Modular Equations and Evaluations of Ramanujan Quantities

Nikos Bagis
Stenimahou 5 Edessa Pellas Greece 58200

abstract

In this article we continue a previous work, in which we have generalized the Rogers Ramanujan continued fraction (RR) introducing what we call, the Ramanujan-Quantities (RQ). In the present paper we use the Mathematica package to give several modular equations for certain cases of Ramanujan Qunatities-(RQ). We give also new modular equations of degree 2 and 3 for the complete evaluation of the first derivative of (RR).
Also for certain class of (RQ)'s we show how we can found the coresponding continued fraction expansions-S, in which we are able to evaluate with numerical methods some lower degree modular equations and values of this fraction.

keywords: Ramanujan; Continued Fractions; Quantities; Modular Equations; Derivatives; Evaluations

1 Definitions and Introductory Results

In this article we will define and study expressions that rise from continued fractions, analogous to that of Rogers-Ramanujan (RR), Ramanujan's Cubic (RC), Ramanujan-Gollnitz-Gordon (RGG). The results are new since no work have been done in this area and most of them are experimental observations.
The focused quantities are

$$q^{-(a-b)/2+(a^2-b^2)/(2p)}\frac{\prod_{n=0}^{\infty}(1-q^aq^{np})(1-q^{p-a}q^{np})}{\prod_{n=0}^{\infty}(1-q^bq^{np})(1-q^{p-b}q^{np})}, \tag{1}$$

where a, b, p are positive rationals such that $a+b<p$. As someone can see these quantities are behave as (RR), the (RC) and (RGG) continued fractions identities. For example when $q=e^{-\pi\sqrt{r}}$, r positive rational, they are algebraic numbers and satisfy modular equations. Their derivatives, also are all obey the same nome.
Let now

$$(a;q)_k := \prod_{n=0}^{k-1}(1-aq^n) \tag{2}$$

The Rogers Ramanujan continued fraction is

$$R(q) := \frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+}\cdots \tag{3}$$

which satisfies the famous Roger's-Ramanujan identity:

$$R^*(q) := q^{-1/5}R(q) = \frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty} = \prod_{n=1}^{\infty}(1-q^n)^{X_2(n)} \tag{4}$$

where $X_2(n)$ is the Legendre symbol $\left(\frac{n}{5}\right)$.
Also hold

$$R(e^{-x}) = e^{-x/5}\frac{\vartheta_4(3ix/4, e^{-5x/2})}{\vartheta_4(ix/4, e^{-5x/2})}, x > 0 \tag{5}$$

Where $\vartheta_4(a,q)$ is the 4th kind Elliptic Theta function (see [9]).
The concept of formulation (1) is described below. We first begin with the rewriting of (5) into the form

$$R(e^{-x}) = \exp\left(-x/5 - \sum_{n=1}^{\infty}\frac{1}{n}\frac{e^{4nx}-e^{3nx}-e^{2nx}+e^{nx}}{e^{5nx}-1}\right), x > 0 \tag{6}$$

The Ramanujan-Gollnitz-Gordon continued fraction is

$$H(q) = \frac{q^{1/2}}{(1+q)+}\frac{q^2}{(1+q^3)+}\frac{q^4}{(1+q^5)+}\frac{q^6}{(1+q^7)+}\cdots \tag{7}$$

Also for this continued fraction holds

$$H(e^{-x}) = \exp\left(-x/2 - \sum_{n=1}^{\infty}\frac{1}{n}\frac{e^{7nx}-e^{5nx}-e^{3nx}+e^{nx}}{e^{8nx}-1}\right), x > 0 \tag{8}$$

$$H(e^{-x}) = e^{-x/2}\frac{\vartheta_4(3ix/2, e^{-4x})}{\vartheta_4(ix/2, e^{-4x})}, x > 0 \tag{9}$$

Is true that exists generalizations for these expansions, but there is no theory developed, especialy for evaluations and modular equations.

