\big(-\varepsilon\frac{8\pi}{3}+\varepsilon^2\frac{\pi^2}{2}\big)\big[|\boldsymbol{m}_p^o|^2-2|m_3^o|^2\big]$$
$$=\frac{16\pi}{3}|m_3^o|^2-\frac{8\pi}{3}|\boldsymbol{m}_p^o|^2.$$

The $\limsup$ and $\liminf$ inequalities together give our result. □

Set

$$P_1=\frac{16\pi}{3}|m_3^o|^2-\frac{8\pi}{3}|\boldsymbol{m}_p^o|^2. \tag{4.11}$$

In the limit we get that $\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon,\boldsymbol{u}^\varepsilon)$ converges to a fixed quantity P_1 depending only on $\boldsymbol{m}^o$. P_1 consists of the mutual interaction of the poles generated by $\boldsymbol{m}^o$ on one end $\omega(0)$ of the wire domain Ω with the other end $\omega(1)$ giving the term $\frac{16}{3}\pi|m_3^o|^2$, and the self-interaction of the poles created by $\boldsymbol{m}^o$ on the curved surface $\partial\omega\times(0,1)$ giving the term $-\frac{8}{3}\pi|\boldsymbol{m}_p^o|^2$. To see this, note that $\boldsymbol{m}^\varepsilon\to\boldsymbol{m}^o$, also $\widehat{\boldsymbol{m}}^\varepsilon\to\boldsymbol{m}^o$ in $H^1(\Omega)$, and $\nabla\boldsymbol{m}^o=\boldsymbol{0}$. Thus we have,

$$P_1=\frac{1}{\varepsilon}\big[J_{22}(\boldsymbol{m}^o)+J_{211}(\boldsymbol{m}^o)-2\int_\Omega|\boldsymbol{m}_p^o|^2\big]=\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\big[J_{22}(\widehat{\boldsymbol{m}}^\varepsilon)+J_{211}(\widehat{\boldsymbol{m}}^\varepsilon)-2\int_\Omega|\widehat{\boldsymbol{m}}_p^\varepsilon|^2\big].$$

Let $\widehat{\boldsymbol{m}}^\varepsilon-\boldsymbol{m}^o=\boldsymbol{m}_1^\varepsilon$. Then using Proposition A.6,

$$J_{22}(\widehat{\boldsymbol{m}}^\varepsilon)+J_{211}(\widehat{\boldsymbol{m}}^\varepsilon)-2\int_\Omega|\widehat{\boldsymbol{m}}_p^\varepsilon|^2\geq P_1-K_{48}\varepsilon\|\boldsymbol{m}_1^\varepsilon\|_{H^1(0,1)}-K_{49}\varepsilon\|\boldsymbol{m}_1^\varepsilon\|^2_{H^1(0,1)}.$$

5 Third variational limit problem

In this section we assume for the sake of simplicity of calculation that $\boldsymbol{m}^o=(0,0,m_s)$. This assumption though not necessary, makes the calculations shorter. The spontaneous strain due to this $\boldsymbol{m}^o$ is given by,

$$\boldsymbol{E}_s(\boldsymbol{m}^o) = \frac{1}{3}\begin{bmatrix} -\lambda_{100} & 0 & 0 \\ 0 & -\lambda_{100} & 0 \\ 0 & 0 & 2\lambda_{100} \end{bmatrix}, \tag{5.1}$$

and $E'_{s_{33}}(\boldsymbol{m}^o)$ which is derivative of $E_{s_{33}}(\boldsymbol{m})$ at $\boldsymbol{m} = \boldsymbol{m}^o$ is given by,

$$E'_{s_{33}}(\boldsymbol{m}^o) = \begin{bmatrix} 0 & 0 & 2\lambda_{100} m_3^o \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 2\lambda_{100} m_s \end{bmatrix}^T. \tag{5.2}$$

As in the previous section we first define $\mathscr{I}_2^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) := \dfrac{1}{\varepsilon^2}(\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) - P_0 - \varepsilon P_1)$. We will show that and note $\mathscr{I}_2^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$ is bounded above and below independent of ε. Then we define $\boldsymbol{w}^\varepsilon$ in (5.14) and prove a weak compactness result for it. The convergence is improved to strong in Theorem 5.1 where we also define a new variational problem $\mathscr{I}_2^o$ and show its relation with $\mathscr{I}_2^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$. Recalling $\mathfrak{A}^\varepsilon, \mathfrak{B}^\varepsilon$ and $\mathfrak{C}^\varepsilon$ from equation (4.1) in § 4 , we note

$$\begin{aligned} &\mathscr{I}^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) - P_0 - \varepsilon P_1 \\ &= \Big[\int_\Omega \frac{d}{\varepsilon^2} |\nabla_p \boldsymbol{m}^\varepsilon|^2 \Big] + \Big[\mathscr{E}_d^\varepsilon(\boldsymbol{m}^\varepsilon) - \int_\Omega 2|\boldsymbol{m}_p^\varepsilon|^2 - \varepsilon P_1 \Big] + \int_\Omega \Big[d\, |\partial_3 \boldsymbol{m}^\varepsilon|^2 + \varphi(\boldsymbol{m}^\varepsilon) - \boldsymbol{h}_a \cdot \boldsymbol{m}^\varepsilon \\ &\qquad + 2|\boldsymbol{m}_p^\varepsilon|^2 + \frac{1}{2}\mathbb{C}[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^\varepsilon] - \boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)]^2 - P_0 \Big] \\ &= \mathfrak{A}^\varepsilon + (\mathfrak{B}^\varepsilon - \varepsilon P_1) + \mathfrak{C}^\varepsilon. \end{aligned} \tag{5.3}$$

5.1 Boundedness of $\mathscr{I}_2^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$

To get an upper bound on $\mathscr{I}_2^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$ we use (4.2) and subtract εP_1 from both sides to get

$$\begin{aligned} \mathscr{I}_2^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) = \frac{1}{\varepsilon}\big[\mathscr{I}_1^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) - \varepsilon P_1\big] &\le \frac{1}{\varepsilon}\big[\mathscr{I}_1^\varepsilon(\boldsymbol{m}^o, \boldsymbol{u}^o) - \varepsilon P_1\big] \\ &\le \frac{\pi^2}{2}\big[\,|\boldsymbol{m}_p^o|^2 - 2|m_3^o|^2\big]. \end{aligned} \tag{5.4}$$

Recall we defined in (3.11), $\boldsymbol{M}^\varepsilon = \boldsymbol{m}^\varepsilon - \boldsymbol{m}^o$ and $\boldsymbol{m}^o = (0,0,m_s)$.

$$|\boldsymbol{m}^\varepsilon|^2 = m_s^2 = |\boldsymbol{m}^o + \boldsymbol{M}^\varepsilon|^2 = m_s^2 + |\boldsymbol{M}^\varepsilon|^2 + 2m_s M_3^\varepsilon, \qquad \Rightarrow |\boldsymbol{M}^\varepsilon|^2 = -2m_s M_3^\varepsilon. \tag{5.5}$$

Recall the strong second variation condition on $\mathscr{I}^o(\boldsymbol{m}, v^o)$ and Lemma 4.1 gives,

$$\mathfrak{C}^\varepsilon \ge \Lambda\Big(\|\partial_3 \boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)} + \|\boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)} \Big) = \Lambda\Big(\|\partial_3 \boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega)} + \|\boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)} \Big). \tag{5.6}$$

Then

$$\begin{aligned} &\frac{\Lambda}{2}\big\|\partial_3 \widehat{\boldsymbol{m}}^\varepsilon\big\|^2_{L^2(\Omega)} - \frac{J_1^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon)}{2} - J_3^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon) \ge \frac{\Lambda}{2}\big\|\partial_3 \widehat{\boldsymbol{m}}^\varepsilon\big\|^2_{L^2(\Omega)} - \varepsilon D_8 \big\|\partial_3 \widehat{\boldsymbol{m}}^\varepsilon\big\|^2_{L^2(\Omega)} - \varepsilon D_9 \big\|\partial_3 \widehat{\boldsymbol{m}}^\varepsilon\big\|_{L^2(\Omega)} \\ &\ge \big(\frac{\Lambda}{2} - \varepsilon D_8\big)\big\|\partial_3 \widehat{\boldsymbol{m}}^\varepsilon\big\|^2_{L^2(\Omega)} - \varepsilon D_9 \big\|\partial_3 \widehat{\boldsymbol{m}}^\varepsilon\big\|_{L^2(\Omega)}. \end{aligned}$$

Using Proposition A.6 and Young's inequality

$$\Lambda\int_\Omega\left[|\widehat{\boldsymbol{M}}^\varepsilon|^2+|\partial_3\widehat{\boldsymbol{M}}^\varepsilon|^2\right]+\frac{1}{2}J_{22}(\widehat{\boldsymbol{m}}^\varepsilon)+\frac{1}{2}J_{211}(\widehat{\boldsymbol{m}}^\varepsilon)-\int_\Omega 2|\widehat{\boldsymbol{m}}_p^\varepsilon|^2-\varepsilon P_1+J_{24}(\widehat{\boldsymbol{m}}^\varepsilon)$$
$$\geq \Lambda\,|\omega|\,\|\widehat{\boldsymbol{M}}^\varepsilon\|^2_{H^1(0,1)}-\varepsilon(D_9+D_{11})\|\widehat{\boldsymbol{M}}^\varepsilon\|^2_{H^1(0,1)}-\varepsilon(D_{10}+D_{12})\|\widehat{\boldsymbol{M}}^\varepsilon\|_{H^1(0,1)}$$
$$\geq \frac{\Lambda}{2}\,|\omega|\,\|\widehat{\boldsymbol{M}}^\varepsilon\|^2_{H^1(0,1)}-K_{11}\,\varepsilon^2.$$

Adding the two estimates together gives,

$$(\mathfrak{B}^\varepsilon-\varepsilon P_1)\geq -\frac{\Lambda}{2}|\omega|\|\widehat{\boldsymbol{M}}^\varepsilon\|_{H^1(0,1)}-K_{12}\varepsilon^2 \tag{5.7}$$

and using (**??**)

$$\mathfrak{A}^\varepsilon+(\mathfrak{B}^\varepsilon-\varepsilon P_1)+\mathfrak{C}^\varepsilon$$
$$\geq \frac{d}{\varepsilon^2}\|\nabla_p\boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega)}+\Lambda|\omega|\;\|\widehat{\boldsymbol{M}}^\varepsilon\|^2_{H^1((0,1)}+\mathscr{E}_d^\varepsilon(\boldsymbol{m}^\varepsilon)-\int_\Omega 2|\boldsymbol{m}_p^\varepsilon|^2-\varepsilon P_1$$
$$\geq -\frac{K_4}{2d}\varepsilon^2+\frac{d}{2\varepsilon^2}\|\nabla_p\boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega)}-K_{10}\,\varepsilon^2+\frac{\Lambda}{2}\,|\omega|\,\|\widehat{\boldsymbol{M}}^\varepsilon\|^2_{H^1(0,1)}-K_{11}\,\varepsilon^2 \tag{5.8a}$$
$$\geq -K_{12}\,\varepsilon^2. \tag{5.8b}$$

Dividing through by ε^2 we get a lower bound. Also note using the upper bound (5.4) on (5.8a) and rearranging terms we get,

$$\frac{\varepsilon^2\pi^2}{2}\left[|\boldsymbol{m}_p^o|^2-2|m_3^o|^2\right]+\frac{K_4}{2d}\varepsilon^2+K_{10}\,\varepsilon^2+K_{11}\,\varepsilon^2\geq\frac{d}{2\varepsilon^2}\|\nabla_p\boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega)}+\frac{\Lambda}{2}\,|\omega|\,\|\widehat{\boldsymbol{M}}^\varepsilon\|^2_{H^1(0,1)}$$

which gives us that $K_{13}\varepsilon^2\geq\mathfrak{A}^\varepsilon$ and $K_{13}\varepsilon^2\geq\|\widehat{\boldsymbol{M}}^\varepsilon\|_{H^1(0,1)}$ and implies using (5.7) that $\mathfrak{A}^\varepsilon+\mathfrak{B}^\varepsilon(\boldsymbol{m}^\varepsilon)-\varepsilon P_1\geq -K_{14}\varepsilon^2$. Finally we also then get that $K_{15}\varepsilon^2\geq\mathfrak{C}^\varepsilon$. Thus Note that

$$K_{13}\varepsilon^2\geq\mathfrak{A}^\varepsilon\qquad\qquad \mathfrak{B}^\varepsilon(\boldsymbol{m}^\varepsilon)-\varepsilon P_1\geq -K_{14}\varepsilon^2\qquad\qquad K_{15}\varepsilon^2\geq\mathfrak{C}^\varepsilon \tag{5.9}$$

5.2 Weak convergence of $\boldsymbol{w}^\varepsilon$

Using a truncated Taylor Expansion

$$\boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)=\boldsymbol{E}_s(\boldsymbol{m}^o)-\boldsymbol{E}_s'(\boldsymbol{m}^o)\cdot\boldsymbol{M}^\varepsilon-\frac{1}{2}\boldsymbol{E}_s''(\boldsymbol{m}^*)\boldsymbol{M}^\varepsilon\cdot\boldsymbol{M}^\varepsilon=\boldsymbol{E}_s(\boldsymbol{m}^o)-\Delta(\boldsymbol{M}^\varepsilon) \tag{5.10}$$

where $\Delta(\boldsymbol{M}^\varepsilon)$ is defined as the last 2 terms. Since $\boldsymbol{E}_s(\boldsymbol{m})$ is a polynomial function of $\boldsymbol{m}$, then $\boldsymbol{E}_s'$ and $\boldsymbol{E}_s''$ are both bounded in L^∞ for $|\boldsymbol{m}|=m_s$.

$$\Delta(\boldsymbol{M}^\varepsilon)\leq K_{15}|\boldsymbol{M}^\varepsilon|+K_{16}|\boldsymbol{M}^\varepsilon|^2\qquad\Rightarrow\|\Delta(\boldsymbol{M}^\varepsilon)\|^2_{L^2(\Omega)}\leq K_{17}\|\boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)}\leq K_{18}\varepsilon^2. \tag{5.11}$$

