An identity on the $2m$-th power mean value of the generalized Gauss sums*

Feng Liu and Quan-Hui Yang†

School of Mathematical Sciences, Nanjing Normal University,

Nanjing 210046, P. R. China

Abstract. In this paper, using combinatorial and analytic methods, we prove an exact calculating formula on the $2m$-th power mean value of the generalized quadratic Gauss sums for $m \geq 2$. This solves a conjecture of He and Zhang ['On the $2k$-th power mean value of the generalized quadratic Gauss sums', Bull. Korean Math. Soc. 48 (2011), No.1, 9-15].

2010 Mathematics Subject Classification: Primary 11M20.

Keywords and phrases: $2m$-th power mean, exact calculating formula, generalized quadratic Gauss sums.

1 Introduction

Let $q \geq 2$ be an integer and χ be a Dirichlet character modulo q. For any integer n, the classical quadratic Gauss sums $G(n;q)$ and the generalized quadratic Gauss sums $G(n,\chi;q)$ are defined respectively by

$$G(n;q) = \sum_{a=1}^{q} e\left(\frac{na^2}{q}\right),$$

*This work was supported by the National Natural Science Foundation of China, Grant No. 11071121.

†Corresponding author. E-mail addresses: fliu_1986@126.com; yangquanhui01@163.com.

and

$$G(n,\chi;q)=\sum_{a=1}^{q}\chi(a)e\left(\frac{na^2}{q}\right),$$

where $e(x)=e^{2\pi ix}$.

The study of $G(n,\chi;q)$ is important in number theory, since it is a generation of $G(n,q)$. In [5], Weil proved that if $p\geq 3$ is a prime, then

$$|G(n,\chi;p)|\leq 2\sqrt{p}.$$

In fact, Cochrane and Zheng [2] generalized this result to any integer. That is, for any integer n with $(n,q)=1$, we have

$$|G(n,\chi;q)|\leq 2^{\omega}(q)\sqrt{q},$$

where $\omega(q)$ is the number of all distinct prime divisors of q.

Beside the upper bound of $G(n,\chi;q)$, the power mean value of $|G(n,\chi;q)|$ had also been studied by some authors. W. Zhang (see [6]) proved that if p is an odd prime and n is an integer with $(n,p)=1$, then

$$\sum_{\chi \mod p}|G(n,\chi;p)|^4=\begin{cases}(p-1)[3p^2-6p-1+4\left(\frac{n}{p}\right)\sqrt{p}]\ , & p\equiv 1 \mod 4\ ;\\ (p-1)(3p^2-6p-1)\ , & p\equiv 3 \mod 4\ .\end{cases}$$

and

$$\sum_{\chi \mod p}|G(n,\chi;p)|^6=(p-1)(10p^3-25p^2-4p-1),\quad \text{if}\ \ p\equiv 3 \mod 4,$$

where $(\frac{n}{p})$ is the Legendre symbol. For $p\equiv 1 \mod 4$, it is still an open question to calculate the exact value of $\sum_{\chi \mod p}|G(n,\chi;p)|^6$.

In 2005, W. Zhang and H. Liu [7] proved that if $q\geq 3$ is a square-full number, then for any integer n,k with $(nk,q)=1, k\geq 1$, we have

$$\sum_{\chi \mod q}|G(n,\chi;q)|^4=q\cdot\phi^2(q)\prod_{p|q}(k,p-1)^2\cdot\prod_{\substack{p|q\\(k,p-1)=1}}\frac{\phi(p-1)}{p-1},$$

where $\phi(q)$ is the Euler funtion.

Recently, Y. He and W. Zhang [3] proved the following result.

Let odd number $q > 1$ be a square-full number. Then for any integer n with $(n,q)=1$ and k=3 or 4, we have the identity

$$\sum_{\chi \bmod q} |G(n,\chi;q)|^{2k} = 4^{(k-1)\omega(q)} \cdot q^{k-1} \cdot \phi^2(q).$$

Besides, they conjectured the above identity also holds for $k \geq 5$.

In this paper, we prove this conjecture in the following.