2 Theorems on Rogers Ramanujan Quantities

Definition 1.
In general if $q = e^{-\pi\sqrt{r}}$ where $a, p, r > 0$ we denote 'Agile' the quantity

$$[a,p;q] = (q^{p-a};q^p)_\infty(q^a;q^p)_\infty \tag{10}$$

Definition 2.
We call

$$R(a,b,p;q) := q^{-(a-b)/2+(a^2-b^2)/(2p)} \frac{[a,p;q]}{[b,p;q]} \tag{11}$$

'Ramanujan's Quantity' because many of Ramanujan's continued fractions can be put in this form.
Also

$$R^*(a,b,p;q) := \frac{[a,p;q]}{[b,p;q]}$$

Observation 1.(Unproved)
If $q = e^{-\pi\sqrt{r}}$, a, b, p, r positive rationals then

$$q^{p/12-a/2+a^2/(2p)}[a,p;q] \stackrel{?}{=} Algebraic \tag{12}$$

Note.
The mark "?" means that we have no proof.

Lemma 1.

$$\sum_{k=1}^{\infty} \frac{\cosh(2tk)}{k \sinh(\pi a k)} = \log(P_0) - \log(\vartheta_4(it, e^{-a\pi})) \text{ , where } |2t| < |\pi a| \tag{13}$$

and $P_0 = \prod_{n=1}^{\infty}(1 - e^{-2n\pi a})$ and $\vartheta_4(u,q) = 1 + \sum_{n=1}^{\infty}(-1)^n q^{n^2} \cos(2nu)$
Proof.
From ([2] pg.170 relation (13-2-12)) and the definition of theta functions we have

$$\vartheta_4(z,q) = \prod_{n=0}^{\infty}(1 - q^{2n+2})(1 - q^{2n-1}e^{2iz})(1 - q^{2n-1}e^{-2iz}) \tag{14}$$

By taking the logarithm of both sides and expanding the logarithm of the individual terms in a power series it is simple to show (13) from (14), where $q = e^{-\pi a}$, a positive real.

Theorem 1.
If a, b, p, r are positive rationals, then

$$R(a,b,p;q) := q^{-(a-b)/2+(a^2-b^2)/(2p)} R^*(a,b,p;q) = Algebraic \tag{15}$$

Proof.
Eq.(15) follows easy from the Observation 1 and the Definitions 1,2.

One example is the Rogers-Ramanujan continued fraction

$$q^{1/5}R^*(1,2,5;q) = R^*(q)q^{1/5} = R(q) \tag{16}$$

Theorem 2.
For all positive reals a, b, p, x

$$R(a,b,p;e^{-x}) = \exp\left(-x\frac{a^2-b^2}{2p} + x\frac{a-b}{2}\right)\frac{\vartheta_4((p-2a)ix/4, e^{-px/2})}{\vartheta_4((p-2b)ix/4, e^{-px/2})} = \tag{17}$$

$$= \exp\left[-x\left(\frac{a^2-b^2}{2p} - \frac{a-b}{2}\right) - \sum_{n=1}^{\infty}\frac{1}{n}\frac{e^{anx}+e^{(p-a)nx}-e^{(p-b)nx}-e^{bnx}}{e^{pnx}-1}\right] \tag{18}$$

Proof.
From Definitions 1, 2 and the relations (13), (14) we can rewrite R in the form

$$R(a,b,p;e^{-x}) = \exp\left(-x\frac{a^2-b^2}{2p} + x\frac{a-b}{2}\right)\frac{\exp\left(\sum_{n=1}^{\infty}\frac{\cosh(nx(p-2b)/2)}{n\sinh(pnx/2)}\right)}{\exp\left(\sum_{n=1}^{\infty}\frac{\cosh((p-2a)nx/2)}{n\sinh(pnx/2)}\right)}$$

from which as one can see (17) and (18) follow.

