VERSALDEFORMATIONS — A PACKAGE FOR COMPUTING VERSAL DEFORMATIONS AND LOCAL HILBERT SCHEMES

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Abstract. We provide an overview of the Macaulay2 package *VersalDeformations*, which algorithmically computes versal deformations of isolated singularities, as well as local (multi)graded Hilbert schemes.

1. Introduction

Deformation theory provides mathematicians with the tools to describe (local) parameter spaces for various algebraic geometric objects, for example, for isolated singularities, or for invertible sheaves on a projective variety, see [Har10]. However, computing such spaces in practice can be quite difficult, and by hand often intractable. The Macaulay2 package *VersalDeformations*, available online [Ilt11], aims to facilitate such calculations for two concrete deformation problems: versal deformations of isolated singularities, and local (possibly multigraded) Hilbert schemes.

The package *VersalDeformations* provides several functions which may be used to calculate tangent and obstruction spaces for the above-mentioned deformation problems. The function `normalMod` may be used to calculate a basis for any degree of the normal module of some (multi)homogeneous ideal in a polynomial ring. The scripted functor `CT` may be used to calculate bases of the first and second cotangent cohomology modules T^1_A and T^2_A of some algebra A over a field k, assuming that these modules are finite dimensional vector spaces. In the (multi)homogeneous case, `CT` may also be used to calculate bases of homogeneous pieces of these modules.

The main contribution of the package is the method `versalDeformation`, which uses the Massey product algorithm to iteratively lift solutions of a deformation equation to higher and higher order; we describe this in more detail in the following section. This can be used to find power series descriptions of versal Deformations and local Hilbert schemes. Since such a description may not be polynomial, the package provides an interface allowing the user to control at what point the lifting should terminate. The package also implements a more time-consuming lifting algorithm (via the option `SmartLift`) which seeks to minimize the number of higher order terms appearing in the equations for the parameter space.

There are a number of other software packages which provide related functionality. J. Stevens has written scripts for the original Macaulay to calculate T^1 and T^2 for isolated singularities [Ste95]. There is a library for Singular by B. Martin which calculates the versal deformation of an isolated singularity as well as of modules [Mar99]. B. Hovinen has written a package for Macaulay2 which computes versal deformations of maximal Cohen-Macaulay modules on hypersurfaces [Hov10]. Finally, J. Boehm is developing a package for computations involving deformations of Stanley-Reisner rings [Boe19].

2. Solving the Deformation Equation

In the following, we briefly describe the Massey product algorithm as we have implemented it. For more details and mathematical background, see [Ste95] or [Ste03]. For simplicity, we restrict to the case of the versal deformation of an isolated singularity, although our approach for Hilbert schemes is similar.

First we fix some notation. Let S be a polynomial ring over some field k, and let I be an ideal of S defining a scheme $X = \operatorname{Spec} S/I$ with isolated singularity at the origin. Consider a free resolution of S/I:

$$\cdots \longrightarrow S^l \xrightarrow{R^0} S^m \xrightarrow{F^0} S \longrightarrow S/I \longrightarrow 0.$$

Let $\phi_i \in \operatorname{Hom}(S^m/\operatorname{Im} R^0, S)$ $i = 1, \dots, n$ represent a basis of

$$T^1_{S/I} \cong \operatorname{Hom}(S^m/\operatorname{Im} R^0, S)/\operatorname{Jac} F^0.$$

We introduce deformation parameters $t_1, \dots, t_n$ and consider the map $F^1 \colon S[\,\underline{t}\,]^m \to S[\,\underline{t}\,]$ defined as

$$F^1 = F^0 + \sum_{i=1}^{n} t_i \phi_i.$$

Let $\mathfrak{m}$ be the ideal generated by $t_1, \dots, t_n$. It follows that there is a map $R^1 \colon S[\,\underline{t}\,]^l \to S[\,\underline{t}\,]^m$ with $R^1 \equiv R^0 \mod \mathfrak{m}$ satisfying the first order deformation equation

$$F^1 R^1 \equiv 0 \mod \mathfrak{m}^2.$$

Our goal is to lift the above equation to higher order, that is, for each $i > 0$, to find $F^i \colon S[\,\underline{t}\,]^m \to S[\,\underline{t}\,]$ with $F^i \equiv F^{i-1} \mod \mathfrak{m}^i$ and $R^i \colon S[\,\underline{t}\,]^l \to S[\,\underline{t}\,]^m$ with $R^i \equiv R^{i-1} \mod \mathfrak{m}^i$ satisfying $F^i R^i \equiv 0 \mod \mathfrak{m}^{i+1}$. In general, there are obstructions to doing this, governed by the d-dimensional k vector space $T^2_{S/I}$. Thus, we instead aim to solve

$$(F^i R^i)^{\mathrm{tr}} + C^{i-2} G^{i-2} \equiv 0 \mod \mathfrak{m}^{i+1}. \tag{1}$$

