On the orthogonality of q-classical polynomials of the Hahn class

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Abstract

The central idea behind this article is to discuss in a unified sense the orthogonality of all possible polynomial solutions of the q-hypergeometric difference equation on a q-linear lattice by means of a qualitative analysis of the relevant q-Pearson equation. To be more specific, a geometrical approach has been used by taking into account every possible rational form of the polynomial coefficients, together with various relative positions of their zeros, in the q-Pearson equation to describe a desired q-weight function on a suitable orthogonality interval. Therefore, our method differs from the standard ones which are based on the Favard theorem, the three-term recurrence relation and the difference equation of hypergeometric type. As a matter of fact, our approach enables to introduce some new orthogonality relations which have not been reported previously.

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1 Introduction

The so-called q-polynomials are of great interest inside the class of special functions since they play an important role in the treatment of several problems such as Eulerian series and continued fractions [8, 13], q-algebras and quantum groups [21, 22, 32] and q-oscillators [9, 4, 16], and references therein, among others.

A q-analog of the Chebychev polynomials is due to Markov in 1884 [11], which can be regarded as the first example of a q-polynomial family. In 1949, Hahn introduced the q-Hahn class [17] including the most popular orthogonal polynomials, which are known as the big q-Jacobi polynomials, on the exponential lattice although he did not use this terminology. During the last decades the q-polynomials have been studied by many authors from different points of view. Nevertheless, there are two most recognized trends. The first approach is based on the basic hypergeometric series [8, 15], which is initiated after the work by Andrews and Askey [7]. The second trend is due to Nikiforov and Uvarov [27, 28] and uses the analysis of difference equations written on non-uniform lattices. The readers are also referred to the surveys [10, 26, 29, 31] as well. These trends are associated with the so-called q-Askey scheme [19] and the Nikiforov-Uvarov scheme [29], respectively. Another approach was published in [25] where the authors proved several characterizations of the q-polynomials [2] starting from the so-called distributional q-Pearson equation (for the non q-case see e.g. [14, 24] and references therein).

In particular, in [25] a classification of all possible families of orthogonal polynomials on the exponential lattice was established, and latter on in [5] the comparison with the q-Askey and Nikiforov-Uvarov schemes was performed, finding two new families of orthogonal polynomials. Furthermore, important contribution to the theory of the q-polynomials, and in particular, to the theory of the q-polynomials on the linear exponential lattice, which is generally called the q-Hahn tableau after the work of Koornwinder [22], was appeared in the very recent book [19]. In [19], the authors presented a unified study of the orthogonality of q-polynomials based on the Favard Theorem. For more details on the q-polynomials of the q-Hahn tableau we refer the readers to the works [1, 2, 5, 3, 12, 19, 22, 23, 26, 27, 28, 29, 31], and references therein.

In this paper, we deal with the orthogonality properties of the q-polynomials of the q-Hahn tableau from a different view point. The main idea is to provide a relatively simpler geometrical analysis of the q-Pearson equation by taking into account every possible rational form of the polynomial coefficients of the q-difference equation. Such a qualitative analysis implies all possible orthogonality relations among the polynomial solutions of the q-difference equation in question. In fact, another attempt of using a similar geometrical approach has been introduced also in [12]. However, the study is far to being complete where only some partial results were obtained.

An obvious way of the derivation of q-difference equation whose bounded solutions are the q-polynomials of the Hahn class, is to consider the discretization of the classical differential equation of hypergeometric type (EHT)

$$\sigma(x)y'' + \tau(x)y' + \lambda y = 0, \tag{1.1}$$

where $\sigma(x)$ and $\tau(x)$ are polynomials of at most second and first degree, respectively, and λ is a constant [1, 24, 26, 28]. To this end, we can use the asymptotic approximations

$$y'(x) \sim \frac{1}{1+q}[D_q y(x) + qD_{q^{-1}}y(x)] \quad \text{and} \quad y''(x) \sim \frac{2q}{1+q}D_q D_{q^{-1}}y(x) \quad \text{as} \quad q \to 1$$

for the derivatives in (1.1), where we use the stardard notation for the q and q^{-1}-Jackson derivatives of $y(x)$ [5, 15, 18], i.e.,

$$D_\zeta y(x) = \frac{y(x) - y(\zeta x)}{(1-\zeta)x}, \qquad \zeta \in \mathbb{C} \setminus \{0, \pm 1\}$$

for $x \neq 0$ and $D_\zeta y(0) = y'(0)$, provided that $y'(0)$ exists. This leads to the q-difference equation of hypergeometric type (q-EHT)

$$\sigma_1(x;q)D_{q^{-1}}D_q y(x,q) + \tau(x,q)D_q y(x,q) + \lambda(q)y(x,q) = 0 \tag{1.2}$$

where

$$\sigma_1(x;q) := \frac{2}{1+q}\left[\sigma(x) - \frac{1}{2}(q-1)x\tau(x)\right], \;\; \tau(x,q) := \tau(x), \;\; \lambda(q) := \lambda, \;\; y(x,q) := y(x).$$

Notice here the relations

$$D_q = D_{q^{-1}} + (q-1)xD_q D_{q^{-1}} \quad \text{and} \quad D_q D_{q^{-1}} = q^{-1}D_{q^{-1}}D_q \tag{1.3}$$

from which (1.2) can be rewritten in the equivalent form

$$\sigma_2(x;q)D_q D_{q^{-1}}y(x,q) + \tau(x,q)D_{q^{-1}}y(x,q) + \lambda(q)y(x,q) = 0, \tag{1.4}$$

where

$$\sigma_2(x,q) := q\left[\sigma_1(x,q) + (1-q^{-1})x\tau(x,q)\right]. \tag{1.5}$$

Therefore, difference equations (1.2) and (1.4) are called the q-EHT of the 1st and 2nd kind, respectively. Clearly, the coefficients $\sigma_1(x;q)$ and $\sigma_2(x;q)$ are polynomials of at most 2nd degree and $\tau(x,q)$ is a 1st degree polynomial in x. Since $\sigma_2(0,q) = q\sigma_1(0,q)$, σ_1 and σ_2 both have the same constant terms. Moreover, substituting (1.3) and (1.5) into (1.2) or (1.4) we may also write an alternative difference equation of the form

$$\sigma_2(x,q)D_q y(x,q) - q\sigma_1(x,q)D_{q^{-1}}y(x,q) + (q-1)x\lambda(q)y(x,q) = 0 \tag{1.6}$$

containing both σ_1 and σ_2. It should be noted that the q-EHT of the 1st and 2nd kind correspond to the second order linear difference equations of hypergeometric type on the linear exponential lattices $x(s) = c_1 q^s + c_2$ and $x(s) = c_1 q^{-s} + c_2$, respectively [1, 26]. In what follows the solutions of (1.2), (1.4) or (1.6) are referred to as the q-classical orthogonal polynomials or, simply, q-polynomials.

In accordance with [1, 5, 25, 29] the q-polynomials can be classified by means of the degree of the polynomial coefficients σ_1 and σ_2 and the fact that either $\sigma_1(0,q), \sigma_2(0,q) \neq 0$ or $q\sigma_1(0,q) = \sigma_2(0,q) = 0$ at the same time. Therefore, we define two classes, namely, the non-zero ($\emptyset$) and zero (0) classes corresponding to the cases where $\sigma_1(0,q), \sigma_2(0,q) \neq 0$ and $q\sigma_1(0,q) = \sigma_2(0,q) = 0$, respectively. In each class we consider all possible degrees of the polynomial coefficients $\sigma_1(x,q)$ and $\sigma_2(x,q)$ as shown in [25, page 182]. For example, the statement q-Jacobi/q-Laguerre implies that $\deg\sigma_1 = 2$ and $\deg\sigma_2 = 1$, or q-Hermite/q-Jacobi indicates that $\deg\sigma_1 = 0$ and $\deg\sigma_2 = 2$.

It is convenient to consider the Taylor polynomial representations of the coefficients

$$\sigma_1(x,q) = \tfrac{1}{2}\sigma_1''(0,q)x^2 + \sigma_1'(0,q)x + \sigma_1(0,q), \quad \sigma_2(x,q) = \tfrac{1}{2}\sigma_2''(0,q)x^2 + \sigma_2'(0,q)x + \sigma_2(0,q)$$

and $\tau(x,q) = \tau'(0,q)x + \tau(0,q)$ with $\tau'(0,q) \neq 0$, and rewrite the q-EHT of the 1st kind in the self-adjoint form

$$D_q\left[\sigma_1(x,q)\rho(x,q)D_{q^{-1}}y(x)\right] + q^{-1}\lambda(q)\rho(x,q)y(x) = 0,$$

where ρ is a function satisfying the so-called q-Pearson equation $D_q[\sigma_1(x,q)\rho] = q^{-1}\tau(x,q)\rho$ or, explicitly,

$$\frac{\rho(qx,q)}{\rho(x,q)} = \frac{\sigma_1(x,q) + (1-q^{-1})x\tau(x,q)}{\sigma_1(qx,q)} = \frac{q^{-1}\sigma_2(x,q)}{\sigma_1(qx,q)}. \tag{1.7}$$

In this paper it is worth mentioning, without loss of generality, that we study the q-EHT of the 1st kind, assume $0 < q < 1$ and take $\lambda(q)$ as

$$\lambda(q) := \lambda_n(q) = -[n]_q\left[\tau'(0,q) + \tfrac{1}{2}[n-1]_{q^{-1}}\sigma_1''(0,q)\right], \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \tag{1.8}$$

since we are interested only in the polynomial solutions [1, 26]. Moreover, we restrict ourselves to the cases of $\emptyset$-Jacobi/Jacobi and 0-Jacobi/Jacobi families (for the other cases see [6, 30]). Our main goal is to study each orthogonal polynomial system or sequence (OPS), which is orthogonal with respect to (w.r.t.) a q-weight function $\rho(x,q) > 0$ satisfying the q-Pearson equation as well as certain boundary conditions (BCs) to be introduced in Section 2. For each family of polynomial solutions of (1.2) we search for all possible orthogonality intervals depending on the range of the parameters coming from the coefficients of (1.2) and the corresponding orthogonality relations. Hence, in Section 2, we present the candidate intervals by inspecting the possibilities to satisfy the BCs for ρ to have orthogonal solutions. A theorem which helps to calculate q-weight functions is given in Section 3. Section 4 deals with the qualitative analysis including the theorems stating the main results of our article. Roughly, our qualitative analysis is concerned with the examination of the behaviour of the graphs of the ratio $\rho(qx,q)/\rho(x,q)$ by means of the definite right hand side (r.h.s.) of (1.7) in order to find out a suitable q-weight function. The last Section concludes the paper with some remarks.

2 The orthogonality and preliminary results

We first introduce the well known theorem for the orthogonality of polynomial solutions of (1.2) in order to make the article self-contained [1, 11, 26].

Theorem 1 *Let ρ be a function satisfying the q-Pearson equation (1.7) in such a way that the BCs*

$$\sigma_1(x,q)\rho(x,q)x^k\Big|_{x=a,b} = \sigma_2(q^{-1}x,q)\rho(q^{-1}x,q)x^k\Big|_{x=a,b} = 0, \qquad k\in\mathbb{N}_0 \tag{2.1}$$

also hold. Then the sequence $\{P_n(x)\}$ of polynomial solutions are orthogonal on (a,b) w.r.t $\rho(x,q)$ in the sense that

$$\int_a^b P_n(x,q)P_m(x,q)\rho(x,q)d_qx = d_n^2(q)\delta_{mn},$$

where $d_n(q)$ and δ_{mn} denote the norm of the polynomials P_n and the Kronecker delta, respectively. Analogously, if the conditions

$$\sigma_2(x,q)\rho(x,q)x^k\Big|_{x=a,b} = \sigma_1(qx,q)\rho(qx,q)x^k\Big|_{x=a,b} = 0, \qquad k\in\mathbb{N}_0 \tag{2.2}$$

are fulfilled, the q-polynomials then satisfy the relation

$$\int_a^b P_n(x,q)P_m(x,q)\rho(x,q)d_{q^{-1}}x = d_n^2(q)\delta_{mn}.$$

In this theorem, the q-Jackson integrals [15, 18] are defined by

$$\int_0^a f(x)d_qx = (1-q)a\sum_{j=0}^{\infty} q^j f(q^ja) \quad\text{and}\quad \int_a^0 f(x)d_qx = (1-q)(-a)\sum_{j=0}^{\infty} q^j f(q^ja) \tag{2.3}$$

if $a>0$ and $a<0$, respectively. Therefore, we have

$$\int_a^b f(x)d_qx := \int_0^b f(x)d_qx - \int_0^a f(x)d_qx \quad\text{and}\quad \int_a^b f(x)d_qx := \int_a^0 f(x)d_qx + \int_0^b f(x)d_qx \tag{2.4}$$

when $0<a<b$ and $a<0<b$, respectively. Furthermore, we make use of the *improper* q-Jackson integrals

$$\int_0^{\infty} f(x)d_qx = (1-q)\sum_{j=-\infty}^{\infty} q^j f(q^j) \;\text{and}\; \int_{-\infty}^{\infty} f(x)d_qx = (1-q)\sum_{j=-\infty}^{\infty} q^j[f(q^j)+f(-q^j)] \tag{2.5}$$

where the second one is sometimes called the *bilateral* q-integral. The q^{-1}*-Jackson integrals* may be defined similarly. For instance, the improper q^{-1}*-Jackson integral* on (a,∞) is given by

$$\int_a^{\infty} f(x)d_{q^{-1}}x = (q^{-1}-1)a\sum_{j=0}^{\infty} q^{-j}f(q^{-j}a), \quad a>0 \tag{2.6}$$

provided that $\lim_{j\to\infty} q^{-j}f(q^{-j}a)=0$ and the series is convergent.

