On the orthogonality of q-classical polynomials of the Hahn class II

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Abstract

In this article, the study of the orthogonality properties of q-polynomials of the Hahn class started in the initial article by R. Álvarez-Nodarse, R. Sevinik-Adıgüzel, and H. Taşeli, *On the orthogonality of q-classical polynomials of the Hahn class I* is proceeded. To be more specific, the orthogonality properties of the q-polynomials belonging to the $\emptyset$-Hermite-Laguerre/Jacobi, $\emptyset$-Jacobi/Hermite-Laguerre, 0-Laguerre/Jacobi-Bessel and 0-Jacobi/Laguerre-Bessel cases are studied by taking into account the idea considered in the initial paper. In particular, a new orthogonality relation for the q-Meixner polynomials is established.

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1 Introduction

In our previous article [1], we have started the study of the orthogonality properties of polynomial solutions of the q-difference equation of hypergeometric type (q-EHT)

$$\sigma_1(x;q)D_{q^{-1}}D_q y(x,q) + \tau(x,q)D_q y(x,q) + \lambda(q)y(x,q) = 0, \tag{1.1}$$

where $\sigma_1(x,q)$ and

$$\sigma_2(x,q) := q\left[\sigma_1(x,q) + (1-q^{-1})x\tau(x,q)\right] \tag{1.2}$$

are both quadratic polynomials. Our main tool was the q-Pearson equation that the weight function ρ satisfy

$$\frac{\rho(qx,q)}{\rho(x,q)} = \frac{\sigma_1(x,q) + (1-q^{-1})x\tau(x,q)}{\sigma_1(qx,q)} = \frac{q^{-1}\sigma_2(x,q)}{\sigma_1(qx,q)}. \tag{1.3}$$

In this paper, we continue the study of the orthogonality properties of q-polynomials of the q-Hahn tableau started in [1]. The main idea is to provide a relatively simpler geometrical analysis of the q-Pearson equation by taking into account every possible rational form of the polynomial coefficients of the q-difference equation. Such a qualitative analysis implies all possible orthogonality relations among the polynomial solutions of the q-difference equation in question. A previous attempt of using a similar geometrical approach has been introduced also in [4] but it is far to being complete where only some partial results were obtained.

In this respect, we have considered $\emptyset$-Jacobi/Jacobi and 0-Jacobi/Jacobi cases in [1] and we deal with the $\emptyset$-Hermite/Jacobi, $\emptyset$-Laguerre/Jacobi, $\emptyset$-Jacobi/Hermite, $\emptyset$-Jacobi/Laguerre, 0-Laguerre/Jacobi, 0-Laguerre/Bessel, 0-Jacobi/Laguerre and 0-Jacobi/Bessel cases in this paper. The above classification of the q-polynomials is based on the degrees of the polynomial coefficients σ_1 and σ_2 and the fact that either $q\sigma_1(0,q) = \sigma_2(0,q) = 0$ or $\sigma_1(0,q), \sigma_2(0,q) \neq 0$. For example, the statement q-Jacobi/q-Laguerre implies that $\deg[\sigma_1] = 2$ and $\deg[\sigma_2] = 1$, or q-Hermite/q-Jacobi indicates that $\deg[\sigma_1] = 0$ and $\deg[\sigma_2] = 2$. For further details on this classification see [2, 3, 9, 10].

In the framework considered in [1] the paper is organized as follows: In Section 2, we give a theorem in order to calculate q-weight functions for every degree of σ_1 and σ_2 of the families that were not considered in [1]. In Section 3, we present the qualitative analysis for the families of $\emptyset$-Hermite/Jacobi, $\emptyset$-Laguerre/Jacobi, $\emptyset$-Jacobi/Hermite, $\emptyset$-Jacobi/Laguerre, 0-Laguerre/Jacobi, 0-Laguerre/Bessel, 0-Jacobi/Laguerre, 0-Jacobi/Bessel polynomials. In fact, we study each orthogonal polynomial sequence (OPS), which is orthogonal with respect to (w.r.t.) a q-weight function $\rho(x, q) > 0$ satisfying the q-Pearson equation as well as certain boundary conditions (BCs). In order to save space we discuss cases, leading to an OPS, in detail while we mention briefly for the others. A more detailed discussion can be found in [11].

2 The q-Weight function

In this section we include the analytic representations of q-weight functions satisfying (1.3) for each $\emptyset$-Hermite-Laguerre/Jacobi, $\emptyset$-Jacobi/Hermite-Laguerre, 0-Laguerre/Jacobi-Bessel and 0-Jacobi/Laguerre-Bessel cases by considering the polynomial coefficients σ_1 and σ_2 of at most 2nd degree and τ of 1st degree in x,

$$\begin{aligned}
\tau(x,q) &= \tau'(0,q)x + \tau(0,q),\ \tau'(0,q) \neq 0,\\
\sigma_1(x,q) &= \tfrac{1}{2}\sigma_1''(0,q)x^2 + \sigma_1'(0,q)x + \sigma_1(0,q) = \tfrac{1}{2}\sigma_1''(0,q)[x - a_1(q)][x - b_1(q)], \\
\sigma_2(x,q) &= \tfrac{1}{2}\sigma_2''(0,q)x^2 + \sigma_2'(0,q)x + \sigma_2(0,q) = \tfrac{1}{2}\sigma_2''(0,q)[x - a_2(q)][x - b_2(q)].
\end{aligned} \tag{2.1}$$

In the following we follow the notations introduced in [1]. For the sake of completeness we include here the Theorems 1 and 2 of [1].

Theorem 1 *[1, Theorem 1] Let ρ be a function satisfying the q-Pearson equation (1.3) in such a way that the BCs*

$$\sigma_1(x,q)\rho(x,q)x^k\Big|_{x=a,b} = \sigma_2(q^{-1}x,q)\rho(q^{-1}x,q)x^k\Big|_{x=a,b} = 0, \qquad k \in \mathbb{N}_0 \tag{2.2}$$

also hold. Then the sequence $\{P_n(x)\}$ of polynomial solutions are orthogonal on (a, b) w.r.t $\rho(x, q)$ in the sense that

$$\int_a^b P_n(x,q)P_m(x,q)\rho(x,q)d_qx = d_n^2(q)\delta_{mn},$$

where $d_n(q)$ and δ_{mn} denote the norm of the polynomials P_n and the Kronecker delta, respectively. Analogously, if the conditions

$$\sigma_2(x,q)\rho(x,q)x^k\Big|_{x=a,b} = \sigma_1(qx,q)\rho(qx,q)x^k\Big|_{x=a,b} = 0, \qquad k \in \mathbb{N}_0 \tag{2.3}$$

are fulfilled, the q-polynomials then satisfy the relation

$$\int_a^b P_n(x,q)P_m(x,q)\rho(x,q)d_{q^{-1}}x = d_n^2(q)\delta_{mn}.$$

Theorem 2 *[1, Theorem 2] Let $a_1(q)$, $b_1(q)$ and $a_2(q)$, $b_2(q)$ denote the zeros of $\sigma_1(x, q)$ and $\sigma_2(x, q)$, respectively. Let ρ be a bounded and non-negative function satisfying the q-Pearson equation (1.3). Such a function ρ can satisfy the BCs (2.2) or (2.3) and, therefore, it may be a desired weight function for the polynomial solutions $P_j(x, q)$ of (1.1) only in the following cases:*

1. *Let $a < 0 < b$, where $a = a_1(q)$ and $b = b_1(q)$. Then ρ is supported at the points $q^k a$ and $q^k b$ for $k \in \mathbb{N}_0$ on $[a, b]$ such that*

$$\int_{a_1(q)}^{b_1(q)} P_n(x,q)P_m(x,q)\rho(x,q)d_qx = d_n^2(q)\delta_{mn}. \tag{2.4}$$

where the q-Jackson integral is of type (2.13).

2. *Let $a = 0 < b$, where $a = a_1(q)$ and $b = b_1(q)$. Then ρ is supported at the points $q^k b$ for $k \in \mathbb{N}_0$ on $(0, b]$ such that*

$$\int_0^{b_1(q)} P_n(x,q)P_m(x,q)\rho(x,q)d_qx = d_n^2(q)\delta_{mn}, \tag{2.5}$$

where the q-Jackson integral is of type (2.12).

3. *Let $0 = a < b$, where $a = a_2(q)$ and $b = a_1(q)$. Then ρ is supported at the points $q^k b$ for $k \in \mathbb{N}_0$ on $(0, b]$ such that*

$$\int_0^{a_1(q)} P_n(x,q)P_m(x,q)\rho(x,q)d_qx = d_n^2(q)\delta_{mn}, \tag{2.6}$$

where the q-Jackson integral is of type (2.12).

4. *Let $0 < a < b$, where $a = a_2(q)$ and $b = a_1(q)$. Then ρ is supported at the points $q^k b$; $a = q^N b < \cdots < q^2 b < qb < b$ or, equivalently, $q^{-k}a$; $a < aq^{-1} < aq^{-2} < \cdots < q^{-N}a = b$ such that*

$$\int_{qa_2(q)=q^N a_1(q)}^{a_1(q)} P_n(x,q)P_m(x,q)\rho(x,q)d_qx = d_n^2(q)\delta_{mn}, \tag{2.7}$$

where the q-Jackson integral is of type (2.13) and equivalent to the finite sum

$$\int_{q^N a_1(q)}^{a_1(q)} [\cdot]d_qx = (1-q)a_1(q)\sum_{k=0}^{N-1} P_n(q^k a_1(q),q)P_m(q^k a_1(q),q)\rho(q^k a_1(q),q).$$

5. *Let $a < b = 0$, where $a = a_1(q)$ and $b = 0$. Then ρ is supported at the points $q^k a$ for $k \in \mathbb{N}_0$ on $[a, 0)$ such that*

$$\int_{a_1(q)}^{0} P_n(x,q)P_m(x,q)\rho(x,q)d_qx = d_n^2(q)\delta_{mn},$$

where the q-Jackson integral is of type (2.12).

6. *Let $a = a_1(q) = 0$ and $b \to \infty$. Then ρ is supported at the points $q^{\pm k}\alpha$ for arbitrary $\alpha > 0$ and $k \in \mathbb{N}_0$ on $(0, \infty)$ such that*

$$\int_0^{\infty} P_n(x,q)P_m(x,q)\rho(x,q)d_qx = d_n^2(q)\delta_{mn},$$

where the q-Jackson integral is of type (2.14).

7. *Let $a = a_1(q) < 0$ and $b \to \infty$. Then ρ is supported at the points $q^k a$ and $q^{\mp k}\alpha$ for arbitrary $\alpha > 0$ and $k \in \mathbb{N}_0$ on $[a, \infty)$ such that*

$$\int_{a_1(q)}^{\infty} P_n(x,q)P_m(x,q)\rho(x,q)d_qx := \int_{a_1(q)}^{0} [\cdot]d_qx + \int_0^{\infty} [\cdot]d_qx = d_n^2(q)\delta_{mn}, \tag{2.8}$$

where the first q-Jackson integral is of type (2.12) and the second one is of type (2.14), respectively.

8. *Let $a = a_2(q) > 0$ and $b \to \infty$. Then ρ is supported at the points $q^{-k}a$ for $k \in \mathbb{N}_0$ on $[a, \infty)$ such that*

$$\int_{a_2(q)}^{\infty} P_n(x,q)P_m(x,q)\rho(x,q)d_{q^{-1}}x = d_n^2(q)\delta_{mn}, \tag{2.9}$$

where the q^{-1}-Jackson integral is of type (2.15).

9. *Let $a = a_2(q) = 0$ and $b \to \infty$. Then ρ is supported at the points $q^{\pm k}\alpha$ for arbitrary $\alpha > 0$ and $k \in \mathbb{N}_0$ on $(0, \infty)$ such that*

$$\int_0^{\infty} P_n(x,q)P_m(x,q)\rho(x,q)d_qx = d_n^2(q)\delta_{mn}, \tag{2.10}$$

where the q-Jackson integral is of type (2.14).