Set $\boldsymbol{u}^\varepsilon = \boldsymbol{u}^o + \boldsymbol{U}^\varepsilon$. Write $\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^\varepsilon] = \boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^o] + \boldsymbol{\kappa}^\varepsilon[\boldsymbol{U}^\varepsilon]$. Using Young's inequality, $\mathbb{C}[\Delta(\boldsymbol{M}^\varepsilon)] : \boldsymbol{\kappa}^\varepsilon[\boldsymbol{U}^\varepsilon] \le \Gamma|\Delta(\boldsymbol{M}^\varepsilon)|\,|\boldsymbol{\kappa}^\varepsilon[\boldsymbol{U}^\varepsilon]|$ and (5.11) we get the following,

$$\mathbb{C}[\Delta(\boldsymbol{M}^\varepsilon)] : \boldsymbol{\kappa}^\varepsilon[\boldsymbol{U}^\varepsilon] \;\le\; \Gamma|\Delta(\boldsymbol{M}^\varepsilon)|\,|\boldsymbol{\kappa}^\varepsilon[\boldsymbol{U}^\varepsilon]| \;\le\; \frac{4\Gamma^2}{\gamma}|\Delta(\boldsymbol{M}^\varepsilon)|^2 + \frac{\gamma}{4}|\boldsymbol{\kappa}^\varepsilon[\boldsymbol{U}^\varepsilon]|^2. \tag{5.12}$$

From (3.22) note that $\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^o] - \boldsymbol{E}_s(\boldsymbol{m}^o) = \mathbf{0}$ which along with (3.9) and (5.12) gives

$$\begin{aligned}
&\frac{\mathbb{C}}{2}[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^\varepsilon] - \boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)]^2\\
&= \frac{\mathbb{C}}{2}[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^o] - \boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)]^2 + \frac{\mathbb{C}}{2}[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{U}^\varepsilon]]^2 + \mathbb{C}[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^o] - \boldsymbol{E}_s(\boldsymbol{m}^o)] : \boldsymbol{\kappa}^\varepsilon[\boldsymbol{U}^\varepsilon] - \mathbb{C}[\Delta(\boldsymbol{M}^\varepsilon)] : \boldsymbol{\kappa}^\varepsilon[\boldsymbol{U}^\varepsilon]\\
&\ge \frac{1}{2} f_0(\partial_3 v^o - E_{s_{33}}(\boldsymbol{m}^\varepsilon)) + \frac{\gamma}{4}|\boldsymbol{\kappa}^\varepsilon[\boldsymbol{U}^\varepsilon]|^2 - \frac{4\Gamma^2}{\gamma}|\Delta(\boldsymbol{M}^\varepsilon)|^2.
\end{aligned}$$

Integrating over Ω and using (5.11) we get,

$$\begin{aligned}
\mathfrak{E}^\varepsilon &= \int_\omega \mathscr{I}^o(\boldsymbol{m}^\varepsilon, V)\boldsymbol{d\sigma} - P_0 + \int_\Omega \frac{\mathbb{C}}{2}[\boldsymbol{\kappa}^\varepsilon[\boldsymbol{u}^\varepsilon] - \boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)]^2 - \frac{1}{2} f_0(\partial_3 v^o - E_{s_{33}}(\boldsymbol{m}^\varepsilon))\\
&\ge \Lambda\|\partial_3 \boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)} + \Lambda\|\boldsymbol{M}^\varepsilon\|^2_{L^2(\Omega)} + \int_\Omega \frac{\gamma}{4}|\boldsymbol{\kappa}[\boldsymbol{U}^\varepsilon]|^2 - K_{18}\varepsilon^2\\
&\Rightarrow K_{19}\varepsilon^2 \ge \int_\Omega |\boldsymbol{\kappa}^\varepsilon(\boldsymbol{u}^1)|^2.
\end{aligned} \tag{5.13}$$

Set

$$\boldsymbol{w}^\varepsilon = (\boldsymbol{u}_p^\varepsilon - \boldsymbol{u}_p^o, \frac{u_3^\varepsilon - u_3^o}{\varepsilon}) = (\boldsymbol{U}_p^\varepsilon, \varepsilon^{-1}U_3^\varepsilon) \tag{5.14}$$

and note

$$\begin{aligned}
\boldsymbol{\kappa}^\varepsilon(\boldsymbol{U}^\varepsilon) &= \begin{bmatrix}
\frac{1}{\varepsilon}\partial_1 w_1^\varepsilon & \frac{1}{2\varepsilon}(\partial_1 w_2^\varepsilon + \partial_2 w_1^\varepsilon) & \frac{1}{2}(\partial_1 w_3^\varepsilon + \partial_3 w_1^\varepsilon)\\
\frac{1}{2\varepsilon}(\partial_1 w_2^\varepsilon + \partial_2 w_1^\varepsilon) & \frac{1}{\varepsilon}\partial_2 w_2^\varepsilon & \frac{1}{2}(\partial_2 w_3^\varepsilon + \partial_3 w_2^\varepsilon)\\
\frac{1}{2}(\partial_1 w_3^\varepsilon + \partial_3 w_1^\varepsilon) & \frac{1}{2}(\partial_2 w_3^\varepsilon + \partial_3 w_2^\varepsilon) & \varepsilon\partial_3 w_3^\varepsilon
\end{bmatrix}\\
&=: \boldsymbol{\chi}(\boldsymbol{w}^\varepsilon).
\end{aligned} \tag{5.15}$$

Note $\left|\dfrac{\boldsymbol{\chi}(\boldsymbol{w}^\varepsilon)}{\varepsilon}\right| \ge \left|\boldsymbol{E}(\boldsymbol{w}^\varepsilon)\right|$. Using Korn's inequality, (5.13) becomes,

$$K_{14} \ge \int_\Omega \left|\frac{\boldsymbol{\kappa}^\varepsilon(\boldsymbol{u}^1)}{\varepsilon}\right|^2 \boldsymbol{dx} = \int_\Omega \left|\frac{\boldsymbol{\chi}(\boldsymbol{w}^\varepsilon)}{\varepsilon}\right|^2 \boldsymbol{dx} \ge \int_\Omega |\boldsymbol{E}(\boldsymbol{w}^\varepsilon)|^2 \boldsymbol{dx} \ge \alpha \int_\Omega \left[|\nabla \boldsymbol{w}^\varepsilon|^2 + |\boldsymbol{w}^\varepsilon|^2\right]\boldsymbol{dx}$$

where $\alpha(\Omega) > 0$ being the Korn's constant. These results together imply for some unrelabeled subsequence

$$\boldsymbol{w}^\varepsilon \rightharpoonup \boldsymbol{w}^o \text{ in } H^1(\Omega;\mathbb{R}^3) \qquad \boldsymbol{w}^\varepsilon \to \boldsymbol{w}^o \text{ in } L^2(\Omega;\mathbb{R}^3) \tag{5.16a}$$

$$\boldsymbol{E}(\boldsymbol{w}^\varepsilon) \rightharpoonup \boldsymbol{E}(\boldsymbol{w}^o) \text{ in } L^2(\Omega;\mathbb{R}^3) \qquad \frac{\boldsymbol{\chi}^\varepsilon}{\varepsilon} \rightharpoonup \boldsymbol{\nu}^0 \text{ in } L^2(\Omega;\mathbb{R}^3). \tag{5.16b}$$

Note from (5.15),

$$\frac{\chi^\varepsilon_{ij}}{\varepsilon} = \frac{1}{\varepsilon^2} E_{ij}(\boldsymbol{w}^\varepsilon) \text{ for } (i,j) \in \{1,2\}, \qquad \frac{\chi^\varepsilon_{i3}}{\varepsilon} = \frac{1}{\varepsilon} E_{i3}(\boldsymbol{w}^\varepsilon) \text{ for } i \in \{1,2\}$$

which together imply after using lower semi-continuity of norm w.r.t weak convergence

$$\|E_{ij}(\boldsymbol{w}^o)\|_{L^2(\Omega)} \le \liminf \|E_{ij}(\boldsymbol{w}^\varepsilon)\|_{L^2(\Omega)} = 0 \tag{5.17}$$

when $i \in \{1,2\}$ and $j \in \{1,2,3\}$.

Lemma 5.1. *Let the strain corresponding to a displacement field $\boldsymbol{w}^o$, $\boldsymbol{E}(\boldsymbol{w}^o)$ be such that $E_{ij}(\boldsymbol{w}^o) = 0$ for $i \in \{1,2\}$ and $j \in \{1,2,3\}$, the displacement $\boldsymbol{w}^o$ can be expressed as,*

$$\begin{aligned}
w_1^o(x_1,x_2,x_3) &= w_1^o(x_3),\\
w_2^o(x_1,x_2,x_3) &= w_2^o(x_3),\\
w_3^o(x_1,x_2,x_3) &= -x_1\, \partial_3 w_1^o(x_3) - x_2\, \partial_3 w_2^o(x_3) + \gamma(x_3).
\end{aligned}$$

Proof. Given $E_{11}(\boldsymbol{w}^o) = \partial_1 w_1^o(x_1,x_2,x_3) = 0$ and $E_{11}(\boldsymbol{w}^o) = \partial_2 w_2^o(x_1,x_2,x_3) = 0$ together gives $w_1^o(x_1,x_2,x_3) = \alpha_1(x_2,x_3)$ and $w_2^o(x_1,x_2,x_3) = \alpha_2(x_1,x_3)$. $E_{12}(\boldsymbol{w}^o) = 0$ gives us,

$$\partial_2 w_1^o(x_1,x_2,x_3) + \partial_1 w_2^o(x_1,x_2,x_3) = \partial_2 \alpha_1(x_2,x_3) + \partial_1 \alpha_2(x_1,x_3) = 0,$$

which implies $\partial_2 \alpha_1(x_2,x_3) = -\partial_1 \alpha_2(x_1,x_3) = \beta(x_3)$. Thus, $w_1^o(x_1,x_2,x_3) = \gamma_1(x_3) + x_2\beta(x_3)$ and $w_2^o(x_1,x_2,x_3) = \gamma_2(x_3) - x_1\beta(x_3)$. Also given $E_{1,3}(\boldsymbol{w}^o) = E_{2,3}(\boldsymbol{w}^o) = 0$,

$$\partial_2 E_{1,3}(\boldsymbol{w}^o) - \partial_1 E_{2,3}(\boldsymbol{w}^o) = \frac{1}{2}(w^o_{1,32} + w^o_{3,12} - w^o_{2,31} - w^o_{3,12}) = \partial_3\beta(x_3) = 0.$$

This gives us $\beta(x_3) = K_{14}$ (constant). Using the Dirichlet boundary conditions at the base $x_3 = 0$, we have $w_1^o(x_2,0) = \gamma_1(0) + x_2 K = 0$ which gives us $K = 0$. So $w_1^o(x_1,x_2,x_3) = \gamma_1(x_3)$ and $w_1^o(x_1,x_2,x_3) = \gamma_2(x_3)$. We finally have,

$$\begin{aligned}
&E_{1,3}(\boldsymbol{w}^o) = 0 \Rightarrow \partial_1 w_3^o(x_1,x_2,x_3) = -\partial_3 w_1^o = -\partial_3\gamma_1(x_3),\\
&E_{2,3}(\boldsymbol{w}^o) = 0 \Rightarrow \partial_2 w_3^o(x_1,x_2,x_3) = -\partial_3 w_2^o = -\partial_3\gamma_2(x_3),\\
&w_3^o(x_1,x_2,x_3) = -x_1\gamma_1'(x_3) - x_2\gamma_2'(x_3) + \gamma_3(x_3) = -x_1\, \partial_3 w_1^o(x_3) - x_2\, \partial_3 w_2^o(x_3) + \gamma_3(x_3),\\
&\partial_3 w_3^o(x_1,x_2,x_3) = -x_1\, \partial_{33} w_1^o(x_3) - x_2\, \partial_{33} w_2^o(x_3) + \partial_3\gamma_3(x_3).
\end{aligned} \tag{5.19}$$

□

Note also that $\boldsymbol{w}^o \in H^1(\Omega)$ gives $\partial_3 w_3^o \in L^2(\Omega)$. Equation (5.19) gives then that $\partial_{33} w_i^o(x_3) \in L^2(\Omega)$ for $i = 1,2$ and $\partial_3\gamma_3 \in L^2(\Omega)$. Thus $w_i^o(x_3) \in H^2(\Omega)$ and $\gamma \in H^1(\Omega)$.

5.3 Strong convergence of $\boldsymbol{w}^\varepsilon$

Define $\mathscr{I}_2^o(w_1(x_3), w_2(x_3), v(x_3))$ in $\mathscr{A}_2$ as

$$\mathscr{I}_2^o(w_1(x_3), w_2(x_3), v(x_3)) = \int_\Omega \frac{1}{2}\left[f_0(x_1\partial_{33}w_1) + f_0(x_2\partial_{33}w_2) + f_0(\partial_3 v)\right] - 2\pi^2 m_s^2 \tag{5.20}$$

and $\mathscr{A}_2 := \left\{(w_1(x_3), w_2(x_3), v(x_3)) \in H^2_\sharp(0,1) \times H^2_\sharp(0,1) \times H^1_\sharp(0,1)\right\}$.

