Deformation Expression for Elements of Algebras (III) –Generic product formula for $*$-exponentials of quadratic forms–

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In a noncommutative algebra there is no canonical way to express elements in univalent way, which is often called "ordering problem". In this note we give product formula of the Weyl algebra in generic ordered expression. In particular, the generic product formula of $*$-exponential functions of quadratic forms will be given.

In differential geometry, it is widely accepted that geometrical notion should have coordinate free expression. Obviously, algebraic structure of $(\mathbb{C}[\boldsymbol{u}], *_\Lambda)$ depends only on the skew part of Λ. It seems reasonable to accept the independence of ordering principle as a basic principle that the physical implication should be independent of ordered expressions.

In the last section, we mention the independence of ordering principle (IOP), and how this principle breaks down in the system containing $*$-exponential functions of quadratic forms.

As a result, we obtain a kind of "double covering group" of $Sp(m;\mathbb{C})$ which is simply connected, but this contains the double covering group (meta-plectic group) of $Sp(m;\mathbb{R})$. Several extraordinary properties of $*$-exponential functions of quadratic forms will be given.

In these calculus, we found peculiar elements, called polar elements, each of which has infinitely many square roots.

1 General product formula and intertwiners

We start with a little general setting as follows: Let $\mathfrak{S}(n)$ and $\mathfrak{A}(n)$ be the spaces of complex symmetric matrices and skew-symmetric matrices respectively, and $\mathfrak{M}(n)=\mathfrak{S}(n) \oplus \mathfrak{A}(n)$. For an arbitrary fixed $n{\times}n$-complex matrix $\Lambda{\in}\mathfrak{M}(n)$, we define a product $*_{_\Lambda}$ on the space of polynomials $\mathbb{C}[\boldsymbol{u}]$ by the formula

$$f *_{_\Lambda} g = f e^{\frac{i\hbar}{2}(\sum \overleftarrow{\partial_{u_i}}\Lambda^{ij}\overrightarrow{\partial_{u_j}})} g = \sum_k \frac{(i\hbar)^k}{k!2^k}\Lambda^{i_1j_1}\cdots\Lambda^{i_kj_k}\partial_{u_{i_1}}\cdots\partial_{u_{i_k}} f\ \partial_{u_{j_1}}\cdots\partial_{u_{j_k}} g. \tag{1.1}$$

It is known and not hard to prove that $(\mathbb{C}[\boldsymbol{u}], *_{_\Lambda})$ is an associative algebra. Clearly, if Λ is symmetric, then the algebra obtained is commutative and is isomorphic to the standard polynomial algebra with $\hbar$.

For every Λ, ∂_{u_i} acts as a derivation of the algebra $(\mathbb{C}[\boldsymbol{u}], *_{_\Lambda})$. Noting this, we define for any other constant symmetric matrix K a new product $*_{_{\Lambda,K}}$ by the formula

$$\begin{aligned} f *_{_{\Lambda,K}} g =& f e^{\frac{i\hbar}{2}(\sum \overleftarrow{\partial_{u_i}} K^{ij} *_{_\Lambda} \overrightarrow{\partial_{u_j}})} g \\ =& \sum_k \frac{(i\hbar)^k}{k!2^k} K^{i_1 j_1} \cdots K^{i_k j_k} (\partial_{u_{i_1}} \cdots \partial_{u_{i_k}} f) *_{_\Lambda} (\partial_{u_{j_1}} \cdots \partial_{u_{j_k}} g). \end{aligned}$$

This is also an associative algebra $(\mathbb{C}[\boldsymbol{u}], *_{_{\Lambda,K}})$. Since Λ, K are constant matrices and the non-commutativity of matrix algebra is not used in the calculation of the product formula, the new product formula can be rewritten as

$$f *_{_{\Lambda,K}} g = \sum_k \frac{(i\hbar)^k}{k!2^k} (\Lambda{+}K)^{i_1 j_1} \cdots (\Lambda{+}K)^{i_k j_k} \partial_{u_{i_1}} \cdots \partial_{u_{i_k}} f \partial_{u_{j_1}} \cdots \partial_{u_{j_k}} g$$

by noting that the exchanging indexes of $\partial_{u_{i_1} \cdots u_{i_k}}$ is permitted. That is, $*_{_{\Lambda,K}} = *_{_{\Lambda+K}}$.

This formula may be written as

$$f e^{\frac{i\hbar}{2}(\sum \overleftarrow{\partial_{u_i}} (\Lambda+K)^{ij} \overrightarrow{\partial_{u_j}})} g = f e^{\frac{i\hbar}{2}(\sum \overleftarrow{\partial_{u_i}} K^{ij} e^{\frac{i\hbar}{2}(\sum \overleftarrow{\partial_{u_k}} \Lambda^{kl} \overrightarrow{\partial_{u_k}})} \overrightarrow{\partial_{u_j}})} g. \tag{1.2}$$

Using a symmetric matrix K, we compute $\frac{1}{k!}(\frac{i\hbar}{4} \sum K^{ij} \partial_{u_i} \partial_{u_j})^k (f *_{_K} g)$ by noting that this is written as follows:

$$\begin{aligned} \sum_{p+q+r=k} \frac{(i\hbar)^r}{r!2^r} K^{i_1 j_1} \cdots K^{i_r j_r} \partial_{u_{i_1}} \cdots \partial_{u_{i_r}} \frac{1}{p!} (\frac{i\hbar}{4} \sum K^{ij} \partial_{u_i} \partial_{u_j})^p f \\ \times \partial_{u_{j_1}} \cdots \partial_{u_{j_r}} \frac{1}{q!} (\frac{i\hbar}{4} \sum K^{ij} \partial_{u_i} \partial_{u_j})^q g. \end{aligned}$$

Using this formula, we have the following formula:

$$\begin{aligned} & e^{\frac{i\hbar}{4} \sum K^{ij} \partial_{u_i} \partial_{u_j}} \Big(\big(e^{-\frac{i\hbar}{4} \sum K^{ij} \partial_{u_i} \partial_{u_j}} f \big) *_{_\Lambda} \big(e^{-\frac{i\hbar}{4} \sum K^{ij} \partial_{u_i} \partial_{u_j}} g \big) \Big) \\ & = f e^{\frac{i\hbar}{2}(\sum \overleftarrow{\partial_{u_i}} *_\Lambda K^{ij} *_{_\Lambda} \overrightarrow{\partial_{u_j}})} g = f *_{_{\Lambda+K}} g. \end{aligned} \tag{1.3}$$

Set $\Lambda = K{+}J$ where K, J are the symmetric part and the skew-part of Λ respectively. Since the commutator $[u_i, u_j] = i\hbar J^{ij}$ is given by the skew-part of Λ, the algebraic structure of $(\mathbb{C}[\boldsymbol{u}], *_{_\Lambda})$ depends only on J, whose isomorphism class may be denoted by $(\mathbb{C}[\boldsymbol{u}], *_{_J})$ or simply by $(\mathbb{C}[\boldsymbol{u}], *)$ by noticing this class consists of a *single* algebra.

This is confirmed directly by the formula (1.3). Namely, we see the following:

Corollary 1.1 *Let* $I_0^K(f) = e^{\frac{i\hbar}{4} \sum K^{ij} \partial_{u_i} \partial_{u_j}}$, *and* $I_K^0(f) = e^{-\frac{i\hbar}{4} \sum K^{ij} \partial_{u_i} \partial_{u_j}}$. *Then* I_0^K *is an isomorphism of* $(\mathbb{C}[\boldsymbol{u}]; *_{_\Lambda})$ *onto* $(\mathbb{C}[\boldsymbol{u}]; *_{_{\Lambda+K}})$.

It is clear that the product $f *_{_\Lambda} g$ is defined if one of f, g is a polynomial and another is a smooth function.

Let $Hol(\mathbb{C}^n)$ be the space of all holomorphic functions on the complex n-plane $\mathbb{C}^n$ with the uniform convergence topology on each compact domain. The next one gives a useful remark:

Lemma 1.1 $Hol(\mathbb{C}^n)$ *with the topology above is a Fréchet space defined by a countable family of seminorms.*

Proposition 1.1 *For every* $p(\boldsymbol{u}) \in \mathbb{C}[\boldsymbol{u}]$, *the left-multiplication* $f \to p(\boldsymbol{u}) *_{_\Lambda} f$ *and the right-multiplication* $f \to f *_{_\Lambda} p(\boldsymbol{u})$ *are both continuous linear mapping of* $Hol(\mathbb{C}^n)$ *into itself.*

If two of f, g, h *are polynomials, then associativity* $(f *_{_\Lambda} g *_{_\Lambda}) h = f *_{_\Lambda} (g *_{_\Lambda} h)$ *holds.*

1.1 Expression parameters and intertwiners

In what follows we treat the case of $2m$ variables, and we use notations

$$\boldsymbol{u} = (u_1, u_2, \cdots, u_{2m}) = (\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}}), \quad \tilde{\boldsymbol{u}} = (\tilde{u}_1, \cdots, \tilde{u}_m),\ \tilde{\boldsymbol{v}} = (\tilde{v}_1, \cdots, \tilde{v}_m). \tag{1.4}$$

The skew part J is fixed to be the standard skew-symmetric matrix $J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$. The algebra is called the **Weyl algebra** and the isomorphism class is denoted by $W_\hbar(2m)$.

We use sometimes notations $(u_1, \cdots, u_m, v_1, \cdots, v_m)$ instead of $(\tilde{u}_1, \cdots, \tilde{u}_m, \tilde{v}_1, \cdots, \tilde{v}_m)$ when no confusion is suspected.

For the case of a universal enveloping algebra of a Lie algebra, Poincaré-Birkhoff-Witt theorem ensures that this is realized on the space of ordinary polynomials by giving a new associative product. However, there is no standard way of unique expressing elements of algebra.

Note that if the generator system is fixed, then Proposition 1.1 gives a representation of the algebra. The product formula (1.1) gives also the unique expression of elements of this algebra by the usual polynomials. For instance, computing $u^i{*}u^j{*}u^k$ by using (1.1) gives the expression of $u^i{*}u^j{*}u^k$ as a polynomial. Thus, the product formula (1.1) will be referred to K-*ordered expression* (or K-*ordering*), i.e. if generators are fixed, giving an ordering expression is nothing but giving a product formula on the space of polynomials which defines the Weyl algebra $W_\hbar$.