According to Theorem 1, we have to determine an interval (a,b) in which ρ is q-integrable and $\rho>0$ on the lattice points of the types $\alpha q^{\pm k}$ and $\beta q^{\pm k}$ for $k\in\mathbb{N}_0$ and $\alpha,\beta\in\mathbb{R}$. Such a weight function will clearly be a solution of the q-Pearson equation in (1.7) or

$$\frac{\rho(q^{-1}x,q)}{\rho(x,q)} = \frac{\sigma_2(x,q)+(1-q)x\tau(x,q)}{\sigma_2(q^{-1}x,q)} = \frac{q\sigma_1(x,q)}{\sigma_2(q^{-1}x,q)} \tag{2.7}$$

in alternative form, which imply the equivalence of the following relations

$$\sigma_2(x,q)\rho(x,q) = q\sigma_1(qx,q)\rho(qx,q) \quad \text{and} \quad \sigma_2(q^{-1}x,q)\rho(q^{-1}x,q) = q^{-1}\sigma_1(x,q)\rho(x,q). \tag{2.8}$$

To this end, a qualitative analysis of the q-Pearson equation is presented by a detailed inspection of the right hand sides of (1.7) and (2.7). Note that the right hand sides of (1.7) and (2.7) consist of the polynomial coefficients σ_1 and σ_2 of the q-EHT which can be made definite for possible forms of the coefficients. As a result, the possible behaviour of ρ on the left hand side (l.h.s.) of (1.7) and (2.7) and the candidate orthogonality intervals can be obtained accordingly.

OPSs on finite (a,b) intervals

First assume that (a,b) denotes a finite interval. We list the following possibilities for finding ρ which obeys the BCs in (2.1) or in (2.2).

PI. This is the simplest case where σ_1 vanishes at both $x=a$ and b, i.e., $\sigma_1(a,q)=\sigma_1(b,q)=0$. Using (2.7) rewritten of the form

$$\rho(q^{-1}x,q) = \frac{q\sigma_1(x,q)}{\sigma_2(q^{-1}x,q)}\rho(x,q) \tag{2.9}$$

we see that the function $\rho(x,q)$ becomes zero at the points $q^{-k}a$ and $q^{-k}b$ for $k\in\mathbb{N}$. However, we have to take into consideration three different situations.

(i) Let $a<0<b$. Since the points $q^{-k}a$ and $q^{-k}b$ lie outside the interval (a,b) and BCs are fulfilled at $x=a$ and b, there could be an OPS on (a,b) w.r.t. a measure supported at the points aq^k and bq^k if ρ is positive.

(ii) Let $0<a<b$. In this case $\rho(x,q)$ vanishes at the points $q^{-k}a$ in (a,b) and $q^{-k}b$ out of (a,b). Then, the only possibility to have an OPS on (a,b) depends on the existence of N such that $q^Nb=a$. This condition, however, implies that $bq^k=aq^{-(N-k)}$ and that ρ vanishes at bq^k for $k=0,1,\ldots,N$, and, therefore, it must be rejected. The similar statement is true when $a<b<0$, which can be obtained by a simple linear scaling transformation so that it does not represent an independent case.

(iii) Let $a=0<b$ (or, $a<b=0$). This case is much more involved. First of all, if $a=0$ is a zero of $\sigma_1(x,q)$ then it is a zero of $\sigma_2(x,q)$ as well, both containing a factor x. Therefore, the r.h.s. of q-Pearson equation (1.7) or (2.7) may be simplified and **PI**(i)-(ii) are not valid anymore. In fact, in this case an OPS may be defined on $(0,b)$ supported at the points bq^k for $k\in\mathbb{N}_0$.

PII. The relations in (2.8) suggest an alternative possibility to define an OPS on (a,b). Namely, if $q^{-1}a$ and $q^{-1}b$ are both zeros of $\sigma_2(x,q)$, by using (1.7) rewritten of the form

$$\rho(qx,q) = \frac{q^{-1}\sigma_2(x,q)}{\sigma_1(qx,q)}\rho(x,q), \tag{2.10}$$

it follows that $\rho(x,q)$ vanishes at the points q^ka and q^kb for $k\in\mathbb{N}_0$. Then two different situations appear depending on whether $a<0<b$ or $0<a<b$. In the first case, $\rho(x,q)=0$ at the points q^ka and q^kb in (a,b), which is not interesting. In the second case, the q^kb are in (a,b) whereas the q^ka remain out of (a,b), so that we could have an OPS if there exists N such that $q^{-N}a=q^{-1}b$. However, since $q^{-k}a=q^{N-k-1}b$ ρ vanishes at the $q^{-k}a$ implying that the second case is also not interesting.

PIII. Let $q^{-1}a$ and b be the roots of σ_2 and σ_1, respectively. Then we see, from (2.9) and (2.10), that $\rho=0$ at $x=q^{-k}b$ for $k\in\mathbb{N}$ and at $x=q^ka$ for $k\in\mathbb{N}_0$. That is, if $a<0<b$, $\rho=0$ on $x\in(a,0)$ and, therefore, an OPS can not be constructed on (a,b) unless $a\to 0^-$. In this limiting case of $x\in(0,b)$, it may be possible to introduce a desired weight function supported at the points bq^k for $k\in\mathbb{N}_0$. If

$0 < a < b$, on the other hand, ρ vanishes for $x < a$ and $x > b$. Thus there could be an OPS defined on (a, b) with a weight function supported at the points $q^k b$ marched in the direction of negative x-axis provided that $q^N b = a$ for some finite N integer. Note that in the limiting case of $a \to 0^+$ the number of lattice points $q^k b$ becomes infinity, the point at the origin $a = 0$ being an accumulation point. Alternatively, we can define an equivalent OPS on $(q^{-1}a, q^{-1}b)$ at $q^{-k}a$ increasing in the direction of positive x-axis provided now that $q^{-N}a = b$, where N is a finite integer.

PIV. Assume that a and $q^{-1}b$ are the roots of $\sigma_1(x, q)$ and $\sigma_2(x, q)$, respectively. Then, from (2.9) and (2.10), it follows that $\rho(x, q)$ vanishes at the points $q^{-k}a$, $k \in \mathbb{N}$ and $q^k b$, $k \in \mathbb{N}_0$. So, if $a < 0 < b$, it is not possible to find a weight function satisfying the BCs. Nevertheless, as in **PIII**, in the limiting case of $b \to 0^+$ an OPS at the points $q^k a$ for $k \in \mathbb{N}_0$ on $(a, 0)$ may be constructed. If $0 < a < b$, there is no possibility to introduce an OPS. Note that when $a = 0 < b$, an OPS does also not exist.

OPSs on infinite intervals

Assume now that (a, b) is an infinite interval. Without any loss of generality, let a be a finite number and $b \to \infty$. In fact, the system on the infinite interval $(-\infty, b)$ is not independent which may be transformed into (a, ∞) on replacing x by $-x$. Obviously one BC in (2.1) reads as

$$\lim_{b\to\infty} \sigma_1(b, q)\rho(b, q)b^k = 0 \quad \text{or} \quad \lim_{b\to\infty} \sigma_2(b, q)\rho(b, q)b^k = 0, \quad k \in \mathbb{N}_0,$$

and there are additional cases to be considered as a function of the l.h.s. boundary point at $x = a$.

PV. If $x = a \neq 0$ is root of $\sigma_1(x, q)$ then, from (2.9), $\rho(x, q)$ vanishes at the points $q^{-k}a$ for $k \in \mathbb{N}$ which are interior points of (a, ∞) when $a > 0$. Therefore there is no OPS on (a, ∞) for $a > 0$. If $a < 0$ we may find a q-weight function on (a, ∞) supported at the points of the form $q^k a$ for $a < 0$ and $q^{\pm k}\alpha$ for arbitrary $\alpha > 0$ where α can be taken as unity. If $a = 0$, on the other hand, then a weight function on $(0, \infty)$ may be defined at $q^{\pm k}\alpha$ for arbitrary $\alpha > 0$ and $k \in \mathbb{N}_0$.

PVI. If $x = q^{-1}a$ is a root of $\sigma_2(x, q)$, as we have already discussed, ρ is zero at $q^k a$ for $k \in \mathbb{N}_0$. Therefore, for $a > 0$ a q-weight function may exist on $(q^{-1}a, \infty)$ supported at the points $q^{-k}a$, $k \in \mathbb{N}$. An OPS does not exist if $a < 0$. Finally, if $a = 0$ it is possible to find a ρ on $(0, \infty)$ at $q^{\pm k}\alpha$ for arbitrary $\alpha > 0$ and $k \in \mathbb{N}_0$.

PVII. Finally, we consider the possibility of satisfying the BC

$$\lim_{a\to-\infty} \sigma_1(a, q)\rho(a, q)a^k = 0$$

in the limiting case as $a \to -\infty$. If this condition holds a weight function and, hence, an OPS on $(-\infty, \infty)$ at the points $\pm q^{\pm k}\alpha$ for arbitrary $\alpha > 0$ and $k \in \mathbb{N}_0$ can be defined.

The aforementioned considerations are expressible as a theorem.

Theorem 2 *Let $a_1(q)$, $b_1(q)$ and $a_2(q)$, $b_2(q)$ denote the zeros of $\sigma_1(x, q)$ and $\sigma_2(x, q)$, respectively. Let ρ be a bounded and non-negative function satisfying the q-Pearson equation (1.7). Such a function ρ can satisfy the BCs (2.1) or (2.2) and, therefore, it may be a desired weight function for the polynomial solutions $P_j(x, q)$ of (1.2) only in the following cases:*

1. *Let $a < 0 < b$, where $a = a_1(q)$ and $b = b_1(q)$. Then ρ is supported at the points $q^k a$ and $q^k b$ for $k \in \mathbb{N}_0$ on $[a, b]$ such that*

$$\int_{a_1(q)}^{b_1(q)} P_n(x, q)P_m(x, q)\rho(x, q)d_q x = d_n^2(q)\delta_{mn}. \tag{2.11}$$

where the q-Jackson integral is of type (2.4).

2. *Let $a = 0 < b$, where $a = a_1(q)$ and $b = b_1(q)$. Then ρ is supported at the points $q^k b$ for $k \in \mathbb{N}_0$ on $(0, b]$ such that*

$$\int_0^{b_1(q)} P_n(x,q)P_m(x,q)\rho(x,q)d_qx = d_n^2(q)\delta_{mn}, \tag{2.12}$$

where the q-Jackson integral is of type (2.3).

3. *Let $0 = a < b$, where $a = a_2(q)$ and $b = a_1(q)$. Then ρ is supported at the points $q^k b$ for $k \in \mathbb{N}_0$ on $(0, b]$ such that*

$$\int_0^{a_1(q)} P_n(x,q)P_m(x,q)\rho(x,q)d_qx = d_n^2(q)\delta_{mn}, \tag{2.13}$$

where the q-Jackson integral is of type (2.3).

4. *Let $0 < a < b$, where $a = a_2(q)$ and $b = a_1(q)$. Then ρ is supported at the points $q^k b$; $a = q^N b < \cdots < q^2 b < qb < b$ or, equivalently, $q^{-k}a$; $a < aq^{-1} < aq^{-2} < \cdots < q^{-N}a = b$ such that*

$$\int_{qa_2(q)=q^N a_1(q)}^{a_1(q)} P_n(x,q)P_m(x,q)\rho(x,q)d_qx = d_n^2(q)\delta_{mn}, \tag{2.14}$$

where the q-Jackson integral is of type (2.4) and equivalent to the finite sum

$$\int_{q^N a_1(q)}^{a_1(q)} [\cdot]d_qx = (1-q)a_1(q)\sum_{k=0}^{N-1} P_n(q^k a_1(q),q)P_m(q^k a_1(q),q)\rho(q^k a_1(q),q).$$

The orthogonality relation can also be written, at least, formally in terms of the q^{-1}-integral of type (2.6)

$$\int_{a_2(q)}^{q^{-1}a_1(q)=q^{-N}a_2(q)} P_n(x,q)P_m(x,q)\rho(x,q)d_{q^{-1}}x = d_n^2(q)\delta_{mn}, \tag{2.15}$$

which is again a finite sum of the form

$$\int_{a_2(q)}^{q^{-N}a_2(q)} [\cdot]d_{q^{-1}}x = (1-q^{-1})a_2(q)\sum_{k=0}^{N-1} P_n(q^{-k}a_2(q),q)P_m(q^{-k}a_2(q),q)\rho(q^{-k}a_2(q),q).$$

5. *Let $a < b = 0$, where $a = a_1(q)$ and $b = 0$. Then ρ is supported at the points $q^k a$ for $k \in \mathbb{N}_0$ on $[a, 0)$ such that*

$$\int_{a_1(q)}^{0} P_n(x,q)P_m(x,q)\rho(x,q)d_qx = d_n^2(q)\delta_{mn},$$

where the q-Jackson integral is of type (2.3).