Theorem 5.1. *There exists a subsequence* $\boldsymbol{w}^\varepsilon$ *not relabeled such that* $\boldsymbol{w}^\varepsilon \to \boldsymbol{w}^o$ *strongly in* $H^1(\Omega, \mathbb{R}^3)$*.* $\boldsymbol{w}^o$ *is of the form given in Lemma 5.1 and* $(w_1^o(x_3), w_2^o(x_3), \gamma(x_3))$ *minimizes* $\mathscr{I}_2^0$ *in* $\mathscr{A}_2$ *and* $\lim_{\varepsilon\to 0} \mathscr{I}_2^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) = \mathscr{I}_2^o(w_1^o, w_2^o, \gamma)$ *where* (w_1^o, w_2^o, γ) *minimizes* $\mathscr{I}_2^o(w_1(x_3), w_2(x_3), v(x_3)$ *in* $\mathscr{A}_2$*.*

Proof. By continuity of magnetostatic energy w.r.t. strong convergence in (3.4), $\boldsymbol{m}^\varepsilon \to \boldsymbol{m}^o$ we have that,

$$\begin{aligned}\lim_{\varepsilon\to 0} \frac{\mathfrak{B}^\varepsilon - \varepsilon P_1}{\varepsilon^2} &\geq \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\left[\mathscr{E}_d^\varepsilon(\boldsymbol{m}^\varepsilon) - \int_\Omega 2|\boldsymbol{m}_p^\varepsilon|^2 - \varepsilon P_1\right] \\ &= \frac{1}{\varepsilon^2}\left[\mathscr{E}_d^\varepsilon(\boldsymbol{m}^o) - \int_\Omega 2|\boldsymbol{m}_p^o|^2 - \varepsilon P_1\right] = \frac{\pi^2}{2}|\boldsymbol{m}_p^o|^2 - \pi^2|m_3^o|^2 \end{aligned}\tag{5.21}$$

where $\dfrac{\mathfrak{B}^\varepsilon - \varepsilon P_1}{\varepsilon^2}$ makes sense because of (5.9). The truncated Taylor expansion from (5.10) $E_{s_{33}}(\boldsymbol{m}^\varepsilon) = E_{s_{33}}(\boldsymbol{m}^o) + \Delta_{33}(\boldsymbol{M}^\varepsilon)$ along with (5.2) and (5.5) gives $\Delta_{33}(\boldsymbol{M}^\varepsilon) = 2\lambda_{100} m_s M_3^\varepsilon + O(|\boldsymbol{M}^\varepsilon|^2) = -\lambda_{100} m_s |\boldsymbol{M}^\varepsilon|^2 + O(|\boldsymbol{M}^\varepsilon|^2)$. Then

$$\Delta_{33}(\boldsymbol{M}^\varepsilon) = O(|\boldsymbol{M}^\varepsilon|^2) \tag{5.22}$$

$$\int_\Omega Y\,\Delta_{33}(\boldsymbol{M}^\varepsilon)\,\chi_{33}^\varepsilon(\boldsymbol{w}^\varepsilon) \leq Y\,\|\Delta_{33}(\boldsymbol{M}^\varepsilon)\|_{L^2}\|\chi_{33}^\varepsilon(\boldsymbol{w}^\varepsilon)\|_{L^2} = O(\varepsilon^2)\|\chi_{33}^\varepsilon(\boldsymbol{w}^\varepsilon)\|_{L^2(\Omega)}. \tag{5.23}$$

Noting $\kappa_{33}^\varepsilon(\boldsymbol{u}^\varepsilon) = \partial_3 \boldsymbol{u}_3^o + \chi_{33}^\varepsilon(\boldsymbol{w}^\varepsilon)$ and $\kappa_{33}^\varepsilon(\boldsymbol{u}^o) - E_{s_{33}}(\boldsymbol{m}^\varepsilon) = \partial_3\boldsymbol{u}_3^o - E_{s_{33}}(\boldsymbol{m}^o) - \Delta_{33}(\boldsymbol{M}^\varepsilon) = -\Delta_{33}(\boldsymbol{M}^\varepsilon)$ gives

$$\begin{aligned}\int_\Omega f_0(\kappa_{33}^\varepsilon(\boldsymbol{u}^\varepsilon) - E_{s_{33}}(\boldsymbol{m}^\varepsilon)) &= \int_\Omega f_0(\kappa_{33}^\varepsilon(\boldsymbol{u}^o) - E_{s_{33}}(\boldsymbol{m}^\varepsilon)) + f_0(\chi_{33}^\varepsilon(\boldsymbol{w}^\varepsilon)) + Y\,\Delta_{33}(\boldsymbol{M}^\varepsilon)\,\chi_{33}^\varepsilon(\boldsymbol{w}^\varepsilon) \\ &\geq \int_\Omega f_0(\kappa_{33}^\varepsilon(\boldsymbol{u}^o) - E_{s_{33}}(\boldsymbol{m}^\varepsilon)) + f_0(\chi_{33}^\varepsilon(\boldsymbol{w}^\varepsilon)) - O(\varepsilon^2)\|\chi_{33}^\varepsilon(\boldsymbol{w}^\varepsilon)\|_{L^2}.\end{aligned}$$

Then using (3.9)

$$\begin{aligned}
\mathfrak{C}^\varepsilon &= \int_\Omega d\ |\partial_3 \boldsymbol{m}^\varepsilon|^2 + \varphi(\boldsymbol{m}^\varepsilon) - \boldsymbol{h}_a \cdot \boldsymbol{m}^\varepsilon + 2|\boldsymbol{m}_p^\varepsilon|^2 + \frac{1}{2}\mathbb{C}[\boldsymbol{\kappa}^\varepsilon(\boldsymbol{u}^\varepsilon) - \boldsymbol{E}_s(\boldsymbol{m}^\varepsilon)]^2 - P_0 \\
&\geq \int_\Omega d\ |\partial_3 \boldsymbol{m}^\varepsilon|^2 + \varphi(\boldsymbol{m}^\varepsilon) - \boldsymbol{h}_a \cdot \boldsymbol{m}^\varepsilon + 2|\boldsymbol{m}_p^\varepsilon|^2 + \frac{1}{2} f_0(\kappa_{33}^\varepsilon(\boldsymbol{u}^\varepsilon) - E_{s_{33}}(\boldsymbol{m}^\varepsilon)) - P_0 \\
&\geq \int_\Omega d\ |\partial_3 \boldsymbol{m}^\varepsilon|^2 + \varphi(\boldsymbol{m}^\varepsilon) - \boldsymbol{h}_a \cdot \boldsymbol{m}^\varepsilon + 2|\boldsymbol{m}_p^\varepsilon|^2 + \frac{1}{2} f_0(\kappa_{33}^\varepsilon(\boldsymbol{u}^o) - E_{s_{33}}(\boldsymbol{m}^\varepsilon)) - P_0 \\
&\qquad\qquad + \int_\Omega f_0(\chi_{33}^\varepsilon(\boldsymbol{w}^\varepsilon)) - O(\varepsilon^2)\|\chi_{33}^\varepsilon(\boldsymbol{w}^\varepsilon)\|_{L^2(\Omega)} \\
&= \int_\omega \mathscr{I}^o(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^o)\boldsymbol{d\sigma} - P_0 + \int_\Omega f_0(\chi_{33}^\varepsilon(\boldsymbol{w}^\varepsilon)) - O(\varepsilon^2)\|\chi_{33}^\varepsilon(\boldsymbol{w}^\varepsilon)\|_{L^2(\Omega)} \\
&\geq \int_\Omega \frac{1}{2} f_0(\chi_{33}^\varepsilon(\boldsymbol{w}^\varepsilon)) - O(\varepsilon^2)\|\chi_{33}^\varepsilon(\boldsymbol{w}^\varepsilon)\|_{L^2(\Omega)}.
\end{aligned}$$

Dividing by ε^2 and using the L^2 boundedness of $\varepsilon^{-1}\chi_{33}^\varepsilon(\boldsymbol{w}^\varepsilon)$ from (5.16b) we get

$$\frac{\mathfrak{C}^\varepsilon}{\varepsilon^2} \geq \int_\Omega \frac{1}{2} f_0(\frac{\chi_{33}^\varepsilon(\boldsymbol{w}^\varepsilon)}{\varepsilon}) + O(\varepsilon) = \int_\Omega \frac{1}{2} f_0(\partial_3 w_3^\varepsilon) + O(\varepsilon). \tag{5.24}$$

Thus we have,

$$\begin{aligned}
\liminf_{\varepsilon\to 0} \mathscr{I}_2^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) &\geq \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\left[\mathfrak{B}^\varepsilon - \varepsilon P_1 + \mathfrak{C}^\varepsilon\right] \\
&\geq \pi^2 |\boldsymbol{m}_p^o|^2 - 2\pi^2 |m_3^o|^2 + \liminf_{\varepsilon\to 0}\left\{ \int_\Omega \frac{1}{2} f_0(\partial_3 w_3^\varepsilon) + O(\varepsilon) \right\} \\
&= -2\pi^2 m_s^2 + \int_\Omega \frac{1}{2} f_0(\partial_3 w_3^0).
\end{aligned} \tag{5.25}$$

using (5.21) and (5.24) and $\boldsymbol{m}^o = (0,0,m_s)$. The last term comes from the fact that $\partial_3 w_3^0 \rightharpoonup \partial_3 w_3^\varepsilon$ in $L^2(\Omega)$ which means $\|\partial_3 w_3^0\|_{L^2(\Omega)} \leq \liminf \|\partial_3 w_3^\varepsilon\|_{L^2(\Omega)}$ and from (3.9), $\int_\Omega f_0(\partial_3 w_3^\varepsilon) = Y\|\partial_3 w_3^\varepsilon\|^2_{L^2(\Omega)}$. To get a the reverse inequality compare energy of $\mathscr{I}_2^\varepsilon$ at its minimizer $(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon)$ against a test function $(\boldsymbol{m}^o, \boldsymbol{U} = \boldsymbol{u}^o + (\boldsymbol{W}_p, \varepsilon W_3))$ where

$$\begin{aligned}
W_1 &= w_1^o(x_3) - \varepsilon^2 \sigma x_1 \partial_3 \gamma(x_3) + \varepsilon^2 \frac{\sigma}{2}(x_1^2 \partial_{33} w_1^o - x_2^2 \partial_{33} w_1^o + 2x_1 x_2 \partial_{33} w_2^o) \\
W_2 &= w_2^o(x_3) - \varepsilon^2 \sigma x_2 \partial_3 \gamma(x_3) + \varepsilon^2 \frac{\sigma}{2}(x_2^2 \partial_{33} w_2^o - x_1^2 \partial_{33} w_2^o + 2x_1 x_2 \partial_{33} w_1^o) \\
W_3 &= \gamma(x_3) - x_1 \partial_3 w_1^o - x_2 \partial_3 w_2^o - \varepsilon\sigma V''(x^2 + y^2) = w_3^o - \varepsilon\sigma V''(x^2 + y^2).
\end{aligned}$$

Then $\boldsymbol{\kappa}^\varepsilon(\boldsymbol{U}) - \boldsymbol{E}_s(\boldsymbol{m}^o) = \boldsymbol{\kappa}^\varepsilon(\boldsymbol{u}^o) - \boldsymbol{E}_s(\boldsymbol{m}^o) + \boldsymbol{\chi}^\varepsilon(\boldsymbol{W}) = \boldsymbol{\chi}^\varepsilon(\boldsymbol{W})$ where $\varepsilon^{-1}\boldsymbol{\chi}^\varepsilon(\boldsymbol{W})$ converges as:

$$\varepsilon^{-1}\boldsymbol{\chi}_{11}^\varepsilon(\boldsymbol{W}) = \varepsilon^{-1}\boldsymbol{\chi}_{22}^\varepsilon(\boldsymbol{W}) = -\sigma(\gamma(x_3) - x_1\partial_3 w_1^o - x_2 \partial_3 w_2^o), \qquad \varepsilon^{-1}\boldsymbol{\chi}_{12}^\varepsilon(\boldsymbol{W}) = 0$$

$$\varepsilon^{-1}\chi^\varepsilon_{12}(\boldsymbol{W}) = \frac{1}{2}\Big\{-\varepsilon\sigma x_1\partial_{33}\gamma(x_3) + \varepsilon\frac{\sigma}{2}(x_1^2\partial_{333}w_1^o - x_2^2\partial_{333}w_1^o + 2x_1x_2\partial_{333}w_2^o)\Big\} = O(\varepsilon)$$

$$\varepsilon^{-1}\chi^\varepsilon_{13}(\boldsymbol{W}) = \frac{1}{2}\Big\{-\varepsilon\sigma x_2\partial_{33}\gamma(x_3) + \varepsilon\frac{\sigma}{2}(x_2^2\partial_{333}w_2^o - x_1^2\partial_{333}w_2^o + 2x_1x_2\partial_{333}w_1^o)\Big\} = O(\varepsilon)$$

$$\varepsilon^{-1}\chi^\varepsilon_{33}(\boldsymbol{W}) = \partial_3 W_3 = \partial_3\gamma(x_3) - x_1\partial_{33}w_1^o - x_2\partial_{33}w_2^o.$$

Noting that $\mathbb{C}[\boldsymbol{\kappa}^\varepsilon(\boldsymbol{U}) - \boldsymbol{E}_s(\boldsymbol{m}^o)]^2 = f_0(\varepsilon\partial_3 W_3) + O(\varepsilon)$ gives

$$\mathscr{I}_2^\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{u}^\varepsilon) \le \mathscr{I}_2^\varepsilon(\boldsymbol{m}^o, \boldsymbol{U}) = \frac{\mathscr{I}^\varepsilon(\boldsymbol{m}^o, \boldsymbol{U}) - \varepsilon P_1 - P_0}{\varepsilon^2}$$
$$= -2\pi^2 m_s^2 + \int_\Omega f_0(\partial_3\gamma(x_3) - x_1\partial_{33}w_1^o - x_2\partial_{33}w_2^o) + O(\varepsilon).$$

Taking $\limsup$ as $\varepsilon \to 0$ we get our result. □

6 Summary

Figure 3: Bent wires of Galfenol

From the form of second variational limit and the third variational limit, it is clear that the magnetization remains strongly stabilized at $\boldsymbol{m}^o$, which minimizes the first order limit variational problem $\mathscr{I}^o(\boldsymbol{m}, v)$. Higher order theories do not add correctors to $\boldsymbol{m}^o$ within the framework of geometrically linear theory of magnetostriction. The displacement solution $\boldsymbol{u}^o$ corresponding to the first variational problem is however corrected due to the appearance of the bending energy terms in the third variational limit $\mathscr{I}_2^o(w_1, w_2, v)$. Nontrivial correctors to $\boldsymbol{m}^o$ may appear if we start with a geometrically nonlinear theory for magnetostriction. For geometrically nonlinear deformations however, the problem is significantly harder as the magnetic energies in the starting energy (1.7) will be defined on the deformed configuration, while typically in nonlinear elasticity, the free energy is defined over the reference configuration.

Recall the energy $\mathscr{I}_2^o(w_1(x_3), w_2(x_3), v(x_3))$ was defined in the previous section as

$$\mathscr{I}_2^o(w_1(x_3), w_2(x_3), v(x_3)) = \int_\Omega \frac{1}{2}\left[f_0(x_1\partial_{33}w_1) + f_0(x_2\partial_{33}w_2) + f_0(\partial_3\gamma)\right] - 2\pi^2 m_s^2.$$

Note that the first and second term are exactly the bending energy that appears in classical Euler-Bernoulli theory. To see this note that from the definition of f_o in (3.10) we get,

$$\begin{aligned}\int_\Omega f_0(x_1\partial_{33}w_1(x_3))\boldsymbol{dx} = \int_0^1 \int_\omega Y x_1^2 \left|\partial_{33}w_1(x_3)\right|^2 &= \int_0^1 Y\left\{\int_\omega x_1^2 \boldsymbol{dx}_p\right\}\left|\partial_{33}w_1(x_3)\right|^2 dx_3 \\ &= \int_0^1 Y I_{22}\left|\partial_{33}w_1(x_3)\right|^2 dx_3\end{aligned}$$

where I_{22} is the moment of inertia.