By this formulation of orderings, the intertwiner between K-ordered expression and K'-ordered expression is explicitly given as follows:

Proposition 1.2 *For every $K, K' \in \mathfrak{S}(n)$, the intertwiner is defined by*

$$I_{_K}^{^{K'}}(f) = \exp\Big(\frac{i\hbar}{4}\sum_{i,j}(K'^{ij}{-}K^{ij})\partial_{u_i}\partial_{u_j}\Big)f \ \ (= I_{_0}^{^{K'}}(I_{_0}^{^K})^{-1}(f)), \tag{1.5}$$

and by (1.3) *it gives an isomorphism $I_{_K}^{^{K'}} : (\mathbb{C}[\boldsymbol{u}]; *_{_{K+J}}) \to (\mathbb{C}[\boldsymbol{u}]; *_{_{K'+J}})$. Namely, the following identity holds for any $f, g \in \mathbb{C}[\boldsymbol{u}]$:*

$$I_{_K}^{^{K'}}(f *_{_\Lambda} g) = I_{_K}^{^{K'}}(f) *_{_{\Lambda'}} I_{_K}^{^{K'}}(g), \tag{1.6}$$

where $\Lambda = K{+}J$, $\Lambda' = K'{+}J$.

Intertwiners do not change the algebraic structure $*$, but do change the expression of elements by the ordinary commutative structure.

$\boxed{\text{If the skew part } J \text{ is fixed, we often use notation } *_{_K} \text{ instead of } *_{_\Lambda}}$

In what follows, we use the notation $*_{_K}$ instead of $*_{_\Lambda}$, since the skew-part J is fixed as the standard skew-matrix. We use notations

$$\boldsymbol{u} = (u_1, u_2, \cdots, u_{2m}) = (\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}}), \quad \tilde{\boldsymbol{u}} = (\tilde{u}_1, \cdots, \tilde{u}_m),\ \tilde{\boldsymbol{v}} = (\tilde{v}_1, \cdots, \tilde{v}_m).$$

As in the case of one variable, infinitesimal intertwiner

$$dI_{_K}(K') = \frac{d}{dt}\Big|_{t=0} I_{_K}^{^{K+tK'}} = \frac{i\hbar}{4}K'_{ij}\partial_{u_i}\partial_{u_j}$$

is viewed as a flat connection on the trivial bundle $\coprod_{K\in\mathfrak{S}(n)} Hol(\mathbb{C}^n)$. The equation of parallel translation along a curve $K(t)$ is given by

$$\frac{d}{dt}f_t = dI_{\dot K(t)}(\dot K(t))f_t, \quad \dot K(t) = \frac{d}{dt}K(t), \tag{1.7}$$

but this may not have a solution for some initial function.

Note that according to the choice of $K = 0, K_0, -K_0, I$, where

$$(0,\ K_0, -K_0, I) = \left(\begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&I\\I&0\end{bmatrix}, \begin{bmatrix}0&-I\\-I&0\end{bmatrix}, \begin{bmatrix}I&0\\0&I\end{bmatrix}\right),$$

Choice of K	(name of ordering)
$K=0$	Weyl ordered expression
$K_0 = \begin{bmatrix}0&I\\I&0\end{bmatrix}$	Normal ordered expression
$-K_0$	Anti-normal ordered expression
$\begin{bmatrix}I&0\\0&I\end{bmatrix}$	Unit ordered expression
General K	K-ordered expression

the product formulas (1.1) give the Weyl ordered expression and the normal ordered expression, the antinormal ordered expression respectively, but the unit ordered expression is not so familiar in physics.

For each ordered expression, the product formulas are given respectively by the following formula:

$$\begin{aligned} f(\boldsymbol{u}){*_0}g(\boldsymbol{u}) &= f \exp \frac{\hbar i}{2}\{\overleftarrow{\partial_v}{\wedge}\overrightarrow{\partial_u}\}g, &&\text{(Moyal product formula)}\\ f(\boldsymbol{u}){*_{K_0}}g(\boldsymbol{u}) &= f \exp \hbar i\{\overleftarrow{\partial_v}\ \overrightarrow{\partial_u}\}g, &&\text{(ΨDO product formula)}\\ f(\boldsymbol{u}){*_{-K_0}}g(\boldsymbol{u}) &= f \exp -\hbar i\{\overleftarrow{\partial_u}\ \overrightarrow{\partial_v}\}g, &&\text{($\overline{\Psi}$DO product formula)}\end{aligned} \tag{1.8}$$

where $\overleftarrow{\partial_v}{\wedge}\overrightarrow{\partial_u} = \sum_i(\overleftarrow{\partial_{\tilde v_i}}\overrightarrow{\partial_{\tilde u_i}} - \overleftarrow{\partial_{\tilde u_i}}\overrightarrow{\partial_{\tilde v_i}})$ and $\overleftarrow{\partial_v}\ \overrightarrow{\partial_u} = \sum_i \overleftarrow{\partial_{\tilde v_i}}\ \overrightarrow{\partial_{\tilde u_i}}$.

The product formula for the unit ordered expression is a bit complicated to write down, but it is easy to obtain. For instance

$$u^2_{*_I}{=}u^2{+}\frac{i\hbar}{2}, \quad u{*_I}e^{-\frac{1}{i\hbar}u^2}{=}0{=}e^{-\frac{1}{i\hbar}u^2}{*_I}u \quad e.t.c.$$

while the Weyl ordered expression gives

$$v{*_0}e^{-\frac{2}{i\hbar}uv}{=}0{=}e^{-\frac{2}{i\hbar}uv}{*_0}u.$$

The next one is trivial, but an important remark:

Proposition 1.3 *Every entire function $f(u,v) = \sum a_{kl}u^kv^l$ can be viewed as a K-ordered expression of an element of extended Weyl algebra.*

The relations between two different expressions are given by intertwiners, but computations in the algebra can be done by using only the associativity and the fundamental commutation relations. Note for instance that $u{*}v{-}v{*}u = -i\hbar$ give for every polynomial $p(v{*}v)$ of $v{*}u$ that

$$u{*}p(v{*}u) = p(u{*}v){*}u, \quad \text{(bumping identity)}.$$

Let $u{\circ}v=\frac{1}{2}(u{*}v{+}v{*}u)$; the symmetric product. The bumping identity gives

$$u{*}(u{*}v){*}v=u{*}(u{\circ}v{+}\frac{1}{2}i\hbar){*}v=(u{\circ}v{+}\frac{1}{2}i\hbar){*}(u{\circ}v{+}\frac{3}{2}i\hbar).$$

Throughout this series, we use notation $:\bullet:_{_K}$ to indicate the expression parameter for elements of $W_\hbar$. For instance, we write

$$:u_i{*}u_j:_{_K}{=}u^2{+}\frac{i\hbar}{2}(K{+}J)_{ij},\quad :u_j{*}u_j:_{_I}=u_j^2{+}\frac{i\hbar}{2}\quad \textit{etc.}$$

A remarkable feature of the first three formulas of (1.8) is seen as follows:

$$\begin{aligned}
&:\tilde u_j{*}\tilde v_j:_0=\tilde u_j\tilde v_j{-}\frac{1}{2}i\hbar,\quad :\tilde v_j{*}\tilde u_j:_0=\tilde u_j\tilde v_j{+}\frac{1}{2}i\hbar,\quad \text{(Weyl ordering)}\\
&:\sum a_{kl}\tilde u_*^k{*}\tilde v_*^l:_{K_0}=\sum a_{kl}\tilde u^k\tilde v^l,\qquad \text{(normal ordering)},\\
&:\sum a_{kl}\tilde v_*^k{*}\tilde u_*^l:_{-K_0}=\sum a_{kl}\tilde u^k\tilde v^l,\quad \text{(anti-normal ordering)},
\end{aligned}$$

but concrete product formulas will be used to extend the algebra transcendentally.

Weyl ordered expression. In general, define $w_*(u_kv_l)$ by $\frac{1}{(k+l)!}\sum x_1{*}x_2{*}\cdots{*}x_{k+l}$, where x_i is $\tilde u_k$ or $\tilde v_l$ and the summation runs through all possible rearrangement of $u^k{*}v^l$.

(1.9)
$$(\tilde u{+}\tilde v)_*^n=\sum_k {}_nC_kw_*(\tilde u^k\tilde v^{n-k}),\quad :w_*(\tilde u^k\tilde v^l):_0=\tilde u^k\tilde v^l.$$

Special expression parameter K_s. In [12], we introduced the special ordered expression K_s to control the distribution of singular points of $*$-exponential functions of quadratic forms.

By K_s-product formula, we see that $(\mathit{Hol}(\mathbb{C}^n),*_{_{K_s}})$ contains a subalgebra which is isomorphic to the Clifford algebra $\mathit{Cliff}(m)$.

Siegel class of expression parameters. In [12], we introduced the class $\mathfrak{S}_+(\mathbb{R}^n)$ of expression parameters and gave several remarks. This is

$$\mathfrak{S}_+(\mathbb{R}^n)=\{K;\mathrm{Re}\frac{1}{\hbar}\langle\xi(iK),\xi\rangle\geq c_K|\xi|^2,\quad \exists c_k>0,\ \forall\xi\in\mathbb{R}^n\}$$

which will be called the *imaginary positive definite class* or the *Siegel class*. $\mathfrak{S}_+(\mathbb{R}^n)$ is $GL(n,\mathbb{R})$-invariant. Expressions in this class is easy to treat up to $*$-exponential functions of linear forms and their integrals. Further remarks will be given in the last section.

1.1.1 Linear change of generators

Next, we consider the effect of linear changes of generators such as

$${u'}_i=\sum u_kS_i^k,\quad S\in GL(n,\mathbb{C}),\quad (\boldsymbol{u}'=\boldsymbol{u}S).$$

Since $\partial_{u_i}=\sum S_i^k\partial_{u'_k}$, the product formula is rewritten by using new generators as

(1.10)
$$f*_{_\Lambda}g=fe^{\frac{i\hbar}{2}(\sum\overleftarrow{\partial_{u'_i}}({}^tS\Lambda S)^{ij}\overrightarrow{\partial_{u'_j}})}g.$$

Hence the notation $*_{_\Lambda}$ is better to be replaced $*_{_{\Lambda'}}$ where $\Lambda'={}^t\!S\Lambda S$.