6. *Let $a = a_1(q) = 0$ and $b \to \infty$. Then ρ is supported at the points $q^{\pm k}\alpha$ for arbitrary $\alpha > 0$ and $k \in \mathbb{N}_0$ on $(0, \infty)$ such that*

$$\int_0^{\infty} P_n(x,q)P_m(x,q)\rho(x,q)d_qx = d_n^2(q)\delta_{mn},$$

where the q-Jackson integral is of type (2.5).

7. *Let $a = a_1(q) < 0$ and $b \to \infty$. Then ρ is supported at the points $q^k a$ and $q^{\mp k}\alpha$ for arbitrary $\alpha > 0$ and $k \in \mathbb{N}_0$ on $[a, \infty)$ such that*

$$\int_{a_1(q)}^{\infty} P_n(x,q)P_m(x,q)\rho(x,q)d_qx := \int_{a_1(q)}^{0} [\cdot]d_qx + \int_0^{\infty} [\cdot]d_qx = d_n^2(q)\delta_{mn},$$

where the first q-Jackson integral is of type (2.3) and the second one is of type (2.5), respectively.

8. *Let $a = a_2(q) > 0$ and $b \to \infty$. Then ρ is supported at the points $q^{-k}a$ for $k \in \mathbb{N}_0$ on $[a, \infty)$ such that*

$$\int_{a_2(q)}^{\infty} P_n(x,q)P_m(x,q)\rho(x,q)d_{q^{-1}}x = d_n^2(q)\delta_{mn},$$

where the q^{-1}-Jackson integral is of type (2.6).

9. *Let $a = a_2(q) = 0$ and $b \to \infty$. Then ρ is supported at the points $q^{\pm k}\alpha$ for arbitrary $\alpha > 0$ and $k \in \mathbb{N}_0$ on $(0, \infty)$ such that*

$$\int_{0}^{\infty} P_n(x,q)P_m(x,q)\rho(x,q)d_q x = d_n^2(q)\delta_{mn},$$

where the q-Jackson integral is of type (2.5).

10. *Let $a \to -\infty$ and $b \to \infty$. Then ρ is supported at the points $\mp q^{\pm k}\alpha$ for arbitrary $\alpha > 0$ and $k \in \mathbb{N}_0$ on $(-\infty, \infty)$ such that*

$$\int_{-\infty}^{\infty} P_n(x,q)P_m(x,q)\rho(x,q)d_q x = d_n^2(q)\delta_{mn},$$

where the bilateral q-Jackson integral is of type (2.5).

3 The q-weight function

The explicit form of a q-weight function may be deduced by means of Theorem 3.

Theorem 3 *Let f satisfy the difference equation*

$$\frac{f(qx;q)}{f(x;q)} = \frac{a(x;q)}{b(x;q)}, \tag{3.1}$$

such that the limits $\lim_{x\to 0} f(x;q) = f(0,q)$ and $\lim_{x\to\infty} f(x;q) = f(\infty,q)$ exist, where $a(x;q)$ and $b(x;q)$ are definite functions. Then $f(x;q)$ admits the two q-integral representations

$$f(x,q) = f(0,q)\exp\left[\int_0^x \frac{1}{(q-1)t}\ln\left[\frac{a(t,q)}{b(t,q)}\right]d_q t\right] \tag{3.2}$$

and

$$f(x,q) = f(\infty,q)\exp\left[\int_x^\infty \frac{1}{(1-q^{-1})t}\ln\left[\frac{a(t,q)}{b(t,q)}\right]d_{q^{-1}} t\right] \tag{3.3}$$

provided that the integrals are convergent.

Proof Taking the logarithms of both sides of (3.1), multiplying by $1/(q-1)t$ and then integrating from 0 to x, we have

$$\int_0^x \frac{1}{(q-1)t}\ln\left[\frac{f(qt,q)}{f(t,q)}\right]d_q t = \int_0^x \frac{1}{(q-1)t}\ln\left[\frac{a(t,q)}{b(t,q)}\right]d_q t.$$

The l.h.s. is expressible as

$$\begin{aligned}\int_0^x \frac{1}{(q-1)t}\ln\left[\frac{f(qt,q)}{f(t,q)}\right]d_q t &= \lim_{n\to\infty}\sum_{j=0}^{n}\left[\ln\left(f(q^j x,q)\right) - \ln\left(f(q^{j+1}x,q)\right)\right] \\ &= \ln\left[f(x,q)\right] - \ln\left[f(0,q)\right],\end{aligned}$$

which completes the proof, on using the fact that $f(q^{n+1}x,q) \to f(0,q)$ as $n \to \infty$ for $0 < q < 1$. The second representation (3.3) can be proven in a similar way. ■

This theorem can be used to derive the q-weight function for every σ_1 and σ_2. However, here we take into account the quadratic coefficients leading to $\emptyset$-Jacobi/Jacobi and 0-Jacobi/Jacobi cases. The results, some of which may be found in [5], are stated by the next theorem.

Theorem 4 *In the $\emptyset$-Jacobi/Jacobi case, let $\sigma_1(x,q)$ and $\sigma_2(x,q)$ be of forms*

$$\sigma_1(x,q)=\frac{1}{2}\sigma_1''(0,q)[x-a_1(q)][(x-b_1(q))] \text{ and } \sigma_2(x,q)=\frac{1}{2}\sigma_2''(0,q)[x-a_2(q)][x-b_2(q)]$$

in which $\sigma_1''(0,q)a_1(q)b_1(q)\neq 0$ and $\sigma_2''(0,q)a_2(q)b_2(q)\neq 0$. And let, in 0-Jacobi/Jacobi case,

$$\sigma_1(x,q)=\frac{1}{2}\sigma_1''(0,q)x[x-a_1(q)] \text{ and } \sigma_2(x,q)=\frac{1}{2}\sigma_2''(0,q)x[x-a_2(q)]$$

where $\sigma_1''(0,q)\neq 0$ and $\sigma_2''(0,q)\neq 0$. Then a solution $\rho(x,q)$ of q-Pearson equation (1.7) is expressible in the equivalent forms shown in Table 1.

Table 1: Expressions for the q-weight function $\rho(x,q)$

$\emptyset$-Jacobi/Jacobi case
1. $\dfrac{(a_1^{-1}qx,\, b_1^{-1}qx;q)_\infty}{(a_2^{-1}x,\, b_2^{-1}x;q)_\infty}$
2. $\lvert x\rvert^\alpha \dfrac{(b_1^{-1}qx,a_2q/x;q)_\infty}{(a_1/x,b_2^{-1}x;q)_\infty}$ where $q^\alpha=\dfrac{q^{-2}\sigma_2''(0,q)b_2}{\sigma_1''(0,q)b_1}$
0-Jacobi/Jacobi case
3. $\lvert x\rvert^\alpha \dfrac{(a_1^{-1}qx;q)_\infty}{(a_2^{-1}x;q)_\infty}$ where $q^\alpha=\dfrac{q^{-2}\sigma_2''(0,q)a_2}{\sigma_1''(0,q)a_1}$
4. $\lvert x\rvert^\alpha \sqrt{x^{\log_q x-1}}(qa_1^{-1}x,qa_2/x;q)_\infty$ where $q^\alpha=\dfrac{q^{-2}\sigma_2''(0,q)}{\sigma_1''(0,q)a_1}$

Proof We start proving the first expression in Table 1. Keeping in mind that $q^{-1}\sigma_2(0,q)=\sigma_1(0,q)$ and that $q^{-1}\sigma_2''(0,q)a_2(q)b_2(q)=\sigma_1''(0,q)a_1(q)b_1(q)$ we have from (1.7)

$$\frac{\rho(qx,q)}{\rho(x,q)}=\frac{q^{-1}\sigma_2''(0,q)[x-a_2(q)][x-b_2(q)]}{\sigma_1''(0,q)[qx-a_1(q)][qx-b_1(q)]}=\frac{[1-a_2^{-1}(q)x][1-b_2^{-1}(q)x]}{[1-a_1^{-1}(q)qx][1-b_1^{-1}(q)qx]} \tag{3.4}$$

which gives

$$\rho(x,q)=\rho(0,q)\exp\left\{\int_0^x\frac{1}{(q-1)t}\Big[\ln(1-a_2^{-1}t)+\ln(1-b_2^{-1}t)-\ln(1-a_1^{-1}qt)-\ln(1-b_1^{-1}qt)\Big]d_qt\right\}$$

on using (3.2). By definition (2.3) of the q-integral, we first obtain

$$\rho(x,q)=\rho(0,q)\exp\left\{\ln\prod_{k=0}^{\infty}(1-a_1^{-1}q^{k+1}x)(1-b_1^{-1}q^{k+1}x)-\ln\prod_{k=0}^{\infty}(1-a_2^{-1}q^kx)(1-b_2^{-1}q^kx)\right\}$$

and, therefore,

$$\rho(x,q)=\rho(0,q)\frac{(a_1^{-1}qx,b_1^{-1}qx;q)_\infty}{(a_2^{-1}x,b_2^{-1}x;q)_\infty},\qquad \rho(0,q)\neq 0. \tag{3.5}$$

This implies that $a_1(q)q^{-1-k}$ and $b_1(q)q^{-1-k}$ for $k\geq 0$ are zeros of ρ. Furthermore, $a_2(q)q^{-j}$ and $b_2(q)q^{-j}$ for $j\geq 0$ stand for the simple poles of ρ. Note here that $\rho(0,q)$ can be made unity, and $a_1(q)$,

$b_1(q)$, $a_2(q)$ and $b_2(q)$ are non-zero constants. Therefore the solution in (3.5) is continuous everywhere except at the simple poles.

To show 4., we rewrite the the q-Pearson equation in the form

$$\frac{\rho(qx,q)}{\rho(x,q)} = \frac{ax[1-a_2(q)/x]}{[1-a_1^{-1}(q)qx]}, \qquad a = \frac{q^{-2}\sigma_2''(0,q)}{\sigma_1''(0,q)a_1(q)} \tag{3.6}$$

and assume that ρ is a product of three functions $\rho(x,q) = f(x,q)g(x,q)h(x,q)$. Hence, if f, g and h are solutions of

$$\frac{f(qx,q)}{f(x,q)} = ax, \quad \frac{g(qx,q)}{g(x,q)} = \frac{1}{[1-a_1^{-1}(q)qx]} \text{ and } \frac{h(qx,q)}{h(x,q)} = \left[1 - \frac{a_2(q)}{x}\right] \tag{3.7}$$

respectively, then $\rho = fgh$ is a solution of (3.6). A solution of (3.7) for $f(x,q)$ is of the form $f(x,q) = |x|^{\alpha} H^{(1)}(x)$, which may be verified by direct substitution. Here, the function $H^{(\beta)}(x) = \sqrt{x^{\log_q^{x^\beta}} - \beta}$ with $\beta \neq 0$ was first defined in [5], and $\alpha \neq 0$ is such that $q^\alpha = a$. The equation in (3.7) for $g(x,q)$ can be solved in a way similar to that of (3.4). So we find that $g(x,q) = g(0,q)(a_1^{-1}qx;q)_\infty$, where $g(0,q) = 1$. The expression (3.2) is not suitable in finding $h(x,q)$ which gives a divergent infinite product. Instead, we employ (3.3) so that the equation for $h(x,q)$ becomes

$$\frac{h(q^{-1}x,q)}{h(x,q)} = \frac{1}{[1-qa_2(q)/x]}$$

whose solution is of the form $h(x,q) = h(\infty,q)(qa_2/x;q)_\infty$, where $h(\infty,q)$ may be taken again as unity without loss of generality. Clearly $h(x,q)$ is uniformly convergent in any compact subset of the complex plane that does not contain the point at the origin. Moreover, the product converges to an arbitrary constant c, which has been set to unity, as $x \to \infty$. Thus,

$$\rho(x,q) = f(x,q)g(x,q)h(x,q) = |x|^\alpha \sqrt{x^{\log_q x - 1}}(qa_1^{-1}x, qa_2/x; q)_\infty.$$

In order to obtain the expressions 2 and 3 in table 1 for the weight function we use the same procedure as before, but starting from the q-Pearson equation written in the forms

$$\frac{\rho(qx,q)}{\rho(x,q)} = a\frac{[1-a_2(q)/x][1-b_2^{-1}(q)x]}{[1-a_1(q)q^{-1}/x][1-b_1^{-1}(q)qx]}, \quad a = \frac{q^{-2}\sigma_2''(0,q)b_2(q)}{\sigma_1''(0,q)b_1(q)} \tag{3.8}$$

and

$$\frac{\rho(qx,q)}{\rho(x,q)} = a\frac{[1-a_2^{-1}(q)x]}{[1-a_1^{-1}(q)qx]}, \quad a = \frac{q^{-2}\sigma_2''(0,q)a_2(q)}{\sigma_1''(0,q)a_1(q)}, \tag{3.9}$$

respectively. This complete the proof. ■

Remark Notice that for getting the expressions of the weight function we have used the q-Pearson equation rewritten in different forms, namely (3.4), (3.6), (3.8) and (3.9), and different solution procedure in each case, therefore, it is not surprising that ρ has several equivalent representations displayed in Table 1. However, they all satisfy the same equation and, therefore, they differ only by a multiplicative constant.