Here, $G^{i-2} \colon k[\,\underline{t}\,] \to k[\,\underline{t}\,]^d$ and $C^{i-2} \colon S[\,\underline{t}\,]^d \to S[\,\underline{t}\,]^l$ are congruent modulo $\mathfrak{m}^i$ to G^{i-3} and C^{i-3}, respectively. Furthermore, we require that G^i and C^i vanish for $i < 0$, and C^0 is of the form $V \cdot D$, where $V \in \operatorname{Hom}(S^d, S^l)$ gives representatives of a basis for $T^2_{S/I}$ and $D \in \operatorname{Hom}(S^d, S^d)$ is a diagonal matrix. The G^i now give equations for the miniversal base space of X.

Our implementation solves (1) step by step. Given a solution $(F^i, R^i, G^{i-2}, C^{i-2})$ modulo $\mathfrak{m}^{i+1}$, the package uses Macaulay2's built in matrix quotients to first solve for F^{i+1} and G^{i-1} (by working over the ring $S[\,\underline{t}\,]/I + \operatorname{Im}(G^{i-2})^{\mathrm{tr}} + \mathfrak{m}^{i+2}$) and then solve for R^{i+1} and C^{i-1}. For the actual computation, we avoid working over quotient rings involving high powers of $\mathfrak{m}$ by representing the $(F^i, R^i, G^{i-2}, C^{i-2})$ as lists of matrices which keep track of the orders of the t_j involved.

3. Examples

We provide two examples: a versal deformation and a multigraded Hilbert scheme. We begin with the classical example of the miniversal deformation of the cone over the rational normal curve of degree 4, see [Pin74].

```
i1 : loadPackage "VersalDeformations";

i2 : S=QQ[x_0..x_4];

i3 : I=minors(2,matrix {{x_0,x_1,x_2,x_3},{x_1,x_2,x_3,x_4}});

o3 : Ideal of S

i4 : F0=gens I;

             1       6
o4 : Matrix S  <--- S

i5 : transpose F0

o5 = {-2} | -x_1^2+x_0x_2  |
     {-2} | -x_1x_2+x_0x_3 |
     {-2} | -x_2^2+x_1x_3  |
     {-2} | -x_1x_3+x_0x_4 |
     {-2} | -x_2x_3+x_1x_4 |
     {-2} | -x_3^2+x_2x_4  |

             6       1
o5 : Matrix S  <--- S
```

We see that the tangent space $T^1_{S/I}$ of the miniversal deformation is four-dimensional, and the obstruction space $T^2_{S/I}$ is three-dimensional:

```
i6 : CT^1(F0)

o6 = {-2} | x_1  x_0  0   0    |
     {-2} | 0    0    0   x_0  |
     {-2} | -x_3 -x_2 0   x_1  |
     {-2} | 0    0    x_2 0    |
     {-2} | -x_4 -x_3 x_3 0    |
     {-2} | 0    0    x_4 -x_3 |

             6       4
o6 : Matrix S  <--- S

i7 : CT^2(F0)

o7 = {-3} | 0    0   0   |
     {-3} | 0    0   0   |
     {-3} | x_3  x_4 0   |
     {-3} | x_2  x_3 0   |
     {-3} | x_1  x_2 0   |
     {-3} | -x_4 0   x_3 |
```

```
     {-3} | 0    x_4 x_2 |
     {-3} | 0    x_3 x_1 |

             8       3
o7 : Matrix S  <--- S
```

In this example, our algorithm gives a polynomial solution to the deformation equation:

```
i8 : (F,R,G,C)=versalDeformation(F0);
Calculating first order deformations and obstruction space
Calculating first order relations
Starting lifting
Order 2
Order 3
Solution is polynomial
```

The miniversal base space is the union of $\mathbb{A}^3$ and $\mathbb{A}^1$, intersecting in a point, as can be seen from the following equations:

```
i9 : sum G

o9 = | t_2t_3-t_3^2 |
     | t_1t_3       |
     | t_3t_4       |

                                  3                         1
o9 : Matrix (S[t , t , t , t ])  <--- (S[t , t , t , t ])
                1   2   3   4             1   2   3   4
```