Although we have not included any external applied force in our analysis, it can be included with very minor changes to our presentation. The galfenol wires in bending behave like purely elastic beams with additional magnetic term which comes thorough the interaction of the positive and negative poles created at the two ends of the wire by the magnetization $\boldsymbol{m}^0 = (0, 0, m_s)$. This contribution is a fixed energy at the order at which bending elastic terms appear.

The strong stabilization of the magnetization is borne out by experiments where nanowires have been bend using an AFM tip. The Figure 3 shows the MFM scan for a galfenol wire in bent shape. The details of the experiment are available from [Downey, 2008]. The MFM scan shows the same bright and dark spots at the two ends of the wire characteristic of axially magnetized wires as seen in Figure 2. The bright spot in the middle was detected to be a topological defect. It is clear that even the large bending is unable to alter the axial magnetization, which can be interpreted as being equal to $\boldsymbol{m}^o$.

From the point of view of using Galfenol as a potential material for sensor application, the strong stabilization is not encouraging, as a designer would hope that the magnetization would change drastically from $\boldsymbol{m}^o$ on imposing any bending deformation. Newer proposals for sensor design using Galfenol have been made which replace the wire array of Galfenol with an array where each wire is multilayered with fine layers of magnetic Galfenol and non-magnetic Copper (cf. [Park et al., 2010]). With other magnetostrictive materials, for eg. materials with strong crystalline anisotropy with wells not along the axis of the wire, the derived theory could give very different results.

A Magnetostatic calculations

A.1 Introduction

Recall in (2.10) we defined $\mathscr{E}_d^\varepsilon(\boldsymbol{m}) = \frac{1}{8\pi}\int_{\mathbb{R}^3} |\boldsymbol{h}_{\boldsymbol{m}}^\varepsilon(\boldsymbol{x})|^2 \boldsymbol{dx} = \frac{1}{\varepsilon^2}\frac{1}{8\pi}\int_{\mathbb{R}^3} |\widetilde{\boldsymbol{h}}_{\widetilde{\boldsymbol{m}}}^\varepsilon(\boldsymbol{y})|^2 \boldsymbol{dy}$. In this section we will work in the unrescaled magnetization $\widetilde{\boldsymbol{m}}$ and demag field $\widetilde{\boldsymbol{h}}_{\boldsymbol{m}}^\varepsilon$. We define

$$\widehat{\boldsymbol{m}}^\varepsilon(y_3) = |\omega_\varepsilon|^{-1}\int_{\omega_\varepsilon} \widetilde{\boldsymbol{m}}^\varepsilon(\boldsymbol{y}_p, y_3)\, \boldsymbol{dy}_p \tag{A.1}$$

Note unlike in (3.1), the average defined here is on the unrescaled variable $\widetilde{\boldsymbol{m}}^\varepsilon$ and the unscaled cross-section ω_ε. Similiar to Proposition 3.1 we have using Jensen inequality

$$\begin{aligned}\int_0^1 |\partial_3^{\boldsymbol{y}}\widehat{\boldsymbol{m}}^\varepsilon|^2 dy_3 &= \int_0^1 \left|\partial_3^{\boldsymbol{y}} \fint_{\omega_\varepsilon} \widetilde{\boldsymbol{m}}^\varepsilon\, \boldsymbol{dy}_p\right|^2 dy_3 = \int_0^1 \left|\fint_{\omega_\varepsilon} \partial_3^{\boldsymbol{y}}\widetilde{\boldsymbol{m}}^\varepsilon\, \boldsymbol{dy}_p\right|^2 dy_3 \\ &\le \int_0^1 \fint_{\omega_\varepsilon} \left|\partial_3^{\boldsymbol{y}}\widetilde{\boldsymbol{m}}^\varepsilon\right|^2 \boldsymbol{dy}_p dy_3 = \frac{1}{|\omega_\varepsilon|}\|\partial_3^{\boldsymbol{y}}\widetilde{\boldsymbol{m}}^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 = \frac{1}{|\omega|}\|\partial_3 \boldsymbol{m}^\varepsilon\|_{L^2(\Omega)}^2\end{aligned} \tag{A.2}$$

where in the last step we have rescaled $\widetilde{\boldsymbol{m}}^\varepsilon$ to $\boldsymbol{m}^\varepsilon$. Thus we get estimate $\|\widetilde{\boldsymbol{m}}^\varepsilon\|_{L^2(0,1)}$ in terms of $\|\boldsymbol{m}^\varepsilon\|_{L^2(\Omega)}$.

Let $\widetilde{\boldsymbol{h}}_{\widehat{\boldsymbol{m}}^\varepsilon}^\varepsilon$ solve Maxwell equation for $\widehat{\boldsymbol{m}}^\varepsilon$. To simplify magnetostatic estimates we need the following Lemma.

Lemma A.1. *The following inequality holds:*

$$\frac{1}{8\pi\varepsilon^2}\left|\int_{\mathbb{R}^3} |\widetilde{\boldsymbol{h}}_{\widetilde{\boldsymbol{m}}^\varepsilon}^\varepsilon(\boldsymbol{y})|^2\, \boldsymbol{dy} - \int_{\mathbb{R}^3} |\widetilde{\boldsymbol{h}}_{\widehat{\boldsymbol{m}}^\varepsilon}^\varepsilon(\boldsymbol{y})|^2\, \boldsymbol{dy}\right| \le D_0\|\nabla_p^{\boldsymbol{y}}\widetilde{\boldsymbol{m}}^\varepsilon\|_{L^2(\Omega_\varepsilon)} = D_0\|\nabla_p \boldsymbol{m}^\varepsilon\|_{L^2(\Omega)}. \tag{A.3}$$

Proof. We have $\|\widetilde{\boldsymbol{m}}^\varepsilon - \widehat{\boldsymbol{m}}^\varepsilon\|_{L^2(\omega^\varepsilon)}^2 \le R_1\varepsilon^2\|\nabla_p^{\boldsymbol{y}}\widetilde{\boldsymbol{m}}^\varepsilon\|_{L^2(\omega^\varepsilon)}^2$ using Poincaré inequality on a cross-section plane $\omega_\varepsilon(y_3)$, which on integrating on $(0,1)$ gives,

$$\|\widetilde{\boldsymbol{m}}^\varepsilon - \widehat{\boldsymbol{m}}^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 < R_1\,\varepsilon^2\int_{\Omega_\varepsilon} |\nabla_p^{\boldsymbol{y}}\widetilde{\boldsymbol{m}}^\varepsilon(\boldsymbol{y})|^2 \boldsymbol{dy}. \tag{A.4}$$

Since $\widetilde{\boldsymbol{h}}_{\widehat{\boldsymbol{m}}^\varepsilon}^\varepsilon$ satisfies Maxwell equation for $\widehat{\boldsymbol{m}}^\varepsilon$, by linearity $(\widetilde{\boldsymbol{h}}_{\widetilde{\boldsymbol{m}}^\varepsilon}^\varepsilon - \widetilde{\boldsymbol{h}}_{\widehat{\boldsymbol{m}}^\varepsilon}^\varepsilon)$ satisfies Maxwell equation for $(\widetilde{\boldsymbol{m}}^\varepsilon - \widehat{\boldsymbol{m}}^\varepsilon)$. Then using basic bound (1.3) for Maxwell equation we have,

$$\frac{1}{8\pi}\left\|\widetilde{\boldsymbol{h}}_{\widetilde{\boldsymbol{m}}^\varepsilon}^\varepsilon - \widetilde{\boldsymbol{h}}_{\widehat{\boldsymbol{m}}^\varepsilon}^\varepsilon\right\|_{L^2(\mathbb{R}^3)}^2 \le \frac{1}{2}\left\|\widetilde{\boldsymbol{m}}^\varepsilon - \widehat{\boldsymbol{m}}^\varepsilon\right\|_{L^2(\Omega_\varepsilon)}^2 < \frac{R_1}{2}\,\varepsilon^2\left\|\nabla_p^{\boldsymbol{y}}\widetilde{\boldsymbol{m}}^\varepsilon\right\|_{L^2(\Omega_\varepsilon)}^2.$$

Using triangle inequality we also have,

$$\left|\,\left\|\widetilde{\boldsymbol{h}}_{\widetilde{\boldsymbol{m}}^\varepsilon}^\varepsilon\right\|_{L^2(\mathbb{R}^3)} - \left\|\widetilde{\boldsymbol{h}}_{\widehat{\boldsymbol{m}}^\varepsilon}^\varepsilon\right\|_{L^2(\mathbb{R}^3)}\,\right| \le \left\|\widetilde{\boldsymbol{h}}_{\widetilde{\boldsymbol{m}}^\varepsilon}^\varepsilon - \widetilde{\boldsymbol{h}}_{\widehat{\boldsymbol{m}}^\varepsilon}^\varepsilon\right\|_{L^2(\mathbb{R}^3)} < R_2\,\varepsilon\left\|\nabla_p^{\boldsymbol{y}}\widetilde{\boldsymbol{m}}^\varepsilon\right\|_{L^2(\Omega_\varepsilon)}.$$

Jensen's inequality gives $|\widehat{\boldsymbol{m}}^{\varepsilon}| \leq m_s$ and using (1.3) for $\widetilde{\boldsymbol{m}}^{\varepsilon}$ and $\widehat{\boldsymbol{m}}^{\varepsilon}$ we have,

$$\Big|\, \|\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widetilde{\boldsymbol{m}}^{\varepsilon}}\|_{L^2(\mathbb{R}^3)} + \|\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widehat{\boldsymbol{m}}^{\varepsilon}}\|_{L^2(\mathbb{R}^3)} \,\Big| \leq \|\widetilde{\boldsymbol{m}}^{\varepsilon}\|_{L^2(\Omega_\varepsilon)} + \|\widehat{\boldsymbol{m}}^{\varepsilon}\|_{L^2(\Omega_\varepsilon)} < R_3\,\varepsilon.$$

Then

$$\frac{1}{8\pi\varepsilon^2}\Big|\, \|\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widetilde{\boldsymbol{m}}^{\varepsilon}}\|^2_{L^2(\mathbb{R}^3)} - \|\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widehat{\boldsymbol{m}}^{\varepsilon}}\|^2_{L^2(\mathbb{R}^3)} \Big| \leq \frac{R_2\,\varepsilon}{8\pi\varepsilon^2}\|\nabla^{\boldsymbol{y}}_p\widetilde{\boldsymbol{m}}^{\varepsilon}\|_{L^2(\Omega_\varepsilon)}\cdot R_3\,\varepsilon = D_0\,\|\nabla^{\boldsymbol{y}}_p\widetilde{\boldsymbol{m}}^{\varepsilon}\|_{L^2(\Omega_\varepsilon)}.$$

Rescaling $\widetilde{\boldsymbol{m}}^{\varepsilon}$ to $\boldsymbol{m}^{\varepsilon}$ we complete the proof. □

Remark A.1. *From* (3.3) *we know that the exchange energy of magnetization* $\boldsymbol{m}^{\varepsilon}$ *is bounded as* $K_5 \geq \frac{d}{\varepsilon^2}\|\nabla_p \boldsymbol{m}^{\varepsilon}\|_{L^2(\Omega)}$. *Since* $\mathscr{E}^{\varepsilon}_d(\boldsymbol{m}) = \frac{1}{8\pi}\|\boldsymbol{h}^{\varepsilon}_{\boldsymbol{m}}\|^2_{L^2(\mathbb{R}^3)} = \frac{1}{8\pi\varepsilon^2}\|\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widetilde{\boldsymbol{m}}}\|_{L^2(\mathbb{R}^3)}$, *we then get from the Lemma*

$$\begin{aligned}\big|\mathscr{E}^{\varepsilon}_d(\boldsymbol{m}^{\varepsilon}) - \mathscr{E}^{\varepsilon}_d(\widehat{\boldsymbol{m}}^{\varepsilon})\big| = \frac{1}{8\pi\varepsilon^2}\Big|\, \|\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widetilde{\boldsymbol{m}}^{\varepsilon}}\|^2_{L^2(\mathbb{R}^3)} - \|\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widehat{\boldsymbol{m}}^{\varepsilon}}\|^2_{L^2(\mathbb{R}^3)}\Big| &\leq D_0\,\|\nabla_p\boldsymbol{m}^{\varepsilon}\|_{L^2(\Omega)} \\ &\leq D_0\frac{K_5\varepsilon^2}{d} = O(\varepsilon^2).\end{aligned}$$

Thus the difference in magnetostatic energy between $\mathscr{E}^{\varepsilon}_d(\boldsymbol{m}^{\varepsilon})$ *and* $\mathscr{E}^{\varepsilon}_d(\widehat{\boldsymbol{m}}^{\varepsilon})$ *is of order* $O(\varepsilon^2)$. *Hence for the convergence arguments we will only estimate* $\mathscr{E}^{\varepsilon}_d(\widehat{\boldsymbol{m}}^{\varepsilon}) = \frac{1}{8\pi\varepsilon^2}\|\widetilde{\boldsymbol{h}}^{\varepsilon}_{\widehat{\boldsymbol{m}}^{\varepsilon}}\|^2_{L^2(\mathbb{R}^3)}$.