4 The qualitative analysis and characterization of q-polynomials

The rational function on the r.h.s. of the q-Pearson equation (1.7) has been examined in detail. Since it is the ratio of two polynomials σ_1 and σ_2 of at most second degree, we deal with a definite rational function having at most two zeros and two poles. In the analysis of the unknown quantity

$\rho(qx,q)/\rho(x,q)$ on the l.h.s. of (1.7), we sketch roughly its graph by using every possible form of the definite rational function in question. In particular, we split the x-interval into subintervals according to whether $\rho(qx,q)/\rho(x,q) < 1$ or $\rho(qx,q)/\rho(x,q) > 1$, which yields valuable information about the monotonicity of $\rho(x,q)$. Other significant properties of ρ are provided by the asymptotes, if there exist any, of $\rho(qx,q)/\rho(x,q)$. A full analysis along these lines is sufficient for a complete characterization of the orthogonal q-polynomials. A similar characterization is made in a very recent book [19] based on the three-term recursion and the Favard theorem.

Let us remind once more that, because of a large number of possibilities, we discuss here only the cases in which both σ_1 and σ_2 are of second degree.

4.1 The non-zero case

Let the coefficients σ_1 and σ_2 be quadratic polynomials in x such that $\sigma_1(0,q) \neq 0$ and $\sigma_2(0,q) \neq 0$. If σ_1 is written in terms of its roots, i.e., $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)[x-a_1(q)][x-b_1(q)]$ then, from (1.5),

$$\sigma_2(x,q) = \Big[\tfrac{1}{2}\sigma_1''(0,q)+(1-q^{-1})\tau'(0,q)\Big]qx^2 - \Big[\tfrac{1}{2}\sigma_1''(0,q)(a_1+b_1)-(1-q^{-1})\tau(0,q)\Big]qx + \tfrac{1}{2}q\sigma_1''(0,q)a_1b_1$$

where $\frac{1}{2}\sigma_1''(0,q)+(1-q^{-1})\tau'(0,q) \neq 0$ by hypothesis. Then q-Pearson equation (1.7) takes the form

$$f(x,q) := \frac{\rho(qx,q)}{\rho(x,q)} = \frac{q^{-1}\sigma_2(x,q)}{\sigma_1(qx,q)} = \left[1+\frac{(1-q^{-1})\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right]\frac{[x-a_2(q)][x-b_2(q)]}{[qx-a_1(q)][qx-b_1(q)]} \tag{4.1}$$

provided that the discriminant denoted by Δ_q,

$$\Delta_q := \Big[a_1(q)+b_1(q)-\frac{(1-q^{-1})\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\Big]^2 - 4a_1(q)b_1(q)\Big[1+\frac{(1-q^{-1})\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\Big],$$

of the quadratic polynomial in the nominator of $f(x,q)$ in (4.1) is non-zero. Here $x=a_2$ and $x=b_2$ denote the zeros of f, and they are constant multiples of the roots of $\sigma_2(x,q)$.

We see that the lines $x=q^{-1}a_1$ and $x=q^{-1}b_1$ stand for the vertical asymptotes of $f(x,q)$ and the point $y=1$ is always its y-intercept since $\sigma_2(0,q)=q\sigma_1(0,q)$. Moreover, the locations of the zeros of f are determined by the straightforward lemma.

Lemma 5 *Define the parameter*

$$\Lambda_q = \frac{1}{q^2}\left[1+\frac{(1-q^{-1})\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right] \neq 0 \tag{4.2}$$

so that the line $y=\Lambda_q$ denotes the horizontal asymptote of $f(x,q)$. Then we encounter the following cases for the roots of the equation $f(x,q)=0$.

Case 1. *If $\Lambda_q>0$ and $a_1(q)<0<b_1(q)$, f has two real and distinct roots with opposite signs.*

Case 2. *If $\Lambda_q>0$ and $0<a_1(q)<b_1(q)$, there exist three possibilities*

- **(a)** *if $\Delta_q>0$, f has two real roots with the same signs*
- **(b)** *if $\Delta_q=0$, f has a double root*
- **(c)** *if $\Delta_q<0$, f has a pair of complex conjugate roots.*

Case 3. *If $\Lambda_q<0$ and $a_1(q)<0<b_1(q)$, there exist three possibilities*

- **(a)** *if $\Delta_q>0$, f has two real roots with the same signs*
- **(b)** *if $\Delta_q=0$, f has a double root*
- **(c)** *if $\Delta_q<0$, f has a pair of complex conjugate roots.*

Case 4. *If $\Lambda_q<0$ and $0<a_1(q)<b_1(q)$, f has two real distinct roots with opposite signs.*

Now, our strategy consists of sketching first the graphs of $f(x,q)$ depending on all possible relative positions of the zeros of σ_1 and σ_2. To obtain the behaviours of q-weight functions ρ from the graphs of $f(x,q) = \rho(qx,q)/\rho(x,q)$, we divide the real line into subintervals in which ρ is either monotonic decreasing or increasing. We take into consideration only the subintervals where $\rho > 0$. Note that if ρ is initially positive then we have $\rho > 0$ everywhere in an interval where $\rho(qx,q)/\rho(x,q) > 0$. Then we find suitable orthogonality intervals in cooperation with Theorem 2 as well.

As another remark, we need to examine the cases $\Lambda_q > 1$ and $0 < \Lambda_q < 1$ separately.

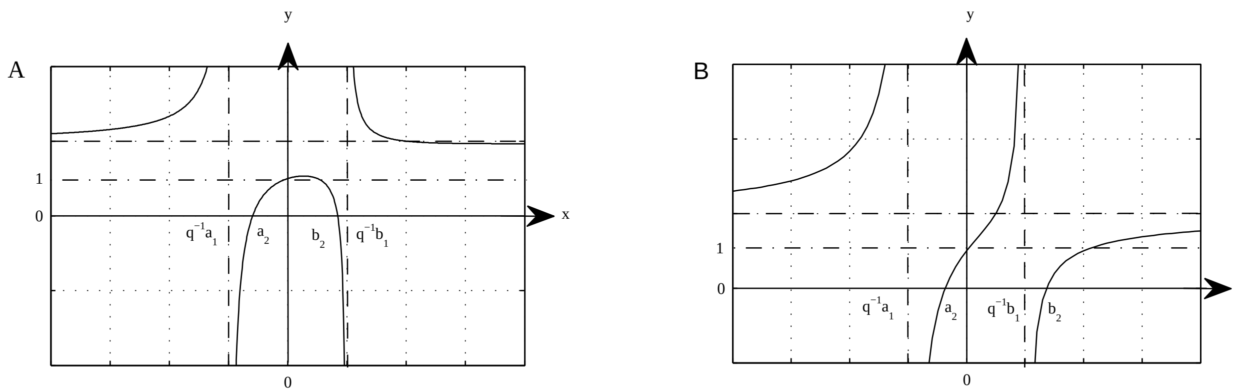


Figure 1: The graph of $f(x,q)$ in **Case 1** with $\Lambda_q > 1$. In A, the zeros are in order $q^{-1}a_1 < a_2 < 0 < b_2 < q^{-1}b_1$, and in B, $q^{-1}a_1 < a_2 < 0 < q^{-1}b_1 < b_2$.

In Figure 1A, the intervals $(q^{-1}a_1, a_2)$ and $(b_2, q^{-1}b_1)$ are rejected immediately since f is negative there. The subinterval (a_2, b_2) should also be rejected in which $\rho = 0$ by **PII**. For the same reason $(q^{-1}b_1, \infty)$ and $(-\infty, q^{-1}a_1)$, by symmetry, are not suitable by **PV**. Therefore, an OPS fails to exist.

Let us analyse the problem presented in Figure 1B. The positivity of ρ implies that the intervals $(q^{-1}a_1, a_2)$ and $(q^{-1}b_1, b_2)$ should be eliminated. With the transformation $x = -t$, we eliminate also $(-\infty, q^{-1}a_1)$ by **PV**. The interval $(a_2, q^{-1}b_1)$ is not suitable too, by **PIII**. So it remains only (b_2, ∞) to examine which coincides with $8th$ case in Theorem 2. Since $\rho(qx,q)/\rho(x,q) = 1$ at $x_0 = -\tau(0,q)/\tau'(0,q) > b_2(q)$, then ρ is increasing on (b_2, x_0) and decreasing on (x_0, ∞). As is shown from the figure f has a finite limit as $x \to +\infty$ so that it may be the case that $\rho \to 0$ as $x \to \infty$. Even if $\rho \to 0$ as $x \to \infty$, we must show also that $\sigma_1(x,q)\rho(x,q)x^k \to 0$ as $x \to \infty$ to satisfy the BC. In fact, instead of the usual q-Pearson equation we have to consider the equation

$$g(x,q) := \frac{\sigma_1(qx,q)\rho(qx,q)(qx)^k}{\sigma_1(x,q)\rho(x,q)x^k} = q^k \frac{\sigma_1(x,q) + (1-q^{-1})x\tau(x,q)}{\sigma_1(x,q)} = q^k \frac{q^{-1}\sigma_2(x,q)}{\sigma_1(x,q)} \tag{4.3}$$

in case of an infinite interval, what we call it here the *extended* q-Pearson equation to determine the behaviour of the quantity $\sigma_1(x,q)\rho(x,q)x^k$ as $x \to \infty$, which has been easily derived from (1.7). It is obvious that the extended q-Pearson equation is the difference equation not for the weight function $\rho(x,q)$ but for $\sigma_1(x,q)\rho(x,q)x^k$.

In Figure 2 we draw the graph of a typical g for some $0 < q < 1$, where k is large enough. From this figure we see that $g < 1$ for $x > b_2$ so that $\sigma_1(x,q)\rho(x,q)x^k$ does not vanish at ∞ since it is increasing as x increases. Thus we cannot find a weight function ρ on (b_2, ∞).

From Figure 3A, we first eliminate the intervals $(a_2, q^{-1}a_1)$ and $(q^{-1}b_1, b_2)$ because of the positivity of ρ. The interval (b_2, ∞) coincides again with $8th$ case in Theorem 2. However, $f(x,q) < 1$ on this interval so that ρ is increasing on (b_2, ∞) which implies that ρ can not vanish as $x \to \infty$. Thus $\sigma_1(x,q)\rho(x,q)x^k$ is never zero as $x \to \infty$ for some $k \in \mathbb{N}_0$. The same is true for $(-\infty, a_2)$ by symmetry. For the last subinterval $(q^{-1}a_1, q^{-1}b_1)$, we face the $1th$ case in Theorem 2. Since $\rho(qx,q)/\rho(x,q) = 1$ at $q^{-1}a_1 < x_0 = -\tau(0,q)/\tau'(0,q) < q^{-1}b_1$, then ρ is increasing on $(q^{-1}a_1, x_0)$ and decreasing on $(x_0, q^{-1}b_1)$. Furthermore, $\rho(qx,q)/\rho(x,q) \to \infty$, and hence $\rho \to 0$, as $x \to q^{-1}a_1^+$ and $x \to q^{-1}b_1^-$.

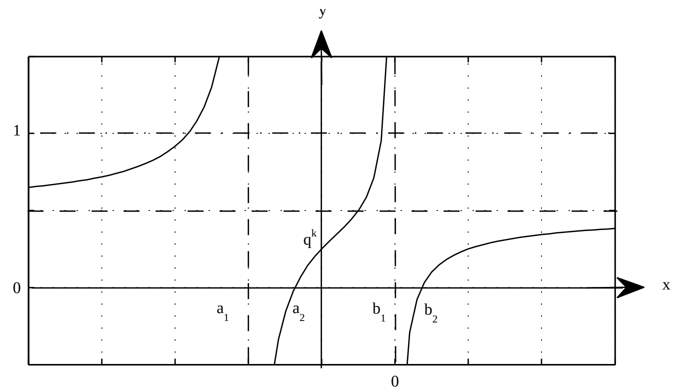


Figure 2: The graph of $g(x,q)$ corresponding to Figure 1B.