10. *Let $a \to -\infty$ and $b \to \infty$. Then ρ is supported at the points $\mp q^{\pm k}\alpha$ for arbitrary $\alpha > 0$ and $k \in \mathbb{N}_0$ on $(-\infty, \infty)$ such that*

$$\int_{-\infty}^{\infty} P_n(x,q)P_m(x,q)\rho(x,q)d_qx = d_n^2(q)\delta_{mn}, \tag{2.11}$$

where the bilateral q-Jackson integral is of type (2.14).

In the above theorem, the q-Jackson integrals [5, 6] are defined by

$$\int_0^a f(x)d_qx = (1-q)a\sum_{j=0}^{\infty} q^j f(q^j a) \quad \text{and} \quad \int_a^0 f(x)d_qx = (1-q)(-a)\sum_{j=0}^{\infty} q^j f(q^j a) \tag{2.12}$$

if $a > 0$ and $a < 0$, respectively. Therefore, we have

$$\int_a^b f(x)d_qx := \int_0^b f(x)d_qx - \int_0^a f(x)d_qx \quad \text{and} \quad \int_a^b f(x)d_qx := \int_a^0 f(x)d_qx + \int_0^b f(x)d_qx \tag{2.13}$$

when $0 < a < b$ and $a < 0 < b$, respectively. Furthermore, we make use of the *improper* q-Jackson integrals

$$\int_0^{\infty} f(x)d_qx = (1-q)\sum_{j=-\infty}^{\infty} q^j f(q^j) \ \text{and} \ \int_{-\infty}^{\infty} f(x)d_qx = (1-q)\sum_{j=-\infty}^{\infty} q^j[f(q^j)+f(-q^j)] \tag{2.14}$$

where the second one is sometimes called the *bilateral* q-integral. The q^{-1}-*Jackson integrals* may be defined similarly. For instance, the improper q^{-1}-*Jackson integral* on (a,∞) is given by

$$\int_a^{\infty} f(x)d_{q^{-1}}x = (q^{-1}-1)a\sum_{j=0}^{\infty} q^{-j} f(q^{-j}a), \quad a > 0 \tag{2.15}$$

provided that $\lim_{j\to\infty} q^{-j}f(q^{-j}a) = 0$ and the series is convergent.

The next theorem is the extension of [1, Theorem 4] for the other q- polynomials (see also [3]).

Theorem 3 *Let σ_1 and σ_2 be polynomials of at most 2nd degree in x as the form (2.1). Then a solution $\rho(x,q)$ of q-Pearson equation (1.3) for each $\emptyset$-Hermite-Laguerre/Jacobi, $\emptyset$-Jacobi/Hermite-Laguerre, 0-Laguerre/Jacobi-Bessel and 0-Jacobi/Laguerre-Bessel cases is expressible in the equivalent forms shown in Table 1.*

Table 1: Expressions for the q-weight function $\rho(x,q)$

$\emptyset$-Jacobi/Laguerre	1. $\dfrac{(a_1^{-1}qx,\, b_1^{-1}qx;q)_\infty}{(a_2^{-1}x;q)_\infty}$ 2. $\lvert x\rvert^\alpha \dfrac{(qa_2/x, qb_1^{-1}x;q)_\infty}{(a_1/x;q)_\infty}$, $q^\alpha = -\dfrac{q^{-2}\sigma_2'(0,q)}{\frac12\sigma_1''(0,q)b_1}$
$\emptyset$-Jacobi/Hermite	$(a_1^{-1}qx, b_1^{-1}qx;q)_\infty$
$\emptyset$-Laguerre/Jacobi	1. $\dfrac{(a_1^{-1}qx;q)_\infty}{(a_2^{-1}x,\, b_2^{-1}x;q)_\infty}$ 2. $\lvert x\rvert^\alpha \sqrt{x^{\log_q x-1}}\dfrac{(a_2q/x, b_2q/x;q)_\infty}{(a_1/x;q)_\infty}$, $q^\alpha = \dfrac{\frac12\sigma_2''(0,q)q^{-2}}{\sigma_1'(0,q)}$
	3. $\lvert x\rvert^\alpha x^{\log_q x}(qa_1^{-1}x, qa_2/x, qb_2/x;q)_\infty$, $\quad q^\alpha = -\dfrac{q^{-2}\frac12\sigma_2''(0,q)}{\sigma_1'(0,q)a_1}$
$\emptyset$-Hermite/Jacobi	1. $\dfrac{1}{(a_2^{-1}x, b_2^{-1}x;q)_\infty}$ 2. $\lvert x\rvert^\alpha x^{\log_q x-1}(a_2q/x,\, b_2q/x;q)_\infty$, $q^\alpha = \dfrac{\frac12\sigma_2''(0,q)q^{-1}}{\sigma_1(0,q)}$
0-Jacobi/Laguerre	$\lvert x\rvert^\alpha\,(a_1^{-1}qx;q)_\infty$, $q^\alpha = -\dfrac{q^{-2}\sigma_2'(0,q)}{\frac12\sigma_1''(0,q)a_1}$
0-Jacobi/Bessel	$\lvert x\rvert^\alpha \sqrt{x^{\log_q x-1}}(a_1^{-1}qx;q)_\infty$, $q^\alpha = -\dfrac{q^{-2}\frac12\sigma_2''(0,q)}{\frac12\sigma_1''(0,q)a_1}$
0-Laguerre/Jacobi	1. $\lvert x\rvert^\alpha \dfrac{1}{(a_2^{-1}x;q)_\infty}$, $q^\alpha = -\dfrac{q^{-2}\frac12\sigma_2''(0,q)a_2}{\sigma_1'(0,q)}$ 2. $\lvert x\rvert^\alpha \sqrt{x^{\log_q x-1}}(qa_2/x;q)_\infty$, $\dfrac{q^{-2}\frac12\sigma_2''(0,q)}{\sigma_1'(0,q)}$
0-Laguerre/Bessel	$\lvert x\rvert^\alpha \sqrt{x^{\log_q x-1}}$, $q^\alpha = \dfrac{q^{-2}\frac12\sigma_2''(0,q)}{\sigma_1'(0,q)}$

Proof: The proof is similar to the [1, Theorem 4]. To obtain the second formula for the $\emptyset$-Laguerre/Jacobi family we rewrite the q-Pearson equation (1.3) in the form

$$\frac{\rho(qx,q)}{\rho(x,q)} = ax\frac{[1-a_2/x][1-b_2/x]}{[1-a_1q^{-1}/x]}, \quad a = \frac{q^{-2}\frac12\sigma_2''(0,q)}{\sigma_1'(0,q)}$$

and then apply the same procedure described in [1]. ■

3 The analysis of the orthogonality of the q-polynomials

This section includes the main analysis of the $\emptyset$-Hermite-Laguerre/Jacobi, $\emptyset$-Jacobi/Hermite-Laguerre, 0-Laguerre/Jacobi-Bessel and 0-Jacobi/Laguerre-Bessel cases by taking into account the rational function on the r.h.s. of the q-Pearson equation (1.3) along the same lines with the $\emptyset$-Jacobi/Jacobi and 0-Jacobi/Jacobi cases handled in [1]. Therefore, in order to follow the paper we highly recommend the reader to read first paper [1]. In particular, Section 2 and Theorem 2 where different kinds of orthogonality relations of the form

$$\int_a^b P_n(x,q)P_m(x,q)\rho(x,q)d_qx = d_n^2(q)\delta_{mn}$$

for different kinds of q-Jackson integrals have been established.

3.1 q-Classical $\emptyset$-Hermite/Jacobi Polynomials

Let the coefficients σ_1 and σ_2 be constant and quadratic polynomials in x, respectively, such that $\sigma_1(0,q) \neq 0$ and $\sigma_2(0,q) \neq 0$. If $\sigma_1(x,q) = \sigma_1(0,q) \neq 0$ then, from (1.2),

$$\sigma_2(x,q) = q\left[\sigma_1(x,q) + (1-q^{-1})x\tau(x,q)\right] = (q-1)\tau'(0,q)x^2 + (q-1)\tau(0,q)x + q\sigma_1(0,q)$$

where $\tau'(0,q) \neq 0$ by hypothesis. Then the q-Pearson equation (1.3) takes the form

$$f(x,q) := \frac{\rho(qx,q)}{\rho(x,q)} = \frac{q^{-1}\sigma_2(x,q)}{\sigma_1(qx,q)} = (1-q^{-1})\frac{\tau'(0,q)}{\sigma_1(0,q)}[x-a_2(q)][x-b_2(q)] \tag{3.1}$$

provided that the discriminant denoted by Δ_q,

$$\Delta_q := \left[(1-q^{-1})\frac{\tau(0,q)}{\sigma_1(0,q)}\right]^2 - 4(1-q^{-1})\frac{\tau'(0,q)}{\sigma_1(0,q)}$$

of f in (3.1) is non-zero. Notice that y-intercept of f is $y=1$ since $\sigma_2(0,q) = q\sigma_1(0,q)$. Moreover, $x=a_2$ and $x=b_2$ indicate its zeros which are constant multiples of the roots of σ_2. The following straightforward lemma allows us to determine the locations of the zeros of f.

Lemma 4 *Let $\Lambda_q = \frac{\tau'(0,q)}{\sigma_1(0,q)} \neq 0$. Then we encounter the following cases for the roots of the equation $f(x,q)=0$.*

Case 1. *If $\Lambda_q > 0$, f has two real distinct roots with opposite signs.*

Case 2. *If $\Lambda_q < 0$, there exist three possibilities*

- **(a)** *if $\Delta_q > 0$, f has two real roots with same signs*
- **(b)** *if $\Delta_q = 0$, f has a double root*
- **(c)** *if $\Delta_q < 0$, f has a pair of complex conjugate roots.*

The next step is sketching roughly all graphs of f by taking into account all possible relative positions of the zeros of f in question. As a result of analysis of the graphs of f, we determine a suitable $\rho > 0$ satisfying the q-Pearson equation (1.3) with BCs (2.2), (2.3).

In Figure 1A, let us consider the possible intervals in which we can have a suitable weight function ρ which are defined by the zeros of the polynomials σ_1 and σ_2. First of all, notice that since ρ should be a positive weight function and f is negative in the intervals $(-\infty, a_2)$ and (b_2, ∞), they are not suitable. On the other hand, the interval (a_2, b_2) is also eliminated in which $\rho = 0$ due to **PII** in [1, page 5]. As a result, an OPS fails to exist.

Let us analyse the case in Figure 1B. The positivity of ρ implies that the interval (a_2, b_2) should be eliminated. On the other hand, $(-\infty, a_2)$ is not suitable since $\rho = 0$ in $(0, a_2)$ (this situation is similar to the one described in [1, **PVI** page 6]). The interval (b_2, ∞) coincides with $8th$ case of Theorem 2. Notice that $\rho(qx,q)/\rho(x,q) = 1$ at $x_0 = -\tau(0,q)/\tau'(0,q) > b_2$, then ρ is decreasing on (x_0, ∞). Since f has infinite limit as $x \to +\infty$, $\rho \to 0$ as $x \to \infty$. As a result, the typical shape of ρ is constructed in Figure 2 assuming a positive initial value of ρ in each subinterval.