Therefore the algebraic structure of $(\mathbb{C}[\boldsymbol{u}], *_{_\Lambda})$ depends only on the conjugate class of the skew part J. If ${}^tSJS = J$, that is, S is a symplectic linear change of generators such as

$$u'_i = \sum u_k S_i^k, \quad S \in Sp(m, \mathbb{C}),$$

the mapping $\boldsymbol{u} \to \boldsymbol{u'}$ does not change the algebraic structure. Change of generators are viewed often as coordinate transformations, but note here that $I_{_K}^{{}^tSKS}$ is something like the "square root" of symplectic coordinate transformations.

Since $\det S = 1$ for $S \in Sp(m, \mathbb{C})$, we see $\det {}^tSKS = \det K$, hence the isomorphic change by the intertwiner $I_{_K}^{K'}$ can not be covered by a coordinate transformation if $\det K \neq \det K'$.

1.2 Star-exponential functions of linear functions

For $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}^{2m}$, we set $\langle \boldsymbol{a}\varGamma, \boldsymbol{b}\rangle = \sum_{ij=1}^{2m} \varGamma^{ij} a_i b_j$, $\langle \boldsymbol{a}, \boldsymbol{u}\rangle = \sum_{i=1}^{2m} a_i u_i$. These will be denoted also by $\boldsymbol{a}\varGamma\, {}^t\boldsymbol{b}$ and $\langle \boldsymbol{a}, \boldsymbol{u}\rangle = \boldsymbol{a}\, {}^t\boldsymbol{u}$. For $f(\boldsymbol{u}) \in Hol(\mathbb{C}^{2m})$, the direct calculation via the product formula (1.1) by using Taylor expansion gives the following:

$$\begin{aligned} &e^{s\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle} *_{_K} f(\boldsymbol{u}) = e^{s\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle} f(\boldsymbol{u}+\frac{s}{2}\boldsymbol{a}(K+J)), \\ &f(\boldsymbol{u}) *_{_K} e^{-s\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle} = f(\boldsymbol{u}+\frac{s}{2}\boldsymbol{a}(-K+J)) e^{-s\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle} \end{aligned} \tag{1.11}$$

as natural extension of the product formula. This gives also the associativity

$$\Big(e^{s\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle} *_{_K} f(\boldsymbol{u})\Big) *_{_K} e^{t\frac{1}{i\hbar}\langle \boldsymbol{b},\boldsymbol{u}\rangle} = e^{s\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle} *_{_K} \Big(f(\boldsymbol{u}) *_{_K} e^{t\frac{1}{i\hbar}\langle \boldsymbol{b},\boldsymbol{u}\rangle}\Big), \quad f(\boldsymbol{u}) \in Hol(\mathbb{C}^{2m}). \tag{1.12}$$

By a direct calculation of intertwiner, we see that

$$I_{_K}^{K'}(e^{\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}) = e^{\frac{1}{4i\hbar}\langle \boldsymbol{a}(K'-K),\boldsymbol{a}\rangle} e^{\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}. \tag{1.13}$$

Hence, $\{e^{\frac{1}{4i\hbar}\langle \boldsymbol{a}K,\boldsymbol{a}\rangle} e^{\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}; K \in \mathfrak{S}_{\mathbb{C}}(2m)\}$ is a parallel section of $\coprod_{K\in\mathfrak{S}_{\mathbb{C}}(2m)} Hol(\mathbb{C}^{2m})$.

We denote this element symbolically by $e_*^{\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}$. Namely we denote

$$:e_*^{\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}:_{_K} = e^{\frac{1}{4i\hbar}\langle \boldsymbol{a}K,\boldsymbol{a}\rangle} e^{\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle} = e^{\frac{1}{4i\hbar}\langle \boldsymbol{a}K,\boldsymbol{a}\rangle + \frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}. \tag{1.14}$$

By using the product formula in K-ordered expression, we have easily the exponential law

$$:e_*^{s\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}:_{_K} *_{_K} :e_*^{t\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}:_{_K} = :e_*^{(s+t)\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}:_{_K}, \ \forall K \in \mathfrak{S}(2m).$$

The exponential law may be written by omitting the suffix K as

$$e_*^{s\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle} * e_*^{t\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle} = e_*^{(s+t)\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}, \quad e^s e_*^{t\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle} = e_*^{s+t\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}$$

together with the exponential law with the ordinary exponential functions.

Furthermore note also that $:\langle \boldsymbol{a}, \boldsymbol{u}\rangle:_K = \langle \boldsymbol{a}, \boldsymbol{u}\rangle$ for every K, and $e_*^{\frac{s}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}$ is the solution of the evolution equation

$$\frac{d}{dt}:e_*^{\frac{t}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}:_K = \frac{1}{i\hbar}:\langle \boldsymbol{a}, \boldsymbol{u}\rangle:_K *_K :e_*^{\frac{t}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}:_K \text{ with initial data } :e_*^{\frac{0}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}:_K = 1.$$

$e_*^{s\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle} = \{e^{s^2\frac{1}{4i\hbar}\langle \boldsymbol{a}K\boldsymbol{a}\rangle}e^{s\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}; K \in \mathfrak{S}(2m)\}$ forms a one parameter group of parallel sections.

By applying (1.11) to $:e_*^{\pm s\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}:_K$, we have for every $f \in Hol(\mathbb{C}^n)$ the associativity

$$:(e_*^{s\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}*f_*(\boldsymbol{u}))*e_*^{-s\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}:_K = :f_*(\boldsymbol{u}+s\boldsymbol{a}J):_K = :e_*^{s\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}*(f_*(\boldsymbol{u})*e_*^{-s\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}):_K. \tag{1.15}$$

This gives also the real analyticity of $e_*^{s\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}*f_*(\boldsymbol{u})*e_*^{-s\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}$ in s.

It is remarkable that if $K = 0$, then $:e_*^{\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}:_K = e^{\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}$, that is, $*$-exponential functions of linear functions are ordinary exponential functions in Weyl ordered expression. On the other hand, if $K \in \mathfrak{S}_+(\mathbb{R}^n)$ then $:e_*^{\pm s\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}:_K$ has a very strong property that

$$:e_*^{\pm s\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}:_K = e^{\frac{s^2}{4i\hbar}\langle \boldsymbol{a}K,\boldsymbol{a}\rangle}e^{\pm\frac{s}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}$$

is rapidly decreasing in $s \in \mathbb{R}$.

1.2.1 Extension of products

For every positive real number p, we set

$$\mathcal{E}_p(\mathbb{C}^n) = \{f \in Hol(\mathbb{C}^n)\,;\, \|f\|_{p,s} = \sup |f|\, e^{-s|\xi|^p} < \infty,\ \forall s > 0\} \tag{1.16}$$

where $|\xi| = (\sum_i |u_i|^2)^{1/2}$. The family of seminorms $\{||\cdot||_{p,s}\}_{s>0}$ induces a topology on $\mathcal{E}_p(\mathbb{C}^n)$ and $(\mathcal{E}_p(\mathbb{C}^n), \cdot)$ is an associative commutative Fréchet algebra, where the dot $\cdot$ is the ordinary product for functions in $\mathcal{E}_p(\mathbb{C}^n)$. It is easily seen that for $0 < p < p'$, there is a continuous embedding

$$\mathcal{E}_p(\mathbb{C}^n) \subset \mathcal{E}_{p'}(\mathbb{C}^n) \tag{1.17}$$

as commutative Fréchet algebras (cf. [4]), and that $\mathcal{E}_p(\mathbb{C}^n)$ is $GL(n,\mathbb{C})$-invariant.

We denote

$$\mathcal{E}_{p+}(\mathbb{C}^n) = \bigcap_{p'>p} \mathcal{E}_{p'}(\mathbb{C}^n), \quad \text{(with the intersection topology)} \tag{1.18}$$

It is obvious that every polynomial is contained in $\mathcal{E}_p(\mathbb{C}^n)$, that is $p(\boldsymbol{u}) \in \mathcal{E}_{0+}(\mathbb{C}^n)$, and $\mathbb{C}[\boldsymbol{u}]$ is dense in $\mathcal{E}_p(\mathbb{C}^n)$ for any $p > 0$ in the Fréchet topology defined by the family of seminorms $\{||\ ||_{p,s}\}_{s>0}$.

We easily see that $e^{\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle} \in \mathcal{E}_{1+}(\mathbb{C}^n)$. Moreover, it is not difficult to show that an exponential function $e^{p(\boldsymbol{u})}$ of a polynomial of degree d is contained in $\mathcal{E}_{d+}(\mathbb{C}^n)$, but not in $\mathcal{E}_d(\mathbb{C}^n)$.

Theorems 1.1 and 1.2 stated below give basic tools to study $*$-functions (cf. [9] for the proof), although most of the concrete formulas can be obtained without these theorems.

Theorem 1.1 *For $0 < p \leq 2$, the product formula* (1.1) *extends to give the following:*
(1) *The space $(\mathcal{E}_p(\mathbb{C}^n), *_{_K})$ forms a complete noncommutative topological associative algebra over $\mathbb{C}$.*
(2) *The intertwiner $I_{_K}^{K'}$ extends to give an isomorphism of $(\mathcal{E}_p(\mathbb{C}^n), *_{_K})$ onto $(\mathcal{E}_p(\mathbb{C}^n), *_{_{K'}})$.*

Remark For the second statement, it is enough to prove that $I_{_K}^{K'}$ extends to give a linear isomorphism of $\mathcal{E}_p(\mathbb{C}^n)$ onto itself. The property (2) shows that if $p \leq 2$, $\coprod_{K\in\mathfrak{S}(n)} \mathcal{E}_p(\mathbb{C}^n)$ is a trivial subbundle, and this is in fact an algebra bundle

$$\coprod_{K\in\mathfrak{S}(n)} (\mathcal{E}_p(\mathbb{C}^n), *_{_K}). \quad (0 < p \leq 2)$$

The equation of parallel translation (1.7) has a unique solution for the initial function f is in $\mathcal{E}_p(\mathbb{C}^n)$, $0 < p \leq 2$.

It is easily seen that the following identities hold on $\mathcal{E}_p(\mathbb{C}^n)$, $p \leq 2$

$$I_{_{K'}}^{K} I_{_K}^{K'} = 1, \quad I_{_{K'}}^{K''} I_{_K}^{K'} = I_{_K}^{K''}. \tag{1.19}$$

Hence, for every $f \in \mathcal{E}_p(\mathbb{C}^n)$, the set $f_*(\boldsymbol{u}) = \{I_{0}^{K}(f); K \in \mathfrak{S}_{\mathbb{C}}(n)\}$ is a globally defined parallel section.