For the continued fraction (7) we give some evaluations with the command 'Recognize' of Mathematica:

$$H\left(e^{-\pi}\right) \stackrel{?}{=} \sqrt{4-2\sqrt{2}} - 1 - \sqrt{2}$$

$$H\left(e^{-\pi\sqrt{2}}\right) \stackrel{?}{=} (1-8t-12t^2-8t^3+38t^4+8t^5-12t^6+8t^7+t^8)_3$$

Theorem 3. (The Rogers Ramanujan Identity of the Quantities)
If a, b, p are positive integers and $p-a \neq p-b$ $|q|<1$, then

$$R^*(a,b,p;q) = \prod_{n=1}^{\infty}(1-q^n)^{X(n)} \tag{19}$$

where

$$X(n) = \left\{\begin{array}{c} 1,\ n \equiv (p-a) mod p \\ -1,\ n \equiv (p-b) mod p \\ 1,\ n \equiv a mod p \\ -1,\ n \equiv b mod p \\ 0,\ p|n \end{array}\right\} \tag{20}$$

Proof.
Use Theorem 2. Take the logarithms and expand the product (19). The proof is easy.

Theorem 4.
Let $|q|<1$, then

$$\log(R^*(a,b;p;q)) = -\sum_{n=1}^{\infty}\frac{q^n}{n}\sum_{d|n}X(d)d \tag{21}$$

Proof
Follows from Theorem 3.

Proposition.(See [7] pg. 24)
Suppose that a, b and q are complex numbers with $|ab|<1$ and $|q|<1$ or that $a=b^{2m+1}$ for some integer m. Then

$$P(a,b,q) := \frac{(a^2q^3;q^4)_\infty (b^2q^3;q^4)_\infty}{(a^2q;q^4)_\infty (b^2q;q^4)_\infty} =$$

$$= \frac{1}{(1-ab)+}\frac{(a-bq)(b-aq)}{(1-ab)(q^2+1)+}\frac{(a-bq^3)(b-aq^3)}{(1-ab)(q^4+1)+}\frac{(a-bq^5)(b-aq^5)}{(1-ab)(q^5+1)+\ldots} \tag{22}$$

Theorem 5.
If $a=2A+3p/4$, $b=2B+p/4$ and $p=4(A+B)$, $|q|<1$

$$R^*(a,b,p;q) = (1-q^{B-A})P(q^A,q^B,q^{A+B}) \tag{23}$$

Proof.
One can see that

$$P(q^A,q^B,q^{A+B}) = \frac{(q^a;q^p)_\infty (q^{2p-a};q^p)_\infty}{[b,p;q]} \tag{24}$$

where $a=2A+3p/4$, $b=2B+p/4$ and $p=4(A+B)$.
Define

$${}_2\phi_1[a,b;c;q,z] := \sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(c;q)_n}\frac{z^n}{(q;q)_n} \tag{25}$$

and

$$\psi(a,q,z) := \sum_{n=0}^{\infty}\frac{(a;q)_n}{(q,q)_n}z^n = {}_2\phi_1[a,0,0,q,z] \tag{26}$$

Then

$$\psi(q^p,q^p,q^{p-a})R^*(a,b,p;q) = P[q^A,q^B,q^{A+B}] \tag{27}$$

The proof of (23) follows easily from (27) and the q-binomial theorem (see [7]):

$$\psi(a,q,z)=\prod_{n=0}^{\infty}\frac{1-azq^n}{1-zq^n}$$

Note.
Relation (23) is an expansion of a Ramanujan Quantity in continued fraction.