Theorem 1. *Let odd number $q > 1$ be a square-full number, $m \geq 2$ be an integer. Then for any integer n with $(n,q)=1$, we have the identity*

$$\sum_{\chi \bmod q} |G(n,\chi;q)|^{2m} = 4^{(m-1)\omega(q)} \cdot q^{m-1} \cdot \phi^2(q).$$

2 Proofs

Let $p \geq 3$ be a prime, and let k, n, a be three integers with $1 \leq k \leq n$. Write

$$T_p(n,k,a) = \sum_{\substack{x_1=1 \\ x_1+\cdots+x_n \equiv a}}^{p-1} \sum_{x_2=1}^{p-1} \cdots \sum_{\substack{x_n=1 \\ \bmod p}}^{p-1} \left(\frac{x_1 x_2 \cdots x_k}{p}\right).$$

In order to prove Theorem 1, we need some lemmas on the value of $T_p(n,k,a)$.

Lemma 1. *(See [4, Theorem 8.2].) Let $p \geq 3$ be a prime. Then for any integer a, we have*

$$\sum_{x=1}^{p-1} \left(\frac{x^2+ax}{p}\right) = \begin{cases} -1, & \text{if } p \nmid a; \\ p-1, & \text{if } p \mid a. \end{cases}$$

This is a basic lemma which we will use to calculate the value of $T_p(n,k,a)$.

Lemma 2. *Let $p \geq 3$ be a prime. Then for any integer $n \geq 1$, we have*

$$T_p(n,n,0) = \begin{cases} 0\ , & \text{if } 2 \nmid n; \\ p^{(n-2)/2}(p-1)\left(\frac{-1}{p}\right)^{n/2}\ , & \text{if } 2 \mid n. \end{cases}$$

Proof. By the definition of $T_p(n,k,a)$ and Lemma 1, for $n \geq 3$, we have

$$\begin{aligned}
& T_p(n,n,0) \\
= & \sum_{\substack{x_1=1 \\ x_1+\cdots+x_n\equiv 0 \mod p}}^{p-1} \sum_{x_2=1}^{p-1} \cdots \sum_{x_{n-1}=1}^{p-1} \sum_{x_n=1}^{p-1} \left(\frac{x_1x_2\cdots x_{n-1}x_n}{p}\right) \\
= & \sum_{x_1=1}^{p-1} \sum_{x_2=1}^{p-1} \cdots \sum_{x_{n-1}=1}^{p-1} \left(\frac{x_1x_2\cdots x_{n-1}(-x_1-\cdots-x_{n-1})}{p}\right) \\
= & \left(\frac{-1}{p}\right) \sum_{x_1=1}^{p-1} \sum_{x_2=1}^{p-1} \cdots \sum_{x_{n-2}=1}^{p-1} \left(\frac{x_1x_2\cdots x_{n-2}}{p}\right) \\
& \cdot \sum_{x_{n-1}=1}^{p-1} \left(\frac{x_{n-1}^2+(x_1+\cdots+x_{n-2})x_{n-1}}{p}\right) \\
= & \left(\frac{-1}{p}\right) \sum_{\substack{x_1=1 \\ x_1+\cdots+x_{n-2}\not\equiv 0 \mod p}}^{p-1} \sum_{x_2=1}^{p-1} \cdots \sum_{x_{n-2}=1}^{p-1} \left(\frac{x_1x_2\cdots x_{n-2}}{p}\right) \cdot (-1) \\
& + \left(\frac{-1}{p}\right) \sum_{\substack{x_1=1 \\ x_1+\cdots+x_{n-2}\equiv 0 \mod p}}^{p-1} \sum_{x_2=1}^{p-1} \cdots \sum_{x_{n-2}=1}^{p-1} \left(\frac{x_1x_2\cdots x_{n-2}}{p}\right) \cdot (p-1) \\
= & \left(\frac{-1}{p}\right) (-1) \sum_{x_1=1}^{p-1} \sum_{x_2=1}^{p-1} \cdots \sum_{x_{n-2}=1}^{p-1} \left(\frac{x_1x_2\cdots x_{n-2}}{p}\right) \\
& - \left(\frac{-1}{p}\right) (-1) \sum_{\substack{x_1=1 \\ x_1+\cdots+x_{n-2}\equiv 0 \mod p}}^{p-1} \sum_{x_2=1}^{p-1} \cdots \sum_{x_{n-2}=1}^{p-1} \left(\frac{x_1x_2\cdots x_{n-2}}{p}\right) \\
& + \left(\frac{-1}{p}\right) (p-1) \sum_{\substack{x_1=1 \\ x_1+\cdots+x_{n-2}\equiv 0 \mod p}}^{p-1} \sum_{x_2=1}^{p-1} \cdots \sum_{x_{n-2}=1}^{p-1} \left(\frac{x_1x_2\cdots x_{n-2}}{p}\right) \\
= & \left(\frac{-1}{p}\right) p \sum_{\substack{x_1=1 \\ x_1+\cdots+x_{n-2}\equiv 0 \mod p}}^{p-1} \sum_{x_2=1}^{p-1} \cdots \sum_{x_{n-2}=1}^{p-1} \left(\frac{x_1x_2\cdots x_{n-2}}{p}\right) \\
= & \left(\frac{-1}{p}\right) p \cdot T_p(n-2,n-2,0).
\end{aligned}$$