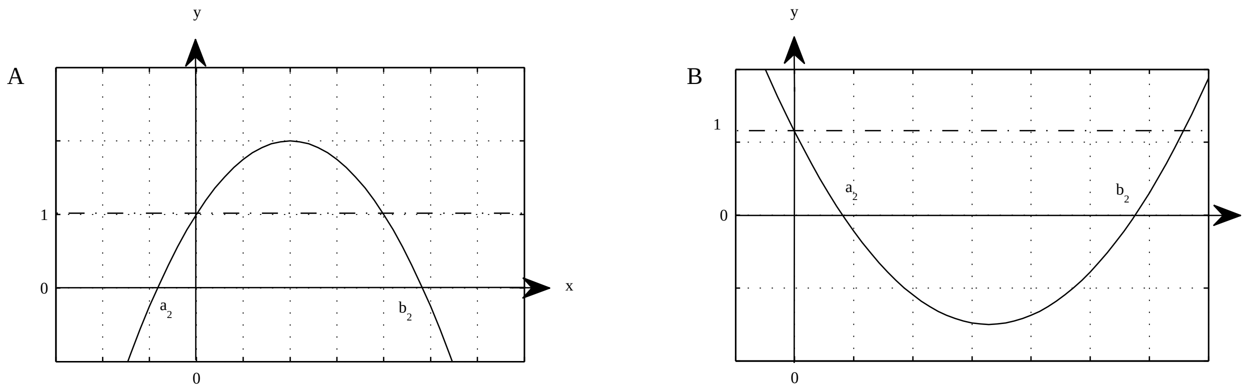


Figure 1: The graph of $f(x,q)$. In A, we have **Case 1** with $\Lambda_q > 0$ and $a_2 < 0 < b_2$, and in B, **Case 2(a)** with $\Lambda_q < 0$ and $0 < a_2 < b_2$.

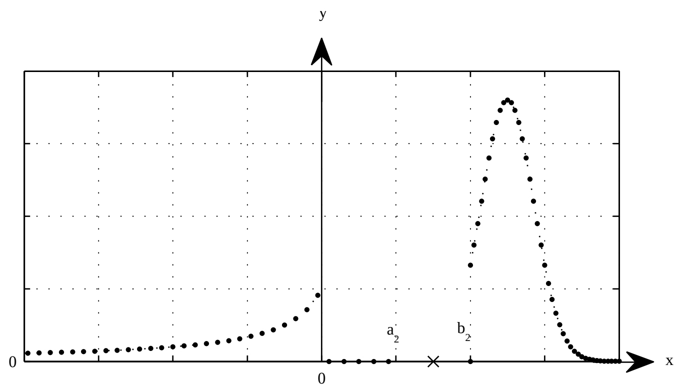


Figure 2: The graph of $\rho(x,q)$ associated with the case in Figure 1B.

However, it is not enough to assure that ρ satisfies the BC at $+\infty$. In fact, even if $\rho \to 0$ as $x \to \infty$ we should check that $\sigma_1(x,q)\rho(x,q)x^k \to 0$ as $x \to \infty$ by using instead of the q-Pearson equation (1.3), the following *extended* q-Pearson equation [1]

$$g(x,q) := \frac{\sigma_1(qx,q)\rho(qx,q)(qx)^k}{\sigma_1(x,q)\rho(x,q)x^k} = q^k \frac{\sigma_1(x,q) + (1-q^{-1})x\tau(x,q)}{\sigma_1(x,q)} = q^k \frac{q^{-1}\sigma_2(x,q)}{\sigma_1(x,q)} \tag{3.2}$$

which is represented in Figure 3 for some $0 < q < 1$, where k is large enough.

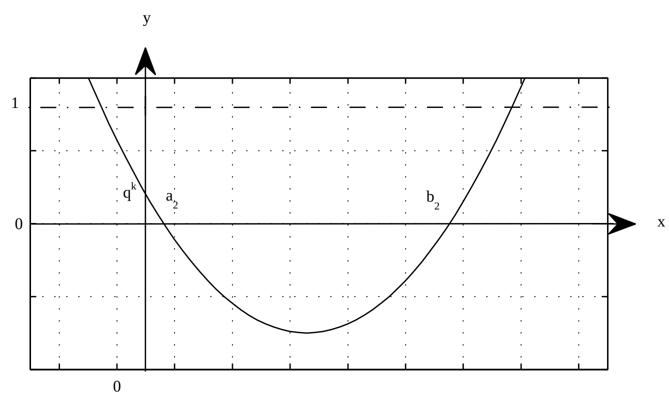


Figure 3: The graph of $g(x,q)$ corresponding to Figure 1B.

If we now provide a similar anaysis for g in (3.2), we see from Figure 3 that, g has the same property with f. Therefore, $\sigma_1(x,q)\rho(x,q)x^k \to 0$ as $x \to \infty$. That is, an OPS, to be stated in Theorem 5, on (b_2,∞) supported at the points b_2q^{-k} for $k \in \mathbb{N}_0$ exists.

Theorem 5 *Let $0 < a_2 \le b_2$ and $q^2\Lambda_q < 0$. Let $a = b_2(q)$ be a zero of $\sigma_2(x,q)$ and $b \to \infty$. Then, there exists a sequence of polynomials $(P_n)_n$ for $n \in \mathbb{N}_0$ orthogonal on (a,b) w.r.t. the weight function (see expression 2 for the $\emptyset$-Hermite/Jacobi case in Table 1)*

$$\rho(x,q) = |x|^{\alpha}\, x^{\log_q x - 1}(qa_2/x, qa/x; q)_\infty, \quad q^\alpha = \frac{q^{-1}\frac{1}{2}\sigma_2''(0,q)}{\sigma_1(0,q)}$$

*in the sense (2.9) of Theorem 2-**8**.*

The OPS in Theorem 5 coincides with the case Ia1 in Chapter 11 of [7, pages 335 and 357]. In fact, a typical example of this family is the Al-Salam-Carlitz II polynomials $V_n^{(\alpha)}(x;q)$ on $(1,\infty)$ satisfying the q-EHT with

$$\sigma_1(x,q) = aq^{-1}, \quad \sigma_2(x,q) = (x-a_2)(x-b_2),$$

$$\tau(x,q) = \frac{1}{q-1}x - \frac{1+a}{q-1} \quad \text{and} \quad \lambda_n(q) = \frac{1}{1-q}[n]_q$$

where $a_2 = a$, $b_2 = 1$ in our notation [7]. The conditions $q^2\Lambda_q < 0$ and $0 < a_2 \le b_2$ give the restriction $0 < a \le 1$ on the parameters of $V_n^{(\alpha)}(x;q)$ which is orthogonal on $(1,\infty)$ in the sense (2.9) with

$$d_n^2 = (q^{-1}-1)q^{-\alpha n - n^2}(q;q)_n(q;q)_\infty.$$

In the literature, this relation is usually written as a finite sum [7, page 357].

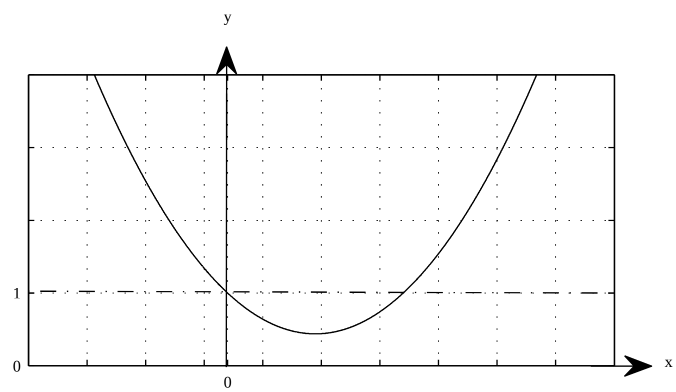


Figure 4: The graph of $f(x,q)$ in **Case2(c)** with $\Lambda_q < 0$ and $a_2(q), b_2(q) \in \mathbb{C}$.

In Figure 4, the only interval is $(-\infty,\infty)$ which corresponds to $10th$ case of Theorem 2. Notice that $\rho(qx,q)/\rho(x,q) = 1$ at $x_0 = -\tau(0,q)/\tau'(0,q)$, then it follows that ρ is increasing on $(-\infty, x_0)$ and decreasing on (x_0,∞). Moreover, $\rho \to 0$ as $x \to \mp\infty$ since $\rho(qx,q)/\rho(x,q) \to \infty$. Then, there may be an OPS on $(-\infty,\infty)$. But we should analyse the extended q-Pearson equation (3.2) to check $\sigma_1(x,q)\rho(x,q)x^k \to 0$ as $x \to \mp\infty$ which leads to similar figure as Figure 4. Then ρ and $\sigma_1(x,q)\rho(x,q)x^k$ have same property that $q\sigma_1(x,q)\rho(x,q)x^k = \sigma_2(q^{-1}x,q)\rho(q^{-1}x,q)x^k \to 0$ as $x \to \mp\infty$ for $k \in \mathbb{N}_0$. Thus we can find a suitable ρ on $(-\infty,\infty)$ supported at the points $q^{\mp k}$ for $k \in \mathbb{N}$. Therefore, we have the following theorem.

Theorem 6 *Let $q^2\Lambda_q < 0$ and $a_2, b_2 \in \mathbb{C}$. Let $a \to -\infty$ and $b \to \infty$. Then, there exists a sequence of polynomials $(P_n)_n$ for $n \in \mathbb{N}_0$ orthogonal on (a,b) w.r.t. the weight function (see expression 1 for the $\emptyset$-Hermite/Jacobi in Table 1)*

$$\rho(x,q) = \frac{1}{(a_2^{-1}x, b_2^{-1}x; q)_\infty}$$

*in the sense (2.11) of Theorem 2-**10**.*

The OPS in Theorem 6 corresponds to the case Ia1 in Chapter 11 and case Va2 in chapter 10 of [7, pages 335, 357, 283 and 315]. An example of this family is the discrete q^{-1}-Hermite II polynomials $\widetilde{h}_n(x;q)$ whose q-EHT has the coefficients

$$\sigma_1(x,q) = q^{-1}, \quad \sigma_2(x,q) = (x-a_2)(x-b_2),$$

$$\tau(x,q)=\frac{1}{q-1}x \quad \text{and} \quad \lambda_n(q)=\frac{1}{1-q}[n]_q$$

where $a_2=-i, b_2=i\in\mathbb{C}$ in our notation [7]. Discrete q^{-1}-Hermite II polynomials are orthogonal on $(-\infty,\infty)$ with

$$d_n^2=(1-q)q^{-n^2}(q;q)_n\frac{(q,-q,-1,-1,-q;q)_\infty}{(i,-i,-iq,iq,-i,i,iq,-iq;q)_\infty}$$

and the conditions $q^2\Lambda_q<0$ and $a_2,b_2\in\mathbb{C}$ hold.

3.2 q-Classical $\emptyset$-Laguerre/Jacobi Polynomials

Let the coefficients σ_1 and σ_2 be linear and quadratic polynomials in x, respectively, such that $\sigma_1(0,q)\neq 0$ and $\sigma_2(0,q)\neq 0$. If σ_1 is written in terms of its roots, i.e., $\sigma_1(x,q)=\sigma_1'(0,q)[x-a_1(q)]$, $a_1(q)=-\frac{\sigma_1(0,q)}{\sigma_1'(0,q)}$ then from (1.2)

$$\sigma_2(x,q)=(q-1)\tau'(0,q)x^2+\left[q\sigma_1'(0,q)+(q-1)\tau(0,q)\right]x-q\sigma_1'(0,q)a_1(q)$$

where $\tau'(0,q)\neq 0$ by hypothesis. Then the q-Pearson equation (1.3) takes the form

$$f(x,q):=\frac{\rho(qx,q)}{\rho(x,q)}=\frac{q^{-1}\sigma_2(x,q)}{\sigma_1(qx,q)}=\frac{(1-q^{-1})\frac{\tau'(0,q)}{\sigma_1'(0,q)}[x-a_2(q)][x-b_2(q)]}{qx-a_1(q)} \tag{3.3}$$

provided that the discriminant denoted by Δ_q,

$$\Delta_q:=\left[1+\frac{(1-q^{-1})\tau(0,q)}{\sigma_1'(0,q)}\right]^2+4a_1(q)(1-q^{-1})\frac{\tau'(0,q)}{\sigma_1'(0,q)}$$

of the quadratic polynomial in the nominator of f in (3.3) is non-zero. Note that here $x=a_2$ and $x=b_2$ are roots of f which are constant multiplies of the roots of σ_2. Moreover, $x=q^{-1}a_1$ is the vertical asymptote of f and $y=1$ is its y-intercept since $\sigma_2(0,q)=q\sigma_1(0,q)$. On the other hand, the locations of the zeros of f are introduced by the following straightforward lemma.