3 The first Order Derivatives of Ramanujan's Quantities

Observe that if

$$R_1(q)=\frac{q^{1/5}}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+}\cdots=q^{1/5}\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty} \tag{28}$$

$$R_2(q)=\frac{q^{1/3}}{1+}\frac{q+q^2}{1+}\frac{q^2+q^4}{1+}\frac{q^3+q^6}{1+}\cdots=q^{1/3}\frac{(q;q^6)_\infty(q^5;q^6)_\infty}{(q^3;q^6)^2_\infty} \tag{29}$$

$$R_3(q)=\frac{q^{1/2}}{(1+q)+}\frac{q^2}{(1+q^3)+}\frac{q^4}{(1+q^5)+}\cdots=q^{1/2}\frac{(q;q^8)_\infty(q^7;q^8)_\infty}{(q^3;q^8)_\infty(q^5;q^8)_\infty} \tag{30}$$

$R_1(q)=R(q)=R(1,3,5;q)$, $R_2(q)=V(q)=R(1,3,6;q)$, $R_3(q)=H(q)=R(1,3,8;q)$ are respectively the Rogers-Ramanujan, Ramanujan's Cubic and Ramanujan-Gollnitz-Gordon continued fractions. All these have derivatives

$$R'_{1,2,3}(q)\frac{q\pi^2}{K(k_r)^2}\overset{?}{=}Algebraic \tag{31}$$

whenever $q=e^{-\pi\sqrt{r}}$ and r is a positive rational.

Observation 2.
If a,b,p,r are positive rationals with $a,b<p$, then

$$\frac{d}{dq}R(a,b,p;q)\overset{?}{=}\frac{K(k_r)^2}{q\pi^2}Algebraic \tag{32}$$

$$\frac{d}{dq}\left(q^{p/12-a/2+a^2/(2p)}[a,p;q]\right)\overset{?}{=}\frac{K(k_r)^2}{q\pi^2}Algebraic \tag{33}$$

Let now $k=k_r$ be the Elliptic singular moduli (see [9], [16]).
In [12] we prove the following relation, for $r>0$

$$\frac{dr}{dk}=\frac{\pi\sqrt{r}}{K(k)^2k(1-k^2)} \tag{34}$$

Hence

$$\frac{dq}{dk} = \frac{-q\pi^2}{2k(1-k^2)K(k)^2} \tag{35}$$

This observation along with observation (2) lead us to the concluding remark

$$\frac{dR(a,b,p;q)}{dk} =$$

$$= \frac{dR(a,b,p;q)}{dq}\frac{dq}{dk} = \frac{K(k)^2}{q\pi^2}\frac{-q\pi^2}{2K(k)^2 2k(1-k^2)} \cdot Algebraic$$

Hence

Proposition 1.
When $q = e^{-\pi\sqrt{r}}$, a, b, p, r positive rationals, then

$$\frac{dR(a,b,p;q)}{dk} = Algebraic \tag{36}$$

Theorem 6.
If $q = e^{-\pi\sqrt{r}}$, then

$$\frac{dH(q)}{dq} = \frac{-q\pi^2}{2k(1-k^2)K(k)^2}\frac{\sqrt{1-k'}}{k'(k\sqrt{2}+2\sqrt{1-k'})} \tag{37}$$

Proof
In [10], we have proved that

$$H(q) = -t + \sqrt{t^2+1}\ ,\ t = \frac{k_r}{(1-k'_r)} \tag{38}$$

which gives

$$\frac{dH(q)}{dk} = \frac{\sqrt{1-k'}}{k'(k\sqrt{2}+2\sqrt{1-k'})} \tag{39}$$

using now (35) we get the result.

Note.
In [12] we have derive the first order derivative for the Cubic Continued fraction:
Let $q = e^{-\pi\sqrt{r}}$, $r > 0$ then

$$V'(q) = \frac{dV(q)}{dq} = \frac{4K^2(k_r)k_r'^2(V(q)+V^4(q))}{3q\pi^2\sqrt{r}\sqrt{1-8V^3(q)}} \tag{40}$$

Hence (29) and (30) have proved. In [10], we give a formula for the (RR) first derivative involving equations that can not solve in radicals (higher than 4).

Also (see [13]) for (RR) we have given a formula but contains the function $k^{(-1)}(x)$, which is the inverse function of k_r.