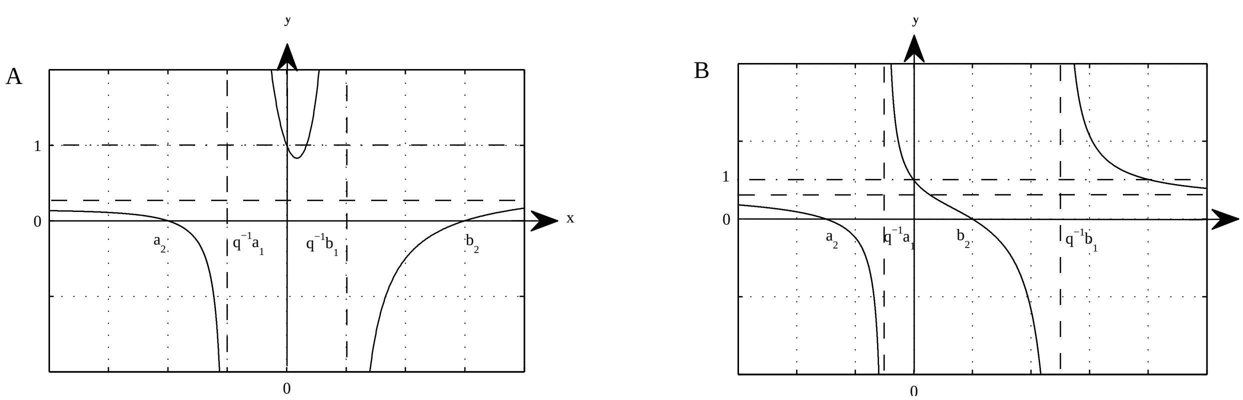


Figure 3: The graph of $f(x,q)$ in **Case 1** with $0 < \Lambda_q < 1$. In A, the zeros are in order $a_2 < q^{-1}a_1 < 0 < q^{-1}b_1 < b_2$ and in B, $a_2 < q^{-1}a_1 < 0 < b_2 < q^{-1}b_1$.

As a result, the typical shape of ρ is shown in Figure 4 assuming a positive initial value of ρ in each subinterval. That is, an OPS, to be stated in Theorem 6 below, on (a_1, b_1) with such a weight function in Figure 4 is sure to exist supported at the points $q^k a_1(q)$ and $q^k b_1(q)$ for $k \in \mathbb{N}_0$ according to Theorem 2-**1**.

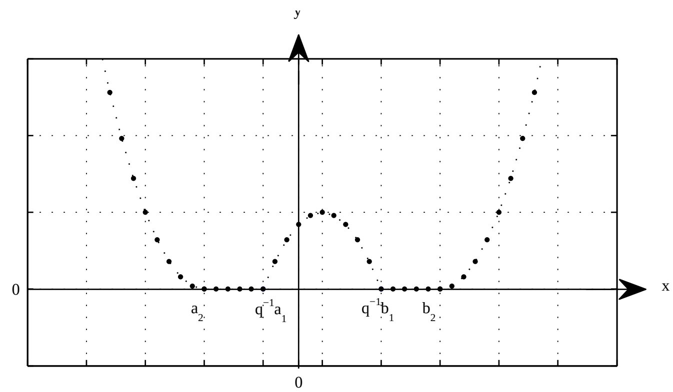


Figure 4: The graph of $\rho(x, q)$ associated with the case in Figure 3A.

Theorem 6 *Consider the case where $a_2 < a_1 < 0 < b_1 < b_2$ and $0 < q^2\Lambda_q < 1$. Let $a = a_1(q)$ and $b = b_1(q)$ be zeros of $\sigma_1(x, q)$. Then there exists a sequence of polynomials $\{P_n\}$ for $n \in \mathbb{N}_0$ orthogonal on (a, b) w.r.t. the weight function (see Eq. 1 in Table 1)*

$$\rho(x, q) = \frac{(qa^{-1}x, qb^{-1}x; q)_\infty}{(a_2^{-1}x, b_2^{-1}x; q)_\infty} > 0, \quad x \in (a, b)$$

*in the sense (2.11) of Theorem 2-**1**.*

The OPS in Theorem 6 coincides with the case VIIa1 in Chapter 10 of [19, pages 292 and 318]. In fact, a typical example of this family is the celebrated big q-Jacobi polynomials $P_n(x; a, b, c; q)$ on (cq, aq) satisfying the q-EHT for which

$$\sigma_1(x, q) = q^{-2}(x - a_1)(x - b_1), \quad \sigma_2(x, q) = abq(x - a_2)(x - b_2),$$

$$\tau(x, q) = \frac{1 - abq^2}{(1 - q)q}x + \frac{a(bq - 1) + c(aq - 1)}{1 - q} \quad \text{and} \quad \lambda_n(q) = q^{-n}[n]_q \frac{1 - abq^{n+1}}{q - 1} \tag{4.4}$$

where $a_1 = cq$, $b_1 = aq$, $a_2 = b^{-1}c$ and $b_2 = 1$ in our notation [19]. The conditions $0 < q^2\Lambda_q < 1$ and $a_2 < a_1 < 0 < b_1 < b_2$ give the restrictions $c < 0$, $0 < b < q^{-1}$ and $0 < a < q^{-1}$ on the parameters of $P_n(x; a, b, c; q)$ which is orthogonal on (cq, aq) in the sense (2.11) with

$$d_n^2 = (a - c)q(1 - q)\frac{(q, abq^2, a^{-1}cq, ac^{-1}q; q)_\infty}{(aq, bq, cq, abc^{-1}q; q)_\infty} \frac{(q, abq; q)_n}{(abq, abq^2; q)_{2n}}(aq, bq, cq, abc^{-1}q; q)_n(-ac)^n q^{n(n+3)/2}.$$

It should be noted that the difference between these conditions and those of Figure 3 comes from the fact that we have considered not only the conditions on ρ but also on $\sigma_1\rho$ in Theorem 6. Finally, the analysis of the case in Figure 3B does not yield an OPS.

The case in Figure 5B is inappropriate to define an OPS. On the other hand, in Figure 5A, the intervals $(q^{-1}a_1, a_2)$ and $(q^{-1}b_1, b_2)$ are rejected by the positivity of ρ. The intervals $(-\infty, q^{-1}a_1)$ and (b_2, ∞) coincide with $7th$, by symmetry, and $8th$ cases in Theorem 2. However, $f(x, q) < 1$ on $(-\infty, 0)$ and on (b_2, ∞) so that ρ is decreasing on $(-\infty, 0)$ and increasing on (b_2, ∞) which implies that ρ can not vanish as $x \to -\infty$ and $x \to \infty$. Thus $\sigma_1(x, q)\rho(x, q)x^k$ is never zero as $x \to -\infty$ and $x \to \infty$ for

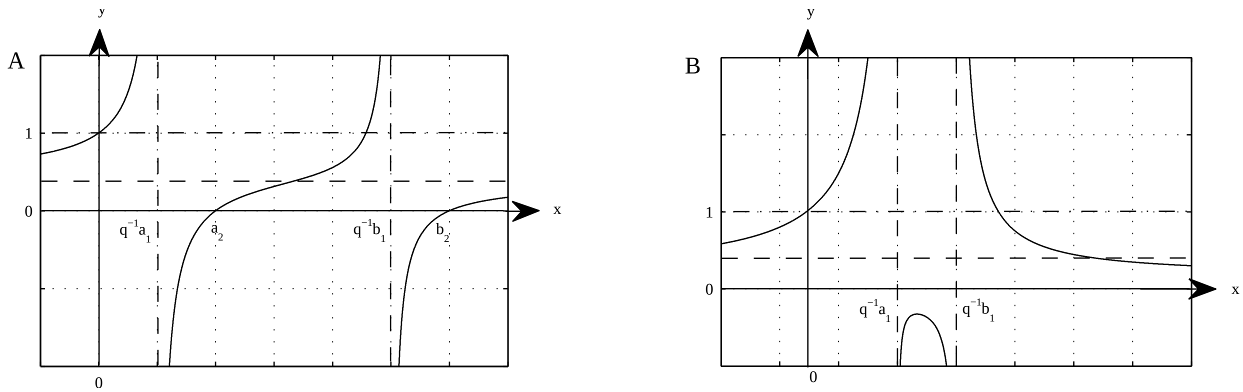


Figure 5: The graph of $f(x,q)$ in **Case 2** with $0 < \Lambda_q < 1$. In A, we have **Case 2(a)** with $0 < q^{-1}a_1 < a_2 < q^{-1}b_1 < b_2$. In B, we have **Case 2(c)** with $0 < q^{-1}a_1 < q^{-1}b_1$ and $a_2, b_2 \in \mathbb{C}$.

some $k \in \mathbb{N}_0$. The only possible interval for the case in Figure 5A is $(a_2, q^{-1}b_1)$ which corresponds to the $4th$ case of Theorem 2. Note that $\rho(qx,q)/\rho(x,q) = 1$ at $a_2 < x_0 = -\tau(0,q)/\tau'(0,q) < q^{-1}b_1$ and ρ is increasing on (a_2, x_0) and decreasing on $(x_0, q^{-1}b_1)$. Furthermore, $\rho(qa_2,q) = 0$ and $\rho \to 0$ as $x \to q^{-1}b_1^-$ since $\rho(qa_2,q)/\rho(a_2,q)$=0 and $\rho(qx,q)/\rho(x,q) \to \infty$ as $x \to q^{-1}b_1^-$, respectively. It is clear that the BCs are satisfied at $x = a_2$ and $x = q^{-1}b_1$. Thus we can find a suitable ρ on $(a_2, q^{-1}b_1)$ or on (qa_2, b_1) supported at the points $q^{-k}a_2$ or $q^k b_1$ for $k \in \mathbb{N}_0$, respectively. Therefore, we state the following theorem.

Theorem 7 *Consider the case where $0 < a_1 < a_2 < b_1 < b_2$ and $0 < q^2\Lambda_q < 1$. Let $a = a_2(q)$ and $b = q^{-1}b_1(q)$ be zeros of $\sigma_2(x,q)$ and $\sigma_1(qx,q)$, respectively. Then there exists a sequence of polynomials $\{P_n\}$ for $n \in \mathbb{N}_0$ orthogonal on (a,b) supported at the points $q^{-k}a$ and on (qa,qb) at $q^{k+1}b$ for $k \in \mathbb{N}_0$ w.r.t. the weight function (see Eq. 2 in Table 1)*

$$\rho(x,q) = |x|^{\iota}\,\frac{(\frac{qa}{x}, b^{-1}x; q)_\infty}{(\frac{a_1}{x}, b_2^{-1}x; q)_\infty}, \quad q^{\iota} = \frac{q^{-3}\sigma_2''(0,q)b_2}{\sigma_1''(0,q)b}$$

in the senses (2.15) and (2.14), respectively.

The OPS in Theorem 7 corresponds to the case IIIb9 in Chapter 11 of [19, page 366]. A well known example of this family is the q-Hahn polynomials on $(1, q^{-N-1})$ whose q-EHT has the coefficients

$$\sigma_1(x,q) = q^{-2}(x-a_1)(x-b_1), \quad \sigma_2(x,q) = \alpha\beta q(x-a_2)(x-b_2),$$

$$\tau(x,q) = \frac{1-\alpha\beta q^2}{(1-q)q}x + \frac{\alpha q^{-N} + \alpha\beta q - \alpha - q^{-N-1}}{1-q} \quad \text{and} \quad \lambda_n(q) = -q^{-n}[n]_q\frac{1-\alpha\beta q^{n+1}}{1-q} \tag{4.5}$$

where $a_1 = \alpha q$, $b_1 = q^{-N}$, $a_2 = 1$ and $b_2 = \beta^{-1}q^{-N-1}$ in our notation [19]. The conditions $0 < q^2\Lambda_q < 1$ and $0 < a_1 < a_2 < b_1 < b_2$ result in the restrictions $0 < \alpha < q^{-1}$ and $0 < \beta < q^{-1}$ on the parameters of $Q_n(x; \alpha, \beta, N|q)$ which forms an orthogonal set on $(1, q^{-N-1})$ in the sense (2.15) with

$$d_n^2 = \frac{(q, q^{N+1}, \beta^{-1}, \alpha^{-1}\beta^{-1}q^{-N-1}; q)_\infty}{(\alpha q, \beta q^{N+1}, \beta^{-1}q^{-N}, \alpha^{-1}\beta^{-1}q^{-1}; q)_\infty}\frac{(q, \alpha\beta q, \alpha q, q^{-N}, \beta q, \alpha\beta q^{N+2}; q)_n}{(\alpha\beta q, \alpha\beta q^2; q)_{2n}}(-\alpha q^{-N})^n q^{n(n+1)/2}(q^{-1}-1). \tag{4.6}$$

In literature, this relation can be found as a finite sum [19, page 367].