Henceforth we drop the $\boldsymbol{y}$ superscript on the derivative operator and ε superscript on $\widehat{\boldsymbol{m}}^{\varepsilon}$. Then $\widehat{\boldsymbol{m}}(y_3) \in H^1(0,1)$ and $\nabla\cdot\widehat{\boldsymbol{m}}(\boldsymbol{y}) = \partial_3\widehat{m}_3(y_3)$. It is well know that for magnetization $\widehat{\boldsymbol{m}}$, the energy $\frac{1}{8\pi\varepsilon^2}\|\widehat{\boldsymbol{h}}^{\varepsilon}_{\widehat{\boldsymbol{m}}}\|^2_{L^2(\mathbb{R}^3)}$ can be written as a convolution of fundamental solutions with $\widehat{\boldsymbol{m}}$,

$$\begin{aligned}\frac{\varepsilon^{-2}}{8\pi}\|\widehat{\boldsymbol{h}_{\boldsymbol{m}}}\|^2_{L^2} &= \frac{1}{2}\int_{\Omega_\varepsilon}\int_{\Omega_\varepsilon}\frac{\nabla\cdot\widehat{\boldsymbol{m}}^{\varepsilon}(\boldsymbol{y})\;\nabla\cdot\widehat{\boldsymbol{m}}^{\varepsilon}(\boldsymbol{z})}{\varepsilon^2|\boldsymbol{y}-\boldsymbol{z}|} + \frac{1}{2}\int_{\partial\Omega_\varepsilon}\int_{\partial\Omega_\varepsilon}\frac{\widehat{\boldsymbol{m}}^{\varepsilon}(\boldsymbol{y})\cdot\tilde{\boldsymbol{n}}(\boldsymbol{y})\;\widehat{\boldsymbol{m}}^{\varepsilon}(\boldsymbol{z})\cdot\tilde{\boldsymbol{n}}(\boldsymbol{z})}{\varepsilon^2|\boldsymbol{y}-\boldsymbol{z}|} \\ &\quad - \int_{\Omega_\varepsilon}\int_{\partial\Omega_\varepsilon}\frac{\nabla\cdot\widehat{\boldsymbol{m}}^{\varepsilon}(\boldsymbol{y})\;\widehat{\boldsymbol{m}}^{\varepsilon}(\boldsymbol{z})\cdot\tilde{\boldsymbol{n}}(\boldsymbol{z})}{\varepsilon^2|\boldsymbol{y}-\boldsymbol{z}|} \\ &= \frac{1}{2}\int_{\Omega_\varepsilon}\int_{\Omega_\varepsilon}\frac{\partial_3\widehat{m}^{\varepsilon}_3(y_3)\;\partial_3\widehat{m}^{\varepsilon}_3(z_3)}{\varepsilon^2|\boldsymbol{y}-\boldsymbol{z}|} + \frac{1}{2}\int_{\partial\Omega_\varepsilon}\int_{\partial\Omega_\varepsilon}\frac{\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{y})\;\widehat{\boldsymbol{m}}^{\varepsilon}(z_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{z})}{\varepsilon^2|\boldsymbol{y}-\boldsymbol{z}|} \\ &\quad - \int_{\Omega_\varepsilon}\int_{\partial\Omega_\varepsilon}\frac{\partial_3\widehat{m}^{\varepsilon}_3(y_3)\;\widehat{\boldsymbol{m}}^{\varepsilon}(z_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{z})}{\varepsilon^2|\boldsymbol{y}-\boldsymbol{z}|} \\ &= \frac{1}{2}J^{\varepsilon}_1(\widehat{\boldsymbol{m}}^{\varepsilon}) + \frac{1}{2}J^{\varepsilon}_2(\widehat{\boldsymbol{m}}^{\varepsilon}) + J^{\varepsilon}_3(\widehat{\boldsymbol{m}}^{\varepsilon}).\end{aligned}$$

Note that $J^{\varepsilon}_1, J^{\varepsilon}_2$ and J^{ε}_3 respectively represent that "Bulk-Bulk", the "Boundary-Boundary" and the "Bulk-Boundary" terms of the magnetostatic energy. The body $\Omega_\varepsilon = \omega_\varepsilon \times (0,1)$ and the boundary $\partial\Omega_\varepsilon$ can be decomposed as as $\partial\Omega_\varepsilon = \big\{\partial\omega_\varepsilon \times (0,1)\big\} \bigcup \omega_\varepsilon(y_3 = 0) \bigcup \omega_\varepsilon(y_3 = 1)$.

A.2 Estimates of $J_1^\varepsilon(\widehat{\boldsymbol{m}}), J_2^\varepsilon(\widehat{\boldsymbol{m}})$, and $J_3^\varepsilon(\widehat{\boldsymbol{m}})$

The magnetostatic estimates in this section are inspired by similiar estimates in other works like [Kohn and Slastikov, 2005] and [Carbou, 2001]. We use the following integral inequality in this section: for arbitrary $a \neq b \in \mathbb{R}$ and $q, L \in \mathbb{R}$ using the fact that $q(q^2+L^2)^{-1/2} \leq 1$ we have

$$\int_a^b \frac{1}{\{L^2+q^2\}^{3/2}} dq = \frac{1}{L^2} \frac{q}{(L^2+q^2)^{1/2}}\Big|_a^b = \frac{1}{L^2}\Big(\frac{b}{(L^2+b^2)^{1/2}} - \frac{a}{(L^2+a^2)^{1/2}}\Big) \leq \frac{2}{L^2}. \quad \text{(A.6)}$$

We also need an estimate of the following term, where we use the change of variable $\boldsymbol{w}_p = \boldsymbol{y}_p - \boldsymbol{z}_p$, $\boldsymbol{dw}_p = \boldsymbol{dy}_p$ to get, $\Big($Recall ω_ε is a ball of radius ε in 2-d$\Big)$

$$\begin{aligned}\int_{\omega_\varepsilon}\int_{\omega_\varepsilon} \frac{\boldsymbol{dy}_p\, \boldsymbol{dz}_p}{|\boldsymbol{y}_p - \boldsymbol{z}_p|} = \int_{\omega_\varepsilon} \boldsymbol{dz}_p \int_{\omega_\varepsilon - \boldsymbol{z}_p} \frac{\boldsymbol{dw}_p}{|\boldsymbol{w}_p|} &\leq \int_{\omega_\varepsilon} \boldsymbol{dz}_p \int_{\omega_{3\varepsilon}} \frac{\boldsymbol{dw}_p}{|\boldsymbol{w}_p|} \\ &= \int_{\omega_\varepsilon} \boldsymbol{dz}_p \int_0^{2\pi}\int_0^{3\varepsilon} \frac{|\boldsymbol{w}_p| \boldsymbol{d}|\boldsymbol{w}_p| d\theta}{|\boldsymbol{w}_p|} \\ &= (\pi\varepsilon^2)\,(2\pi)\,(3\varepsilon) = 6\pi^2\varepsilon^3, \quad \text{(A.7)}\end{aligned}$$

where we have used the fact that $(\omega_\varepsilon - \boldsymbol{z}_p) \subset \omega_{3\varepsilon}$ for $\boldsymbol{z}_p \in \omega_\varepsilon$. All estimates in this section are of the type $J_i^\varepsilon(\widehat{\boldsymbol{m}})$, $\Big(\widehat{\boldsymbol{m}}$ is the cross-section average of the unscaled magnetization $\widetilde{\boldsymbol{m}}$ defined in (A.1)$\Big)$ in terms of the rescaled magnetization $\boldsymbol{m}$ and i is a string of integers.

Proposition A.1.

$$|\, J_1^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon)\,| \leq D_1 \varepsilon \|\partial_3 \boldsymbol{m}^\varepsilon\|^2_{L^2(\Omega)}.$$

Proof. Noting that $|\boldsymbol{y}_p - \boldsymbol{z}_p| \leq |\boldsymbol{y} - \boldsymbol{z}|$ we have

$$\begin{aligned}|\varepsilon^2 J_1^\varepsilon(\widehat{\boldsymbol{m}})| &\leq \int_{\Omega_\varepsilon}\int_{\Omega_\varepsilon} \frac{|\partial_3 \widehat{m}_3^\varepsilon(y_3)\, \partial_3 \widehat{m}_3^\varepsilon(z_3)|}{|\boldsymbol{y}-\boldsymbol{z}|} \boldsymbol{dydz} \leq \int_0^1\int_0^1\int_{\omega_\varepsilon}\int_{\omega_\varepsilon} \frac{|\partial_3 \widehat{m}_3^\varepsilon(y_3)\, \partial_3 \widehat{m}_3^\varepsilon(z_3)|}{|\boldsymbol{y}_p - \boldsymbol{z}_p|} \\ &= \int_0^1\int_0^1 |\partial_3 \widehat{m}_3^\varepsilon(y_3)\, \partial_3 \widehat{m}_3^\varepsilon(z_3)| \int_{\omega_\varepsilon}\int_{\omega_\varepsilon} \frac{\boldsymbol{dy}_p \boldsymbol{dz}_p}{|\boldsymbol{y}_p - \boldsymbol{z}_p|} \leq D_1 \varepsilon^3 \|\partial_3 \widehat{m}_3^\varepsilon\|^2_{L^2(0,1)}\end{aligned}$$

where we have used Hölder's inequality on the term $\int_0^1\int_0^1 |\partial_3 \widehat{m}_3(y_3)\, \partial_3 \widehat{m}_3(z_3)|$. Using equation (A.2) we get our result. □

Proposition A.2.

$$|\, J_3^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon)\,| \leq D_2\, \varepsilon \|\partial_3 \boldsymbol{m}^\varepsilon\|_{L^2(\Omega)}.$$

Proof. We split of $J_3^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon)$ into 2 parts,

$$-\varepsilon^2 J_3^\varepsilon(\widehat{\boldsymbol{m}}) = \int_{\omega_\varepsilon}\int_0^1\int_{\partial\omega_\varepsilon}\int_0^1 \frac{\widehat{\boldsymbol{m}}^\varepsilon(z_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{z}_p)}{|\boldsymbol{y}-\boldsymbol{z}|}\partial_3\widehat{m}_3^\varepsilon(y_3) + \int_{\omega_\varepsilon}\int_0^1 \partial_3\widehat{m}_3^\varepsilon(y_3)\times \Big[\int_{\omega_\varepsilon(0)}\frac{\widehat{\boldsymbol{m}}^\varepsilon(y_3=0)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{z})}{|\boldsymbol{y}-\boldsymbol{z}|} + \int_{\omega_\varepsilon(1)}\frac{\widehat{\boldsymbol{m}}^\varepsilon(y_3=1)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{z})}{|\boldsymbol{y}-\boldsymbol{z}|}\Big] =: J_{31}^\varepsilon + J_{32}^\varepsilon.$$

For J_{31}^ε using divergence theorem on $\partial\omega_\varepsilon(z_3)$ gives,

$$\begin{aligned}
\varepsilon^2 J_{31}^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon) &= \int_{\omega_\varepsilon}\int_0^1\int_{\partial\omega_\varepsilon}\int_0^1 \frac{\widehat{\boldsymbol{m}}^\varepsilon(z_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{z}_p)}{|\boldsymbol{y}-\boldsymbol{z}|}\partial_3\widehat{m}_3^\varepsilon(y_3)\,\boldsymbol{dy}_p\boldsymbol{d\sigma}(\boldsymbol{z}_p)dy_3dz_3 \\
&= \int_{\omega_\varepsilon}\int_0^1\int_0^1 \partial_3\widehat{m}_3^\varepsilon(y_3)\,\boldsymbol{dy}_p dy_3dz_3\int_{\omega_\varepsilon}\nabla_p^z\cdot\Big\{\frac{\widehat{\boldsymbol{m}}^\varepsilon(z_3)}{|\boldsymbol{y}-\boldsymbol{z}|}\Big\}\,\boldsymbol{dz}_p \\
&= -\int_{\omega_\varepsilon}\int_0^1\int_0^1\int_{\omega_\varepsilon}\partial_3\widehat{m}_3^\varepsilon(y_3)\frac{\widehat{\boldsymbol{m}}^\varepsilon(z_3)\cdot(\boldsymbol{y}_p-\boldsymbol{z}_p)}{\big\{|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+(y_3-z_3)^2\big\}^{3/2}}\boldsymbol{dydz}.
\end{aligned} \tag{A.8}$$

Setting $q=(z_3-y_3)$ and $dz_3=dq$ gives,

$$|\varepsilon^2 J_{31}^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon)| \le \sup_{z_3}|\widehat{\boldsymbol{m}}^\varepsilon(z_3)|\int_{\omega_\varepsilon}\int_{\omega_\varepsilon}\int_0^1\big|\partial_3\widehat{m}_3^\varepsilon(y_3)\big|\int_{-y_3}^{1-y_3}\frac{m_s|\boldsymbol{y}_p-\boldsymbol{z}_p|}{\big\{|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+q^2\big\}^{3/2}}dq.$$

Using equation (A.6) on the inner integral gives

$$\begin{aligned}
|\varepsilon^2 J_{31}^\varepsilon| &\le 2\sup_{z_3}|\widehat{\boldsymbol{m}}^\varepsilon(z_3)|\int_{\omega_\varepsilon}\int_{\omega_\varepsilon}\int_0^1\frac{|\partial_3\widehat{m}_3^\varepsilon(y_3)|}{|\boldsymbol{y}_p-\boldsymbol{z}_p|} \\
&\le 2\sup_{z_3}|\widehat{\boldsymbol{m}}^\varepsilon(z_3)|\int_0^1|\partial_3\widehat{m}_3^\varepsilon(y_3)|\,dy_3\int_{\omega_\varepsilon}\int_{\omega_\varepsilon}\frac{\boldsymbol{dy}_p\boldsymbol{dz}_p}{|\boldsymbol{y}_p-\boldsymbol{z}_p|} \\
&\le D_2\varepsilon^3\sup_{z_3}|\widehat{\boldsymbol{m}}^\varepsilon(z_3)|\|\partial_3\widehat{\boldsymbol{m}}^\varepsilon\|_{L^2(0,1)}.
\end{aligned}$$

$$\begin{aligned}
\varepsilon^2|J_{32}^\varepsilon| &\le \big(|\widehat{m}_3^\varepsilon(0)|+|\widehat{m}_3^\varepsilon(1)|\big)\int_{\omega_\varepsilon}\int_0^1\int_{\omega_\varepsilon}\Big[\frac{|\partial_3\widehat{m}_3^\varepsilon(y_3)|}{\sqrt{|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+y_3^2}}+\frac{|\partial_3\widehat{m}_3^\varepsilon(y_3)|}{\sqrt{|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+(1-y_3)^2}}\Big] \\
&\le \big(|\widehat{m}_3^\varepsilon(0)|+|\widehat{m}_3^\varepsilon(1)|\big)\int_0^1|\partial_3\widehat{m}_3^\varepsilon(y_3)|dy_3\int_{\omega_\varepsilon}\int_{\omega_\varepsilon}\frac{\boldsymbol{dy}_p\boldsymbol{dz}_p}{|\boldsymbol{y}_p-\boldsymbol{z}_p|} \\
&\le D_2\varepsilon^3\big(|\widehat{m}_3^\varepsilon(0)|+|\widehat{m}_3^\varepsilon(1)|\big)\,\|\partial_3\widehat{\boldsymbol{m}}^\varepsilon\|_{L^2(0,1)}.
\end{aligned}$$

Combining estimates for $J_{31}^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon)$ and $J_{32}^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon)$, noting $|\widehat{m}_3^\varepsilon|\le m_s$ and using equation (A.2) we get our result. □