It is easy to calculate $T_p(1,1,0)$ and $T_p(2,2,0)$.

$$T_p(1,1,0) = \sum_{x=1}^{p-1} \left(\frac{x}{p}\right) = 0 \ ,$$

$$T_p(2,2,0) = \sum_{\substack{x_1=1 \\ x_1+x_2\equiv 0}}^{p-1} \sum_{\substack{x_2=1 \\ \bmod p}}^{p-1} \left(\frac{x_1x_2}{p}\right) = \sum_{x_1=1}^{p-1}\left(\frac{-x_1^2}{p}\right) = \left(\frac{-1}{p}\right)(p-1).$$

Therefore, we have

$$T_p(2k+1,2k+1,0) = \left(\left(\frac{-1}{p}\right)p\right)^k T_p(1,1,0) = 0 \ ,$$

$$T_p(2k,2k,0) = \left(\left(\frac{-1}{p}\right)p\right)^{k-1} T_p(2,2,0) = \left(\frac{-1}{p}\right)^k p^{k-1}(p-1).$$

This completes the proof of Lemma 2. □

Lemma 3. *Let $p \geq 3$ be a prime and $n \geq 1$ be an integer. Then for any integer a with $(a,p)=1$, we have*

$$T_p(n,n,a) = \begin{cases} \left(\frac{a}{p}\right) p^{(n-1)/2} \left(\frac{-1}{p}\right)^{(n-1)/2}, & \text{if } 2 \nmid n; \\ -p^{(n-2)/2}\left(\frac{-1}{p}\right)^{n/2}, & \text{if } 2 \mid n. \end{cases}$$

Proof. Since $(a,p)=1$, we have

$$\begin{aligned} T_p(n,n,a) &= \sum_{\substack{x_1=1 \\ x_1+\cdots+x_n\equiv a}}^{p-1} \sum_{x_2=1}^{p-1} \cdots \sum_{\substack{x_n=1 \\ \bmod p}}^{p-1} \left(\frac{x_1x_2\cdots x_n}{p}\right) \\ &= \sum_{\substack{ax_1=1 \\ ax_1+\cdots+ax_n\equiv a}}^{p-1} \sum_{ax_2=1}^{p-1} \cdots \sum_{\substack{ax_n=1 \\ \bmod p}}^{p-1} \left(\frac{ax_1ax_2\cdots ax_n}{p}\right) \\ &= \left(\frac{a}{p}\right)^n \sum_{\substack{x_1=1 \\ x_1+\cdots+x_n\equiv 1}}^{p-1} \sum_{x_2=1}^{p-1} \cdots \sum_{\substack{x_n=1 \\ \bmod p}}^{p-1} \left(\frac{x_1x_2\cdots x_n}{p}\right) \\ &= \left(\frac{a}{p}\right)^n T_p(n,n,1). \end{aligned}$$

The calculation of $T_p(n,n,1)$ is very similar to that of $T_p(n,n,0)$ in Lemma 2, so we directly give the result here.