Lemma 7 *Let $\Lambda_q=\frac{\tau'(0,q)}{\sigma_1'(0,q)}\neq 0$. Then, we have the following cases for the roots of the equation $f(x,q)=0$.*

Case 1. *If Λ_q and $a_1(q)$ have opposite signs, then there are two real distinct roots with opposite signs.*

Case 2. *If Λ_q and $a_1(q)$ have same signs, then there exist three possibilities*

- **(a)** *if $\Delta_q>0$, f has two real roots with same signs*
- **(b)** *if $\Delta_q=0$, f has a double root*
- **(c)** *if $\Delta_q<0$, f has a pair of complex conjugate roots.*

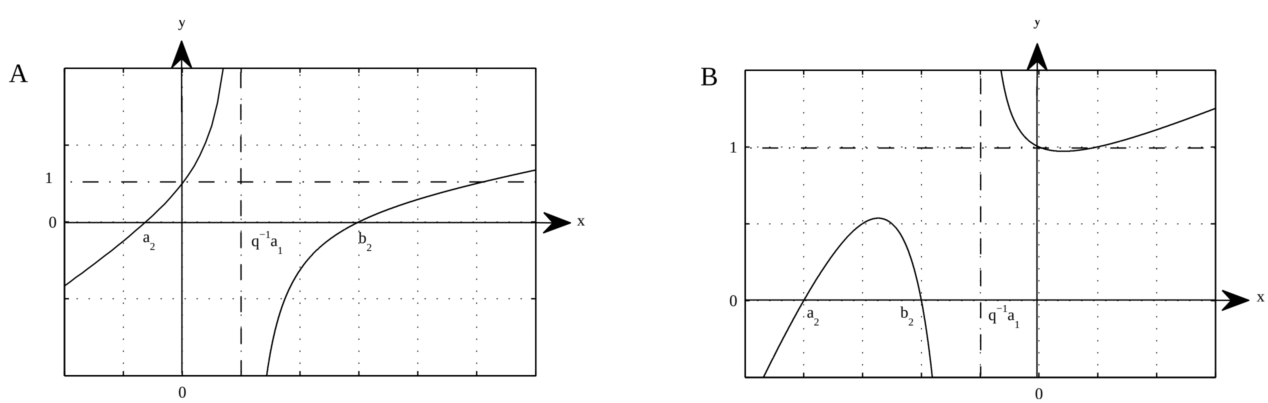


Figure 5: The graph of $f(x,q)$. In A, we have **Case 1** with $\Lambda_q<0$ and $a_2<0<q^{-1}a_1<b_2$, and in B, we have **Case 2(a)** with $\Lambda_q<0$ and $a_2<b_2<q^{-1}a_1<0$.

In Figure 5A, we first start with positivity condition of q-weight function which allows us to exclude the intervals $(-\infty, a_2)$ and $(q^{-1}a_1, b_2)$. Moreover, due to **PIII** in [1, page 5], $(a_2, q^{-1}a_1)$ can not be used. On the other hand, the interval (b_2, ∞) coincides with $8th$ case of Theorem 2. Notice that since $\rho(qx,q)/\rho(x,q) = 1$ at $x_0 = -\tau(0,q)/\tau'(0,q) > b_2$, ρ is decreasing on (x_0, ∞). Moreover, Since $\rho(qx,q)/\rho(x,q)$ has an infinite limit as $x \to +\infty$, we have $\rho \to 0$ as $x \to \infty$. However, since it is infinite interval, we should check that $\sigma_1(x,q)\rho(x,q)x^k \to 0$ as $x \to \infty$ by using extended q-Pearson equation (3.2). The graph of the function g defined in (3.2) looks like the one for f. Then the analysis of the extended q-Pearson equation leads to $\sigma_1(x,q)\rho(x,q)x^k \to 0$ as $x \to \infty$. Therefore, (b_2, ∞) is suitable interval to have ρ. Thus, we have the following theorem.

Theorem 8 *Let $a_2 < 0 < a_1 < b_2$ and $q\Lambda_q < 0$. Let $a = b_2(q)$ be the zero of $\sigma_2(x,q)$ and $b \to \infty$. Then, there exists a sequence of polynomials $(P_n)_n$ for $n \in \mathbb{N}_0$ orthogonal on (a,b) w.r.t. the weight function (see the 2nd expression of the $\emptyset$-Laguerre/Jacobi case in Table 1)*

$$\rho(x,q) = |x|^\alpha \sqrt{x^{\log_q x - 1}} \frac{(qa_2/x, qa/x; q)_\infty}{(a_1/x; q)_\infty}, \quad q^\alpha = \frac{q^{-2}\frac{1}{2}\sigma_2''(0,q)}{\sigma_1'(0,q)} \tag{3.4}$$

*in the sense (2.9) of Theorem 2-**8**.*

The OPS in Theorem 8 coincides with the case IIa2 in Chapter 11 of [7, pages 337 and 358]. An example of this family is the celebrated q-Meixner polynomials $M_n(x;b,c;q)$ on $(1,\infty)$ satisfying the q-EHT with

$$\sigma_1(x,q) = cq^{-2}(x - a_1), \quad \sigma_2(x,q) = (x - a_2)(x - b_2),$$

$$\tau(x,q) = -\frac{1}{1-q}x + \frac{cq^{-1} - bc + 1}{1-q} \quad \text{and} \quad \lambda_n(q) = \frac{[n]_q}{1-q} \tag{3.5}$$

where $a_1 = bq$, $a_2 = -bc$ and $b_2 = 1$ in our notation [7]. The conditions $q^2\Lambda_q < 0$ and $a_2 < 0 < a_1 < b_2$ give us the restrictions $c > 0$ and $0 < b < q^{-1}$ on the parameters of $M_n(x;b,c;q)$ which is orthogonal on $(1,\infty)$ and

$$d_n^2 = (q^{-1} - 1)c^{2n}q^{-n(2n+1)}(q, -c^{-1}q, bq; q)_n \frac{(q, -c; q)_\infty}{(bq; q)_\infty}.$$

In the literature, this relation can be found as a finite sum [7, page 360].

In Figure 5B, the only possible interval is $(q^{-1}a_1, \infty)$ which is the one identified in Theorem 2-**7**. Notice that $\rho(qx,q)/\rho(x,q) = 1$ at $x_0 = -\tau(0,q)/\tau'(0,q) > q^{-1}a_1$, then ρ is increasing on $(q^{-1}a_1, x_0)$ and decreasing on (x_0, ∞) which leads to $\rho \to 0$ as $x \to \infty$ since $\rho(qx,q)/\rho(x,q) \to \infty$. But we still need to show $\sigma_1(x,q)\rho(x,q)x^k \to 0$ as $x \to \infty$ by using the extended q-Pearson equation (3.2). By applying the same procedure to the extended q-Pearson equation (3.2) whose graph looks like the one for f, we get $\sigma_1(x,q)\rho(x,q)x^k \to 0$ as $x \to \infty$. Consequently, we have a suitable ρ on the interval (a_1, ∞) supported at the points a_1q^k and $q^{\mp k}$ for $k \in \mathbb{N}_0$.

Theorem 9 *Let $a_2 \le b_2 < a_1 < 0$, $q\Lambda_q < 0$. Let $a = a_1$ be a zero of $\sigma_1(x,q)$ and $b \to \infty$. Then, there exists a sequence of polynomials $(P_n)_n$ for $n \in \mathbb{N}_0$ orthogonal on (a,b) w.r.t. the weight function (see the 1st expression of the $\emptyset$-Laguerre/Jacobi case in Table 1)*

$$\rho(x,q) = \frac{(a^{-1}qx; q)_\infty}{(a_2^{-1}x, b_2^{-1}x; q)_\infty}$$

*in the sense (2.8) of Theorem 2-**7** with*

$$d_n^2 = (1-q)q^{-n(2n-1)}\left(a_2b_2a_1^{-1}\right)^{2n}(q, a_2^{-1}a_1, b_2^{-1}a_1; q)_n \frac{(q, a_1, qa_1^{-1}, a_2^{-1}b_2^{-1}a_1, qa_2b_2a_1^{-1}; q)_\infty}{(a_2^{-1}a_1, b_2^{-1}a_1, a_2^{-1}, b_2^{-1}, qa_2, qb_2; q)_\infty}. \tag{3.6}$$

The OPS in Theorem 9 coincides with the case VIa2 in Chapter 10 of [7, pages 285 and 315]. This case leads to the new orthogonality relation on the interval (a_1, ∞)

$$\int_{a_1}^{\infty} \frac{(a^{-1}qx; q)_\infty}{(a_2^{-1}x, b_2^{-1}x; q)_\infty} P_n(x)_q P_m(x)_q d_qx = d_n^2\delta_{mn}, \quad a_2 < b_2 < q^{-1}a_1 < 0, \tag{3.7}$$

where d_n^2 is given by (3.6) and coincide with the value [7, page 316]. Note that this family does not appear in the q-Askey scheme [7, 8].

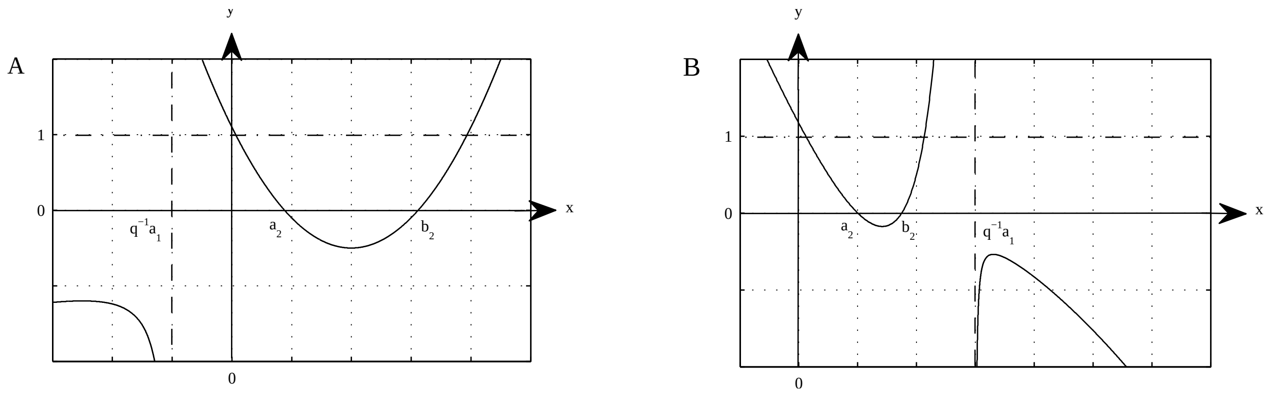


Figure 6: The graph of $f(x,q)$ in **Case 2(a)**. In A, we have $\Lambda_q < 0$ and $q^{-1}a_1 < 0 < a_2 < b_2$ and in B, $\Lambda_q > 0$ and $0 < a_2 < b_2 < q^{-1}a_1$.

In Figure 6A, the only possible interval is (b_2, ∞). An analogous analysis as the one that has been done for the case in Figure 5A yields $\rho \to 0$ as $x \to \infty$. Moreover, since from (3.2) $\sigma_1(x,q)\rho(x,q)x^k \to 0$ as $x \to \infty$ for $k \in \mathbb{N}_0$, then there exists a q-weight function on (b_2, ∞) supported at the points $b_2 q^{-k}$ for $k \in \mathbb{N}_0$. Thus we have the following result.