Next we write $J_2^\varepsilon = J_{21}^\varepsilon + J_{22}^\varepsilon + J_{23}^\varepsilon + J_{24}^\varepsilon$ where,

$$\varepsilon^2 J_{21}^{\varepsilon} = \int_{\partial\omega_\varepsilon}\int_0^1\int_{\partial\omega_\varepsilon}\int_0^1 \frac{\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)\cdot\widetilde{\boldsymbol{n}}(\boldsymbol{y})\,\widehat{\boldsymbol{m}}^{\varepsilon}(z_3)\cdot\widetilde{\boldsymbol{n}}(\boldsymbol{z})}{\left\{\,|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+(y_3-z_3)^2\,\right\}^{1/2}},$$

$$\varepsilon^2 J_{22}^{\varepsilon} = \int_{\omega_\varepsilon(0)}\int_{\omega_\varepsilon(0)} \frac{\widehat{\boldsymbol{m}}^{\varepsilon}(0)\cdot\widetilde{\boldsymbol{n}}(\boldsymbol{y}_p)\,\widehat{\boldsymbol{m}}^{\varepsilon}(0)\cdot\widetilde{\boldsymbol{n}}(\boldsymbol{z}_p)}{|\boldsymbol{y}_p-\boldsymbol{z}_p|} + \int_{\omega_\varepsilon(1)}\int_{\omega_\varepsilon(1)} \frac{\widehat{\boldsymbol{m}}^{\varepsilon}(1)\cdot\widetilde{\boldsymbol{n}}(\boldsymbol{y}_p)\,\widehat{\boldsymbol{m}}^{\varepsilon}(1)\cdot\widetilde{\boldsymbol{n}}(\boldsymbol{z}_p)}{|\boldsymbol{y}_p-\boldsymbol{z}_p|},$$

$$\varepsilon^2 J_{23}^{\varepsilon} = 2\int_{\omega_\varepsilon(0)}\int_{\omega_\varepsilon(1)} \frac{\widehat{\boldsymbol{m}}^{\varepsilon}(0)\cdot\widetilde{\boldsymbol{n}}(\boldsymbol{y}_p)\,\widehat{\boldsymbol{m}}^{\varepsilon}(1)\cdot\widetilde{\boldsymbol{n}}(\boldsymbol{z}_p)}{\left\{\,|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+1\,\right\}^{1/2}} \quad \text{and,}$$

$$\frac{\varepsilon^2 J_{24}^{\varepsilon}}{2} = \int_{\partial\omega_\varepsilon}\int_0^1 \left[\int_{\omega_\varepsilon(0)} \frac{\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)\cdot\widetilde{\boldsymbol{n}}(\boldsymbol{y})\,\widehat{\boldsymbol{m}}^{\varepsilon}(0)\cdot\widetilde{\boldsymbol{n}}(\boldsymbol{z}_p)}{\left\{\,|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+y_3^2\,\right\}^{1/2}} + \int_{\omega_\varepsilon(1)} \frac{\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)\cdot\widetilde{\boldsymbol{n}}(\boldsymbol{y})\,\widehat{\boldsymbol{m}}^{\varepsilon}(1)\cdot\widetilde{\boldsymbol{n}}(\boldsymbol{z}_p)}{\left\{|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+(1-y_3)^2\right\}^{1/2}}\right].$$

Noting that $\widehat{\boldsymbol{m}}^{\varepsilon}(t)\cdot\widetilde{\boldsymbol{n}}(\boldsymbol{z}) = -\widehat{m}_3^{\varepsilon}(0)$ for $t=0$ and $\widehat{m}_3^{\varepsilon}(1)$ for $t=1$ we have,

$$\varepsilon^2 J_{22}^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) = (|\widehat{m}_3^{\varepsilon}(0)|^2+|\widehat{m}_3^{\varepsilon}(1)|^2)\int_{\omega_\varepsilon}\int_{\omega_\varepsilon} \frac{\boldsymbol{d}\boldsymbol{y}_p\,\boldsymbol{d}\boldsymbol{z}_p}{|\boldsymbol{y}_p-\boldsymbol{z}_p|} = O(\varepsilon^3) \tag{A.9}$$

$$\varepsilon^2 J_{24}^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) = 2\int_{\partial\omega_\varepsilon}\int_0^1\int_{\omega_\varepsilon} \frac{-\widehat{m}_3^{\varepsilon}(0)\,\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)\cdot\widetilde{\boldsymbol{n}}(\boldsymbol{y})}{\left\{\,|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+y_3^2\,\right\}^{1/2}} + \frac{\widehat{m}_3^{\varepsilon}(1)\,\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)\cdot\widetilde{\boldsymbol{n}}(\boldsymbol{y})}{\left\{|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+(1-y_3)^2\right\}^{1/2}}. \tag{A.10}$$

Since $1 \gg \boldsymbol{y}_p-\boldsymbol{z}_p = O(\varepsilon)$, using Binomial expansion $\left(|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+1\right)^{-\frac{1}{2}} = 1+O(\varepsilon^2)$ we get,

$$\varepsilon^2\frac{J_{23}^{\varepsilon}}{2}(\widehat{\boldsymbol{m}}^{\varepsilon}) = -\widehat{m}_3^{\varepsilon}(0)\widehat{m}_3^{\varepsilon}(1)\int_{\omega_\varepsilon}\int_{\omega_\varepsilon}\left(1+O(\varepsilon^2)\right) = -\pi^2\varepsilon^4\widehat{m}_3^{\varepsilon}(0)\,\widehat{m}_3^{\varepsilon}(1)+O(\varepsilon^6). \tag{A.11}$$

Proposition A.3.

$$|\,J_{24}^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon})\,| \le D_3\varepsilon(|\widehat{m}_3^{\varepsilon}(0)|+|\widehat{m}_3^{\varepsilon}(1)|).$$

Proof. As for term the J_{31}^{ε} in Proposition A.2, first using divergence theorem in J_{24}^{ε} from (A.10) on $\partial\omega_\varepsilon(y_3)$ we get

$$\varepsilon^2 J_{24}^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) = -2\int_{\omega_\varepsilon}\int_{\omega_\varepsilon}\int_0^1 \frac{-\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)\cdot(\boldsymbol{y}_p-\boldsymbol{z}_p)\,\widehat{m}_3^{\varepsilon}(0)}{\left\{\,|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+y_3^2\right\}^{3/2}} + \frac{\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)\cdot(\boldsymbol{y}_p-\boldsymbol{z}_p)\,\widehat{m}_3^{\varepsilon}(1)}{\left\{\,|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+(1-y_3)^2\right\}^{3/2}}.$$

Then using $\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)\cdot(\boldsymbol{y}_p-\boldsymbol{z}_p) \le m_s|\boldsymbol{y}_p-\boldsymbol{z}_p|$ and (A.6) we get,

$$\begin{aligned}|\varepsilon^2 J_{24}^{\varepsilon}| &= \left|2m_s\int_{\omega_\varepsilon}\int_{\omega_\varepsilon}\frac{|\boldsymbol{y}_p-\boldsymbol{z}_p|}{|\boldsymbol{y}_p-\boldsymbol{z}_p|^2}\left[\frac{-y_3\widehat{m}_3^{\varepsilon}(0)}{\left\{\,|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+y_3^2\right\}^{1/2}} + \frac{(1-y_3)\widehat{m}_3^{\varepsilon}(1)}{\left\{\,|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+(1-y_3)^2\right\}^{1/2}}\right]\Big|_0^1\right| \\ &\le 2m_s\int_{\omega_\varepsilon}\int_{\omega_\varepsilon}\frac{|\widehat{m}_3^{\varepsilon}(0)|+|\widehat{m}_3^{\varepsilon}(1)|}{|\boldsymbol{y}_p-\boldsymbol{z}_p|} = D_3\varepsilon^3\left(\,|\widehat{m}_3^{\varepsilon}(0)|+|\widehat{m}_3^{\varepsilon}(1)|\,\right).\end{aligned}$$

□

We will now show that $J^{\varepsilon}_{21}(\widehat{\boldsymbol{m}}^{\varepsilon})$ is the largest term in the magnetostatic terms. It contributes energy of $O(1)$ which appears in the first limit problem $\mathscr{I}_0$. We split $J^{\varepsilon}_{21}(\widehat{\boldsymbol{m}}^{\varepsilon})$ as follows:

$$\begin{aligned}
J^{\varepsilon}_{21}(\widehat{\boldsymbol{m}}^{\varepsilon}) &= \int_{\partial\omega_\varepsilon}\int_0^1\int_{\partial\omega_\varepsilon}\int_0^1 \frac{\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)\cdot\widetilde{\boldsymbol{n}}(\boldsymbol{y}_p)\,\widehat{\boldsymbol{m}}^{\varepsilon}(z_3)\cdot\widetilde{\boldsymbol{n}}(\boldsymbol{z}_p)}{\varepsilon^2|\boldsymbol{y}-\boldsymbol{z}|} \\
&= \int_{\partial\omega_\varepsilon}\int_0^1\int_{\partial\omega_\varepsilon}\int_0^1 \frac{\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)\cdot\widetilde{\boldsymbol{n}}(\boldsymbol{y}_p)\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)\cdot\widetilde{\boldsymbol{n}}(\boldsymbol{z}_p)}{\varepsilon^2|\boldsymbol{y}-\boldsymbol{z}|} \\
&\qquad -\int_{\partial\omega_\varepsilon}\int_0^1\int_{\partial\omega_\varepsilon}\int_0^1 \frac{\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)\cdot\widetilde{\boldsymbol{n}}(\boldsymbol{y}_p)\,(\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)-\widehat{\boldsymbol{m}}^{\varepsilon}(z_3))\cdot\widetilde{\boldsymbol{n}}(\boldsymbol{z}_p)}{\varepsilon^2|\boldsymbol{y}-\boldsymbol{z}|} \\
&= J^{\varepsilon}_{211}(\widehat{\boldsymbol{m}}^{\varepsilon}) + J^{\varepsilon}_{212}(\widehat{\boldsymbol{m}}^{\varepsilon}).
\end{aligned}$$

Next we show the following proposition.

Proposition A.4.

$$|J^{\varepsilon}_{212}(\widehat{\boldsymbol{m}}^{\varepsilon})| \le D_5\varepsilon^{3/4}\left\|\partial_3\boldsymbol{m}^{\varepsilon}\right\|_{L^2(\Omega)}.$$

Proof. Using Divergence theorem in $\boldsymbol{y}_p$ variable as in (A.8) and Fubini's theorem we get,

$$\begin{aligned}
\varepsilon^2 J^{\varepsilon}_{212} &= \int_{\partial\omega_\varepsilon}\int_0^1\int_0^1\int_{\partial\omega_\varepsilon} \frac{\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)\cdot\widetilde{\boldsymbol{n}}(\boldsymbol{y}_p)}{\sqrt{|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+(y_3-z_3)^2}}\,(\widehat{\boldsymbol{m}}^{\varepsilon}(z_3)-\widehat{\boldsymbol{m}}^{\varepsilon}(y_3))\cdot\widetilde{\boldsymbol{n}}(\boldsymbol{z}_p)\boldsymbol{d\sigma}(\boldsymbol{y}_p) \\
&= -\int_{\partial\omega_\varepsilon}\int_0^1\int_0^1\int_{\omega_\varepsilon} \frac{\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)\cdot(\boldsymbol{y}_p-\boldsymbol{z}_p)}{\left\{|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+(y_3-z_3)^2\right\}^{3/2}}\,(\widehat{\boldsymbol{m}}^{\varepsilon}(z_3)-\widehat{\boldsymbol{m}}^{\varepsilon}(y_3))\cdot\widetilde{\boldsymbol{n}}(\boldsymbol{z}_p)\boldsymbol{d}\boldsymbol{y}_p \\
&= -\int_{\partial\omega_\varepsilon}\int_{\omega_\varepsilon}\widetilde{\boldsymbol{n}}(\boldsymbol{z}_p)\cdot\int_0^1\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)\cdot(\boldsymbol{y}_p-\boldsymbol{z}_p)\int_0^1\frac{\widehat{\boldsymbol{m}}^{\varepsilon}(z_3)-\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)}{\left\{|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+(y_3-z_3)^2\right\}^{3/2}}\,dz_3.
\end{aligned}$$

Now note that $\frac{|\boldsymbol{y}_p-\boldsymbol{z}_p|}{\left\{|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+(y_3-z_3)^2\right\}^{1/2}} \le 1$ and $|\widetilde{\boldsymbol{n}}(\boldsymbol{z}_p)| = 1$ which gives

$$\left|\varepsilon^2 J^{\varepsilon}_{212}(\widehat{\boldsymbol{m}}^{\varepsilon}(y_3))\right| \le \sup_{y_3}|\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)|\int_{\partial\omega_\varepsilon}\int_{\omega_\varepsilon}\int_0^1\int_0^1\frac{|\widehat{\boldsymbol{m}}^{\varepsilon}(z_3)-\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)|}{\left\{|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+(y_3-z_3)^2\right\}}\,dz_3.$$

Let $\frac{1}{4} > \delta > 0$ by a fixed number. Then we note that,

$$\frac{1}{\left\{|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+(y_3-z_3)^2\right\}} \le \frac{1}{|\boldsymbol{y}_p-\boldsymbol{z}_p|^{1/4}}\frac{1}{|y_3-z_3|^{7/4}} \tag{A.12}$$

Then

$$\begin{aligned}
\int_0^1\int_0^1\frac{|\widehat{\boldsymbol{m}}^{\varepsilon}(z_3)-\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)|dz_3dy_3}{\left\{|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+(y_3-z_3)^2\right\}} &\le \frac{1}{|\boldsymbol{y}_p-\boldsymbol{z}_p|^{1/4}}\int_0^1\int_0^1\frac{|\widehat{\boldsymbol{m}}^{\varepsilon}(z_3)-\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)|}{|y_3-z_3|^{7/4}}\,dz_3dy_3 \\
&\le \frac{1}{|\boldsymbol{y}_p-\boldsymbol{z}_p|^{1/4}}\left\|\partial_3\widehat{\boldsymbol{m}}^{\varepsilon}(y_3)\right\|_{L^1(0,1)}
\end{aligned}$$

because of the fact that $\int_0^1\int_0^1 \frac{|\widehat{\boldsymbol{m}}^\varepsilon(z_3)-\widehat{\boldsymbol{m}}^\varepsilon(y_3)|}{|y_3-z_3|^{7/4}}\,dz_3dy_3$ denotes the seminorm in the fractional Sobolev space $W^{\frac{3}{4},1}(0,1)$ and by the continuous embedding of $W^{1,1}(0,1)\subset W^{\frac{3}{4},1}(0,1)$. Also note using Hölder inequality

$$\int_0^1 \left|\partial_3\widehat{\boldsymbol{m}}^\varepsilon\right| dy_3 = \int_0^1 \left|\partial_3\widehat{\boldsymbol{m}}^\varepsilon\right| \chi_{(0,1)} dy_3 \le \|\chi_{(0,1)}\|_{L^2(0,1)} \left\|\partial_3\widehat{\boldsymbol{m}}^\varepsilon\right\|_{L^2(0,1)} = \left\|\partial_3\widehat{\boldsymbol{m}}^\varepsilon\right\|_{L^2(0,1)}.$$