For $p > 2$, we note the following:

Theorem 1.2 *For $p > 2$, the product formula* (1.1) *gives continuous bi-linear mappings of*

$$\mathcal{E}_p(\mathbb{C}^n) \times \mathcal{E}_{p'}(\mathbb{C}^n) \to \mathcal{E}_p(\mathbb{C}^n), \quad \mathcal{E}_{p'}(\mathbb{C}^n) \times \mathcal{E}_p(\mathbb{C}^n) \to \mathcal{E}_p(\mathbb{C}^n), \tag{1.20}$$

for $\forall p'$ such that $\frac{1}{p} + \frac{1}{p'} \geq 1$.

For $f, g, h \in \mathcal{E}_p(\mathbb{C}^n)$ $(p > 2)$, the associativity $(f *_{_K} g) *_{_K} h = f *_{_K} (g *_{_K} h)$ holds if two of f, g, h are in $\mathcal{E}_{p'}(\mathbb{C}^n)$ such that $\frac{1}{p} + \frac{1}{p'} \geq 1$.

Note that the linear change of coordinate $\boldsymbol{u}' = \boldsymbol{u}S$ by $S \in GL(n, \mathbb{C})$ gives naturally the topological linear isomorphism $\Phi_S : \mathcal{E}_p(\mathbb{C}^n) \to \mathcal{E}_p(\mathbb{C}^n)$, $p \geq 0$, and this is an isomorphism as $\mathbb{C}[\boldsymbol{u}]$-bi-modules for every $p \geq 0$;

$$\Phi_S : (\mathcal{E}_p(\mathbb{C}^n); *_{_\Lambda}) \to (\mathcal{E}_p(\mathbb{C}^n; *_{_{{}^tS\Lambda S}}).$$

This is not an automorphism, but an *outer isomorphism.*

1.2.2 Remarks on elements obtained by integrals

Suppose $f(x)$ is a continuous mapping of a compact domain D into $\mathcal{E}_p(\mathbb{C}^n)$. As $\|f(x)\|_{p,s}$ is bounded on D, its integral over D is bounded. Hence we see $\int_D f(x)dx \in \mathcal{E}_p(\mathbb{C}^n)$. This will be used to compute Fourier series.

Lemma 1.2 *For every compact interval I, the integral $\int_I {:}e_*^{t\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}{:}_{_K} dt$ gives an element of $\mathcal{E}_{1+}(\mathbb{C}^n)$ for every $K \in \mathfrak{S}(n)$, and*

$$I_{_K}^{K'}\Big(\int_I {:}e_*^{t\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}{:}_{_K} dt\Big) = \int_I {:}e_*^{t\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}{:}_{_{K'}} dt$$

$\{\int_I {:}e_*^{t\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}{:}_{_K} dt; K \in \mathfrak{S}(n)\}$ is a parallel section which may be denoted by $\int_I e_*^{t\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle} dt$ without showing expression parameters.

Since $e^{\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle} \in \mathcal{E}_{1+}(\mathbb{C}^n)$, Theorem 1.2 shows that $e^{\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}*_K f$, $f*_K e^{\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}$ are defined for every $f\in\bigcup_{p>2}\mathcal{E}_p(\mathbb{C}^n)$, but in fact these are defined for every $f\in Hol(\mathbb{C}^n)$ by (1.11).

As we have seen above, $*$-product integrals of exponential functions of linear functions are remained in the class. $\mathcal{E}_{1+}(\mathbb{C}^{2m})$. Note that usual integral can be defined for elements of $\mathcal{E}_{1+}(\mathbb{C}^{2m})$. However, we have to be careful to use the integral on noncompact domain, for such integrals often give elements outside the domain where the integrand is considered. Here, we give a typical example.

Suppose $\mathrm{Re}(\frac{1}{i\hbar}\langle \boldsymbol{a}K,\boldsymbol{a}\rangle)<0$, that is K is in the Siegel class. Then, the integral $\int_{-\infty}^{\infty}{:}e_*^{t\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}{:}_K dt$ converges. The formula of Fourier transform gives

$$\int_{-\infty}^{\infty}{:}e_*^{t\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}{:}_K dt=\int_{\mathbb{R}} e^{\frac{t^2}{4i\hbar}\langle \boldsymbol{a}K,\boldsymbol{a}\rangle}e^{t\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}dt=2\Big(\frac{-i\hbar\pi}{\langle \boldsymbol{a}K,\boldsymbol{a}\rangle}\Big)^{1/2}\ e^{-\frac{1}{i\hbar}\frac{1}{\langle \boldsymbol{a}K,\boldsymbol{a}\rangle}\langle \boldsymbol{a},\boldsymbol{u}\rangle^2}.$$

Since $\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle*: Hol(\mathbb{C}^{2m})\to Hol(\mathbb{C}^{2m})$ is continuous, we see that

$$\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle*\int_{-\infty}^{\infty}e_*^{t\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}dt=\lim_{N,N'\to\infty}\int_{-N}^{N'}\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle*e_*^{t\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}dt=\int_{-\infty}^{\infty}\frac{d}{dt}e_*^{t\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}dt=0. \tag{1.21}$$

Since $\langle \boldsymbol{a},\boldsymbol{u}\rangle*_K f(\langle \boldsymbol{a},\boldsymbol{u}\rangle)=\langle \boldsymbol{a},\boldsymbol{u}\rangle f(\langle \boldsymbol{a},\boldsymbol{u}\rangle)+\frac{i\hbar}{2}\langle \boldsymbol{a}K,\boldsymbol{a}\rangle f'(\langle \boldsymbol{a},\boldsymbol{u}\rangle)$, the direct calculation also gives

$$\langle \boldsymbol{a},\boldsymbol{u}\rangle*_K e^{-\frac{1}{i\hbar}\frac{1}{\langle \boldsymbol{a}K,\boldsymbol{a}\rangle}\langle \boldsymbol{a},\boldsymbol{u}\rangle^2}=0.$$

Under the condition $\mathrm{Re}(\frac{1}{i\hbar}\langle \boldsymbol{a}K,\boldsymbol{a}\rangle)<0$, we denote as in [12]

$$\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}e_*^{t\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}dt=\delta_*(\langle \boldsymbol{a},\boldsymbol{u}\rangle)$$

Moreover, we see that integrals $\int_{-\infty}^{0}{:}e_*^{t\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}{:}_K dt$, $-\int_{0}^{\infty}{:}e_*^{t\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}{:}_K dt$ are both inverses of $\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle$. We denote these by

$$\Big(\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle\Big)_{*+}^{-1}=\int_{-\infty}^{0}{:}e_*^{t\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}{:}_K dt,\quad \Big(\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle\Big)_{*-}^{-1}=-\int_{0}^{\infty}{:}e_*^{t\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}{:}_K dt.$$

This apparently breaks associativity

$$\Big(\big(\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle\big)_{*+}^{-1}*_K\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle\Big)*_K\big(\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle\big)_{*-}^{-1}\neq\big(\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle\big)_{*+}^{-1}*_K\Big(\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle*_K\big(\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle\big)_{*-}^{-1}\Big).$$

1.2.3 Remarks on real analyticity and on associativity

A mapping $f:U\to F$ from an open subset U of $\mathbb{R}$ into a Fréchet space F is called to be **real analytic**, if for every $a\in U$ there is $\varepsilon(a)>0$ such that f is written in the form

$$f(a+s)=\sum_k\frac{1}{k!}a_ks^k,\quad a_k\in F,\quad |s|<\varepsilon(a).$$

a_k is given by $a_k=\partial_s^k f|_{s=0}$.

If F is a Banach space and $\sum_k\frac{1}{k!}\|a_k\||s|^k$ converges, then the power series $\sum_k a_ks^k$ is called to converge absolutely under the norm.

If a Fréchet space F is defined by a countable family of seminorms $\{\|f\|_m; m=1,2,3\cdots\}$, then replace this part by the absolute convergence of $\sum_k \frac{1}{k!}\|a_k\|_m|s|^k$ w.r.t. seminorms $\|\cdot\|_m$. A power series $\sum_k a_k s^k$ converges if this converges absolutely under every seminorms.
Radius of convergence Suppose a Fréchet space F is defined by a countable family of seminorms $\{\|f\|_m; m=1,2,3\cdots\}$.

Lemma 1.3 *For a power series $\sum_k a_k s^k$, $a_k \in F$, there exists uniquely a real number R $(0 \le R \le \infty)$ satisfying* (1) *and* (2) *below:*

(1) *If $|s| < R$, then the power series $\sum_k a_k s^k$ converges absolutely under every seminorm $\|\cdot\|_m$.*

(2) *If $|s| > R$, then $\sum_k a_k s^k$ does not converge for some seminorm.*

Proof Suppose $\sum_k a_k s_0^k$ converges at s_0. Then $a_k s_0^k$ is bounded under every seminorm $\|\cdot\|_m$. Set $\sup_k \|a_k s_0^k\|_m \le M_m$. Then for every s such that $|s| < |s_0|$

$$\sum_k \|a_k s^k\|_m \le \sum_k M_m |s/s_0|^k = M_m \frac{1}{1-|s/s_0|} < \infty.$$

It follows the convergence of $\sum_k a_k s^k$. $\square$

Lemma 1.4 *$\sum_{k\ge0} a_k s^k$ and $\sum_{k\ge1} k a_k s^{k-1}$ have same radius of convergence.*

Real analyticity is left invariant under every continuous linear transformation:

Lemma 1.5 *Let F, G be Fréchet spaces and $\varphi : F \to G$ be a continuous linear mapping. If $f : U \to F$ is real analytic, then $\varphi f : U \to G$ is also real analytic.*

Since $X \to p(\boldsymbol{u}){*}X{*}q(\boldsymbol{u})$ is a continuous linear mapping, Lemma 1.1 gives the following:

Lemma 1.6 *Let U be an connected open neighborhood of* 0 *of $\mathbb{R}^\ell$ Suppose $\psi : U \to Hol(\mathbb{C}^n)$ be a real analytic mapping. Then $x \to p(\boldsymbol{u}){*}\psi(x){*}q(\boldsymbol{u})$ is also a real analytic on U for every polynomial $p(\boldsymbol{u})$, $q(\boldsymbol{u})$.*

Remarks on the associativity
Products of exponential functions of quadratic forms may not be defined, and even if the product is defined the associativity may not hold, since these are elements of $\mathcal{E}_{2+}(\mathbb{C}^n)$. In general, we do not have the associativity even for a polynomial $p(\boldsymbol{u})$

$$(e^{H(\boldsymbol{u})}{*}p(\boldsymbol{u})){*}e^{K(\boldsymbol{u})}, \quad e^{H(\boldsymbol{u})}{*}(p(\boldsymbol{u}){*}e^{K(\boldsymbol{u})}),$$

since $p(u)$ has two different $*$-inverses in general.