$$T_p(n,n,1) = \begin{cases} p^{(n-1)/2}\left(\frac{-1}{p}\right)^{(n-1)/2}, & \text{if } 2 \nmid n; \\ -p^{(n-2)/2}\left(\frac{-1}{p}\right)^{n/2}, & \text{if } 2 \mid n. \end{cases}$$

Hence,

$$T_p(n,n,a) = \left(\frac{a}{p}\right)^n T_p(n,n,1) = \begin{cases} \left(\frac{a}{p}\right) p^{(n-1)/2}\left(\frac{-1}{p}\right)^{(n-1)/2}, & \text{if } 2 \nmid n; \\ -p^{(n-2)/2}\left(\frac{-1}{p}\right)^{n/2}, & \text{if } 2 \mid n. \end{cases}$$

This completes the proof of Lemma 3. □

Lemma 4. *Let $p \geq 3$ be a prime, and let k, n, a be three integers with $1 \leq k \leq n$. Then we have*

$$T_p(n,k,a) = \begin{cases} (-1)^{n-k}\left(\frac{a}{p}\right) p^{(k-1)/2}\left(\frac{-1}{p}\right)^{(k-1)/2}, & \text{if } 2 \nmid k \text{ and } p \nmid a; \\ 0, & \text{if } 2 \nmid k \text{ and } p \mid a; \\ (-1)^{n+1-k}\left(\frac{-1}{p}\right)^{k/2} p^{(k-2)/2}, & \text{if } 2 \mid k \text{ and } p \nmid a; \\ (-1)^{n-k}(p-1)\left(\frac{-1}{p}\right)^{k/2} p^{(k-2)/2}, & \text{if } 2 \mid k \text{ and } p \mid a. \end{cases} \tag{1}$$

Proof. For $k \leq n-1$, we have

$$\begin{aligned} & T_p(n,k,a) \\ = & \sum_{\substack{x_1=1 \\ x_1+\cdots+x_n \equiv a}}^{p-1} \sum_{x_2=1}^{p-1} \cdots \sum_{\substack{x_n=1 \\ \bmod p}}^{p-1} \left(\frac{x_1x_2\cdots x_k}{p}\right) \\ = & \sum_{x_1=1}^{p-1}\sum_{x_2=1}^{p-1}\cdots\sum_{x_{n-1}=1}^{p-1}\left(\frac{x_1x_2\cdots x_k}{p}\right) - \sum_{\substack{x_1=1 \\ x_1+\cdots+x_{n-1}\equiv a}}^{p-1}\sum_{x_2=1}^{p-1}\cdots\sum_{\substack{x_{n-1}=1 \\ \bmod p}}^{p-1}\left(\frac{x_1x_2\cdots x_k}{p}\right) \\ = & -T_p(n-1,k,a). \end{aligned}$$

Then by induction on n we have

$$T_p(n,k,a) = (-1)^{n-k}T_p(k,k,a)$$

for all $n \geq k$. By Lemma 2 and Lemma 3, we obtain equation (1), which completes the proof of Lemma 4.

$\square$

Lemma 5. *Let $p \geq 3$ be a prime, $\alpha \geq 2$, a and n be three integers with $1 \leq a \leq p^\alpha - 1$ and $(n, p) = 1$. If $p^{\alpha-1} \parallel a^2 - 1$, then we write $a = rp^{\alpha-1} + \varepsilon$, where $1 \leq r \leq p - 1$ and $\varepsilon = \pm 1$. Then we have*

$$\sum_{b=1}^{p^\alpha}{}' e\left(\frac{nb^2(a^2-1)}{p^\alpha}\right) = \begin{cases} 0 \ , & \text{if } \ p^{\alpha-1} \nmid a^2 - 1; \\ p^{\alpha-1}\left[\left(\frac{2\varepsilon rn}{p}\right)G(1;p) - 1\right] \ , & \text{if } \ p^{\alpha-1} \parallel a^2 - 1; \\ \phi(p^\alpha) \ , & \text{if } \ p^\alpha \mid a^2 - 1 \ . \end{cases}$$

Proof. See the proof of Lemma 4 of [3].