Examples.

$$\left(\frac{d}{dq}R(1,2,4;q)\right)_{q=e^{-\pi}} \stackrel{?}{=} \frac{e^{\pi}\Gamma(1/4)^4}{64\cdot 2^{5/8}\pi^3} \tag{41}$$

$$\left(\frac{d}{dq}R(1,2,5;q)\right)_{q=e^{-\pi}} \stackrel{?}{=} \frac{e^{\pi}\Gamma(1/4)^4}{16\pi^3}p_1 \tag{42}$$

where

$$p_1 = (16 - 240t^2 + 800t^3 - 2900t^4 - 6000t^5 - 6500t^6 + 17500t^7 + 625t^8)_3$$

$$\left(\frac{d}{dq}R(1,3,8;q)\right)_{q=e^{-\pi}} \stackrel{?}{=} \left(2+\sqrt{2}-\sqrt{5-\frac{7}{2}\sqrt{2}}\right)\frac{64e^{\pi}\pi}{\Gamma\left(-\frac{1}{4}\right)^4} \tag{43}$$

$$\left(\frac{d}{dq}R(1,3,8;q)\right)_{q=e^{-2\pi}} \stackrel{?}{=} \frac{(6+4\sqrt{2})e^{2\pi}\Gamma(5/4)^4}{\pi^3}p_2 \tag{44}$$

$$p_2 = (16384 - 1720320t^2 - 6684672t^3 + 143104t^4 - 18432t^5 - 1664t^6 + t^8)_3$$

4 Modular equations and Ramanujan Quantities

With the help of Theorem 2 we can evaluate $R(a,b,p;q)$ in series of q^Q:

$$R(a,b,p;q) = \sum_{n=0}^{M} c_n q^{nQ}, \tag{1a}$$

where M is positive integer and

$$Q = \frac{a^2-b^2}{2p} - \frac{a-b}{2}$$

Setting as in [11]:

$$R_S = \sum_{0\le i+j\le d} a_{i,j}u^i v^j,$$

where d is suitable positive integer, we try to solve $R_S = 0$. Where

$$u = R(a,b,p;q), v = R(a,b,p;q^{\nu})$$

are given from (1a) and ν positive integer. Evaluating the $a_{i,j}$, we obtain the modular equations for $R(a,b,p;q)$.

1) We present some modular equations for the Ramanujan Quantity $R(1,2,4,q)$:

a) If $u = R(1,2,4;q)$ and $v = R(1,2,4;q^2)$, then

$$u^4 - v^2 + 4u^4v^4 \stackrel{?}{=} 0 \tag{45}$$

b) If $u = R(1,2,4;q)$ and $v = R(1,2,4;q^3)$, then

$$u^4 - uv + 4u^3v^3 - v^4 \stackrel{?}{=} 0 \tag{46}$$

c) If $u = R(1,2,4;q)$ and $v = R(1,2,4;q^5)$, then

$$u^6 - uv + 5u^4v^2 - 5u^2v^4 + 16u^5v^5 - v^6 \stackrel{?}{=} 0 \tag{47}$$

d) If $u = R(1,2,4;q)$ and $v = R(1,2,4,q^7)$, then

$$u^8 - uv + 7u^2v^2 - 28u^3v^3 + 70u^4v^4 - 112u^5v^5 + 112u^6v^6 - 64u^7v^7 + v^8 \stackrel{?}{=} 0 \tag{48}$$

2) For the Ramanujan Quantity $R(1,2,6;q)$ we have

a) If $u = R(1,2,6;q)$ and $v = R(1,2,6;q^2)$, then

$$u^4 - v^2 + 3u^4v^2 + v^4 \stackrel{?}{=} 0 \tag{49}$$

...
One can find with the help of Mathematica many relations such above

The 5-degree modular equation of Ramanujan's Cubic continued fraction:
If $u = R(1,3,6;q)$ and $v = R(1,3,6;q^5)$, then

$$u^6 - uv + 5u^4v + 5u^2v^2 - 10u^5v^2 - 20u^3v^3 + 5uv^4 + 20u^4v^4 - 10u^2v^5 - 16u^5v^5 + v^6 \stackrel{?}{=} 0 \tag{50}$$