A versal family is given by

```
i10 : transpose sum F

o10 = {0, -2} | t_1x_1+t_2x_0-x_1^2+x_0x_2                     |
      {0, -2} | t_4x_0-x_1x_2+x_0x_3                           |
      {0, -2} | -t_1t_4-t_1x_3-t_2x_2+t_4x_1-x_2^2+x_1x_3 |
      {0, -2} | t_2t_3-t_3^2+t_3x_2-x_1x_3+x_0x_4             |
      {0, -2} | t_3t_4-t_1x_4-t_2x_3+t_3x_3-x_2x_3+x_1x_4 |
      {0, -2} | t_3x_4-t_4x_3-x_3^2+x_2x_4                     |

                                   6                         1
o10 : Matrix (S[t , t , t , t ])  <--- (S[t , t , t , t ])
                 1   2   3   4             1   2   3   4
```

We now consider our second example: the local description of the Hilbert scheme of the diagonal in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ at the point corresponding to the unique Borel fixed ideal, see [CS10] for more details.

```
i11 : S=QQ[x1,x2,x3,y1,y2,y3,z1,z2,z3,Degrees=>
      {{1,0,0},{1,0,0},{1,0,0},{0,1,0},{0,1,0},
      {0,1,0},{0,0,1},{0,0,1},{0,0,1}}];

i12 : I=ideal {y1*z2, x1*z2, y2*z1, y1*z1, x2*z1, x1*z1, x1*y2, x2*y1,
          x1*y1, x2*y2*z2};
```

```
o12 : Ideal of S

i13 : (F,R,G,C)=versalDeformation(gens I,
                  normalModule({0,0,0},gens I),CT^2({0,0,0},gens I));
Calculating first order relations
Starting lifting
Order 2
Order 3
Order 4
Order 5
Order 6
Solution is polynomial
```

Note that since we were interested in the multigraded Hilbert scheme, the tangent space is just the degree $(0,0,0)$ component of the normal module of I, and an obstruction space is given by the degree $(0,0,0)$ component of $T^2_{S/I}$. In any case, this multigraded Hilbert scheme is locally cut out by 8 cubics:

```
i14 : sum G

o14 = | t_2t_3t_4-t_2t_4t_7-t_1t_3t_8+t_1t_7t_8+t_1t_3t_13-...
      | t_1t_3t_4-t_2t_3t_4-t_1t_7t_8+t_2t_7t_8-t_1t_3t_13+...
      | t_1t_3t_16-t_2t_7t_16-t_1t_14t_16+t_2t_14t_16-...
      | t_1t_3t_18-t_2t_7t_18-t_1t_14t_18+t_2t_14t_18-...
      | t_2t_4t_17-t_1t_8t_17+t_1t_13t_17-t_2t_13t_17-...
      | t_2t_4t_18-t_1t_8t_18+t_1t_13t_18-t_2t_13t_18-...
      | t_3t_4t_17-t_7t_8t_17-t_3t_13t_17+t_7t_13t_17-...
      | t_3t_4t_16-t_7t_8t_16-t_3t_13t_16+t_7t_13t_16-...
```

There are in fact 7 irreducible components of the Hilbert scheme which pass through this point:

```
i15 : # primaryDecomposition ideal sum G

o15 = 7
```

References

[Boe19] Janko Boehm. SRdeformations. A Macaulay 2 package for deformations of Stanley-Reisner rings and related computations, 2019. http://www.math.uiuc.edu/Macaulay2/doc/Macaulay2-1.4/share/Macaulay2/SRdeformations.

[CS10] Dustin Cartwright and Bernd Sturmfels. The Hilbert scheme of the diagonal in a product of projective spaces. *Int. Math. Res. Not. IMRN*, (9):1741–1771, 2010.

[Har10] Robin Hartshorne. *Deformation theory*, volume 257 of *Graduate Texts in Mathematics*. Springer, New York, 2010.

[Hov10] Bradford Hovinen. Computations of matrix factorizations with Macaulay2. *JSAG*, 2:6–10, 2010. http://j-sag.org/.

[Ilt11] Nathan Owen Ilten. VersalDeformations. A Macaulay 2 package for computing versal deformations and local Hilbert schemes, 2011. http://people.cs.uchicago.edu/~nilten/m2.html/.

[Mar99] Bernd Martin. Computing versal deformations with SINGULAR. In *Algorithmic algebra and number theory (Heidelberg, 1997)*, pages 283–293. Springer, Berlin, 1999.

[Pin74] Henry C. Pinkham. *Deformations of algebraic varieties with G_m action*. Société Mathématique de France, Paris, 1974. Astérisque, No. 20.

[Ste95] Jan Stevens. Computing versal deformations. *Experiment. Math.*, 4(2):129–144, 1995.

[Ste03] Jan Stevens. *Deformations of singularities*, volume 1811 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2003.

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