However, if we can treat elements in $(\mathbb{C}[\boldsymbol{u}][[\hbar]], *_K)$, the space of formal power series of $\hbar$, then $*_\Lambda$-product is always defined by the product formula (1.1) and the associativity holds.

Elements of $\mathcal{E}_{2+}(\mathbb{C}^n)$ are often given as a real analytic function of $\hbar$ defined on certain interval containing $\hbar = 0$. The following is easy to see:

Theorem 1.3 *Suppose $f(\hbar,\boldsymbol{u})$, $g(\hbar,\boldsymbol{u})$ and $h(\hbar,\boldsymbol{u})$ are given as real analytic function of $\hbar$ in some interval $[0,H]$. If*

$$f(\hbar,\boldsymbol{u}){*_{_K}}g(\hbar,\boldsymbol{u}),\ (f(\hbar,\boldsymbol{u}){*_{_K}}g(\hbar,\boldsymbol{u})){*_{_K}}h(\hbar,\boldsymbol{u}),\ g(\hbar,\boldsymbol{u}){*_{_K}}h(\hbar,\boldsymbol{u}),\ f(\hbar,\boldsymbol{u}){*_{_K}}(g(\hbar,\boldsymbol{u}){*_{_K}}h(\hbar,\boldsymbol{u}))$$

are defined as real analytic functions on $[0,H]$, then the associativity

$$(f(\hbar,\boldsymbol{u}){*_{_K}}g(\hbar,\boldsymbol{u})){*_{_K}}h(\hbar,\boldsymbol{u}) = f(\hbar,\boldsymbol{u}){*_{_K}}(g(\hbar,\boldsymbol{u}){*_{_K}}h(\hbar,\boldsymbol{u}))$$

holds.

We refer this theorem to the **formal associativity theorem**.
Remark 1. In what follows, elements are often given in the form $f(\frac{1}{i\hbar}\varphi(t),\boldsymbol{u})$ by using a real analytic function $f(t,\boldsymbol{u})$, $t{\in}[0,T]$, where $\varphi(t)$ is a real analytic function such that $\varphi(0){=}0$. (Cf.(4.19), (2.13)

In such a case, replacing t by $s\hbar$ gives a real analytic function of $\hbar$, and such an element is embedded in $(\mathbb{C}[\boldsymbol{u}][[\hbar]],{*_{_K}})$. Thus, we can apply the above theorem. We call such elements **classical elements**. However, there are many elements in $\mathcal{E}_{2+}(\mathbb{C}^n)$ written in the form $f(\frac{1}{i\hbar}\varphi(t),\boldsymbol{u})$ such that $\varphi(0){\neq}0$.

2 Blurred covering group of $Sp(m,\mathbb{C})$

In this section we first treat the infinitesimal $*$-action of quadratic forms on the space of exponential functions of quadratic forms. We treat this in general expressions by using intertwiners. Since the space of quadratic forms is isomorphic to the Lie algebra of $Sp(m,\mathbb{C})$, i.e.

$$\{\langle\boldsymbol{u}A,\boldsymbol{u}\rangle; A\in\mathfrak{S}(2m)\}\cong\mathfrak{sp}(m,\mathbb{C})=\{\alpha;\alpha J{+}J\,{}^t\!\alpha=0\}$$

as Lie algebra, the natural $*$-action of quadratic forms can be viewed as the **infinitesimal action** of the Lie group $Sp(m,\mathbb{C})$.

In contrast with that infinitesimal intertwiners are viewed as a flat connection on the trivial bundle $\coprod_{K\in\mathfrak{S}(2m)}\mathbb{C}e^{\mathfrak{S}(2m)}$, whose fiber is the space of exponential functions of quadratic forms $\mathbb{C}e^{\mathfrak{S}(2m)}$, all possible infinitesimal actions of quadratic forms gives a tangential distribution on each fiber $\coprod_{K\in\mathfrak{S}(2m)}\mathbb{C}e^{\mathfrak{S}(2m)}$.

2.1 Infinitesimal actions of quadratic forms

On every fiber at K, consider left multiplication

$$:\!\langle\boldsymbol{u}A,\boldsymbol{u}\rangle\!:_{_K}{*_{_K}}:\mathbb{C}e^{\mathfrak{S}(2m)}\to\mathbb{C}e^{\mathfrak{S}(2m)}$$

Since $\frac{1}{i\hbar}{:}\langle\boldsymbol{u}A,\boldsymbol{u}\rangle{:}_{_K}=\frac{1}{i\hbar}\langle\boldsymbol{u}A,\boldsymbol{u}\rangle+\frac{1}{2}\mathrm{Tr}(AK)$, we see

$$(2.1)\qquad \frac{1}{i\hbar}{:}\langle\boldsymbol{u}A,\boldsymbol{u}\rangle{:}_{_K}{*_{_K}}(ge^{\frac{1}{i\hbar}\langle\boldsymbol{u}Q,\boldsymbol{u}\rangle})=\Big(\frac{1}{2}\mathrm{Tr}\big((K{-}J)A(K{+}J)Q+AK\big)+\frac{1}{i\hbar}\langle\boldsymbol{u}Q',\boldsymbol{u}\rangle\Big)ge^{\frac{1}{i\hbar}\langle\boldsymbol{u}Q),\boldsymbol{u}\rangle}$$

where $Q'=A{+}A(K{+}J)Q{+}Q(K{-}J)A{+}Q(K{-}J)A(K{+}J)Q$, and $A\in\mathfrak{S}(2m)$. The term $\frac{1}{i\hbar}\langle\boldsymbol{u}Q',\boldsymbol{u}\rangle$ will be called the *infinitesimal phase* part, and $\frac{1}{2}\mathrm{Tr}\big((K{-}J)A(K{+}J)Q+AK\big)$ will be called the *infinitesimal amplitude* part.

Moving $A \in \mathfrak{S}(2m)$ at every fixed $ge^{\frac{1}{i\hbar}\langle \boldsymbol{u}Q,\boldsymbol{u}\rangle}$, we have a linear subspace of the tangent space of $\mathbb{C}e^{\mathfrak{S}(2m)}$ at $ge^{\frac{1}{i\hbar}\langle \boldsymbol{u}Q,\boldsymbol{u}\rangle}$. We call this the singular distribution of infinitesimal actions of quadratic forms.

On the other hand, there is a natural correspondence between $\mathfrak{sp}(m;\mathbb{C})$ and $\mathfrak{S}(2m)$.

$$\mathfrak{sp}(m;\mathbb{C}) \cong \mathfrak{S}(2m) \quad \text{via} \quad \alpha \in \mathfrak{sp}(m;\mathbb{C}) \Leftrightarrow \alpha J \in \mathfrak{S}(2m), \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

We make the correspondence as follows:

$$A \Leftrightarrow \alpha{=}-AJ, \qquad Q \Leftrightarrow \xi{=}-QJ.$$

We set also $\kappa'{=}JK'$, $\kappa{=}JK$ in $\mathfrak{sp}(m;\mathbb{C})$. Through these, intertwiners $I_K^{K'}$ defined on $\coprod_{K\in\mathfrak{S}(2m)} \mathbb{C}e^{\mathfrak{S}(2m)}$ in § 3.1 of [13] is easily translated on $\coprod_{\kappa\in\mathfrak{sp}(m;\mathbb{C})} \mathbb{C}e^{\mathfrak{sp}(m;\mathbb{C})J}$ as

$$I_\kappa^{\kappa'}(ge^{\langle \boldsymbol{u}(\frac{1}{i\hbar}\alpha J),\boldsymbol{u}\rangle}) = \frac{g}{\sqrt{\det(I{-}\alpha(\kappa'{-}\kappa))}} e^{\langle \boldsymbol{u}(\frac{1}{i\hbar}\frac{1}{I-\alpha(\kappa'-\kappa)}\alpha J),\boldsymbol{u}\rangle}.$$

These intertwiners may be viewed as coordinate transformations: $I_\kappa^{\kappa'} : \mathbb{C}e^{\mathfrak{sp}(m;\mathbb{C})J} \to \mathbb{C}e^{\mathfrak{sp}(m;\mathbb{C})J}$. For the precise treatment of patching by intertwiners, we set

$$I_\kappa^{\kappa'} \frac{1}{\sqrt{\det(I{-}\alpha\kappa)}} e^{\frac{1}{i\hbar}\langle \boldsymbol{u}(\frac{1}{I-\alpha\kappa}\alpha J),\boldsymbol{u}\rangle} = \frac{1}{\sqrt{\det(I{-}\alpha\kappa')}} e^{\frac{1}{i\hbar}\langle \boldsymbol{u}(\frac{1}{I-\alpha\kappa'}\alpha J),\boldsymbol{u}\rangle}$$

$$\widetilde{\mathcal{D}}_\kappa = \{\frac{1}{\sqrt{\det(I{-}\alpha\kappa)}} e^{\frac{1}{i\hbar}\langle \boldsymbol{u}(\frac{1}{I-\alpha\kappa}\alpha J),\boldsymbol{u}\rangle}; \alpha \in \mathcal{D}_\kappa\}, \quad \mathcal{D}_\kappa{=}\{\alpha; \det(I{-}\alpha\kappa) \neq 0\}.$$

As κ moves in the whole space $sp(m,\mathbb{C})$, and for any α, we can find κ such that $\det(I{-}\alpha\kappa){\neq}0$, we easily see that

$$\bigcup_\kappa \mathcal{D}_\kappa = \mathfrak{sp}(m,\mathbb{C}), \quad \bigcap_\kappa \mathcal{D}_\kappa = \{0\}. \tag{2.2}$$

$\widetilde{\mathcal{D}}_\kappa$ is a double cover of $\mathcal{D}_\kappa$. (Recall we set $\sqrt{1} = \{\pm 1\}$ in the case $\kappa = 0$.) Let $\pi : \widetilde{\mathcal{D}}_\kappa \to \mathcal{D}_\kappa$ be the natural projection. As it was seen in in § 3.1 of [13] intertwiners $I_\kappa^{\kappa'}$ give isomorphisms

$$\begin{array}{ccccccc} \widetilde{\mathcal{D}}_\kappa & \supset & \pi^{-1}(\mathcal{D}_\kappa{\cap}\mathcal{D}_{\kappa'}) & \xrightarrow{I_\kappa^{\kappa'}} & \pi^{-1}(\mathcal{D}_{\kappa'}{\cap}\mathcal{D}_\kappa) & \subset & \widetilde{\mathcal{D}}_{\kappa'} \\ \downarrow \pi & & \downarrow \pi & & \downarrow \pi & & \downarrow \pi \\ \mathcal{D}_\kappa & \supset & \mathcal{D}_\kappa{\cap}\mathcal{D}_{\kappa'} & = & \mathcal{D}_{\kappa'}{\cap}\mathcal{D}_\kappa & \subset & \mathcal{D}_{\kappa'} \end{array}$$

However intertwiners $I_\kappa^{\kappa'}$ are 2-to-2 mappings. Thus, the union $\bigcup_\kappa \widetilde{\mathcal{D}}_\kappa$ is a manifold-like object glued by 2-to-2 coordinate transformations.