The 7-degree modular equation of Ramanujan's Cubic continued fraction:
If $u = R(1,3,6;q)$ and $v = R(1,3,6;q^7)$, then

$$u^8 - uv + 7u^4v + 28u^6v^2 - 56u^5v^3 + 7uv^4 + 21u^4v^4 - 56u^7v^4 - 56u^3v^5 + \\ + 28u^2v^6 - 56u^4v^7 - 64u^7v^7 + v^8 \stackrel{?}{=} 0 \tag{51}$$

If $a > b$ then from the definition of the Ramanujan Quantity (RQ) we have

$$R(a, b, p; q) = \frac{1}{R(b, a, p; q)} \tag{52}$$

Suppose that $a = \frac{a_1}{a_2}$, $b = \frac{b_1}{b_2}$, $p = \frac{p_1}{p_2}$, and $u(q) = R(a, b, p; q)$, then

$$u(q^{\frac{1}{a_2 b_2 p_2}}) = R(a_1 b_2 p_2, b_1 a_2 p_2, p_1 a_2 b_2; q) = w(q)$$

if $a_1 b_2 p_2 < b_1 a_2 p_2$, (otherwise we use (52)).
But $w_1 := w(q)$, $w_\nu := w(q^\nu)$ are related by a modular equation $f(w_1, w_\nu) = 0$, or $f(w(q^{a_2 b_2 p_2}), w(q^{\nu \cdot a_2 b_2 p_2})) = 0$. Hence

Theorem 7.
When $a = \frac{a_1}{a_2}$, $b = \frac{b_1}{b_2}$, $p = \frac{p_1}{p_2}$, $a_1, a_2, b_1, b_2, p_1, p_2 \in \mathbf{N}$ and $a_1 b_2 p_2 < b_1 a_2 p_2$ then the modular equation which relates $u_1 := R(a, b, p; q)$ and $u_\nu := R(a, b, p, q^\nu)$, $\nu \in \mathbf{N}$ is that of

$$w(q) = R(a_1 b_2 p_2, b_1 a_2 p_2, p_1 a_2 b_2; q) \text{ and } w(q^\nu). \tag{53}$$

Example
The modular equation between $z_1 = z(q) = R\left(1, \frac{1}{2}, 2; q\right)$ and $z_2 = z(q^2)$ is

$$4 + z_2^4 - z_1^4 z_2^2 = 0 \tag{54}$$

Proof.
We have

$$z_1 = z(q) = R\left(1, \frac{1}{2}, 2, q\right) = R(2, 1, 4; q^{1/2}) = \frac{1}{R(1, 2, 4; q^{1/2})} = \frac{1}{u(q^{1/2})}$$

But $z_2 = z(q^2) = \frac{1}{u(q^{2/2})}$ using (45) we have

$$u(q^{1/2})^4 - u(q)^2 + 4u(q^{1/2})^4 u(q)^4 = 0,$$

from which (54) follows.

From Theorem 4 differentiating (21) and using Observation 2 we have that

$$q \frac{dR(a, b, p; q)}{dq} \frac{1}{R(a, b, p; q)} = Q - \sum_{n=1}^{\infty} q^n \sum_{d|n} X(d) d \tag{55}$$

or from (34) and Proposition 1, along with (see [9]):

$$f(-q)^4 = 2^{4/3}\pi^{-2}q^{-1/6}(k_r)^{1/3}(k'_r)^{4/3}K(k_r)^2$$

$$N(q) = q^{-1/6}f(-q)^{-4}\left(Q - \sum_{n=1}^{\infty} q^n \sum_{d|n} X(d)d\right) R(a,b,p;q) = Algebraic \quad (56)$$