Theorem 10 *Let $q^{-1}a_1 < 0 < a_2 \le b_2$ and $q\Lambda_q < 0$. Let $a = b_2$ be the zero of $\sigma_2(x,q)$ and $b \to \infty$. Then, there exists a sequence of polynomials $(P_n)_n$ for $n \in \mathbb{N}_0$ orthogonal on (a,b) w.r.t. the weight function (3.4) (see Theorem 8) in the sense (2.9) of Theorem 2-**8**.*

A typical example of this family is the q-Meixner polynomials whose q-EHT has the coefficients (3.5) where $a_1 = bq$, $a_2 = -bc$ and $b_2 = 1$ in our notation [7]. This is an interesting set of q-Meixner polynomials having the same orthogonality properties which is defined under the new restrictions $c > 0$, $b < 0$ and $0 < -bc \le 1$ on the parameters.

In Figure 6B, the only possible interval is $(b_2, q^{-1}a_1)$ which concides with $4th$ case of Theorem 2. In fact, $\rho(qx,q)/\rho(x,q) = 1$ at $b_2 < x_0 = -\tau(0,q)/\tau'(0,q) < q^{-1}a_1$, then ρ is increasing on (b_2, x_0) and decreasing on $(x_0, q^{-1}a_1)$. Moreover, $\rho(qb_2,q) = 0$ and $\rho(q^{-1}a_1,q) = 0$ since $\rho(qb_2,q)/\rho(b_2,q) = 0$ and $\rho(qx,q)/\rho(x,q) \to \infty$ as $x \to q^{-1}a_1^-$. Therefore, there is an OPS on (qb_2, a_1) with a weight function supported at the points $a_1 q^k$, or equivalently, an OPS defined on $(b_2, q^{-1}a_1)$ with a weight function supported at $b_2 q^{-k}$ for $k \in \mathbb{N}_0$ exists.

Theorem 11 *Let $0 < a_2 \le b_2 < a_1$ and $q\Lambda_q > 0$. Let $a = qb_2$ be the zero of $\sigma_2(q^{-1}x,q)$ and $b = a_1$ of $\sigma_1(x,q)$. Then, there exists a sequence of polynomials $(P_n)_n$ for $n \in \mathbb{N}_0$ orthogonal on (a,b) w.r.t. the weight function (see the 3th expression of $\emptyset$-Laguerre/Jacobi in Table 1)*

$$\rho(x,q) = |x|^{\alpha}\, x^{\log_q x}(qb^{-1}x, qa_2/x, a/x; q)_\infty, \quad q^{\alpha} = -\frac{q^{-2}\frac{1}{2}\sigma_2''(0,q)}{\sigma_1'(0,q)b}$$

*in the sense (2.7) of Theorem 2-**4**.*

The OPS in Theorem 11 coincides with the case IIb1 in Chapter 11 of [7, pages 337 and 361]. An example of this family is the quantum q-Kravchuk polynomials $K_m^{qtm}(x;p,N;q)$ satisfying the q-EHT with the coefficients

$$\sigma_1(x,q) = -q^{-2}(x-a_1), \quad \sigma_2(x,q) = p(x-a_2)(x-b_2),$$

$$\tau(x,q) = -\frac{p}{1-q}x + \frac{p-q^{-1}+q^{-N-1}}{1-q} \quad \text{and} \quad \lambda_n(q) = \frac{p}{1-q}[n]_q$$

where $a_1 = q^{-N}$, $a_2 = p^{-1}q^{-N-1}$ and $b_2 = 1$. The conditions $q\Lambda_q > 0$ and $0 < a_2 \le b_2 < a_1$ give the restriction $p \ge q^{-N-1}$ on the parameter of $K_m^{qtm}(x;p,N;q)$ which forms an orthogonal set on $(1, q^{-N-1})$ with

$$d_n^2 = (q^{-1}-1)\frac{1}{(p^{-1}q^{-N};q)_N}p^{-2n}q^{-n(2n+1)}(q,pq,q^{-N};q)_n(q,p^{-1}q^{-N},q^{N+1};q)_\infty.$$

In the literature, this relation can be found as a finite sum [7, page 362].

In Figure 7, $(q^{-1}a_1, \infty)$ is the only interval where f is positive. Notice that the graphs of f in the interval $(q^{-1}a_1, \infty)$ in Figures 7 and 5B have the same behaviour. Then, the analysis of Figure 5B is valid for this case and therefore there exists a suitable ρ on $(q^{-1}a_1, \infty)$. Thus, we have the following theorem.

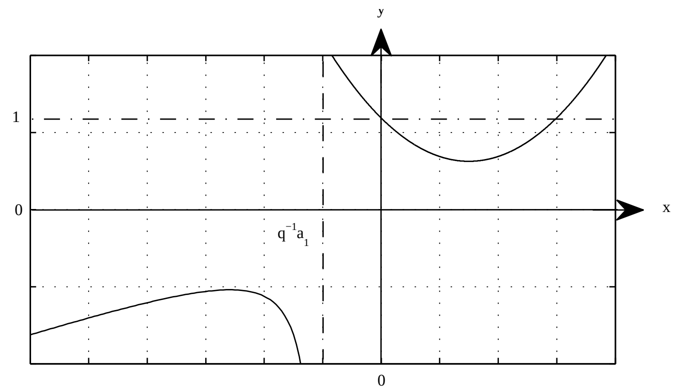


Figure 7: The graph of $f(x,q)$ in **Case 2(c)** with $\Lambda_q < 0$ and $a_1 < 0$, $a_2, b_2 \in \mathbb{C}$.

Theorem 12 *Let $a_1 < 0$, $a_2, b_2 \in \mathbb{C}$ and $q\Lambda_q < 0$. Let $a = a_1$ be the zero of $\sigma_1(x,q)$ and $b \to \infty$. Then, there exists a sequence of polynomials $(P_n)_n$ for $n \in \mathbb{N}_0$ orthogonal on (a,b) w.r.t. the weight function given in Theorem 9, with d_n^2 defined by (3.6), (see the relation (3.7)).*

Notice that the orthogonality relation of this OPS is analogous to the one defined in Theorem 9 but in this case the zeros of σ_2 a_2, b_2 are complex numbers.

For the two cases listed below the OPS fails to exist.

Case 1. $\Lambda_q > 0$ and $q^{-1}a_1(q) < a_2(q) < 0 < b_2(q)$ and **Case 2(a).** $\Lambda_q > 0$ and $0 < a_2(q) < q^{-1}a_1(q) < b_2(q)$.

3.2.1 q-Classical $\emptyset$-Jacobi/Laguerre Polynomials

Let the coefficients σ_1 and σ_2 be quadratic and linear polynomials in x, respectively, such that $\sigma_1(0,q) \neq 0$ and $\sigma_2(0,q) \neq 0$. If σ_1 is written in terms of its roots, i.e., $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)[x - a_1(q)][x - b_1(q)]$, then from (1.2) $\sigma_2(x,q) = \sigma_2'(0,q)x + \sigma_2(0,q)$ where

$$\sigma_2'(0,q) = -q\Big[\tfrac{1}{2}\sigma_1''(0,q)[a_1(q) + b_1(q)] - (1 - q^{-1})\tau(0,q)\Big] \neq 0 \quad \text{and} \quad \sigma_2(0,q) = q\tfrac{1}{2}\sigma_1''(0,q)a_1(q)b_1(q) \neq 0$$

provided that $\tau'(0,q) = -\frac{\frac{1}{2}\sigma_1''(0,q)}{(1-q^{-1})}$. Therefore, the q-Pearson equation (1.3) takes the form

$$f(x,q) := \frac{\rho(qx,q)}{\rho(x,q)} = \frac{-\Big[a_1(q) + b_1(q) - \frac{(1-q^{-1})\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\Big][x - a_2(q)]}{[qx - a_1(q)][qx - b_1(q)]}$$

where $\Big[a_1(q) + b_1(q) - \frac{(1-q^{-1})\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\Big]a_2(q) = a_1(q)b_1(q)$. Let us point out that $f(x,q)$ intercepts the y-axis at the point $y = 1$ since $\sigma_2(0,q) = q\sigma_1(0,q)$. On the other hand, we have the following cases according as the sign of zeros of σ_1 and Λ_q defined as

$$\Lambda_q := \Big[a_1(q) + b_1(q) - \frac{(1 - q^{-1})\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\Big],$$

Case 1. $\Lambda_q < 0$ with $a_1 < 0 < b_1$, **Case 2.** $\Lambda_q > 0$ with $0 < a_1 < b_1$, **Case 3.** $\Lambda_q < 0$ with $0 < a_1 < b_1$.

In Figure 8A, the only possible interval is $(q^{-1}a_1, q^{-1}b_1)$ which is the one described in Theorem 2-**1**. In fact, $\rho(qx,q)/\rho(x,q) = 1$ at $q^{-1}a_1 < x_0 = -\tau(0,q)/\tau'(0,q) < q^{-1}b_1$, Then, ρ is increasing on $(q^{-1}a_1, x_0)$ and decreasing on $(x_0, q^{-1}b_1)$. Moreover, $\rho \to 0$ as $x \to q^{-1}a_1^+$ and $x \to q^{-1}b_1^-$ since $\rho(qx,q)/\rho(x,q) \to \infty$. Then there exists an OPS to be stated in Theorem 13 on (a_1, b_1) w.r.t. a ρ supported at the points $x = q^k a_1$ and $x = q^k b_1$ for $k \in \mathbb{N}_0$.

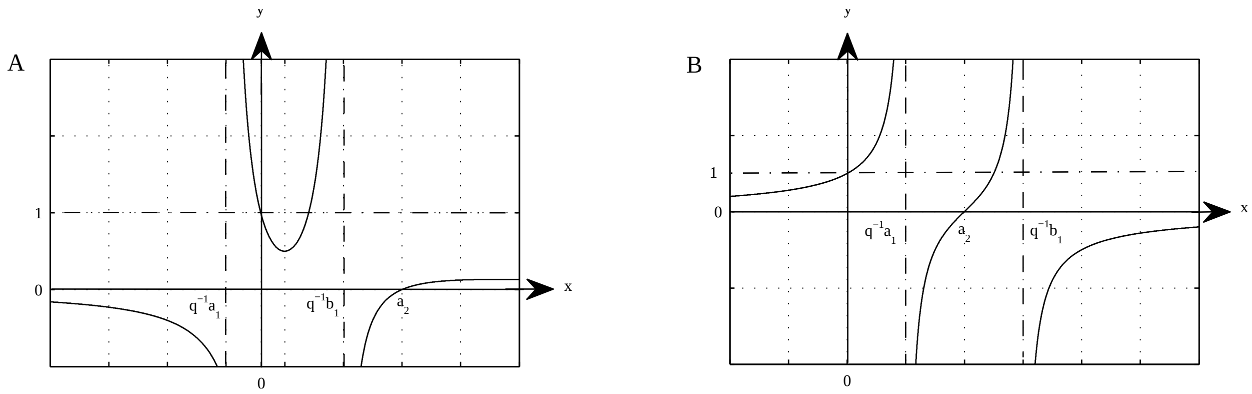


Figure 8: The graph of $f(x,q)$. In A, we have **Case 1** with $\Lambda_q < 0$ and $q^{-1}a_1 < 0 < q^{-1}b_1 < a_2$ and in B, **Case 2** with $\Lambda_q > 0$ and $0 < q^{-1}a_1 < a_2 < q^{-1}b_1$.