$\square$

Lemma 6. *Let $p \geq 3$ be a prime. Then for any two integers $n \geq 1$ and a, we have*

$$\sum_{x_1=1}^{p-1}\sum_{x_2=1}^{p-1}\cdots\sum_{\substack{x_n=1 \\ x_1+x_2+\cdots+x_n \equiv a \bmod p}}^{p-1} 1 = \begin{cases} \big((p-1)^n - (-1)^n\big)/p \ , & \text{if } \ p \nmid a; \\ \big((p-1)^n + (p-1)(-1)^n\big)/p \ , & \text{if } \ p \mid a. \end{cases}$$

Proof.

$$\begin{aligned} \sum_{x_1=1}^{p-1}\sum_{x_2=1}^{p-1}\cdots\sum_{\substack{x_n=1 \\ x_1+\cdots+x_n \equiv a \bmod p}}^{p-1} 1 &= \sum_{x_1=1}^{p-1}\sum_{x_2=1}^{p-1}\cdots\sum_{x_{n-1}=1}^{p-1} 1 - \sum_{x_1=1}^{p-1}\sum_{x_2=1}^{p-1}\cdots\sum_{\substack{x_{n-1}=1 \\ x_1+\cdots+x_{n-1} \equiv a \bmod p}}^{p-1} 1 \\ &= (p-1)^{n-1} - \sum_{x_1=1}^{p-1}\sum_{x_2=1}^{p-1}\cdots\sum_{\substack{x_{n-1}=1 \\ x_1+\cdots+x_{n-1} \equiv a \bmod p}}^{p-1} 1. \end{aligned}$$

Then by induction on n, we have

$$\begin{aligned} \sum_{x_1=1}^{p-1}\sum_{x_2=1}^{p-1}\cdots\sum_{\substack{x_n=1 \\ x_1+\cdots+x_n \equiv a \bmod p}}^{p-1} 1 &= \sum_{k=1}^{n-2}(-1)^{k+1}(p-1)^{n-k} + (-1)^{n-2}\sum_{x_1=1}^{p-1}\sum_{\substack{x_2=1 \\ x_1+x_2 \equiv a \bmod p}}^{p-1} 1 \\ &= \begin{cases} \big((p-1)^n - (-1)^n\big)/p, & \text{if } \ p \nmid a; \\ \big((p-1)^n + (p-1)(-1)^n\big)/p, & \text{if } \ p \mid a. \end{cases} \end{aligned}$$

This completes the proof of Lemma 6. $\square$

Lemma 7. *(See [1, Theorem 9.13].) For any odd prime p, we have*

$$G^2(1;p) = \left(\frac{-1}{p}\right) p \ .$$

Lemma 8. *(See [7, Lemma 6].) Let $m, n \geq 2$ and u be three integers with $(m,n)=1$ and $(u, mn) = 1$. Then for any character $\chi = \chi_1\chi_2$ with χ_1 mod m and χ_2 mod n, we have the identity*

$$G(u, \chi; mn) = \chi_1(n)\chi_2(m)G(un, \chi_1; m)G(um, \chi_2; n).$$

Lemma 9. *Let $p \geq 3$ be a prime, $\alpha \geq 2, m \geq 2$ be two integers . Then for any integer n with $(n,p)=1$, we have the identity*

$$\sum_{\chi \mod p^\alpha} |G(n, \chi; p^\alpha)|^{2m} = 4^{(m-1)} \cdot \phi^2(p^\alpha) \cdot p^{(m-1)\alpha}.$$

Proof. By the definition of $G(n, \chi; p^\alpha)$, we have

$$\begin{aligned} |G(n, \chi; p^\alpha)|^2 &= \sideset{}{'}\sum_{a=1}^{p^\alpha} \sideset{}{'}\sum_{b=1}^{p^\alpha} \chi(a)\overline{\chi(b)} e\left(\frac{n(a^2-b^2)}{p^\alpha}\right) \\ &= \sideset{}{'}\sum_{a=1}^{p^\alpha} \chi(a) \sideset{}{'}\sum_{b=1}^{p^\alpha} e\left(\frac{nb^2(a^2-1)}{p^\alpha}\right) \ . \end{aligned}$$