In Figure 6A, the intervals (a_2, b_2) and $(q^{-1}a_1, q^{-1}b_1)$ are rejected by the positivity of ρ. We also eliminate the intervals $(-\infty, a_2)$ and $(q^{-1}b_1, \infty)$ due to **PVI**, by symmetry, and **PV**, respectively. The last interval $(b_2, q^{-1}a_1)$ coincides with $4th$ case in Theorem 2. Notice that $f(x,q) = 1$ at $b_2 < x_0 = -\tau'(0,q)/\tau(0,q) < q^{-1}a_1$, then ρ is increasing on (b_2, x_0) with $\rho(qb_2,q) = 0$ since $\rho(qb_2,q)/\rho(b_2,q) = 0$

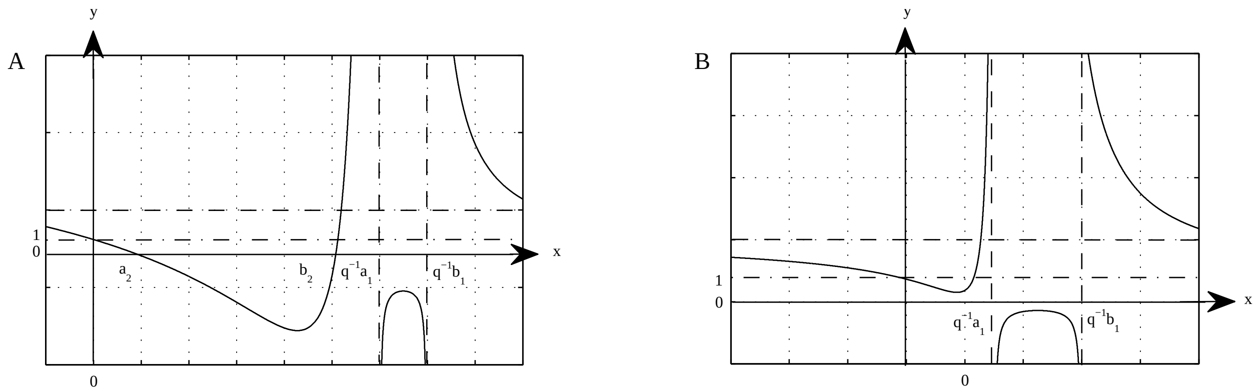


Figure 6: The graph of $f(x,q)$ in **Case 2** with $\Lambda_q > 1$. In A, we have **Case 2(a)** with $0 < a_2 < b_2 < q^{-1}a_1 < q^{-1}b_1$. In B, we have **Case 2(c)** with $0 < q^{-1}a_1 < q^{-1}b_1$ and $a_2, b_2 \in \mathbb{C}$.

and decreasing on $(x_0, q^{-1}a_1)$ with $\rho \to 0$ as $x \to q^{-1}a_1^-$ since $\rho(qx,q)/\rho(x,q) \to \infty$. As a result, (qb_2, a_1) is an interval in which a desired ρ is defined at the supporting points $q^k a_1(q)$ for $k \in \mathbb{N}_0$ such that $q^N a_1 = qb_2$. Notice that the BCs (2.1) and (2.2) hold since a_1 and qb_2 are one of the roots of $\sigma_1(x,q)$ and $\sigma_2(q^{-1}x,q)$, respectively. Notice also, from Theorem 2-**4**, that ρ can also be constructed on $(b_2, q^{-1}a_1)$ having the supporting points $q^{-k}b_2(q)$ for $k \in \mathbb{N}_0$ such that $q^{-N}b_2(q) = q^{-1}a_1(q)$. As a significant remark, observe that the analysis is valid in the limiting cases $a_1 \to b_1$ and $a_2 \to b_2$ as well. Hence, the resulting OPS is presented in Theorem 8. However, the case in Figure 6B does not give any OPS.

Theorem 8 *Consider the case where $0 < a_2 \le b_2 < a_1 \le b_1$ and $q^2\Lambda_q > 1$. Let $a = b_2(q)$ and $b = q^{-1}a_1(q)$ be one of the zeros of $\sigma_2(x,q)$ and $\sigma_1(qx,q)$, respectively. Then there exists a sequence of polynomials $\{P_n\}$, $n \in \mathbb{N}_0$ orthogonal on (a,b) and on (qa,qb) w.r.t. the weight function (see Eq. 2 in Table 1)*

$$\rho(x,q) = |x|^{\iota}\,\frac{(b^{-1}x, \frac{qa}{x}; q)_\infty}{(\frac{b_1}{x}, a_2^{-1}x; q)_\infty}, \quad q^{\iota} = \frac{q^{-3}\sigma_2''(0,q)a_2}{\sigma_1''(0,q)b}$$

in the senses (2.15) and (2.14), respectively.

An example of this family is the q-Hahn polynomials $Q_n(x;\alpha,\beta,N|q)$ on $(1, q^{-N-1})$ with the coefficients in (4.5) where $a_1 = q^{-N}$, $b_1 = \alpha q$, $a_2 = \beta^{-1}q^{-N-1}$ and $b_2 = 1$ in our notation [19]. These polynomials have the orthogonality relation with the same d_n^2 as in (4.6) but the restrictions $\alpha \ge q^{-N-1}$ and $\beta \ge q^{-N-1}$, given in [20], on the parameters are different.

In Figure 7A, the only suitable interval is $(q^{-1}a_1, q^{-1}b_1)$ which coincides with $1st$ case in Theorem 2. In fact, $\rho(qx,q)/\rho(x,q) = 1$ at $q^{-1}a_1 < x_0 = -\tau(0,q)/\tau'(0,q) < q^{-1}b_1$, then it follows that ρ is increasing on $(q^{-1}a_1, x_0)$ and decreasing on $(x_0, q^{-1}b_1)$ with $\rho \to 0$ as $x \to q^{-1}a_1^+$ and $x \to q^{-1}b_1^-$ since $\rho(qx,q)/\rho(x,q) \to \infty$. Therefore since the BCs (2.1) hold at $x = a_1$ and $x = b_1$, (a_1, b_1) is suitable interval and ρ is supported at the points $q^k a_1$ and $q^k b_1$ for $k \in \mathbb{N}_0$. Thus, we have the following result.

Theorem 9 *Consider the case $a_1 < 0 < b_1 < a_2 \le b_2$, and $q^2\Lambda_q < 0$. Let $a = a_1(q)$ and $b = b_1(q)$ be zeros of $\sigma_1(x,q)$. Then there exists a sequence of polynomials $\{P_n\}$ for $n \in \mathbb{N}_0$ orthogonal on (a,b) in the sense (2.11), w.r.t. the weight function in Theorem 6.*

An example of this family is the famous big q-Jacobi polynomials on (cq, aq) whose q-EHT has the coefficients in (4.4) with $a_1 = cq$, $b_1 = aq$, $a_2 = b^{-1}c$ and $b_2 = 1$ in our notation [19]. This is an interesting set of big q-Jacobi polynomials having the same orthogonality properties, which is defined under the new restrictions $c < 0$, $b < 0$, $abc^{-1}q \le 1$ and $0 < a < q^{-1}$ on the parameters.

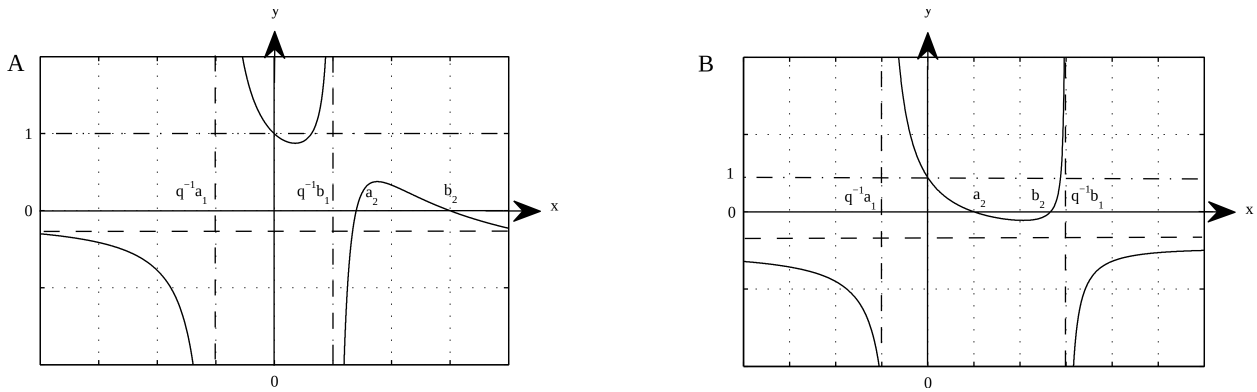


Figure 7: The graph of $f(x,q)$ in **Case 3(a)** with $\Lambda_q < 0$. In A, the zeros are in order $q^{-1}a_1 < 0 < q^{-1}b_1 < a_2 < b_2$, and in B, $q^{-1}a_1 < 0 < a_2 < b_2 < q^{-1}b_1$.

In Figure 7B, the only possible interval is $(b_2, q^{-1}b_1)$ which corresponds to $4th$ case in Theorem 2. In this case $\rho(qx,q)/\rho(x,q) = 1$ at $b_2 < x_0 = -\tau(0,q)/\tau'(0,q) < q^{-1}b_1$, then it follows that ρ is increasing on (b_2, x_0) and decreasing on $(x_0, q^{-1}b_1)$. Furthermore, $\rho(qb_2,q) = 0$ and $\rho(x,q) \to 0$ as $x \to q^{-1}b_1^-$ since $\rho(qb_2,q)/\rho(b_2,q) = 0$ and $\rho(qx,q)/\rho(x,q) \to \infty$ as $x \to q^{-1}b_1^-$. Thus we have a suitable ρ on $(b_2, q^{-1}b_1)$ or on (qb_2, b_1) supported at the points $q^{-k}b_2$ or $q^k b_1$ for $k \in \mathbb{N}_0$ according to the Theorem 2-**4**. Hence the following theorem follows.

Theorem 10 *Consider the case where $a_1 < 0 < a_2 \le b_2 < b_1$ and $q^2\Lambda_q < 0$. Let $a = b_2(q)$ and $b = q^{-1}b_1(q)$ be zeros of $\sigma_2(x,q)$ and $\sigma_1(qx,q)$, respectively. Then there exists a sequence of polynomials $\{P_n\}$, $n \in \mathbb{N}_0$ orthogonal on (a,b) and on (qa,qb) w.r.t. the weight function (see Eq. 2 in Table 1)*

$$\rho(x,q) = |x|^\iota \, \frac{(\frac{qa}{x}, b^{-1}x; q)_\infty}{(\frac{a_1}{x}, a_2^{-1}x; q)_\infty}, \qquad q^\iota = \frac{q^{-3}\sigma_2''(0,q)a_2}{\sigma_1''(0,q)b}$$

in the senses (2.15) and (2.14), respectively.

A typical example of this family is the q-Hahn polynomials on $(1, q^{-N-1})$ given by (4.5), for which $a_1 = \alpha q$, $b_1 = q^{-N}$, $a_2 = \beta^{-1}q^{-N-1}$ and $b_2 = 1$ in our notation [19]. This is also a new orthogonal set of q-Hahn polynomials under the new restrictions $\alpha < 0$ and $\beta \ge q^{-N-1}$ on its parameters.

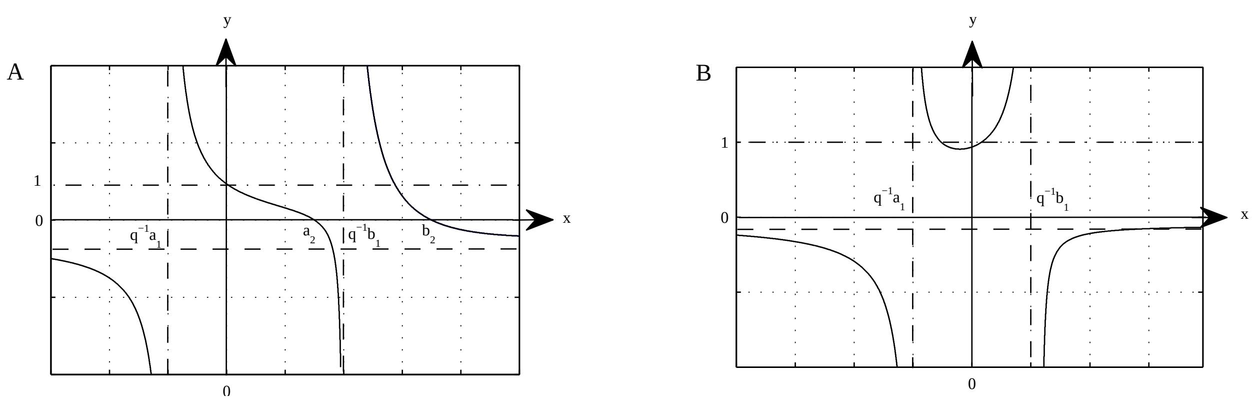


Figure 8: The graph of $f(x,q)$ in **Case 3** with $\Lambda_q < 0$. In A, we have **Case 3(a)** with $q^{-1}a_1 < 0 < a_2 < q^{-1}b_1 < b_2$. In B, we have **Case 3(c)** with $q^{-1}a_1 < 0 < q^{-1}b_1$ and $a_2, b_2 \in \mathbb{C}$.