This is a resulting formula for the first derivative:

$$\frac{dR(a,b,p;q)}{dk}(k_r k'_r)^{2/3} = N(q) \quad (57)$$

The function $N(q)$ take algebraic values when $q = e^{-\pi\sqrt{r}}$, r positive rational and in the case of (RR) satisfies modular equations. With the same method as in $R(a,b,p;q)$ which we use in the begining of the paragraph 4 we have:

The 2-degree Modular equation for the first derivative of RR continued fraction

For $a = 1$, $b = 2$, $p = 5$, we have the case of (RR) and
a) If $u = N(q)$ and $v = N(q^2)$ then

$$5u^6 - u^2v^2 - 125u^4v^4 + 5v^6 \stackrel{?}{=} 0 \quad (58)$$

The 3-degree Modular equation for the first derivative of RR continued fraction

b) If $u = N(q)$ and $v = N(q^3)$ we have

$$125u^{12} + u^3v^3 + 1125u^9v^3 + 1125u^3v^9 + 1953125u^9v^9 - 125v^{12} \stackrel{?}{=} 0 \quad (59)$$

Suppose now that $q_0 = e^{-\pi\sqrt{r_0}}$ and we know $R^{(1)}(q_0) = \left(\frac{dR(q)}{dq}\right)_{q=q_0}$, then from equations (56), (57), (58), (59) and (35) we can evaluate in radicals, any high order values of the first derivative of the (RR) in which $r = 4^n 9^m r_0$, for n, m integers.

Note. If $K(k_r) = K[r]$ then holds:

$$K[4r] = \frac{1 + k'_r}{2}K[r]$$

and

$$K[9r] = m_3(r)K[r]$$

where $m_3(r)$ is solution of

$$27m_3(r)^4 - 18m_3(r)^2 - 8(1-2k_r^2)m_3(r) - 1 = 0$$

The formulas for evaluation of k_{4r} and k_{9r} are in [7].

5 Application in 'almost' random continued fractions

1) The case of $A = 1$, $B = 2$.
Set $a = 11$, $b = 7$ and $p = 12$ then

$$S(q) = S_{1,2}(q) = R(11,7,12;q) = \frac{1}{R(7,11,12;q)}$$

and from Theorem 5 we get

$$S_{1,2}(q) = q\frac{1-q}{1-q^3+}\frac{q^3(1-q^2)(1-q^4)}{(1-q^3)(1+q^6)+}\frac{q^3(1-q^8)(1-q^{10})}{(1-q^3)(1+q^{12})+}\frac{q^3(1-q^{14})(1-q^{16})}{(1-q^3)(1+q^{18})+\ldots} \tag{60}$$

With the above methods we find that Continued fraction $S(q)$ obeys the following modular equations:
1) If we set $u = S_{1,2}(q)$ and $v = S_{1,2}(q^2)$, then

$$-u^2 + v - 2uv + u^2v - v^2 \overset{?}{=} 0 \tag{61}$$

2) If $u = S_{1,2}(q)$ and $v = S_{1,2}(q^3)$, then

$$u^3 - v + 3uv - u^3v + v^2 - 3u^2v^2 + u^3v^2 - v^3 \overset{?}{=} 0 \tag{62}$$

3) If $u = S_{1,2}(q)$ and $v = S_{1,2}(q^5)$, then

$$-u^5 + v - 5uv + 5u^2v + 5u^5v - 10u^3v^2 - 5u^5v^2 + 10u^2v^3 + 10u^4v^3 - 5uv^4 -$$

$$-10u^3v^4 + 5uv^5 + 5u^4v^5 - 5u^5v^5 + u^6v^5 - uv^6 \overset{?}{=} 0 \tag{63}$$