Set $\alpha'{=}-Q'J$, $\xi{=}-QJ$, $\alpha{=}-AJ$, $\kappa{=}JK$. These are $\in \mathfrak{sp}(m;\mathbb{C})$. We want to translate the equality (2.1) by these replacement. First, the infinitesimal phase part is rewritten as

$$\begin{aligned} \alpha' &= \alpha - \alpha(I{-}\kappa)\xi + \xi(I{+}\kappa)\alpha - \xi(I{+}\kappa)\alpha(I{-}\kappa)\xi \\ &= \big(I + \xi(I{+}\kappa)\big)\alpha\big(I - (I{-}\kappa)\xi\big), \end{aligned}$$

and it is easy to see $\big(I+\xi(I{+}\kappa)\big)\alpha\big(I-(I{-}\kappa)\xi\big)\in\mathfrak{sp}(m;\mathbb{C})$ by moving J in the l.h.s of $J\big(I+\xi(I{+}\kappa)\big)\alpha\big(I-(I{-}\kappa)\xi\big)$ to the r.h.s. Hence, the equality (2.1) is translated into

$$(2.3)\qquad (\diamondsuit)\quad \frac{1}{i\hbar}{:}\langle \boldsymbol{u}(\alpha J),\boldsymbol{u}\rangle{:}_{K}*_{K}(ge^{\frac{1}{i\hbar}\langle \boldsymbol{u}\,\xi J,\boldsymbol{u}\rangle})=\Big(\frac{1}{2}\mathrm{Tr}\big((\kappa{+}I)\alpha(\kappa{-}I)\xi{+}\alpha\kappa\big)+\langle \boldsymbol{u}(\alpha' J),\boldsymbol{u}\rangle\Big)ge^{\frac{1}{i\hbar}\langle \boldsymbol{u}\,\xi J),\boldsymbol{u}\rangle}$$
$$\alpha'{=}(I+\xi(I{+}\kappa))\alpha(I-(I{-}\kappa)\xi).$$

By moving $\alpha{\in}\mathfrak{sp}(m,\mathbb{C})$ we make a subspace $D_{(\kappa,ge^{\xi})}$ of the tangent space $T_{ge^{\xi}}\mathbb{C}e^{\mathfrak{S}(2m)}$ of $\mathbb{C}e^{\mathfrak{S}(2m)}$ at $ge^{\frac{1}{i\hbar}\langle \boldsymbol{u}\,\xi J,\boldsymbol{u}\rangle}$. We make also a distribution (a singular subbundle):

$$(2.4)\qquad D_{(\kappa,ge^{\xi})}=\Big\{\Big(\frac{1}{2}\mathrm{Tr}\big((\kappa{+}I)\alpha(\kappa{-}I)\xi{+}\alpha\kappa\big)+\frac{1}{i\hbar}\langle \boldsymbol{u}\,\alpha' J,\boldsymbol{u}\rangle,\ \alpha\Big);\alpha\in sp(m,\mathbb{C})\Big\}$$
$$\text{where}\quad \alpha'=\big(I{+}\xi(I{+}\kappa)\big)\alpha\big(I{-}(I{-}\kappa)\xi\big).$$

on the space $\mathbb{C}e^{\mathfrak{S}(2m)}$. Note the following:

Lemma 2.1 $\det(I{+}\xi(I{+}\kappa)){=}\det(I{-}(I{-}\kappa)\xi){=}\det(I{-}\xi(I{-}\kappa))$.

Proof For the first equality, use $\det J{=}1$ and

$$\det(J(I{+}\xi(I{+}\kappa))){=}\det((I{-}{}^{t}\xi(I{-}{}^{t}\kappa))J)=\det(I{-}(I{-}\kappa)\xi).$$

For the second, we use the standard trick

$$\det(I{-}(I{-}\kappa)\xi)=\det(\xi^{-1}\xi{-}(I{-}\kappa)\xi)=\det(\xi\xi^{-1}{-}\xi(I{-}\kappa))$$

via an appropriate approximation of ξ by nonsingular element. $\square$

First of all, we consider open subsets where the rank of distribution is constant:

Lemma 2.2 *$\alpha\to\alpha'$ is a bijection of $\mathfrak{sp}(m;\mathbb{C})$ onto itself, if and only if $\det(I{+}\xi(I{+}\kappa))\neq 0$. In this case, Lemma 2.1 gives*

$$\alpha=(I{+}\xi(I{+}\kappa))^{-1}\alpha'(I{-}(I{-}\kappa)\xi)^{-1}.$$

That is in K-ordered expression, the infinitesimal action $\{\langle \boldsymbol{u}(\alpha J),\boldsymbol{u}\rangle *;\ \alpha{\in}sp(m;\mathbb{C})\}$ degenerates only at the point ξ such that $\det(I{+}\xi(I{+}\kappa))=0$.

Let $\mathcal{O}_{\kappa}=\{\xi\in\mathfrak{sp}(m;\mathbb{C});\det(I{+}\xi(I{+}\kappa))\neq 0\}$ for every $\kappa{\in}\mathfrak{sp}(m;\mathbb{C})$. Since this distribution is given by the infinitesimal action of a Lie group, we have

Proposition 2.1 *The distribution $D_{(\kappa,ge^{\xi})}$ is constant corank one and involutive on $\mathcal{O}_{\kappa}$.*

The goal of this section is as follows:

Theorem 2.1 *Maximal integral submanifold through $g\in\mathbb{C}_{\times}$ over $\mathcal{O}_{\kappa}$ is given by*

$$(2.5)\qquad \{g\sqrt{\det(I{+}(I{+}\kappa)\alpha)}e^{\langle \boldsymbol{u}(\frac{1}{i\hbar}\alpha J),\boldsymbol{u}\rangle};\alpha\in\mathcal{O}_{\kappa}\}$$

This is a nontrivial double cover of $\mathcal{O}_{\kappa}$.

The proof is given in several steps. Note first that the maximal integral submanifold through 1 must be closed under $*_\kappa$-product.

Step 1 First note that the phase part of the distribution takes arbitrary element. Thus consider elements $g(t)e^{\frac{1}{i\hbar}\langle \boldsymbol{u}(t\tilde{\alpha}J),\boldsymbol{u}\rangle}$ by fixing $\tilde{\alpha}$. We want to make the tangent vectors of this curve are always in the distribution. Taking derivative, we have

$$\Big(\frac{d}{dt}g(t){+}\frac{1}{i\hbar}\langle \boldsymbol{u}(\tilde{\alpha}J),\boldsymbol{u}\rangle g(t)\Big)e^{\frac{1}{i\hbar}\langle \boldsymbol{u}(t\tilde{\alpha}J),\boldsymbol{u}\rangle}.$$

Comparing this with (2.3) at $\xi=t\tilde{\alpha}$, we take $\alpha(t)$ so that

$$\tilde{\alpha}=(I{+}t\tilde{\alpha}(I{+}\kappa))\alpha(t)(I{-}t\tilde{\alpha}(I{-}\kappa)).$$

Then, the infinitesimal action by $\langle \boldsymbol{u}(\alpha(t)J),\boldsymbol{u}\rangle$ satisfies

$$\begin{aligned}
&{:}\frac{1}{i\hbar}\langle \boldsymbol{u}(\alpha(t)J),\boldsymbol{u}\rangle{:}_K{*_K}(g(t)e^{\frac{1}{i\hbar}\langle \boldsymbol{u}(t\tilde{\alpha}J,\boldsymbol{u}\rangle})\\
&=\left\{\frac{1}{2}\mathrm{Tr}\big((\kappa{+}I)\alpha(t)(\kappa{-}I)t\tilde{\alpha}{+}\alpha(t)\kappa\big){+}\frac{1}{i\hbar}\langle \boldsymbol{u}(\tilde{\alpha}J),\boldsymbol{u}\rangle\right\}g(t)e^{\frac{1}{i\hbar}\langle \boldsymbol{u}(t\tilde{\alpha}J),\boldsymbol{u}\rangle}.\\
&=\Big(\frac{d}{dt}g(t){+}\frac{1}{i\hbar}\langle \boldsymbol{u}(\tilde{\alpha}J),\boldsymbol{u}\rangle g(t)\Big)e^{\frac{1}{i\hbar}\langle \boldsymbol{u}(t\tilde{\alpha}J),\boldsymbol{u}\rangle}
\end{aligned}$$

Plugging in $\alpha(t)=(I{+}t\tilde{\alpha}(I{+}\kappa))^{-1}\tilde{\alpha}(I{-}(I{-}\kappa)t\tilde{\alpha})^{-1}$ into the above, $g(t)$ is obtained by solving

$$\frac{d}{dt}g(t)=\frac{1}{2}\mathrm{Tr}\big((\kappa{+}I)\alpha(t)(\kappa{-}I)t\tilde{\alpha}{+}\alpha(t)\kappa\big)g(t),\quad g(0)=g.$$