We could not find an OPS in case of Figure 8A. In Figure 8B, the interval $(q^{-1}a_1, q^{-1}b_1)$ coincides with $1st$ case in Theorem 2. Notice that $\rho(qx,q)/\rho(x,q) = 1$ at $q^{-1}a_1 < x_0 = -\tau(0,q)/\tau'(0,q) < q^{-1}b_1$, then ρ is increasing on $(q^{-1}a_1, x_0)$ and decreasing on $(x_0, q^{-1}b_1)$ with $\rho(x,q) \to 0$ as $x \to q^{-1}a_1^+$ and $x \to q^{-1}b_1^-$ since $\rho(qx,q)/\rho(x,q) \to \infty$. Then $(q^{-1}a_1, q^{-1}b_1)$ is suitable interval for ρ supported at the points $q^k a_1$ and $q^k b_1$, $k \in \mathbb{N}_0$. Therefore, the following theorem holds:

Theorem 11 *Consider the case $a_1 < 0 < b_1$, a_2, $b_2 \in \mathbb{C}$ and $q^2\Lambda_q < 0$. Let $a = a_1(q)$ and $b = b_1(q)$ be zeros of $\sigma_1(x,q)$. Then there exists a sequence of polynomials $\{P_n\}$ for $n \in \mathbb{N}_0$ orthogonal on (a,b), w.r.t. the weight function in Theorem 6, in the sense (2.11) with*

$$\begin{aligned} d_n^2 &= (b_1 - a_1)(1-q)q^{n(n-1)/2}(-a_1b_1)^n \frac{(q, q^{-1}a_2^{-1}b_2^{-1}a_1b_1; q)_n}{(q^{-1}a_2^{-1}b_2^{-1}a_1b_1, a_2^{-1}b_2^{-1}a_1b_1; q)_{2n}} \\ &\times (a_2^{-1}a_1, a_2^{-1}b_1, b_2^{-1}a_1, b_2^{-1}b_1; q)_n \frac{(q, qb_1a_1^{-1}, qa_1b_1^{-1}, a_2^{-1}b_2^{-1}a_1b_1; q)_\infty}{(a_2^{-1}a_1, a_2^{-1}b_1, b_2^{-1}a_1, b_2^{-1}b_1; q)_\infty}. \end{aligned} \tag{4.7}$$

It is worth mentioning that this leads completely to a new orthogonal set which is not included in q-Askey scheme. Actually, this case is similar to the **Cases 1** in Figure 3A and **Case 3(a)** in Figure 7A implying the big q-Jacobi polynomials. The difference lies in the point that the roots $a_2(q)$ and $b_2(q)$ are complex.

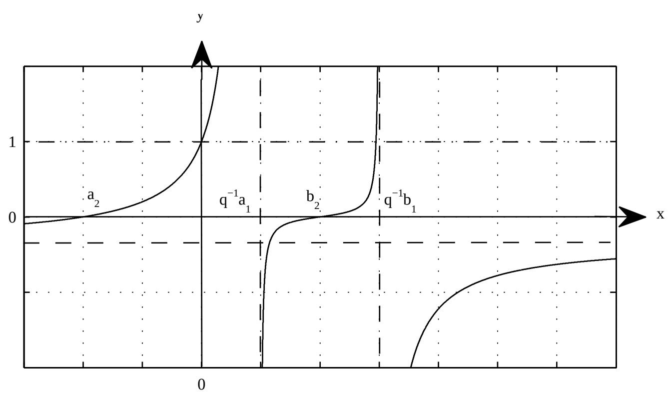


Figure 9: The graph of $f(x,q)$ in **Case 4** with $\Lambda_q < 0$. The zeros are in order $a_2 < 0 < q^{-1}a_1 < b_2 < q^{-1}b_1$.

In Figure 9, the only possible interval is $(b_2, q^{-1}b_1)$ which corresponds to the one described in Theorem 2-**4**. A similar analysis shows that there exists a q-weight function defined on the interval (qb_2, b_1) or on $(b_2, q^{-1}b_1)$ supported at the points $q^k b_1$ or $q^{-k}b_2$ for $k \in \mathbb{N}_0$ which lead to the following theorem:

Theorem 12 *Consider the case where $a_2 < 0 < a_1 < b_2 < b_1$ and $q^2\Lambda_q < 0$. Let $a = a_2(q)$ and $b = q^{-1}b_1(q)$ be zeros of $\sigma_2(x,q)$ and $\sigma_1(qx,q)$, respectively. Then there exists a sequence of polynomials $\{P_n\}$, $n \in \mathbb{N}_0$ orthogonal on (a,b) and on (qa,qb) w.r.t. the weight function in Theorem 7 in the senses (2.15) and (2.14), respectively.*

An example of this family is the q-Hahn polynomials on $(1, q^{-N-1})$ whose q-EHT has the coefficients (4.5) with $a_1 = \alpha q$, $b_1 = q^{-N}$, $a_2 = \beta^{-1}q^{-N-1}$ and $b_2 = 1$ in our notation [19]. This is again a new set of q-Hahn polynomials under the restrictions $0 < \alpha < q^{-1}$ and $\beta < 0$ on the parameters which are not reported previously.

For completeness, we have also examined the cases listed below for which an OPS fails to exist.

Case 2(a) with $0 < a_2 < q^{-1}a_1 < b_2 < q^{-1}b_1$ and $\Lambda_q > 1$.

Case 2(a) with $0 < a_2 < q^{-1}a_1 < q^{-1}b_1 < b_2$ and $\Lambda_q > 1$.

Case 2(a) with $0 < q^{-1}a_1 < a_2 < b_2 < q^{-1}b_1$ and $\Lambda_q > 1$.

Case 2(a) with $a_2 < b_2 < 0 < q^{-1}a_1 < q^{-1}b_1$ and $\Lambda_q > 1$.

Case 2(a) with $0 < q^{-1}a_1 < a_2 < b_2 < q^{-1}b_1$ and $0 < \Lambda_q < 1$.

Case 2(a) with $0 < a_2 < q^{-1}a_1 < q^{-1}b_1 < b_2$ and $0 < \Lambda_q < 1$.

Case 2(a) with $0 < q^{-1}a_1 < q^{-1}b_1 < a_2 < b_2$ and $0 < \Lambda_q < 1$.

Case 2(a) with $a_2 < b_2 < 0 < q^{-1}a_1 < q^{-1}b_1$ and $0 < \Lambda_q < 1$.

Case 4 with $a_2 < 0 < q^{-1}a_1 < q^{-1}b_1 < b_2$ and $\Lambda_q < 0$.

Case 4 with $a_2 < 0 < b_2 < q^{-1}a_1 < q^{-1}b_1$ and $\Lambda_q < 0$.

4.2 The zero case

We make a similar analysis here with the same notations. Let the coefficients σ_1 and σ_2 be quadratic polynomials in x such that $q\sigma_1(0,q) = \sigma_2(0,q) = 0$. If σ_1 is written as $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)x[x - a_1(q)]$ then, from (1.5) $\sigma_2(x,q) = \frac{1}{2}\sigma_2''(0,q)x^2 + \sigma_2'(0,q)x$ where

$$\tfrac{1}{2}\sigma_2''(0,q) = q\left[\tfrac{1}{2}\sigma_1''(0,q) + (1-q^{-1})\tau'(0,q)\right] \neq 0 \quad \text{and} \quad \sigma_2'(0,q) = q(1-q^{-1})\tau(0,q) - \tfrac{1}{2}\sigma_1''(0,q)a_1(q).$$

Then it follows from (1.7) that

$$f(x,q) := \frac{\rho(qx,q)}{\rho(x,q)} = \left[1 + \frac{(1-q^{-1})\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right]\frac{[x - a_2(q)]}{q[qx - a_1(q)]}, \qquad x \neq 0 \tag{4.8}$$

provided that $\left[1 + \frac{(1-q^{-1})\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right]a_2(q) = \left[a_1(q) - \frac{(1-q^{-1})\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right]$. Let us point out that Λ_q defined in (4.2) is also horizontal asymptote of $f(x,q)$ in (4.8). Moreover, f intercepts the y-axis at the point

$$y := y_0 = q^{-1}\left[1 - \frac{(1-q^{-1})}{a_1(q)}\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right].$$

In the zero cases notice that one of the boundary of (a,b) interval could be zero. Therefore for such a case we need to know the behaviour of ρ at the origin.

Lemma 13 *If $0 < y_0 < 1$, then $\rho(x,q) \to 0$ as $x \to 0$. Otherwise it diverges to $\mp\infty$.*

Proof From (2.10), we write $\rho(q^k x,q) = \rho(x,q)\prod_{i=0}^{k-1}\frac{q^{-1}\sigma_2(q^i x,q)}{\sigma_1(q^{i+1}x,q)}$ from which

$$\rho(q^k x,q) = q^{-k}\left[1 - \frac{(1-q^{-1})}{a_1(q)}\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right]^k \frac{(x/a_2(q);q)_k}{(qx/a_1(q);q)_k}\rho(x,q) \tag{4.9}$$

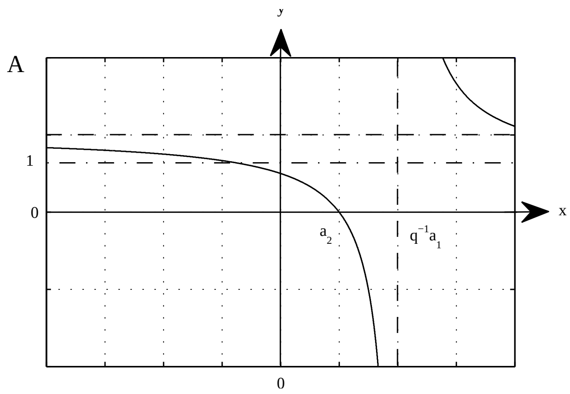
is obtained. ■

In a similar fashion, we introduce the two additional cases.

Case 5. $\Lambda_q > 0$ with **(a)** $0 < y_0 < 1$ or **(b)** $y_0 > 1$ or **(c)** $y_0 < 0$.

Case 6. $\Lambda_q < 0$ with **(a)** $0 < y_0 < 1$ or **(b)** $y_0 > 1$ or **(c)** $y_0 < 0$.

In Figure 10A, we consider all possible intervals in which we can have a suitable q-weight function ρ. By the positivity of ρ, the interval $(a_2, q^{-1}a_1)$ is not suitable. The other intervals $(0, a_2)$ and $(q^{-1}a_1, \infty)$ are both eliminated due to the **PIV** and **PV**, respectively. The interval $(-\infty, 0)$ is the



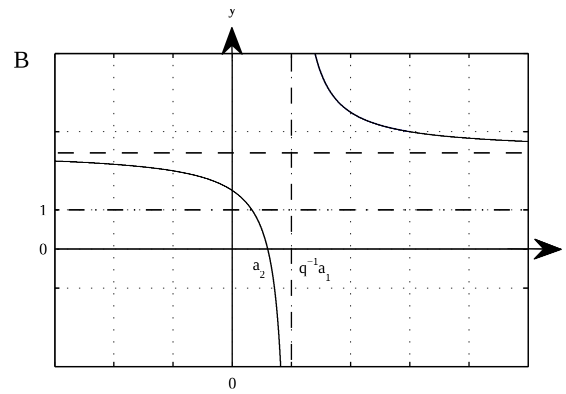
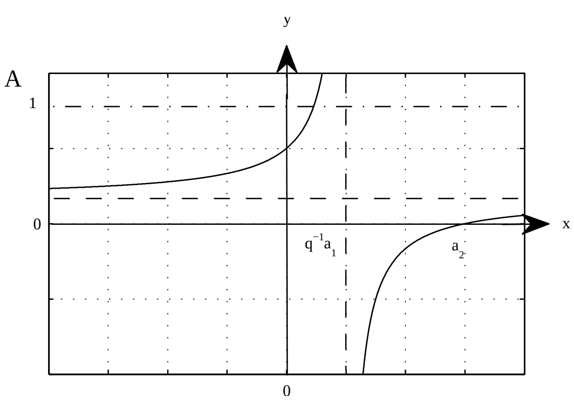


Figure 10: The graph of $f(x, q)$ in **Case 5** with $\Lambda_q > 1$ and $0 < a_2 < q^{-1}a_1$. In A, we have **Case 5(a)**. In B, we have **Case 5(b)**.

one described in Theorem 2-**9** by symmetry. Since $\rho(qx, q)/\rho(x, q) = 1$ at $x_0 = -\tau(0, q)/\tau'(0, q) < 0$, ρ is increasing on $(-\infty, x_0)$ and decreasing on $(x_0, 0)$. Moreover, since $0 < y_0 < 1$ $\rho \to 0$ as $x \to 0^-$ according to Lemma 13. On the other hand, since $\rho(qx, q)/\rho(x, q)$ has a finite limit as $x \to -\infty$, we may have $\rho \to 0$ as $x \to -\infty$, but we should check that $\sigma_1(x, q)\rho(x, q)x^k \to 0$ as $x \to -\infty$ by using the extended q-Pearson equation (4.3). However, the graph of g (4.3) looks like the one represented in Figure 10A together with the property that $g(x, q) < 1$ on $(-\infty, 0)$ for large k which leads to that $\sigma_1(x, q)\rho(x, q)x^k$ is decreasing on $(-\infty, 0)$ with $\sigma_1(x, q)\rho(x, q)x^k \not\to 0$ as $x \to -\infty$. Therefore, this case does not lead to any suitable ρ and, therefore, OPS. The same result is valid for the case in Figure 10B.



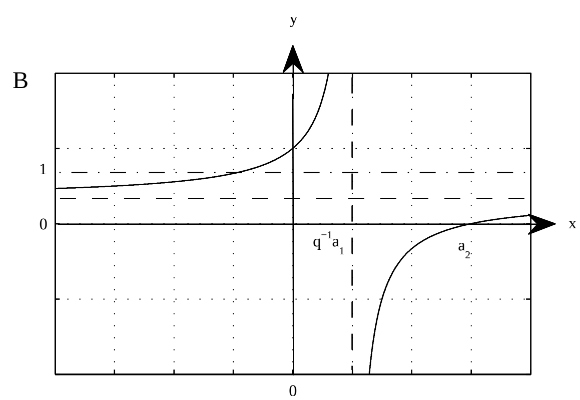


Figure 11: The graph of $f(x, q)$ in **Case 5** with $0 < \Lambda_q < 1$ and $0 < q^{-1}a_1 < a_2$. In A, we have **Case 5(a)**. In B, we have **Case 5(b)**.