Theorem 13 *Let $a_1 < 0 < b_1 < a_2$ and $q^2\Lambda_q < 0$. Let $a = a_1$ and $b = b_1$ be the zeros of $\sigma_1(x,q)$. Then, there exists a sequence of polynomials $(P_n)_n$ for $n \in \mathbb{N}_0$ w.r.t. weight function (see the 1st expression of the $\emptyset$-Jacobi/Laguerre case in Table 1)*

$$\rho(x,q) = \frac{(qa^{-1}x, qb^{-1}x;q)_\infty}{(a_2^{-1}x;q)_\infty}$$

*in the sense (2.4) of Theorem 2-**1**.*

The OPS in Theorem 13 coincides with the case VIIa1 in Chapter 10 of [7, pages 292 and 318]. A typical example of this family is the big q-Laguerre polynomials $P_n(x;a,b;q)$ satisfying the q-EHT with

$$\sigma_1(x,q) = q^{-2}(x-a_1)(x-b_1), \quad \sigma_2(x,q) = -abq(x-a_2),$$

$$\tau(x,q) = -\frac{q^{-1}}{q-1}x + \frac{a+b-abq}{q-1} \quad \text{and} \quad \lambda_n(q) = \frac{q^{-n}}{q-1}[n]_q$$

where $a_1 = bq$, $b_1 = aq$ and $a_2 = 1$. The conditions $q^2\Lambda_q < 0$ and $a_1 < 0 < b_1 < a_2$ give the restrictions $b < 0$ and $0 < a < q^{-1}$ on the parameters of $P_n(x;a,b;q)$ which is orthogonal on (bq, aq) with

$$d_n^2 = (a-b)q(1-q)(-ab)^n q^{n(n+3)/2}(q;q)_n(aq,bq;q)_n \frac{(q, a^{-1}bq, ab^{-1}q;q)_\infty}{(aq,bq;q)_\infty}.$$

In Figure 8B, the only possible interval is $(a_2, q^{-1}b_1)$ which coincides with the one described by Theorem 2-**4**. Notice that $\rho(qx,q)/\rho(x,q) = 1$ at $a_2 < x_0 = -\tau(0,q)/\tau'(0,q) < q^{-1}b_1$. Thus, ρ is increasing on (a_2, x_0) and decreasing on $(x_0, q^{-1}b_1)$. Moreover, $\rho(qa_2,q) = 0$ and $\rho \to 0$ as $x \to q^{-1}b_1^-$ since $\rho(qa_2,q)/\rho(a_2,q) = 0$ and $\rho(qx,q)/\rho(x,q) \to \infty$ as $x \to q^{-1}b_1^-$. Therefore, (qa_2, b_1) is suitable interval in which we have a positive ρ.

Theorem 14 *Let $0 < a_1 < a_2 < b_1$ and $q^2\Lambda_q > 0$. Let $a = qa_2$ be the zero of $\sigma_2(q^{-1}x,q)$ and $b = b_1$ of $\sigma_1(x,q)$. Then, there exists a sequence of polynomials $(P_n)_n$ for $n \in \mathbb{N}_0$ orthogonal on (a,b) w.r.t. the weight function (see the 2nd expression of the $\emptyset$-Jacobi/Laguerre case in Table 1)*

$$\rho(x,q) = |x|^\alpha \, \frac{(a/x, qb^{-1}x;q)_\infty}{(a_1(q)/x;q)_\infty}, \quad q^\alpha = -\frac{q^{-2}\sigma_2'(0,q)}{\frac{1}{2}\sigma_1''(0,q)b}$$

*in the sense (2.7) of Theorem 2-**4**.*

The OPS in Theorem 14 coincides with the case IIIb3 in Chapter 11 of [7, pages 343 and 363]. An example of this family is the affine q-Kravchuk polynomials $K_n^{Aff}(x;p,N;q)$ on $(1, q^{-N-1})$ whose q-EHT has the coefficients

$$\sigma_1(x,q) = q^{-1}(x-a_1)(x-b_1), \quad \sigma_2(x,q) = -pq^{1-N}(x-a_2),$$

$$\tau(x,q) = \frac{1}{1-q}x - \frac{pq + q^{-N} - pq^{1-N}}{1-q} \quad \text{and} \quad \lambda_n(q) = \frac{1}{q-1}[n]_{q^{-1}}$$

where $a_1 = pq$, $b_1 = q^{-N}$ and $a_2 = 1$. The conditions $q^2\Lambda_q > 0$ and $0 < a_1 < a_2 < b_1$ give the restriction $0 < p < q^{-1}$ on the parameter of $K_n^{Aff}(x;p,N;q)$ which forms an orthogonal set on $(1, q^{-N-1})$ with

$$d_n^2 = (-1)^n p^{n-N}(q^{-1}-1)q^{-N(n+1)}q^{n(n+1)/2}(q,pq,q^{-N};q)_n \frac{(q,q^{N+1};q)_\infty}{(pq;q)_\infty}.$$

In the literature, this relation can be found as a finite sum [7, page 364].

The following four cases listed below fail to define an OPS.

Case 1. $\Lambda_q < 0$ and $q^{-1}a_1 < 0 < a_2 < q^{-1}b_1$, **Case 2.** $\Lambda_q > 0$ and $0 < q^{-1}a_1 < q^{-1}b_1 < a_2$, **Case 2.** $\Lambda_q > 0$ and $0 < a_2 < q^{-1}a_1 < q^{-1}b_1$ and **Case 3.** $\Lambda_q < 0$ and $a_2 < 0 < q^{-1}a_1 < q^{-1}b_1$.

3.2.2 q-Classical $\emptyset$-Jacobi/Hermite Polynomials

Let the coefficients σ_1 and σ_2 be quadratic and constant polynomials in x, respectively, such that $\sigma_1(0,q) \neq 0$ and $\sigma_2(0,q) \neq 0$. If σ_1 can be written in terms of its roots, i.e., $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)[x-a_1(q)][x-b_1(q)]$, then, from (1.2)

$$\sigma_2(x,q) = \sigma_2(0,q) = q\tfrac{1}{2}\sigma_1''(0,q)a_1(q)b_1(q)$$

provided that $(1-q^{-1})\tau'(0,q) = -\frac{1}{2}\sigma_1''(0,q)$ and $(1-q^{-1})\tau(0,q) = \frac{1}{2}\sigma_1''(0,q)[a_1(q)+b_1(q)]$. Therefore, the q-Pearson (1.3) becomes

$$f(x,q) := \frac{\rho(qx,q)}{\rho(x,q)} = \frac{a_1(q)b_1(q)}{[qx-a_1(q)][qx-b_1(q)]}.$$

Notice that the point $y=1$ is y-intercept of f. In a similar fashion as before, we introduce the following two cases.

Case1. $a_1(q) < 0 < b_1(q)$. **Case 2.** $0 < a_1(q) < b_1(q)$.

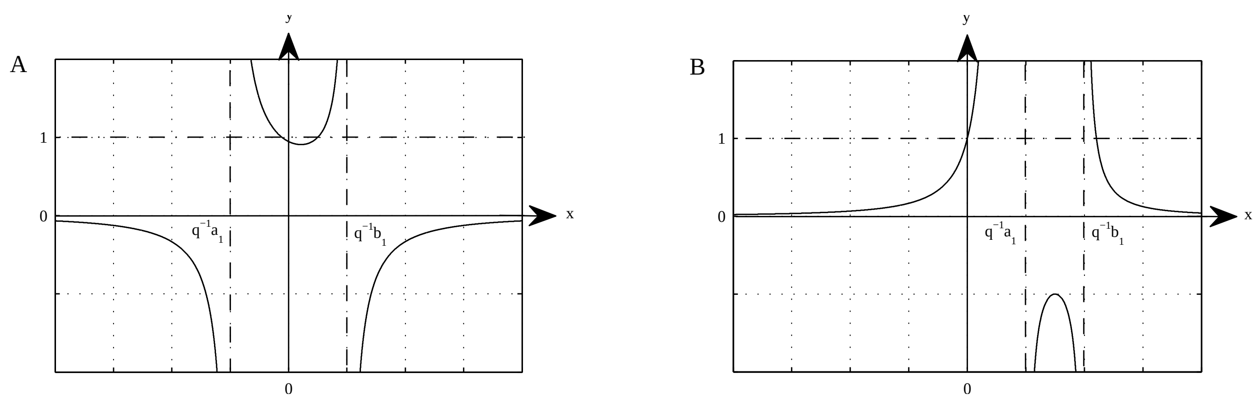


Figure 9: The graph $f(x,q)$ in A, we have Case 1. and in B, Case 2.

In Figure 9A, the only possible interval is $(q^{-1}a_1, q^{-1}b_1)$ which coincides with Theorem 2-**1**. Notice that $\rho(qx,q)/\rho(x,q) = 1$ at $q^{-1}a_1 < x_0 = -\tau(0,q)/\tau'(0,q) < q^{-1}b_1$. Then, ρ is increasing on $(q^{-1}a_1, x_0)$ and decreasing on $(x_0, q^{-1}b_1)$. Moreover, $\rho \to 0$ as $x \to q^{-1}a_1^+$ and $x \to q^{-1}b_1^-$ since $\rho(qx,q)/\rho(x,q) \to \infty$. It is obvious that BC holds at $x = a_1(q)$ and $x = b_1(q)$. Then there exists an OPS with positive q-weight function on (a_1, b_1), as it is stated in the following theorem.

Theorem 15 *Let $a_1 < 0 < b_1$. Let $a = a_1$ and $b = b_1$ be the zeros of $\sigma_1(x,q)$. Then, there exists a sequence of polynomials $(P_n)_n$ for $n \in \mathbb{N}_0$ orthogonal on (a,b) w.r.t. the weight function (see the $\emptyset$-Jacobi/Hermite case in Table 1)*

$$\rho(x,q) = (qa^{-1}x, qb^{-1}x;q)_\infty > 0,\ x \in (a,b)$$

*in the sense (2.4) of Theorem 2-**1**.*

The OPS in Theorem 15 coincides with the case Ia1 in Chapter 11 of [7, pages 335 and 357]. An example of this family is Al-Salam-Carlitz I polynomials $U_n^{(a)}(x;q)$ on $(a,1)$ satisfying the q-EHT with the coefficients

$$\sigma_1(x,q) = q^{-1}(x-a_1)(x-b_1), \quad \sigma_2(x,q) = a,$$

$$\tau(x,q)=\frac{1}{1-q}x-\frac{1+a}{1-q}\quad \text{and}\quad \lambda_n(q)=\frac{q^{1-n}}{q-1}[n]_q$$

where $a_1=a$ and $b_1=1$. The conditions $a_1<0<b_1$ give the restriction $a<0$ on the parameter of $U_n^{(a)}(x;q)$ which forms an orthogonal set on $(a,1)$ with

$$d_n^2=(-a)^n q^{\binom{n}{2}}(1-q)(q;q)_n(q,a,a^{-1}q;q)_\infty .$$

Another example of this family is the discrete q-Hermite I polynomials which are special case of Al-Salam-Carlitz I polynomials (see [8] for further details). Finally, let us mention that the case represented in Figure 9B is inappropriate to define an OPS.

3.2.3 q-Classical 0-Laguerre/Jacobi Polynomials

Let σ_1 and σ_2 be linear and quadratic polynomials in x, respectively, such that $\sigma_2(0,q)=q\sigma_1(0,q)=0$. If $\sigma_1(x,q)=\sigma_1'(0,q)x$, then from (1.2), $\sigma_2(x,q)=\frac{1}{2}\sigma_2''(0,q)x^2+\sigma_2'(0,q)x$ where

$$\tfrac{1}{2}\sigma_2''(0,q)=q(1-q^{-1})\tau'(0,q)\neq 0\quad \text{and}\quad \sigma_2'(0,q)=q[\sigma_1'(0,q)+(1-q^{-1})\tau(0,q)]\neq 0$$

provided that $(1-q^{-1})\tau(0,q)\neq-\sigma_1'(0,q)$. For this case the q-Pearson equation reads

$$f(x,q):=\frac{\rho(qx,q)}{\rho(x,q)}=q^{-1}(1-q^{-1})\frac{\tau'(0,q)}{\sigma_1'(0,q)}[x-a_2(q)] \tag{3.8}$$

where $-(1-q^{-1})\frac{\tau'(0,q)}{\sigma_1'(0,q)}a_2(q)=1+\frac{(1-q^{-1})\tau(0,q)}{\sigma_1'(0,q)}$. Let us point out that f intercepts y-axis at the point

$$y:=y_0=q^{-1}\left[1+\frac{(1-q^{-1})\tau(0,q)}{\sigma_1'(0,q)}\right].$$

Notice that for the zero cases one of the boundary of (a,b) interval could be zero. Therefore it is convenient to know the behaviour of ρ at the point $x=0$.