Hence, by this formula we have

$$\begin{aligned} &\sum_{\chi \mod p^\alpha} |G(n, \chi; p^\alpha)|^{2m} \\ = &\sum_{\chi \mod p^\alpha} \sideset{}{'}\sum_{x_1=1}^{p^\alpha} \sideset{}{'}\sum_{x_2=1}^{p^\alpha} \cdots \sideset{}{'}\sum_{x_m=1}^{p^\alpha} \chi(x_1 \cdots x_m) \prod_{i=1}^{m} \left(\sideset{}{'}\sum_{y_i=1}^{p^\alpha} e\left(\frac{ny_i^2(x_i^2-1)}{p^\alpha}\right)\right) \\ = &\ \phi(p^\alpha) \sideset{}{'}\sum_{\substack{x_1=1 \\ x_1x_2\cdots x_m \equiv 1}}^{p^\alpha} \sideset{}{'}\sum_{x_2=1}^{p^\alpha} \cdots \sideset{}{'}\sum_{\substack{x_m=1 \\ \mod p^\alpha}}^{p^\alpha} \prod_{i=1}^{m} \left(\sideset{}{'}\sum_{y_i=1}^{p^\alpha} e\left(\frac{ny_i^2(x_i^2-1)}{p^\alpha}\right)\right). \end{aligned}$$

Then by Lemma 5 we have

$$\sum_{\chi \mod p^\alpha} |G(n, \chi; p^\alpha)|^{2m} = \phi(p^\alpha) \sum_{k=0}^{m} \binom{m}{k} A(m,k), \tag{2}$$

where

$$A(m,k)=\sideset{}{'}\sum_{\substack{x_1=1\\ p^{\alpha-1}\|x_1^2-1}}^{p^\alpha}\cdots\sideset{}{'}\sum_{\substack{x_k=1\\ p^{\alpha-1}\|x_k^2-1}}^{p^\alpha}\sideset{}{'}\sum_{\substack{x_{k+1}=1\\ p^\alpha|x_{k+1}^2-1}}^{p^\alpha}\cdots\sideset{}{'}\sum_{\substack{x_m=1\\ p^\alpha|x_m^2-1\\ x_1x_2\cdots x_m\equiv 1 \bmod p^\alpha}}^{p^\alpha}\prod_{i=1}^{m}\left(\sideset{}{'}\sum_{y_i=1}^{p^\alpha}e\left(\frac{ny_i^2(x_i^2-1)}{p^\alpha}\right)\right).$$

Now, in order to prove Lemma 9, we need to calculate $A(m,k)$.

$$\begin{aligned}
&A(m,k)\\
=\ &\sideset{}{'}\sum_{\substack{x_1=1\\ p^{\alpha-1}\|x_1^2-1}}^{p^\alpha}\cdots\sideset{}{'}\sum_{\substack{x_k=1\\ p^{\alpha-1}\|x_k^2-1}}^{p^\alpha}\sideset{}{'}\sum_{\substack{x_{k+1}=1\\ p^\alpha|x_{k+1}^2-1}}^{p^\alpha}\cdots\sideset{}{'}\sum_{\substack{x_m=1\\ p^\alpha|x_m^2-1\\ x_1x_2\cdots x_m\equiv 1 \bmod p^\alpha}}^{p^\alpha}\prod_{i=1}^{m}\left(\sideset{}{'}\sum_{y_i=1}^{p^\alpha}e\left(\frac{ny_i^2(x_i^2-1)}{p^\alpha}\right)\right)\\
=\ &2\phi(p^\alpha)\sideset{}{'}\sum_{\substack{x_1=1\\ p^{\alpha-1}\|x_1^2-1}}^{p^\alpha}\cdots\sideset{}{'}\sum_{\substack{x_k=1\\ p^{\alpha-1}\|x_k^2-1}}^{p^\alpha}\sideset{}{'}\sum_{\substack{x_{k+1}=1\\ p^\alpha|x_{k+1}^2-1}}^{p^\alpha}\cdots\sideset{}{'}\sum_{\substack{x_{m-1}=1\\ p^\alpha|x_{m-1}^2-1\\ x_1x_2\cdots x_{m-1}\equiv 1 \bmod p^\alpha}}^{p^\alpha}\prod_{i=1}^{m-1}\left(\sideset{}{'}\sum_{y_i=1}^{p^\alpha}e\left(\frac{ny_i^2(x_i^2-1)}{p^\alpha}\right)\right)\\
=\ &2\phi(p^\alpha)A(m-1,k).
\end{aligned}$$