Then we get,

$$\begin{aligned}
\left|\varepsilon^2 J^\varepsilon_{212}(\widehat{\boldsymbol{m}}^\varepsilon(y_3))\right| &\le \sup_{y_3}|\widehat{\boldsymbol{m}}^\varepsilon(y_3)| \int_{\partial\omega_\varepsilon}\int_{\omega_\varepsilon} \frac{\boldsymbol{dy_p d\sigma(z_p)}}{|\boldsymbol{y}_p-\boldsymbol{z}_p|^{1/4}} \int_0^1\int_0^1 \frac{|\widehat{\boldsymbol{m}}^\varepsilon(z_3)-\widehat{\boldsymbol{m}}^\varepsilon(y_3)|}{|y_3-z_3|^{7/4}} \\
&\le \sup_{y_3}|\widehat{\boldsymbol{m}}^\varepsilon(y_3)| \left\|\partial_3\widehat{\boldsymbol{m}}^\varepsilon\right\|_{L^2(0,1)} \int_{\partial\omega_\varepsilon}\int_{\omega_\varepsilon} \frac{\boldsymbol{dy_p d\sigma(z_p)}}{|\boldsymbol{y}_p-\boldsymbol{z}_p|^{1/4}} \\
&= D_5\varepsilon^2\varepsilon^{3/4}\ \sup_{y_3}|\widehat{\boldsymbol{m}}^\varepsilon(y_3)|\ \left\|\partial_3\widehat{\boldsymbol{m}}^\varepsilon\right\|_{L^2(0,1)}
\end{aligned} \tag{A.13}$$

using a result similar to equation (A.7). We get our result on noting that $\sup_{y_3}|\widehat{\boldsymbol{m}}^\varepsilon(y_3)| = m_s$ and equation (A.2). □

In 2-dimensional micromagnetics on a domain $\Psi \in \mathbb{R}^2$ for a constant magnetization $\boldsymbol{m} \in H^1(\Psi, m_s S^2)$, the demagnetization field is given by,

$$\boldsymbol{h_m}(\boldsymbol{x}) = \int_{\partial\Psi} \frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^2} \boldsymbol{m}\cdot\boldsymbol{n}(\boldsymbol{y})\boldsymbol{dy} \tag{A.14}$$

and magnetostatic energy is given by,

$$\mathscr{E}_{2d} = \int_\Psi\int_{\partial\Psi} \boldsymbol{m}\cdot\frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^2}\ \ \boldsymbol{m}\cdot\boldsymbol{n}(\boldsymbol{y})\boldsymbol{dy}. \tag{A.15}$$

Proposition A.5.

$$\left|\ J^\varepsilon_{211}(\widehat{\boldsymbol{m}}^\varepsilon) - 2\pi|\omega_\varepsilon| \int_0^1 \left|\widehat{\boldsymbol{m}}^\varepsilon_p(y_3)\right|^2 dy_3\ \right| \le D_6\varepsilon. \tag{A.16}$$

Proof. Using the Divergence theorem on $\boldsymbol{z}_p$ as in (A.8) and a subsequent change of variables

$q(z_3) = z_3 - y_3$, followed by (A.6) (as in Proposition A.2) we get

$$\int_{\partial\omega_\varepsilon}\int_0^1\int_0^1\int_{\partial\omega_\varepsilon}\frac{\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{y}_p)\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{z}_p)}{\sqrt{|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+(y_3-z_3)^2}} = \int_{\partial\omega_\varepsilon}\int_0^1\int_0^1\int_{\omega_\varepsilon}\frac{\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{y}_p)\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot(\boldsymbol{y}_p-\boldsymbol{z}_p)}{\left\{\,|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+(y_3-z_3)^2\,\right\}^{3/2}}$$

$$= \int_{\partial\omega_\varepsilon}\int_0^1\int_{\omega_\varepsilon}\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{y}_p)\int_{-y_3}^{1-y_3}\frac{\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot(\boldsymbol{y}_p-\boldsymbol{z}_p)}{\left\{\,|\boldsymbol{y}_p-\boldsymbol{z}_p|^2+q^2\,\right\}^{3/2}}dq$$

$$= \int_{\partial\omega_\varepsilon}\int_0^1\int_{\omega_\varepsilon}\frac{\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot(\boldsymbol{y}_p-\boldsymbol{z}_p)}{|\,\boldsymbol{y}_p-\boldsymbol{z}_p\,|^2}\Big[\frac{y_3\,\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{y}_p)}{\sqrt{y_3^2+|\boldsymbol{y}_p-\boldsymbol{z}_p|^2}}+\frac{(1-y_3)\,\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{y}_p)}{\sqrt{(1-y_3)^2+|\boldsymbol{y}_p-\boldsymbol{z}_p|^2}}\Big]$$

$$=: \varepsilon^2\left(\,J^\varepsilon_{2111}(\widehat{\boldsymbol{m}}^\varepsilon)+J^\varepsilon_{2112}(\widehat{\boldsymbol{m}}^\varepsilon)\,\right).$$

Set $R := \max\ 2\varepsilon^{-1}|\boldsymbol{x}_p-\boldsymbol{y}_p|$, for $\{\boldsymbol{z}_p\in\omega_\varepsilon, \boldsymbol{y}_p\in\partial\omega_\varepsilon\}$, and J_0^ε as

$$J_0^\varepsilon := \int_0^1\int_{\partial\omega_\varepsilon}\int_{\omega_\varepsilon}\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{y}_p)\frac{\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot(\boldsymbol{y}_p-\boldsymbol{z}_p)}{|\boldsymbol{y}_p-\boldsymbol{z}_p|^2}\boldsymbol{d\sigma}(\boldsymbol{y}_p)\boldsymbol{dz}_p dy_3,$$

Note

$$1-\frac{y_3}{\sqrt{y_3^2+|\boldsymbol{y}_p-\boldsymbol{z}_p|^2}} \le \begin{cases}\frac{|\boldsymbol{y}_p-\boldsymbol{z}_p|^2}{2y_3^2}, & \text{for } y_3\ge R_4\varepsilon\ ,\\ 1 & \text{for } y_3\le R_4\varepsilon\ .\end{cases} \tag{A.17}$$

Noting that $|\tilde{\boldsymbol{n}}|=1$, $|\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{y}_p)|\le|\widehat{\boldsymbol{m}}^\varepsilon(y_3)|$ and $|\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot(\boldsymbol{y}_p-\boldsymbol{z}_p)|\le|\widehat{\boldsymbol{m}}^\varepsilon(y_3)|\,|\boldsymbol{y}_p-\boldsymbol{z}_p|$,

$$\int_0^1\frac{\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot(\boldsymbol{y}_p-\boldsymbol{z}_p)}{|\,\boldsymbol{y}_p-\boldsymbol{z}_p\,|^2}\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot\tilde{\boldsymbol{n}}(\boldsymbol{y}_p)\Big(1-\frac{y_3}{\sqrt{y_3^2+|\boldsymbol{y}_p-\boldsymbol{z}_p|^2}}\Big)dy_3$$

$$\le\int_0^{R\varepsilon}\frac{|\widehat{\boldsymbol{m}}^\varepsilon(y_3)|^2}{|\,\boldsymbol{y}_p-\boldsymbol{z}_p\,|}\Big|1-\frac{y_3}{\sqrt{y_3^2+|\boldsymbol{y}_p-\boldsymbol{z}_p|^2}}\Big|+\int_{R\varepsilon}^1\frac{|\widehat{\boldsymbol{m}}^\varepsilon(y_3)|^2}{|\,\boldsymbol{y}_p-\boldsymbol{z}_p\,|}\Big|1-\frac{y_3}{\sqrt{y_3^2+|\boldsymbol{y}_p-\boldsymbol{z}_p|^2}}\Big|$$

$$\le\int_0^{R\varepsilon}\frac{|\widehat{\boldsymbol{m}}^\varepsilon(y_3)|^2}{|\,\boldsymbol{y}_p-\boldsymbol{z}_p\,|}dy_3+\int_{R\varepsilon}^1\widehat{\boldsymbol{m}}^\varepsilon(y_3)|^2\frac{|\boldsymbol{y}_p-\boldsymbol{z}_p|}{2y_3^2}dy_3$$

$$\le\sup_{y_3\in(0,1)}|\widehat{\boldsymbol{m}}^\varepsilon(y_3)|^2\Big[\int_0^{R\varepsilon}\frac{1}{|\,\boldsymbol{y}_p-\boldsymbol{z}_p\,|}+\int_{R\varepsilon}^1\frac{|\boldsymbol{y}_p-\boldsymbol{z}_p|}{2y_3^2}\Big].$$

Noting that $-\partial_3 \frac{1}{y_3} = \frac{1}{y_3^2}$ we get

$$\varepsilon^2 |J_0^\varepsilon - J_{2111}^\varepsilon| \leq \Big| \int_{\partial\omega_\varepsilon} \int_{\omega_\varepsilon} \int_0^1 \frac{\widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot(\boldsymbol{y}_p - \boldsymbol{z}_p)}{|\,\boldsymbol{y}_p - \boldsymbol{z}_p\,|^2} \widehat{\boldsymbol{m}}^\varepsilon(y_3)\cdot\widetilde{\boldsymbol{n}}(\boldsymbol{y}_p)\Big(1 - \frac{y_3}{\sqrt{y_3^2 + |\boldsymbol{y}_p - \boldsymbol{z}_p|^2}}\Big)\Big|$$

$$\leq \sup_{y_3} |\widehat{\boldsymbol{m}}^\varepsilon(y_3)|^2 \int_{\partial\omega_\varepsilon} \int_{\omega_\varepsilon} \Big[\int_0^{R\varepsilon} \frac{1}{|\,\boldsymbol{y}_p - \boldsymbol{z}_p\,|} + \int_{R\varepsilon}^1 \frac{|\boldsymbol{y}_p - \boldsymbol{z}_p|}{2y_3^2}\Big] \boldsymbol{d\sigma}(\boldsymbol{y}_p)\boldsymbol{dz}_p$$

$$\leq \sup_{y_3} |\widehat{\boldsymbol{m}}^\varepsilon(y_3)|^2 \int_{\partial\omega_\varepsilon} \int_{\omega_\varepsilon} \Big\{\frac{R\varepsilon}{|\boldsymbol{y}_p - \boldsymbol{z}_p|} - \frac{|\boldsymbol{y}_p - \boldsymbol{z}_p|}{2}\Big(\frac{1}{y_3}\Big)\Big|_{R\varepsilon}^1\Big\} \boldsymbol{d\sigma}(\boldsymbol{y}_p)\boldsymbol{dz}_p$$

$$\leq \sup_{y_3} |\widehat{\boldsymbol{m}}^\varepsilon(y_3)|^2 \int_{\partial\omega_\varepsilon} \int_{\omega_\varepsilon} \Big\{\frac{R\varepsilon}{|\boldsymbol{y}_p - \boldsymbol{z}_p|} - \frac{|\boldsymbol{y}_p - \boldsymbol{z}_p|}{2} + \frac{|\boldsymbol{y}_p - \boldsymbol{z}_p|}{2R\varepsilon}\Big\} \boldsymbol{d\sigma}(\boldsymbol{y}_p)\boldsymbol{dz}_p$$

Note from equation (A.7), the term $\int_{\omega_\varepsilon} \frac{1}{|\boldsymbol{y}_p - \boldsymbol{z}_p|}\boldsymbol{dz}_p = O(\varepsilon)$ in the first integral. So the full integral $R\varepsilon \int_{\partial\omega_\varepsilon}\int_{\omega_\varepsilon} \frac{1}{|\boldsymbol{y}_p - \boldsymbol{z}_p|}\boldsymbol{dz}_p$ is $O(\varepsilon^3)$. The second integrand is $O(\varepsilon)$ and so its integral is of $O(\varepsilon^4)$. The third integrand is bounded by 1, since by definition $R\varepsilon \geq |\boldsymbol{y}_p - \boldsymbol{z}_p|$. So the third integral is $O(\varepsilon^3)$. So $|J_0^\varepsilon - J_{2111}^\varepsilon| \leq D_6\varepsilon\ \sup_{y_3} |\widehat{\boldsymbol{m}}^\varepsilon(y_3)|^2$. J_{2112}^ε can be treated the same way to give the result

$$\Big|\ J_{211}^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon) - 2J_0^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon)\ \Big| \leq D_6\varepsilon \sup_{y_3} |\widehat{\boldsymbol{m}}^\varepsilon(y_3)|^2. \tag{A.18}$$

We get our result noting that $|\sup_{y_3} |\widehat{\boldsymbol{m}}^\varepsilon(y_3)|^2| = m_s^2$ since $|\widehat{\boldsymbol{m}}^\varepsilon(y_3)| = m_s a.e.$. Note that $J_0^\varepsilon(\widehat{\boldsymbol{m}})$ is exactly the 2-D magnetostatic energy $\mathscr{E}_{2d}$ defined in (A.15) and for a circular cross-section ω_ε it is well know that

$$\mathscr{E}_{2d}(\widehat{\boldsymbol{m}}^\varepsilon) = J_0^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon) = \pi|\omega_\varepsilon| \int_0^1 \left|\widehat{\boldsymbol{m}}_p^\varepsilon(y_3)\right|^2 dy_3 = \varepsilon^2\pi|\omega| \int_0^1 \left|\widehat{\boldsymbol{m}}_p^\varepsilon(x_3)\right|^2 dx_3. \qquad \square$$

A.3 Final Estimate for $\mathscr{E}_d^\varepsilon(\boldsymbol{m}^\varepsilon)$

Recall $(\widetilde{\boldsymbol{m}}^\varepsilon, \widetilde{\boldsymbol{u}}^\varepsilon)$ is a minimizer for $\mathscr{E}(\widetilde{\boldsymbol{m}}, \widetilde{\boldsymbol{u}})$. Also $\widehat{\boldsymbol{m}}^\varepsilon$ is defined as in equation.(A.1). Also equation (3.3) from the estimate for exchange in § 3 gives,

$$K_5 > \int_\Omega \frac{d}{\varepsilon^2} |\nabla_p \boldsymbol{m}^\varepsilon|^2 + d|\partial_3 \boldsymbol{m}^\varepsilon|^2 = d\|\varepsilon^{-1}\nabla_p \boldsymbol{m}^\varepsilon\|_{L^2(\Omega)}^2 + d\|\partial_3 \boldsymbol{m}^\varepsilon\|_{L^2(\Omega)}^2. \tag{A.19}$$