Lemma 16 *If $0<y_0<1$, then $\rho(x,q)\to 0$ as $x\to 0$. Otherwise it diverges to $\mp\infty$.*

Proof: From (3.8) it follows that

$$\rho(q^k x,q)\quad=\quad q^{-k}\left[1+\frac{(1-q^{-1})\tau(0,q)}{\sigma_1'(0,q)}\right]^k(x/a_2(q);q)_k\rho(x,q)$$

where the result follows by iterating. ■

The following cases according to the signs of the zero of σ_2 and $\Lambda_q:=\frac{\tau'(0,q)}{\sigma_1'(0,q)}$ together with $y_0<1$, $y_0>1$ follow:

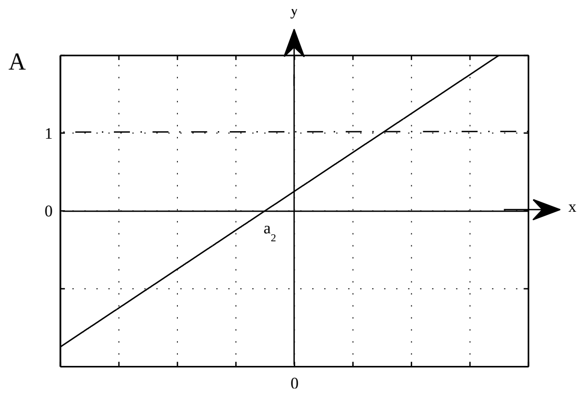
Case 1. $\Lambda_q>0$, $a_2>0$ and $y_0>1$, **Case 2.** $\Lambda_q<0$, $a_2<0$ and $0<y_0<1$, **Case 3.** $\Lambda_q<0$, $a_2>0$ and $y_0<0$.

The **Case 1**, do not lead to any OPS. **Case 2-3** are introduced in Figure 10. In Figure 10A, the only possible interval is $(0,\infty)$ which concides with $9th$ case of Theorem 2. Notice that $\rho(qx,q)/\rho(x,q)=1$ at $x_0=-\tau(0,q)/\tau'(0,q)>0$. Then ρ is increasing on $(0,x_0)$ and decreasing on (x_0,∞). Furthermore, $\rho\to 0$ as $x\to 0^+$ by Lemma 16 since $0<y_0<1$ and $\rho\to 0$ as $x\to\infty$ since $\rho(qx,q)/\rho(x,q)\to\infty$. Therefore, it could be possible to have a suitable ρ on $(0,\infty)$. But we need to check $\sigma_1(x,q)\rho(x,q)x^k\to 0$ as $x\to\infty$ for $k\in\mathbb{N}_0$ by using extended q-Pearson equation (3.2). It is clear from (3.2) that graph of the function g defined in (3.2) looks like the one represented in Figure 10A with y-intercept, $0<q^{k+1}y_0<1$, $k\in\mathbb{N}_0$. Thus $\sigma_1(x,q)\rho(x,q)x^k\to 0$ as $x\to\infty$ for $k\in\mathbb{N}_0$ and therefore, there exists an OPS on $(0,\infty)$ which is established in the next theorem.

Theorem 17 *Let $q\Lambda_q<0$, $a_2<0$ and $0<qy_0<1$. Let $a=0$ and $b\to\infty$. Then, there exists a sequence of polynomials $(P_n)_n$ for $n\in\mathbb{N}_0$ orthogonal on (a,b) w.r.t. the weight function (see the 1st expression of the 0-Laguerre/Jacobi case in Table 1)*

$$\rho(x,q)=|x|^\alpha\frac{1}{(a_2^{-1}x;q)_\infty},\quad q^\alpha=-\frac{q^{-2}\frac{1}{2}\sigma_2''(0,q)a_2}{\sigma_1'(0,q)}$$

*in the sense (2.10) of Theorem 2-**9**.*



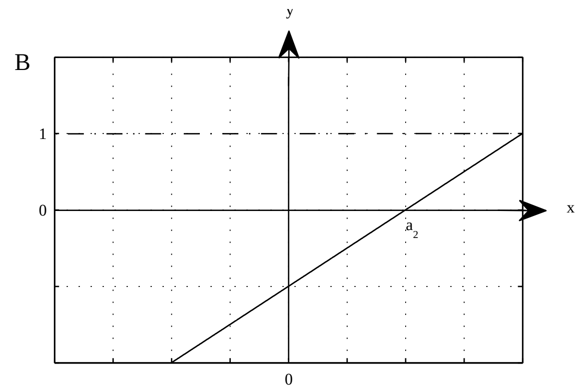


Figure 10: The graph of $f(x,q)$ in A, we have Case 2. and in B, Case 3.

The OPS in Theorem 17 coincides with the case IIIa2 in Chapter 10 of [7, pages 272 and 309]. An example of this family is the q-Laguerre polynomials $L_n^{(\alpha)}(x;q)$ on $(0,\infty)$ whose q-EHT has the coefficients

$$\sigma_1(x,q) = q^{-2}x, \quad \sigma_2(x,q) = q^{\alpha}x(x-a_2),$$

$$\tau(x,q) = -\frac{q^{\alpha}}{1-q}x + \frac{q^{-1}-q^{\alpha}}{1-q} \quad \text{and} \quad \lambda_n(q) = [n]_q \frac{q^{\alpha}}{1-q}$$

where $a_2 = -1$. The conditions $q\Lambda_q < 0$, $a_2 < 0$ and $0 < qy_0 < 1$ result in the restriction $\alpha > -1$ on the parameter of $L_n^{(\alpha)}(x;q)$ which forms an orthogonal set on $(0,\infty)$ with

$$d_n^2 = \frac{1}{2}q^{-n}(1-q)\frac{(q^{\alpha+1};q)_n}{(q;q)_n}\frac{(q,-q^{\alpha+1},-q^{-\alpha};q)_\infty}{(q^{\alpha+1},-q,-q;q)_\infty}.$$

In Figure 10B, the positivity of ρ enables us to skip the intervals $(-\infty,0)$ and $(0,a_2)$. So the only interval is (a_2,∞) which is the one described in Theorem 2-**8**. Notice that $\rho(qx,q)/\rho(x,q) = 1$ at $x_0 = -\tau(0,q)/\tau'(0,q) > a_2$. Therefore, ρ is increasing on (a_2,x_0) and decreasing on (x_0,∞). Moreover, $\rho(qa_2,q) = 0$ since $\rho(qa_2,q)/\rho(a_2,q) = 0$ and $\rho \to 0$ as $x \to \infty$ since $\rho(qx,q)/\rho(x,q) \to \infty$. Furthermore, since the graph of the function g defined in (3.2) looks like the one represented in Figure 10B one can conclude that $\sigma_1(x,q)\rho(x,q)x^k \to 0$ as $x \to \infty$ for $k \in \mathbb{N}_0$ and therefore we have the following theorem.

Theorem 18 *Let $q\Lambda_q < 0$, $a_2 > 0$ and $qy_0 < 0$. Let $a = a_2$ be the zero of $\sigma_2(x,q)$ and $b \to \infty$. Then, there exists a sequence of polynomials $(P_n)_n$ for $n \in \mathbb{N}_0$ orthogonal on (a,b) w.r.t. the weight function (see the 2nd expression of the 0-Laguerre/Jacobi case in Table 1)*

$$\rho(x,q) = |x|^{\alpha}\sqrt{x^{\log_q x - 1}}(qa/x;q)_\infty, \quad q^{\alpha} = \frac{q^{-2}\frac{1}{2}\sigma_2''(0,q)}{\sigma_1'(0,q)}$$

*in the sense (2.9) of Theorem 2-**8**.*

The OPS in Theorem 18 coincides with the case IIa2 in Chapter 11 of [7, pages 337 and 358]. An example of this family is the q-Charlier polynomials $C_n(x;a;q)$ on $(1,\infty)$ satisfying the q-EHT with the coefficients

$$\sigma_1(x,q) = aq^{-2}x, \quad \sigma_2(x,q) = x(x-a_2),$$

$$\tau(x,q) = -\frac{1}{1-q}x + \frac{a+q}{(1-q)q} \quad \text{and} \quad \lambda_n(q) = [n]_q\frac{1}{1-q}$$

where $a_2 = 1$. The conditions $q\Lambda_q < 0$, $a_2 > 0$ and $qy_0 < 0$ give the restriction $a > 0$ on the parameter of $C_n(x;a;q)$ which is orthogonal on $(1,\infty)$ with

$$d_n^2 = a^{2n}q^{-n(2n+1)}(-a^{-1}q,q;q)_n(-a,q;q)_\infty.$$

In the literature, this relation can be found as a finite sum [7, page 360].

3.2.4 q-Classical 0-Laguerre/Bessel Polynomials

Let σ_1 and σ_2 be linear and quadratic polynomials in x, respectively, such that $\sigma_2(0,q) = q\sigma_1(0,q) = 0$. If $\sigma_1(x,q) = \sigma_1'(0,q)x$, then, from (1.2) $\sigma_2(x,q) = \frac{1}{2}\sigma_2''(0,q)x^2 = q(1-q^{-1})\tau'(0,q)x^2$ provided that $(1-q^{-1})\tau(0,q) = -\sigma_1'(0,q)$. As a result, q-Pearson equation becomes

$$f(x,q) := \frac{\rho(qx,q)}{\rho(x,q)} = q^{-1}(1-q^{-1})\frac{\tau'(0,q)}{\sigma_1'(0,q)}x.$$

Let us point out that f intercepts y-axis at the point $y := y_0 = 0$. According to the sign of $\Lambda_q := \frac{\tau'(0,q)}{\sigma_1'(0,q)}$ we have only one possible case.

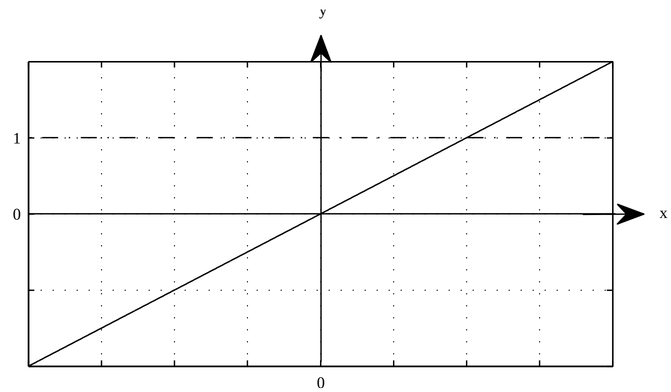


Figure 11: The graph of $f(x,q)$ with $\Lambda_q < 0$, $a_2 = 0$.

From Figure 11 it follows that $(0,\infty)$ is the only possible interval and it coincides with the one described in Theorem 2-**9**. Notice that $\rho(qx,q)/\rho(x,q) = 1$ at $x_0 = -\tau(0,q)/\tau'(0,q) > 0$. Then, ρ is increasing on $(0,x_0)$ and decreasing on (x_0,∞). Moreover, by use of the extended q-Pearson equation (3.2) it is straightforward to see that $\sigma_1(x,q)\rho(x,q)x^k \to 0$ as $x \to +\infty$. Thus, the following theorem holds.