In Figure 11A, the only suitable interval is $(0, q^{-1}a_1)$ which coincides with $2nd$ and $3th$ case in Theorem 2. Notice that $\rho(qx, q)/\rho(x, q) = 1$ at $0 < x_0 = -\tau(0, q)/\tau'(0, q) < q^{-1}a_1$, then ρ is increasing on $(0, x_0)$ with $\rho \to 0$ as $x \to 0^+$ since $0 < y_0 < 1$ and decreasing on $(x_0, q^{-1}a_1)$ with $\rho(x, q) \to 0$ as $x \to q^{-1}a_1^-$. Thus there is a a q-weight function supported at the points $q^k a_1$ for $k \in \mathbb{N}_0$ Hence, the resulting OPS is introduced in Theorem 14. But, the case in Figure 11B does not yield any OPS.

Theorem 14 *Consider the case where $0 < a_1 < a_2$, $0 < qy_0 < 1$ and $0 < q^2\Lambda_q < 1$. Let $a = 0$ and $b = a_1(q)$ be the zeros of $\sigma_1(x, q)$. Then there exists a sequence of polynomials $\{P_n\}$ for $n \in \mathbb{N}_0$ orthogonal on (a, b) w.r.t. the weight function (see Eq. 3 in Table 1)*

$$\rho(x, q) = |x|^\alpha \frac{(qb^{-1}x; q)_\infty}{(a_2^{-1}x; q)_\infty}, \quad q^\alpha = \frac{q^{-2}\sigma_2''(0, q)a_2}{\sigma_1''(0, q)b}$$

in the sense (2.12) and (2.13).

The OPS in Theorem 14 corresponds to the case IVa3 in Chapter 10 of [19, pages 277 and 311]. In fact, a typical example of this family is the little q-Jacobi polynomials $P_n(x;a,b|q)$ on $(0,1)$ satisfying the q-EHT with the coefficients

$$\sigma_1(x,q) = q^{-2}x(x-a_1), \quad \sigma_2(x,q) = abqx(x-a_2),$$

$$\tau(x,q) = \frac{1-abq^2}{(1-q)q}x + \frac{aq-1}{(1-q)q} \quad \text{and} \quad \lambda_n(q) = -q^{-n}[n]_q\frac{1-abq^{n+1}}{1-q} \tag{4.10}$$

where $a_1 = 1$ and $a_2 = b^{-1}q^{-1}$ in our notation [19]. The conditions $0 < q^2\Lambda_q < 1$, $0 < qy_0 < 1$ and $0 < a_1 < a_2$ yield the restrictions $0 < a < q^{-1}$ and $0 < b < q^{-1}$ on the parameters of $P_n(x;a,b|q)$ which is orthogonal on $(0,1)$ in the sense (2.12) and equivalently (2.13) with

$$d_n^2 = a^n q^{n^2}(1-q)\frac{(q,abq;q)_n}{(abq,abq^2;q)_{2n}}(aq,bq;q)_n\frac{(q,abq^2;q)_\infty}{(aq,bq;q)_\infty}. \tag{4.11}$$

In literature, this relation can be found as a finite sum [19, Page 312].

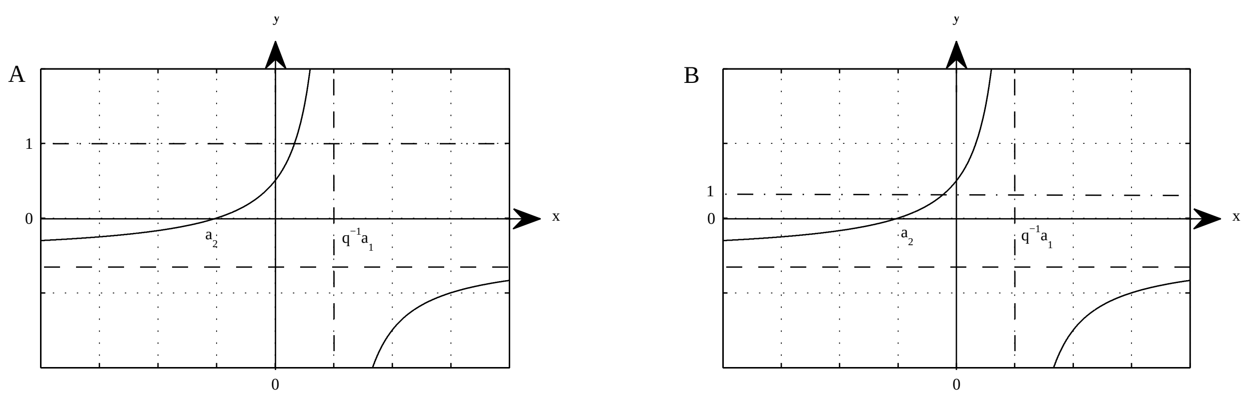


Figure 12: The graph of $f(x,q)$ in **Case 6** with $\Lambda_q < 0$ and $a_2 < 0 < q^{-1}a_1$. In A, we have **Case 6(a)**. In B, we have **Case 6(b)**.

In Figure 12A, the only interval is $(0, q^{-1}a_1)$ which corresponds to the interval described in Theorem 2-**2-3**. Notice that $\rho(qx,q)/\rho(x,q) = 1$ at $0 < x_0 = -\tau'(0,q)/\tau(0,q) < q^{-1}a_1$ then ρ is increasing on $(0, x_0)$ and decreasing on $(x_0, q^{-1}a_1)$. Furthermore, since $0 < y_0 < 1$, $\rho \to 0$ as $x \to 0^+$ according to Lemma 13 and $\rho \to 0$ as $x \to q^{-1}a_1^-$ since $\rho(qx,q)/\rho(x,q) \to +\infty$ as $x \to q^{-1}a_1^-$. Then, there exists a suitable ρ on $(0, a_1)$ supported at the points $q^k a_1$ for $k \in \mathbb{N}_0$ and, therefore, an OPS exists which is stated in the following theorem. However, the case analysed in Figure 12B does not give any OPS.

Theorem 15 *Consider the case where $a_2 < 0 < a_1$, $0 < qy_0 < 1$ and $q^2\Lambda_q < 0$. Let $a = 0$ and $b = a_1(q)$ be zeros of $\sigma_1(x,q)$. Then there exists a sequence of polynomials $\{P_n\}$ for $n \in \mathbb{N}_0$ orthogonal on (a,b) w.r.t. the weight function in Theorem 14, in the sense (2.12) and (2.13).*

The OPS in Theorem 15 coincides with the case IVa4 in Chapter 10 of [19, pages 278 and 312]. An example of this family is the little q-Jacobi polynomials on $(0,1)$ satisfying the q-EHT having the coefficients in (4.10). These polynomials have the same orthogonality relation with the same d_n^2 as in (4.11) but the restrictions $0 < a < q^{-1}$ and $b < 0$ on the parameters are different.

The case in Figure 13A does not yield any OPS. On the other hand, in Figure 13B, the only possible interval is $(a_2, q^{-1}a_1)$ which is $4th$ case in Theorem 2. Note that $\rho(qx,q)/\rho(x,q) = 1$ at $a_2 < x_0 = -\tau'(0,q)/\tau(0,q) < q^{-1}a_1$ and that ρ is increasing on (a_2, x_0) and decreasing on

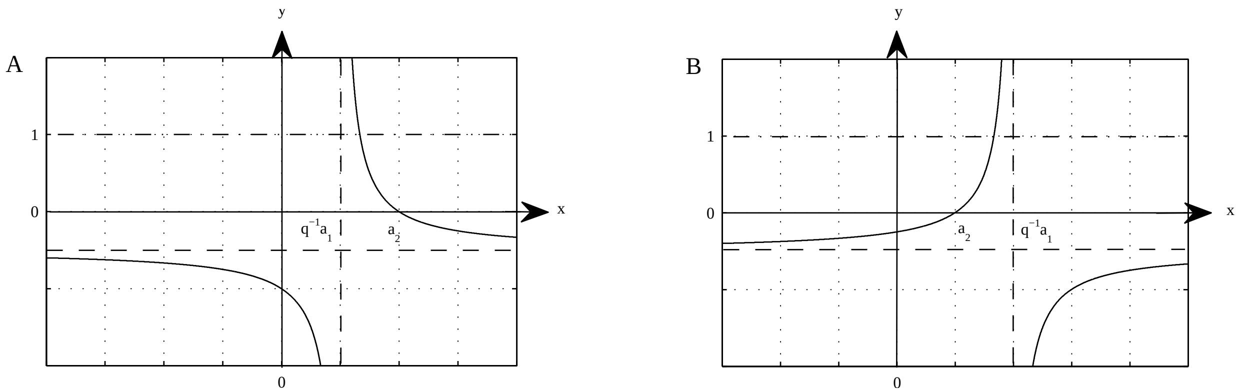


Figure 13: The graph of $f(x,q)$ in **Case 6(c)** with $\Lambda_q < 0$. In A, the zeros are in order $0 < q^{-1}a_1 < a_2$. In B, the zeros are in order $0 < a_2 < q^{-1}a_1$.

$(x_0, q^{-1}a_1)$. Furthermore, $\rho(qa_2, q) = 0$ since $\rho(qa_2, q)/\rho(a_2, q) = 0$ and $\rho \to 0$ as $x \to q^{-1}a_1^-$ since $\rho(qx, q)/\rho(x, q) \to \infty$ as $x \to q^{-1}a_1^-$. Then there is an OPS on $(a_2, q^{-1}a_1)$ or on (qa_2, a_1). Therefore, we have the following theorem.

Theorem 16 *Consider the case where $0 < a_2 < a_1$, $qy_0 < 0$ and $q^2\Lambda_q < 0$. Let $a = a_2(q)$ and $b = q^{-1}a_1(q)$ be zeros of $\sigma_2(x,q)$ and $\sigma_1(qx,q)$, respectively. Then there exists a sequence of polynomials $\{P_n\}$ for $n \in \mathbb{N}_0$ orthogonal on (a,b) at the points $q^{-k}a$ and on (qa,qb) at $q^{k+1}b$ for $k \in \mathbb{N}_0$ w.r.t. the weight function (see Eq. 4 in Table 1)*

$$\rho(x,q) = |x|^\alpha \sqrt{x^{\log_q x - 1}} (\frac{qa}{x}, b^{-1}x; q)_\infty, \quad q^\alpha = -\frac{q^{-3}\sigma_2''(0,q)}{\sigma_1''(0,q)b}$$

in the senses (2.15) and (2.14), respectively.

An example of this family is the q-Kravchuk polynomials $K_n(x; p, N; q)$ on $(1, q^{-N-1})$ for which

$$\sigma_1(x,q) = q^{-2}x(x - a_1), \quad \sigma_2(x,q) = -px(x - a_2),$$

$$\tau(x,q) = \frac{1 + pq}{(1-q)q}x - \frac{p + q^{-N-1}}{1 - q} \quad \text{and} \quad \lambda_n(q) = -q^{-n}[n]_q \frac{1 + pq^n}{1 - q}$$

where $a_1 = q^{-N}$ and $a_2 = 1$ in our notation [20]. The conditions $q^2\Lambda_q < 0$, $qy_0 < 0$ and $0 < a_2 < a_1$ lead to the restriction $p > 0$ on the parameter of $K_n(x; p, N; q)$ which is orthogonal on $(1, q^{-N-1})$ in the sense (2.15) with

$$d_n^2 = (q^{-1} - 1)p^{-N}q^{-\binom{N+1}{2}}(-q^{-N}p)^n q^{n^2} \frac{1 + p}{1 + pq^{2n}}(-pq; q)_N (q, q^{N+1}; q)_\infty \frac{(q, -pq^{N+1}; q)_n}{(-p, q^{-N}; q)_n}. \quad (4.12)$$

In literature, this relation can be found as a finite sum [20, Page 98].

In the following independent cases we fail to define an OPS.

Case 5(a) with $0 < \Lambda_q < 1$ and $0 < a_2 < q^{-1}a_1$.

Case 5(b) with $\Lambda_q > 1$ and $0 < q^{-1}a_1 < a_2$.

Case 5(c) with $\Lambda_q > 1$ and $q^{-1}a_1 < 0 < a_2$.

Case 5(c) with $0 < \Lambda_q < 1$ and $a_2 < 0 < q^{-1}a_1$.

5 Concluding remarks

The q-polynomials of the Hahn class have been revisited by the use of a more direct and a simpler geometrical approach based on the qualitative analysis of solutions of the q-Pearson and the *extended* q-Pearson equations. By this way, it is shown that it is possible to introduce in a unified manner all polynomial solutions of the q-EHT, which are orthogonal on certain intervals. In this article, despite the consideration of the cases $\emptyset$-Jacobi/Jacobi and 0-Jacobi/Jacobi families only, for brevity, we are able to introduce several new orthogonality relations for the big q-Jacobi and q-Hahn polynomials. The appearance of such new relations is due to the fact that we have taken into account the polynomial coefficients of the q-EHT in their full generality dealing with all suitable structures. The rest of the families of q-polynomials of the Hahn class may be studied along the same lines, and results will be reported in due course.

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