Step 2 To solve this equation, we first solve it in the case κ=0. The equation becomes

$$\frac{d}{dt}\log g(t)=\frac{1}{2}\mathrm{Tr}\frac{t\tilde{\alpha}^2}{1-(t\tilde{\alpha})^2}=\frac{1}{4}\frac{d}{dt}\mathrm{Tr}\log(1{-}(t\tilde{\alpha})^2).$$

It follows that

$$g(t)=e^{\mathrm{Tr}\log(1-(t\tilde{\alpha})^2)^{\frac{1}{4}}}=\sqrt[4]{\det(1{-}(t\tilde{\alpha})^2)}.$$

On the other hand, since $\det(1{-}t\tilde{\alpha}){=}\det(1{+}t\tilde{\alpha})$, we have $g(t)=\sqrt{\det(1{+}t\tilde{\alpha})}$, that is,

Lemma 2.3 *$\sqrt{\det(1{+}t\tilde{\alpha})}\ e^{\frac{1}{i\hbar}\langle \boldsymbol{u}(t\tilde{\alpha}J),\boldsymbol{u}\rangle}$ is in an integral submanifold.*

Step 3 The integral manifold for the general κ is obtained by the intertwiner I_0^κ. We have

$$I_0^\kappa\Big(\sqrt{\det(1{+}\tilde{\alpha})}e^{\frac{1}{i\hbar}\langle \boldsymbol{u}(t\tilde{\alpha}J),\boldsymbol{u}\rangle}\Big)=\frac{\sqrt{\det(I{+}\tilde{\alpha})}}{\sqrt{\det(I{-}\tilde{\alpha}\kappa)}}e^{\frac{1}{i\hbar}\langle \boldsymbol{u}(\frac{1}{I-\tilde{\alpha}}\tilde{\alpha}J),\boldsymbol{u}\rangle}.$$

Replacing $\frac{1}{I-\tilde{\alpha}}\tilde{\alpha}=\alpha$ gives

$$\tilde{\alpha}=\alpha\frac{1}{1+\alpha\kappa}{=}\frac{1}{I{+}\alpha\kappa}\alpha.$$

Plugging this and using the algebraic calculation such that $\frac{\sqrt{x}}{\sqrt{x}}=1$, we have the following:

Proposition 2.2 *In κ-ordered expression, the maximal integral submanifold is given by*

$$c\widetilde{\mathcal{O}}_\kappa = \{c\sqrt{\det(I{+}\alpha(I{+}\kappa))}\ e^{\frac{1}{i\hbar}\langle \boldsymbol{u}(\alpha J),\boldsymbol{u}\rangle}; \alpha\in\mathcal{O}_\kappa\}$$

where $\mathcal{O}_\kappa=\{\alpha\in sp(m,\mathbb{C}); \det(I{+}\alpha(I{+}\kappa))\neq 0\}$.

Note that Proposition 2.2 shows that we have only to know the phase part to know the integral submanifold. By definition $\widetilde{\mathcal{O}}_\kappa$ is the maximal integral submanifold through $(1,0)\in\mathbb{C}{\times}sp(m;\mathbb{C})$. Setting $c=1$ in Proposition2.2, we see that

$$\pi_\kappa : \widetilde{\mathcal{O}}_\kappa\to\mathcal{O}_\kappa \tag{2.6}$$

is a nontrivial double cover, which is just the forgetful mapping of the amplitude part. The significance of the set $\mathcal{O}_\kappa$ will be explained in the next section by the Cayley transform.

2.1.1 Integral submanifolds and twisted Cayley transforms

The Cayley transform $C_0(X)=\frac{I-X}{I+X}$ has following properties: For $X\in\mathfrak{sp}(m;\mathbb{C})$ with $\det(I{+}X)\neq 0$, we see $C_0(X)\in Sp(m;\mathbb{C})$ and $\det(I+C_0(X))=(\det(I+X))^{-1}$.

$$X\in\mathfrak{sp}(m;\mathbb{C})\Leftrightarrow C_0(X)\in Sp(m;\mathbb{C}),\quad C_0^2(X)=X. \tag{2.7}$$

Let $\mathcal{O}_0=\{X\in\mathfrak{sp}(m;\mathbb{C}); \det(I{+}X)\neq 0\}$. $C_0:\mathcal{O}_0\to Sp(m;\mathbb{C})$ is viewed as a local coordinate system $Sp(m;\mathbb{C})$, which covers an open dense subset of $Sp(m;\mathbb{C})$.

Let $\mathcal{O}_\kappa=\{\alpha\in\mathfrak{sp}(m;\mathbb{C}); \det(I{+}(I{+}\kappa)\alpha)\neq 0\}$, and define

$$\begin{aligned} C_\kappa(\alpha) &= (I-(I{-}\kappa)\alpha)\frac{1}{I+(I{+}\kappa)\alpha}=\frac{1}{I{+}\alpha(I{+}\kappa)}(I{-}\alpha(I{-}\kappa)),\\ (C_\kappa)^{-1}(Y) &= \frac{1}{I{-}\kappa{+}Y(I{+}\kappa)}(I{-}Y)=(I{-}Y)\frac{1}{I{-}\kappa{+}(I{+}\kappa)Y}. \end{aligned} \tag{2.8}$$

$C_\kappa:\mathcal{O}_\kappa\to Sp(m;\mathbb{C})$ gives also a local coordinate system of $Sp(m;\mathbb{C})$. We call (2.8) the *twisted Cayley transform.* Since $C_\kappa:\mathcal{O}_\kappa\to C_\kappa(\mathcal{O}_\kappa)$ is a diffeomorphism, we often identify $\mathcal{O}_\kappa$ with $C_\kappa(\mathcal{O}_\kappa)$ through the twisted Cayley transform C_κ. The following Lemma is crucial to our purpose:

Lemma 2.4 $\bigcup\{C_\kappa(\mathcal{O}_\kappa);\kappa\in\mathfrak{sp}(m,\mathbb{C})\}=Sp(m,\mathbb{C})$. *On the other hand, if* $\alpha\notin\mathcal{O}_0$*, then* $\frac{1}{I-\alpha\kappa}\alpha\notin\mathcal{O}_\kappa$ *for every* $\kappa\in\mathcal{D}_\alpha{=}\{\kappa{\in}\mathfrak{sp}(m;\mathbb{C}); \det(I-\alpha\kappa)\neq 0\}$.

Proof Suppose there is a $Y{\in}Sp(m,\mathbb{C})$ such that $\det(I{-}\kappa{+}Y(I{+}\kappa)){=}0$ for every $\kappa\in\mathfrak{sp}(m,\mathbb{C})$. Then such a Y must satisfy $\det(\frac{1-\kappa}{I+\kappa}{+}Y){=}0$. Since $\frac{1-\kappa}{I+\kappa}$ moves in an open dense domain of $Sp(m,\mathbb{C})$, it follows $\det(X{+}Y){=}0$ for every $X\in Sp(m,\mathbb{C})$. Set $X{=}Y$ to get a contradiction. Thus we see that for every Y there is κ such that $\det(I{-}\kappa{+}Y(I{+}\kappa)){\neq}0$. Hence $C_\kappa^{-1}(Y)$ exists. The rest of Lemma follows easily. □

Define $T_{\kappa'-\kappa}(\alpha)=\frac{1}{I-\alpha(\kappa'-\kappa)}\alpha$. Then $T^{-1}_{\kappa'-\kappa}(\alpha)=\frac{1}{I+\alpha(\kappa'-\kappa)}\alpha$

$$T_{\kappa'-\kappa}(\alpha)\in\mathfrak{sp}(m;\mathbb{C})\Longleftrightarrow \alpha\in\mathfrak{sp}(m;\mathbb{C}).$$

It is easy to see

$$\frac{1}{I{-}\alpha(\kappa'{-}\kappa)}(I{+}\alpha(I{+}\kappa))=I{+}T_{\kappa'-\kappa}(\alpha)(I{+}\kappa').$$

Hence, we have the following:

The Cayley transform gives the phase part of intertwiners.
$T_{-\kappa}\sim C_0^{-1}C_\kappa,\quad T_{\kappa'-\kappa}\sim C_{\kappa'}^{-1}C_\kappa$ ($\sim$ means equality in algebraic calculations)

On $Sp(m,\mathbb{C})$, the coordinate transformations are given by the phase part of the intertwiners.

Hence, by setting $\mathcal{O}_{\kappa\kappa'}=\mathcal{O}_\kappa\cap\mathcal{O}_{\kappa'}$, intertwiners give 2-to-2 mappings

$$I_\kappa^{\kappa'}:\pi_\kappa^{-1}(\mathcal{O}_{\kappa\kappa'})\to\pi_{\kappa'}^{-1}(\mathcal{O}_{\kappa'\kappa}) \tag{2.9}$$

just as in Proposition 3.1 in § 3.1 of [13]. Since $0\in\bigcap_\kappa\mathcal{O}_\kappa$, and $C_\kappa(0)=I$, we can consider $\bigcup_\kappa\widetilde{\mathcal{O}}_\kappa$ as an object patched by the intertwiners $I_\kappa^{\kappa'}$ as a bundle-like object over $\bigcup_\kappa C_\kappa(\mathcal{O}_\kappa)=Sp(m;\mathbb{C})$.

Computing the derivative of the twisted Cayley transform, we have

$$(dC_\kappa)_\xi(\alpha)=-(I-\kappa)\alpha\frac{1}{I+(I+\kappa)\xi}=-(I-(I-\kappa)\xi)\frac{1}{I+(I+\kappa)\xi}(I+\kappa)\alpha\frac{1}{I+(I+\kappa)\xi}.$$

Using the bumping identity $\frac{1}{I+(I+\kappa)\xi}(I+\kappa)=(I+\kappa)\frac{1}{I+\xi(I+\kappa)}$, we easily see

$$(dC_\kappa)_\xi((I+\xi(I+\kappa))\alpha(1-(I-\kappa)\xi))=-2\alpha C_\kappa(\xi). \tag{2.10}$$

Thus, the phase part of the distribution D_κ is translated by C_κ into the right invariant tangential distribution on $Sp(m;\mathbb{C})$. Recall that $(I+\xi(I+\kappa))\alpha(1-(I-\kappa)\xi)$ appeared already in (2.3), ($\diamondsuit$) as the infinitesimal phase part.