Theorem 19 *Let $q\Lambda_q < 0$, $a_2 = 0$ and $qy_0 = 0$. Let $a = 0$ and $b \to \infty$. Then, there exists a sequence of polynomials $(P_n)_n$ for $n \in \mathbb{N}_0$ orthogonal on (a,b) w.r.t. the weight function (see the 0-Laguerre/Jacobi case in Table 1)*

$$\rho(x,q) = |x|^{\alpha}\sqrt{x^{\log_q x - 1}}, \quad q^{\alpha} = \frac{q^{-2}\frac{1}{2}\sigma_2''(0,q)}{\sigma_1'(0,q)}$$

*in the sense (2.10) of Theorem 2-**9**.*

The OPS in Theorem 19 coincides with the case IIIa2 in Chapter 10 of [7, pages 272 and 309]. An example of this family is Stieltjes-Wigert polynomials $S_n(x;q)$ on $(0,\infty)$ whose q-EHT has the coefficients

$$\sigma_1(x,q) = q^{-2}x, \quad \sigma_2(x,q) = x^2,$$

$$\tau(x,q) = -\frac{1}{1-q}x + \frac{1}{(1-q)q} \quad \text{and} \quad \lambda_n(q) = [n]_q\frac{1}{1-q}$$

where $a_2 = 0$. The conditions $q\Lambda_q < 0$, $a_2 = 0$ and $qy_0 = 0$ are satisfied for $S_n(x;q)$ which forms an orthogonal set on $(0,\infty)$ in the sense (2.10) with

$$d_n^2 = q^{-n}(1-q)\frac{(-tq,-1/t,q;q)_\infty}{(q^2;q)_n}.$$

3.2.5 q-Classical 0-Jacobi/Bessel Polynomials

Let σ_1 and σ_2 be quadratic polynomials in x, respectively, such that $\sigma_2(0,q) = q\sigma_1(0,q) = 0$. If $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)x[x-a_1(q)]$, $\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)} \neq -\frac{1}{(1-q^{-1})}$ and $\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)} = \frac{a_1(q)}{(1-q^{-1})}$, then from (1.2) we have $\sigma_2(x,q) =$

$\frac{1}{2}\sigma_2''(0,q)x^2 = q\left[\frac{1}{2}\sigma_1''(0,q) + (1-q^{-1})\tau'(0,q)\right]x^2$. As a result, the q-Pearson equation (1.3) becomes

$$f(x,q) := \frac{\rho(qx,q)}{\rho(x,q)} \quad = \quad \frac{\left[1 + \frac{(1-q^{-1})\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right]x}{q[qx - a_1(q]}.$$

Let us point out that $y = \Lambda_q := q^{-2}\left[1 + \frac{(1-q^{-1})\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right] \neq 0$, is the horizontal asymptote of $f(x,q)$ and the point $y := y_0 = 0$ is always its y-intercept. Hence, we have the following two cases:

Case 1. $\Lambda_q > 0$ and $a_1 > 0$ and **Case 2.** $\Lambda_q < 0$ and $a_1 > 0$.

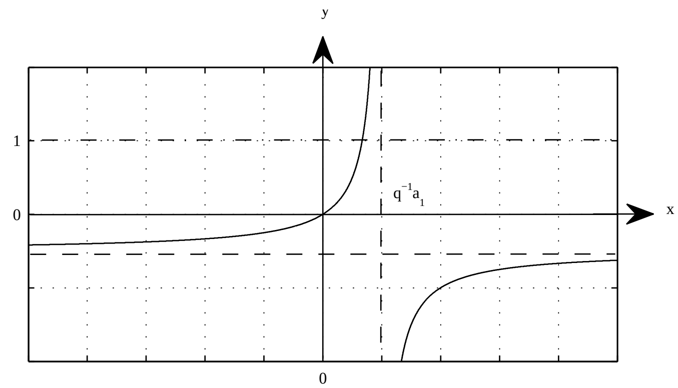


Figure 12: The graph of $f(x,q)$ in Case 2.

The **Case 1** with $\Lambda_q > 1$ and $0 < \Lambda_q < 1$ do not lead to any OPS. The **Case 2** is represented in Figure 12 from where it follows that the only possible interval is $(0, q^{-1}a_1)$ which is the one defined in Theorem 2-**2-3**. Notice also that $\rho(qx,q)/(x,q) = 1$ at $0 < x_0 = -\tau(0,q)/\tau'(0,q) < q^{-1}a_1$. Then, ρ is increasing on $(0,x_0)$ and decreasing on $(x_0, q^{-1}a_1)$. Moreover, $\rho \to 0$ as $x \to 0^+$ and $x \to q^{-1}a_1^-$ since $\rho(qx,q)/(x,q) \to 0$ and $\rho(qx,q)/(x,q) \to \infty$, respectively. Then, there exists an OPS with a suitable ρ defined on $(0,a_1)$ supported at the points a_1q^k for $k \in \mathbb{N}_0$ and the following theorem holds.

Theorem 20 *Let $q^2\Lambda_q < 0$ and $a_1 > 0$. Let $a = 0$ and $b = a_1$ be the zeros of $\sigma_1(x,q)$. Then, there exists a sequence of polynomials $(P_n)_n$ for $n \in \mathbb{N}_0$ orthogonal on (a,b) w.r.t. the weight function (see the 0-Jacobi/Bessel case in Table 1)*

$$\rho(x,q) = |x|^\alpha \sqrt{x^{\log_q x - 1}}(b^{-1}qx;q)_\infty, \quad q^\alpha = -\frac{q^{-2}\frac{1}{2}\sigma_2''(0,q)}{\frac{1}{2}\sigma_1''(0,q)b}$$

*in the sense (2.5) (or (2.6)) of Theorem 2-**2-3**.*

The OPS in Theorem 20 coincides with the case IVa5 in Chapter 10 of [7, pages 278 and 313]. An example of this family is the Alternative q-Charlier polynomials $K_n(x;a;q)$ on $(0,1)$ satisfying the q-EHT with the coefficients

$$\sigma_1(x,q) = -q^{-2}x(x-a_1), \quad \sigma_2(x,q) = ax^2,$$

$$\tau(x,q) = -\frac{1+aq}{(1-q)q}x + \frac{1}{(1-q)q} \quad \text{and} \quad \lambda_n(q) = q^{-n}[n]_q\frac{1+aq^n}{1-q}$$

where $a_1 = 1$. The conditions $q^2\Lambda_q < 0$ and $a_1 > 0$ give the restriction $a > 0$ on the parameter of $K_n(x;a;q)$ which forms an orthogonal set on $(0,1)$ with

$$d_n^2 = a^n q^{n(3n-1)/2}(-aq,q;q)_\infty \frac{(q,-a;q)_n}{(-a,-aq;q)_{2n}}.$$

In the literature, this relation can be found as a finite sum [7, page 314].

3.2.6 q-Classical 0-Jacobi/Laguerre Polynomials

Let σ_1 and σ_2 be quadratic and linear polynomials in x, respectively, such that $\sigma_2(0,q) = q\sigma_1(0,q) = 0$. If $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)x[x-a_1(q)]$ and $\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)} = -\frac{1}{(1-q^{-1})}$, then from (1.2) we get $\sigma_2(x,q) = \sigma_2'(0,q)x = q\left[(1-q^{-1})\tau(0,q) - \frac{1}{2}\sigma_1''(0,q)a_1(q)\right]x$. Therefore, the q-Pearson equation has the form

$$f(x,q) := \frac{\rho(qx,q)}{\rho(x,q)} = \frac{(1-q^{-1})\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)} - a_1(q)}{q[qx-a_1(q]}.$$

Notice that $y=0$ is the horizontal asymptote of the function $f(x,q)$ and its $y$intercept is

$$y := y_0 = q^{-1}\left[1 - \frac{(1-q^{-1})}{a_1(q)}\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right].$$

We have the following two cases: **Case 1.** $y_0 > 0$ and $a_1 > 0$, **Case 2.** $y_0 < 0$ and $a_1 > 0$.

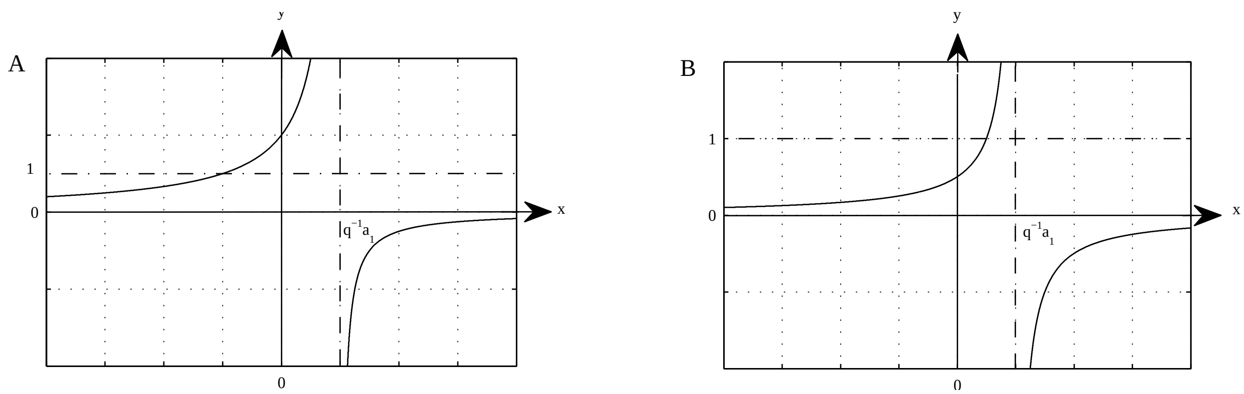


Figure 13: The graph of $f(x,q)$ in **Case 1**. In A, we have $y_0 > 1$ and $a_1 > 0$ and in B, $0 < y_0 < 1$ and $a_1 > 0$.

The Case 1 represented in Figure 13A as well as the **Case 2** do not yield any OPS. From Figure 13B, it follows that the only possible interval is $(0, q^{-1}a_1)$ which coincides with $2nd$ and $3th$ cases of Theorem 2. A completely similar analysis as the one done in the previous case allows us to conclude that in $(0,a_1)$ an OPS can be defined which is orthogonal w.r.t. a suitable ρ supported at the points $q^k a_1$ for $k \in \mathbb{N}_0$. I.e., we have the following Theorem.

Theorem 21 *Let $a_1 > 0$ and $0 < qy_0 < 1$. Let $a = 0$ and $b = a_1$ be the zeros of $\sigma_1(x,q)$. Then, there exists a sequence of polynomials $(P_n)_n$ for $n \in \mathbb{N}_0$ orthogonal on (a,b) w.r.t. the weight function (see the 0-Jacobi/Laguerre case in Table 1)*

$$\rho(x,q) = |x|^\alpha (b^{-1}qx;q)_\infty, \quad q^\alpha = -\frac{q^{-2}\frac{1}{2}\sigma_2''(0,q)}{\frac{1}{2}\sigma_1''(0,q)b}$$

*in the sense (2.5) (or (2.6)) of Theorem 2-**2-3**.*

The OPS in Theorem 21 coincides with the case IVa4 in Chapter 10 of [7, pages 278 and 312]. An example of this family is the little q-Laguerre (Wall) polynomials $P_n(x;\alpha|q)$ on $(0,1)$ satisfying the q-EHT with the coefficients

$$\sigma_1(x,q) = q^{-2}x(a_1 - x), \quad \sigma_2(x,q) = ax,$$

$$\tau(x,q) = -\frac{1}{(1-q)q}x + \frac{1-aq}{(1-q)q} \quad \text{and} \quad \lambda_n(q) = \frac{q^{-n}}{1-q}[n]_q$$

where $a_1 = 1$. The conditions $0 < qy_0 < 1$ and $a_1 > 0$ give the restriction $0 < a < q^{-1}$ on the parameter of $P_n(x;\alpha|q)$ which is orthogonal on $(0,1)$ with

$$d_n^2 = q^{(\alpha+n)n}\frac{(q;q)_\infty}{(q^{\alpha+1};q)_\infty}(q,q^{\alpha+1};q)_n, \quad q^\alpha = a.$$

In the literature, this relation can be found as a finite sum [7, page 312].

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