Hence, by induction on m, we have

$$A(m,k)=2^{m-k}\phi^{m-k}(p^\alpha)A(k,k). \tag{3}$$

Next, we shall calculate $A(k,k)$. By the definition, we have

$$A(k,k)=\sideset{}{'}\sum_{\substack{x_1=1\\ p^{\alpha-1}\|x_1^2-1}}^{p^\alpha}\cdots\sideset{}{'}\sum_{\substack{x_k=1\\ p^{\alpha-1}\|x_k^2-1\\ x_1x_2\cdots x_k\equiv 1 \bmod p^\alpha}}^{p^\alpha}\prod_{i=1}^{k}\left(\sideset{}{'}\sum_{y_i=1}^{p^\alpha}e\left(\frac{ny_i^2(x_i^2-1)}{p^\alpha}\right)\right).$$

Write $x_i=r_ip^{\alpha-1}+\varepsilon_i(1\le r_i\le p-1,\varepsilon_i=\pm 1)$ for $i=1,2,\cdots,k$. Then by Lemma

5, we have

$$\begin{aligned}
&A(k,k)\\
=\ & p^{k(\alpha-1)} \sum_{\substack{r_1=1\\ \varepsilon_1 r_1+\varepsilon_2 r_2+\cdots+\varepsilon_k r_k\equiv 0 \mod p\\ \varepsilon_1\varepsilon_2\cdots\varepsilon_k=1}}^{p-1}\sum_{r_2=1}^{p-1}\cdots\sum_{r_k=1}^{p-1} \prod_{i=1}^{k}\left(\left(\frac{2n\varepsilon_i r_i}{p}\right)G(1;p)-1\right)\\
=\ & p^{k(\alpha-1)} \sum_{\substack{r_1=1\\ r_1+r_2+\cdots+r_k\equiv 0 \mod p\\ \varepsilon_1\varepsilon_2\cdots\varepsilon_k=1}}^{p-1}\sum_{r_2=1}^{p-1}\cdots\sum_{r_k=1}^{p-1} \prod_{i=1}^{k}\left(\left(\frac{2nr_i}{p}\right)G(1;p)-1\right)\\
=\ & 2^{k-1}p^{k(\alpha-1)} \sum_{\substack{r_1=1\\ r_1+r_2+\cdots+r_k\equiv 0 \mod p}}^{p-1}\sum_{r_2=1}^{p-1}\cdots\sum_{r_k=1}^{p-1} \prod_{i=1}^{k}\left(\left(\frac{2nr_i}{p}\right)G(1;p)-1\right)\\
=\ & 2^{k-1}p^{k(\alpha-1)}\cdot \sum_{\substack{r_1=1\\ r_1+r_2+\cdots+r_k\equiv 0 \mod p}}^{p-1}\sum_{r_2=1}^{p-1}\cdots\sum_{r_k=1}^{p-1}\Bigg((-1)^k\\
&+\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}\left(\frac{2n}{p}\right)^j G^j(1;p)\left(\frac{r_1r_2\cdots r_j}{p}\right)\Bigg).
\end{aligned}$$

By Lemma 4 and Lemma 6, the above equality becomes

$$\begin{aligned}
&A(k,k)\\
=\ & 2^{k-1}p^{k(\alpha-1)}(-1)^k\Bigg(\frac{1}{p}\left((p-1)^k+(p-1)(-1)^k\right)\\
&+\sum_{j=1}^{\lfloor k/2\rfloor}(-1)^{2j}\binom{k}{2j}\left(\frac{2n}{p}\right)^{2j}G^{2j}(1;p)(-1)^{k-2j}\left(\frac{-1}{p}\right)^j p^{j-1}(p-1)\Bigg).
\end{aligned}$$