4) If $u = S_{1,2}(q)$ and $v = S_{1,2}(q^7)$, then

$$-u^7 + v - 7uv + 14u^2v - 7u^3v + 7u^5v - 7u^6v + 7u^7v + 7uv^2 - 28u^2v^2 +$$

$$+7u^3v^2 - 28u^5v^2 + 28u^6v^2 - 14u^7v^2 - 7uv^3 + 28u^2v^3 - 7u^3v^3 +$$

$$+35u^4v^3 + 7u^5v^3 - 7u^6v^3 + 7u^7v^3 - 35u^3v^4 - 35u^5v^4 + 7uv^5 -$$

$$-7u^2v^5 + 7u^3v^5 + 35u^4v^5 - 7u^5v^5 + 28u^6v^5 - 7u^7v^5 - 14uv^6 +$$

$$+28u^2v^6-28u^3v^6+7u^5v^6-28u^6v^6+7u^7v^6+7uv^7-7u^2v^7+$$
$$+7u^3v^7-7u^5v^7+14u^6v^7-7u^7v^7+u^8v^7-uv^8\overset{?}{=}0 \tag{64}$$

2) The case of $A=1$, $B=3$.
Set $a=14$, $b=10$ and $p=16$ then

$$S_{1,3}(q)=R(14,10,16;q)=\frac{1}{R(10,14,16;q)}$$

and from Theorem 5 we get

$$S_{1,3}(q)=q\frac{1-q^2}{1-q^4+}\frac{q^4(1-q^2)(1-q^6)}{(1-q^4)(1+q^8)+}\frac{q^4(1-q^{14})(1-q^{10})}{(1-q^4)(1+q^{12})+}\frac{q^4(1-q^{22})(1-q^{18})}{(1-q^4)(1+q^{20})+\ldots} \tag{65}$$

With the above methods we find that Continued fraction $S(q)$ obeys the following modular equations:
1) If we set $u=S_{1,3}(q)$ and $v=S_{1,3}(q^2)$, then

$$u^2-v+u^2v+v^2\overset{?}{=}0 \tag{66}$$

2) If $u=S_{1,3}(q)$ and $v=S_{1,3}(q^3)$, then

$$u^3-v+3u^2v+3uv^2-3u^3v^2-3u^2v^3+u^4v^3-uv^4\overset{?}{=}0 \tag{67}$$

3) If $u=S_{1,3}(q)$ and $v=S_{1,3}(q^5)$, then

$$u^5-v+5u^2v+10u^3v^2-5u^5v^2-10u^2v^3+10u^4v^3+5uv^4-10u^3v^4-$$
$$-5u^4v^5+u^6v^5-uv^6\overset{?}{=}0 \tag{68}$$

4) If $u=S_{1,3}(q)$ and $v=S_{1,3}(q^7)$, then

$$u^8-uv+7u^3v-7u^5v-7u^7v+28u^6v^2+7uv^3-49u^3v^3-7u^5v^3-$$
$$-7u^7v^3+70u^4v^4-7uv^5-7u^3v^5-49u^5v^5+7u^7v^5+28u^2v^6-$$
$$-7uv^7-7u^3v^7+7u^5v^7-u^7v^7+v^8\overset{?}{=}0 \tag{69}$$

As a starting value we can get

$$S_{1,3}(e^{-\pi/2})\overset{?}{=}-1-\sqrt{2}+\sqrt{2\left(2+\sqrt{2}\right)}$$

from (66) we get

$$S_{1,3}(e^{-\pi})=-3-2\sqrt{2}+2\sqrt{2+\sqrt{2}}+\sqrt{4+2\sqrt{2}}-$$

$$-\sqrt{30+22\sqrt{2}-16\sqrt{2+\sqrt{2}}-12\sqrt{2\left(2+\sqrt{2}\right)}}$$

... etc

A value for the derivative according to the Observation 2 is

$$\left(\frac{dS_{1,3}(q)}{dq}\right)_{q=e^{-\pi/2}}=\frac{64\left(4+2\sqrt{2}-\sqrt{2\left(10+7\sqrt{2}\right)}\right)e^{\pi/2}\pi}{\Gamma\left(-\frac{1}{4}\right)^4}$$

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