Remark A.3 gives

$$\Big|\mathscr{E}_d^\varepsilon(\boldsymbol{m}^\varepsilon) - \mathscr{E}_d^\varepsilon(\widehat{\boldsymbol{m}}^\varepsilon)\Big| = \frac{1}{8\pi\varepsilon^2}\Big|\ \|\widetilde{\boldsymbol{h}}_{\widehat{\boldsymbol{m}}^\varepsilon}^\varepsilon\|_{L^2(\mathbb{R}^3)}^2 - \|\widetilde{\boldsymbol{h}}_{\widehat{\boldsymbol{m}}^\varepsilon}^\varepsilon\|_{L^2(\mathbb{R}^3)}^2\Big| = O(\varepsilon^2).$$

So Propositions A.1 , A.2 , A.3 , A.4 , A.5, equations (A.9) and (A.11)

$$\begin{aligned}\mathscr{E}_d^{\varepsilon}(\boldsymbol{m}^{\varepsilon}) &= \mathscr{E}_d^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) + O(\varepsilon^2) = J_0^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) + \mathscr{E}_d^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) - J_0^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) + O(\varepsilon^2)\\ &= J_0^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) + \Big[\frac{1}{2}J_1^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) + \frac{1}{2}\big(J_2^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) - 2J_0^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon})\big) + J_3^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon})\Big] + O(\varepsilon^2)\\ &= J_0^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) + O(\sqrt{\varepsilon})\big\|\partial_3 \boldsymbol{m}^{\varepsilon}\big\|_{L^2(\Omega)} + O(\varepsilon)\Big\{\big\|\partial_3 \boldsymbol{m}^{\varepsilon}\big\|_{L^2(\Omega)} + \big\|\partial_3 \boldsymbol{m}^{\varepsilon}\big\|^2_{L^2(\Omega)}\Big\} + O(\varepsilon^2).\end{aligned} \tag{A.20}$$

Proposition A.1 and A.2 give

$$\big|\ J_1^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon})\ \big| + \big|\ 2J_3^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon})\ \big| \le \varepsilon\big[D_7\|\partial_3 \boldsymbol{m}^{\varepsilon}\|^2_{L^2(\Omega)} + D_8\|\partial_3 \boldsymbol{m}^{\varepsilon}\|_{L^2(\Omega)}\big].$$

Equations (A.9), (A.11) and Propositions A.3 , A.4 , and A.5 also give

$$\begin{aligned}J_2^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) - 2J_0^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) &= J_{211}^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) - 2J_0^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) + J_{22}^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) + J_{23}^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) + J_{24}^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) + J_{212}^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon})\\ &\ge J_{211}^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) - 2J_0^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) + J_{22}^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) - \big|J_{23}^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon})\big| + J_{24}^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) - \big|J_{212}^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon})\big|\\ &\ge J_{211}^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) - 2J_0^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) + J_{22}^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) + J_{24}^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) - D_8\varepsilon^2 - D_9\sqrt{\varepsilon}\big\|\partial_3 \boldsymbol{m}^{\varepsilon}\big\|_{L^2(\Omega)}.\end{aligned}$$

Using this result, we can refine (A.20) to get

$$\begin{aligned}\mathscr{E}_d^{\varepsilon}(\boldsymbol{m}^{\varepsilon}) - J_0^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) &= \Big[\frac{1}{2}J_1^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) + \frac{1}{2}\big(J_2^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) - 2J_0^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon})\big) + J_3^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon})\Big] + O(\varepsilon^2)\\ &\ge J_{211}^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) - 2J_0^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) + J_{22}^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) + J_{24}^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) - D_9\sqrt{\varepsilon}\big\|\partial_3 \boldsymbol{m}^{\varepsilon}\big\|_{L^2(\Omega)}\\ &\quad - \varepsilon\big[D_{11}\|\partial_3 \boldsymbol{m}^{\varepsilon}\|^2_{L^2(\Omega)} + D_{12}\|\partial_3 \boldsymbol{m}^{\varepsilon}\|_{L^2(\Omega)}\big] - D_{10}\varepsilon^2\end{aligned} \tag{A.21}$$

Proposition A.6. *Let* $\widehat{\boldsymbol{m}}^{\varepsilon} \to \boldsymbol{m}^o$ *in* $L^2(0,1)$ *where* $\boldsymbol{m}^o$ *is a constant vector. Then*

$$\frac{1}{\varepsilon}J_{22}^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) \to \frac{16\pi}{3}|m_3^o|^2, \quad \frac{1}{\varepsilon}\big(J_{211}^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) - J_0^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon})\big) \to -\frac{8\pi}{3}|\boldsymbol{m}_p^o|^2, \quad \frac{1}{\varepsilon}J_{24}^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) \to 0.$$

Proof. Since $\widehat{\boldsymbol{m}}^{\varepsilon} \to \boldsymbol{m}^o$, then $\widehat{\boldsymbol{m}}^{\varepsilon} = \boldsymbol{m}^o + \boldsymbol{M}^{\varepsilon}$ where $\boldsymbol{M}^{\varepsilon} \approx o(\varepsilon)$. Using this and noting that $J_{22}^{\varepsilon}, J_{24}^{\varepsilon}$ and $J_{211}^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) - J_0^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon})$ are all quadratic in $\widehat{\boldsymbol{m}}^{\varepsilon}$, we can show using simple expansion of the terms that

$$\begin{aligned}&\varepsilon^2 J_{22}^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) = (|\widehat{m}_3^{\varepsilon}(0)|^2 + |\widehat{m}_3^{\varepsilon}(1)|^2)\int_{\omega_\varepsilon}\int_{\omega_\varepsilon}\frac{d\boldsymbol{y}_p\, d\boldsymbol{z}_p}{|\boldsymbol{y}_p - \boldsymbol{z}_p|}\\ &= \varepsilon^2 J_{22}^{\varepsilon}(\boldsymbol{m}^o) + \Big\{|\boldsymbol{M}_3^{\varepsilon}(0)|^2 + |\boldsymbol{M}_3^{\varepsilon}(1)|^2 + 2\boldsymbol{M}_3^{\varepsilon}(0)\boldsymbol{m}_3^o(0) + 2\boldsymbol{M}_3^{\varepsilon}(1)\boldsymbol{m}_3^o(1)\Big\}\int_{\omega_\varepsilon}\int_{\omega_\varepsilon}\frac{d\boldsymbol{y}_p\, d\boldsymbol{z}_p}{|\boldsymbol{y}_p - \boldsymbol{z}_p|}\\ &= \varepsilon^2 J_{22}^{\varepsilon}(\boldsymbol{m}^o) + O(\varepsilon^3)\big(|\boldsymbol{M}^{\varepsilon}(0)| + |\boldsymbol{M}^{\varepsilon}(1)|\big) + O(\varepsilon^3)\big(|\boldsymbol{M}^{\varepsilon}(0)|^2 + |\boldsymbol{M}^{\varepsilon}(1)|^2\big).\end{aligned}$$

Dividing by ε^3, noting that $\boldsymbol{M}^{\varepsilon} \approx o(\varepsilon)$ and taking $\lim_{\varepsilon}$ gives,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}J_{22}^{\varepsilon}(\widehat{\boldsymbol{m}}^{\varepsilon}) = \lim_{\varepsilon\to 0}\Big\{\frac{1}{\varepsilon}J_{22}^{\varepsilon}(\boldsymbol{m}^o) + o(\varepsilon) + o(\varepsilon^2)\Big\} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}J_{22}^{\varepsilon}(\boldsymbol{m}^o).$$

To show that $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}J^{\varepsilon}_{22}(\boldsymbol{m}^o)=\frac{16\pi}{3}|m^o_3|^2$ we can follow the result in [Joseph, 1966]. The other results can be proved similarly by expanding out the magnetizations. □

Finally for magnetostatic estimate in section § 5 we need the following proposition.

Proposition A.7. *For* $\boldsymbol{m}^o=(0,0,m_s)$ *and* $\widetilde{\boldsymbol{m}}^{\varepsilon}=\widetilde{\boldsymbol{m}}^o+\widetilde{\boldsymbol{M}}^{\varepsilon}$

$$
\begin{aligned}
J^{\varepsilon}_{22}(\widehat{\boldsymbol{m}})+J^{\varepsilon}_{211}(\widehat{\boldsymbol{m}})-2J^{\varepsilon}_{0}(\widehat{\boldsymbol{m}}) \geq J^{\varepsilon}_{22}(\widehat{\boldsymbol{m}}^o)+J^{\varepsilon}_{211}(\widehat{\boldsymbol{m}}^o)-2J^{\varepsilon}_{0}(\widehat{\boldsymbol{m}}^o)\\
-\varepsilon^3 D_6\|\widetilde{\boldsymbol{M}}\|^2_{H^1(0,1)}-\varepsilon^3 D_7\|\widetilde{\boldsymbol{M}}\|_{H^1(0,1)}
\end{aligned}
$$

and

$$
J^{\varepsilon}_{24}(\widehat{\boldsymbol{m}}) \geq -\varepsilon D_8\|\widetilde{\boldsymbol{M}}\|^2_{H^1(0,1)}-\varepsilon D_9\|\widetilde{\boldsymbol{M}}\|_{H^1(0,1)}.
$$

Proof. Since $\widetilde{\boldsymbol{m}}^{\varepsilon}\in H^1(\Omega_\varepsilon;m_sS^2)$ and $\widehat{\boldsymbol{m}}^o$ is a constant means that $\widetilde{\boldsymbol{M}}^{\varepsilon}\in H^1(\Omega_\varepsilon)$. Using Sobolev imbedding we have $\widetilde{\boldsymbol{M}}^{\varepsilon}$ is in $C^{0,\alpha}$ with $\alpha<(0,\frac{1}{2})$ and for any $t\in(0,1)$ we have

$$
\widetilde{M}^{\varepsilon}(t)\leq \sup_{(0,1)}|\,\widetilde{\boldsymbol{M}}^{\varepsilon}\,|\leq K\,\|\,\widetilde{\boldsymbol{M}}^{\varepsilon}\,\|_{H^1(\Omega_\varepsilon)}. \tag{A.22}
$$

Then from definition of $J^{\varepsilon}_{22}(\widehat{\boldsymbol{m}}^{\varepsilon})$ in (A.9)

$$
\begin{aligned}
J^{\varepsilon}_{22}(\widehat{\boldsymbol{m}}^{\varepsilon}) &= (|\widehat{m}^o_3(0)+\widehat{M}^{\varepsilon}_3(0)|^2+|\widehat{m}^o_3(1)+\widehat{M}^{\varepsilon}_3(1)|^2)\int_{\omega_\varepsilon}\int_{\omega_\varepsilon}\frac{\boldsymbol{dy}_p\,\boldsymbol{dz}_p}{|\boldsymbol{y}_p-\boldsymbol{z}_p|}\\
&\geq J^{\varepsilon}_{22}(\widehat{\boldsymbol{m}}^o)-2K\varepsilon^3\|\,\widetilde{\boldsymbol{M}}^{\varepsilon}\,\|^2_{H^1(\Omega_\varepsilon)}-4Km_s\varepsilon^3\|\,\widetilde{\boldsymbol{M}}^{\varepsilon}\,\|_{H^1(\Omega_\varepsilon)}
\end{aligned}
$$

where we have used (A.22). A similiar calculation shows the same easily by noting

$$
\begin{aligned}
J^{\varepsilon}_{211}(\widehat{\boldsymbol{m}})-2J^{\varepsilon}_{0}(\widehat{\boldsymbol{m}}) = \left[J^{\varepsilon}_{211}(\widehat{\boldsymbol{m}}^0)-2J^{\varepsilon}_{0}(\widehat{\boldsymbol{m}}^0)\right]+\left[J^{\varepsilon}_{211}(\widetilde{\boldsymbol{M}})-2J^{\varepsilon}_{0}(\widetilde{\boldsymbol{M}})\right]\\
+\left[\text{cross-terms}\right].
\end{aligned}
$$

Using the result of Proposition A.5 we can show $\left[J^{\varepsilon}_{211}(\widetilde{\boldsymbol{M}})-2J^{\varepsilon}_{0}(\widetilde{\boldsymbol{M}})\right]\leq K\varepsilon\|\,\widetilde{\boldsymbol{M}}\,\|^2_{H^1(0,1)}$ and $\left[\text{cross-terms}\right]\leq 2Km_s\varepsilon\|\widetilde{\boldsymbol{M}}\|_{H^1(0,1)}$. For $\widehat{\boldsymbol{m}}^o$ constant it is easy to check that $J^{\varepsilon}_{24}(\widehat{\boldsymbol{m}}^o)\equiv 0$.

$$
\begin{aligned}
J^{\varepsilon}_{24}(\widehat{\boldsymbol{m}}) &\geq -|J^{\varepsilon}_{24}(\widetilde{\boldsymbol{M}})|-2m_s\int_{\omega_\varepsilon}\int_{\omega_\varepsilon}\frac{|\widetilde{M}_3(0)|+|\widetilde{M}_3(1)|}{\varepsilon^2|\boldsymbol{y}_p-\boldsymbol{z}_p|}-2m_s\sup_{s\in(0,1)}|\widetilde{\boldsymbol{M}}(s)|\int_{\omega_\varepsilon}\int_{\omega_\varepsilon}\frac{\boldsymbol{dx}_p\boldsymbol{dy}_p}{\varepsilon^2|\boldsymbol{y}_p-\boldsymbol{z}_p|}\\
&\geq -\varepsilon D_8\|\widetilde{\boldsymbol{M}}\|^2_{H^1(0,1)}-4m_s\sup_{s\in(0,1)}|\widetilde{\boldsymbol{M}}(s)|\int_{\omega_\varepsilon}\int_{\omega_\varepsilon}\frac{\boldsymbol{dx}_p\boldsymbol{dy}_p}{\varepsilon^2|\boldsymbol{y}_p-\boldsymbol{z}_p|}\\
&= -\varepsilon D_8\|\widetilde{\boldsymbol{M}}\|^2_{H^1(0,1)}-\varepsilon D_9\|\widetilde{\boldsymbol{M}}\|_{H^1(0,1)}.
\end{aligned}
$$

□

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