By Lemma 7, we have

$$\begin{aligned}
&A(k,k)\\
=\ & 2^{k-1}p^{k(\alpha-1)-1}\Bigg((-1)^k(p-1)^k+(p-1)+\sum_{j=1}^{\lfloor k/2\rfloor}\binom{k}{2j}p^{2j}(p-1)\Bigg)\\
=\ & 2^{k-1}p^{k(\alpha-1)-1}\Big((-1)^k(p-1)^k+(p-1)\big((p+1)^k+(1-p)^k\big)/2\Big)\\
=\ & 2^{k-2}p^{k(\alpha-1)-1}\Big((p+1)(1-p)^k+(p-1)(p+1)^k\Big).
\end{aligned}$$

Hence, by (3) we have

$$A(m,k)=2^{m-2}p^{m(\alpha-1)-1}\Big((-1)^k(p+1)(p-1)^m+(p-1)^{m-k+1}(p+1)^k\Big).$$

Finally, by (2) we have

$$
\begin{aligned}
& \sum_{\chi \bmod p^\alpha} |G(n,\chi;p^\alpha)|^{2m} \\
= \ & \phi(p^\alpha)\sum_{k=0}^{m}\binom{m}{k}2^{m-2}p^{m(\alpha-1)-1}\Big((-1)^k(p+1)(p-1)^m+(p-1)^{m-k+1}(p+1)^k\Big) \\
= \ & \phi(p^\alpha)2^{m-2}p^{m(\alpha-1)-1}(p+1)(p-1)^m\sum_{k=0}^{m}\binom{m}{k}(-1)^k \\
& +\phi(p^\alpha)2^{m-2}p^{m(\alpha-1)-1}\sum_{k=0}^{m}\binom{m}{k}(p-1)^{m-k+1}(p+1)^k \\
= \ & 0+\phi(p^\alpha)2^{m-2}p^{m(\alpha-1)-1}(p-1)(2p)^m \\
= \ & 4^{m-1}\phi^2(p^\alpha)p^{\alpha(m-1)}.
\end{aligned}
$$

This completes the proof of Lemma 9. □

Proof of Theorem 1. Since q is an odd square-full number, let $q=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_{\omega(q)}^{\alpha_{\omega(q)}}$, we have $\alpha_i\geq 2, i=1,\cdots\omega(q)$. For any integer n with $(n,q)=1$, by Lemma 8 and Lemma 9, we obtain

$$
\begin{aligned}
& \sum_{\chi \bmod q}|G(n,\chi;q)|^{2m} \\
= \ & \prod_{\substack{i=1\\ p_i^{\alpha_i}\| q}}^{\omega(q)}\sum_{\chi \bmod p_i^{\alpha_i}}|G(nq/p_i^{\alpha_i},\chi;p_i^{\alpha_i})|^{2m} \\
= \ & \prod_{\substack{i=1\\ p_i^{\alpha_i}\| q}}^{\omega(q)}\left(4^{m-1}p_i^{\alpha_i(m-1)}\phi^2(p_i^{\alpha_i})\right) \\
= \ & 4^{(m-1)\omega(q)}\cdot q^{m-1}\cdot\phi^2(q)\ .
\end{aligned}
$$

This completes the proof of theorem 1. □

References

[1] Tom M. Apostol, Introduction to Analytic Number Theory, Spring-Verlag, New York, 1976.

[2] T. Cochrane and Z. Y. Zheng, Pure and mixed exponential sums, Acta Arith, 91 (1999), no.3, 249-278.

[3] Y. He and W. P. Zhang, On the $2k$-th power mean value of the generalized quadratic Gauss sums, Bull. Korean Math. Soc. 48 (2011), No.1, 9-15.

[4] L. K. Hua, Introduction to Number Theory, Science Press, Beijing, 1979.

[5] A. Weil, On some exponential sums, Proc. Nat. Acad. Sci. U.S.A. 34 (1948), 204-207.

[6] W. P. Zhang, Moments of generalized quadratic Gauss sums weighted by L-functions, J. Number Theory 92 (2002), no.2, 304-314.

[7] W. P. Zhang and H. Liu, On the general Gauss sums and their fourth power mean, Osaka J. Math. 42 (2005), no.1 189-199