Proposition 2.3 *The infinitesimal phase part of the infinitesimal action* $:\frac{1}{i\hbar}\langle\boldsymbol{u}\alpha J,\boldsymbol{u}\rangle:_\kappa*_\kappa$ *is translated by the twisted Cayley transform* C_κ *into the right invariant distribution by* $\alpha\in\mathfrak{sp}(m;\mathbb{C})$ *on* $Sp(m;\mathbb{C})$.

As the distribution in the previous section is defined by the infinitesimal action of $*$-exponential functions $e_*^{t\frac{1}{i\hbar}\langle\boldsymbol{u}A,\boldsymbol{u}\rangle}$, the maximal integral submanifold must be closed by the left multiplication $e_*^{t\frac{1}{i\hbar}\langle\boldsymbol{u}A,\boldsymbol{u}\rangle}*$.

Therefore, the joint object $\{\widetilde{\mathcal{O}}_\kappa;\kappa\in sp(m,\mathbb{C})\}$ must have certain "Lie group-like" properties with manifold-like properties patched by 2-to-2 coordinate transformations. A general product formula will be given in the next section.

On the other hand, recall the second statement of Lemma 2.4 gives

Proposition 2.4 *If* $\det(I+\alpha)=0$, *then the infinitesimal $*$-action of the quadratic forms to the parallel section*

$$\Big\{\frac{1}{\sqrt{\det(I-\alpha\kappa)}}e^{\frac{1}{i\hbar}\langle\boldsymbol{u}\frac{1}{I-\alpha\kappa}\alpha,\boldsymbol{u}\rangle};\kappa\in\mathfrak{sp}(m;\mathbb{C})\Big\}$$

degenerates at every κ. *Namely,* $(\frac{1}{\sqrt{\det(I-\alpha\kappa)}};\frac{1}{I-\alpha\kappa}\alpha)\notin\widetilde{\mathcal{O}}_\kappa$ *for every* $\kappa\in\mathcal{D}_\alpha$, *where* $\mathcal{D}_\alpha=\{\kappa\in\mathfrak{sp}(m;\mathbb{C});\det(I-\alpha\kappa)\neq 0\}$.

The most degenerate (minimal rank) orbit will be called the orbit of **vacuums**. An example is given by $\alpha=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ in the case $m=1$. These elements will be used to make matrix representations in the later chapter.

2.2 General product formula

It is rather hard to construct general manifold theory patched together by 2-to-2 coordinate transformations, for such objects do not have underlying topological spaces. In spite of this, there is no difficulty forming a local/classical differential geometry. Hence a certain general theory is easy to construct for a Lie group-like object by using infinitesimal algebraic notions other than point set pictures. It is natural to think this gives an intuitive concrete object of "gerbes".

Proposition 2.3 shows that if one concerns only the phase parts of the $*$-product, then one can compute these via the group structure of $Sp(m;\mathbb{C})$ through twisted Cayley transform.

By this observation, we first investigate the product $*_0$ defined on $\mathbb{C}_\times\times\mathcal{O}_0$ as follows:

$$(g;a){*_0}(g';b)=\Big(gg'\Big(\tfrac{1}{\sqrt{\det(1{+}ab)}}\Big);C_0^{-1}(C_0(a)C_0(b))\Big)$$
$$C_0^{-1}(C_0(a)C_0(b))=\tfrac{1}{1+a}(a{+}b)\tfrac{1}{1+ab}(1{+}a)$$

Note first the following general identity:

Lemma 2.5 $\frac{1}{1+a}(a{+}b)\frac{1}{1+ab}(1{+}a)\sim(1{+}b)\frac{1}{1+ab}(a{+}b)\frac{1}{1+b}$, *where the reason of the notation* $\sim$ *instead of* $=$ *is that algebraic calculation such as* $(1{+}a)\frac{1}{1+a}=1$ *is used in the proof. Hence, one may replace* $\frac{1}{1+a}(a+b)\frac{1}{1+ab}(1+a)$ *by* $(1{+}b)\frac{1}{1+ab}(a{+}b)\frac{1}{1+b}$.

Proof follows immediately by the identity $(a{+}b)\big(1{+}\frac{1}{1+ab}(a{+}b)\big)=\big(1{+}(a{+}b)\frac{1}{1+ab}\big)(a{+}b)$. □

As far as concerning the phase part $C_0(a)$, and forgetting about the singularity, this gives a group which is isomorphic to $Sp(m;\mathbb{C})$.

To consider the amplitude part, we define

$$(g;\alpha){*_\kappa}(g';\beta)\sim I_0^\kappa\big(I_\kappa^0(g;\alpha){*_0}I_\kappa^0(g';\beta)\big).$$

Since $I_\kappa^0(g;\alpha)=\big(g(\det(I{+}\alpha\kappa))^{-\frac{1}{2}};T_{-\kappa}(\alpha)\big)$, the definition of C_κ gives that

$$(g;\alpha)*_\kappa(g';\beta)=\Big(gg'\big(\frac{\det(P{+}Q(I{+}\kappa))}{\det(P(I{+}\alpha(I{+}\kappa))(I{+}\beta(I{+}\kappa)))}\big)^{\frac{1}{2}};C_\kappa^{-1}(C_\kappa(\alpha)C_\kappa(\beta))\Big)\tag{2.11}$$

where $P=I{+}\alpha(I{-}\kappa)\beta(I{+}\kappa),\ \ Q=\alpha{+}\beta{+}2\alpha\kappa\beta,$ and

$$C_\kappa^{-1}(C_\kappa(\alpha)C_\kappa(\beta))=(I{+}\beta(I{+}\kappa))\,\frac{1}{P}\,Q\,\frac{1}{I{+}(I{+}\kappa)\beta}.\tag{2.12}$$

We easily see that $\det(P{+}Q(I{+}\kappa))=\det(I{+}\alpha(I{+}\kappa))(I{+}\beta(I{+}\kappa))$. Hence, the first component of the r.h.s of (2.11) is $gg'(\frac{1}{\det P})^{\frac{1}{2}}$. Hence, we obtain

$$(g;\alpha)*_\kappa(g';\beta)=\Big(gg'(\frac{1}{\det P})^{\frac{1}{2}};(I{+}\beta(I{+}\kappa))\,\frac{1}{P}\,Q\,\frac{1}{I{+}(I{+}\kappa)\beta}\Big).\tag{2.13}$$

The product formula (2.13) works only for α,β such that $\det P\neq 0$, and $\det(I{+}(I{+}\kappa)\beta)\neq 0$. But, one can choose the expression parameter κ so that these conditions are satisfied.

The product formula is classical

Singularities move when κ moves. For every $\forall(g;a),(g';b)$, the product $I_0^\kappa(g;a)*_\kappa I_0^\kappa(g';b)$ is defined in a generic (open dense) expression κ. By this algebraic trick, the product is defined for every pair, which will be denoted by $(g;a)*(g';b)$.

It is remarkable that the product formula does not involve $\hbar$.

This follows from that we treat elements written in the form $e^{\frac{1}{i\hbar}Q(\boldsymbol{u})}$. Therefore, for elements written in the form $e^{Q(\boldsymbol{u})}$ the product must be written in the form $e^{(i\hbar)^2R(\boldsymbol{u})}$, and hence the product formula is real analytic in $\hbar\geq 0$. Hence one can apply the formal associativity Theorem 1.3.

Proposition 2.5 *Associativity holds with $\pm$ ambiguity.*

We call this object a *blurred Lie group*, and denote it by $(Sp^{(\frac{1}{2})}_{\mathbb{C}}(m);*)$. This is not an object in which $\pm a$ is treated simply as a single point, since they can be locally distinguished.

For instance, we first note the following:

$$(1;0) \text{ is the identity with respect to } *_\kappa\text{-product for } \forall\kappa$$

In particular, in the Weyl ordered expression, the integral manifold through $(1;0)$ is

$$\widetilde{\mathcal{O}}_0=\{\sqrt{(\det(1{+}a)};a);a\in\mathcal{O}_0\}.$$

Although the sign ambiguity remains, we obtain the following:

Proposition 2.6 **(a)** *If $A,B\in\widetilde{\mathcal{O}}_0$, and if $A{*_0}B$ is defined, then $A{*_0}B\in\widetilde{\mathcal{O}}_0$.*

(b) $(1;0)$ *is the identity.*

(c) *The inverse $(\sqrt{\det(1{+}a)};a)^{-1}$ is given by $(\sqrt{\det(1{-}a)};-a)$.*

$$(\sqrt{\det(1{+}a)};a)*_0(\sqrt{\det(1{-}a)};-a)=(\sqrt{1};0).$$

In general, $\sqrt{1}$ must be treated as ±1, but concerning the inverse, this should be 1 by **continuous tracing** from the identity $(1;0)$ to the point $(\sqrt{\det(1{+}a)};a)$.

Since $\widetilde{\mathcal{O}}_\kappa$ is a local Lie group with the identity and $\mathfrak{sp}(m;\mathbb{C})$ as its tangent space, we have the following:

Proposition 2.7 *$\mathfrak{sp}(m;\mathbb{C})$ is the Lie algebra of $Sp^{(\frac{1}{2})}_{\mathbb{C}}(m)$.*

By $A=\alpha J$, $\mathfrak{sp}(m;\mathbb{C})$ is naturally identified with the space of expression parameters $\mathfrak{S}(2m)$.

Since the element $(1;0)$ may be viewed as the identity of the blurred Lie group $Sp^{(\frac{1}{2})}_{\mathbb{C}}(m)$, the tangent space of $Sp^{(\frac{1}{2})}_{\mathbb{C}}(m)$ at $(1;0)$ is naturally identified with $\mathfrak{sp}(m,\mathbb{C})$.

In the next section, we define one parameter subgroups of $(Sp^{(\frac{1}{2})}_{\mathbb{C}}(m);*)$, and the $*$-exponential mapping

$$\exp_*:\mathfrak{sp}(m;\mathbb{C})\to Sp^{(\frac{1}{2})}_{\mathbb{C}}(m),$$

and we show that every one parameter subgroup has discrete branched singular points in generic ordered expression.

2.3 Abstract definition of blurred Lie groups

Here we give a *tentative* abstract definition of blurred Lie (covering) groups. As we do not have enough concrete examples, it seems to be too early to give the notion of isomorphisms or the general theory.

Let G be a locally simply arcwise connected topological group and let $\{\mathcal{O}_\alpha; \alpha \in I\}$ be an open covering of G.

(a) For every $\alpha \in I$, $\mathcal{O}_\alpha$ contains the identity e. $\mathcal{O}_\alpha$ is called an **abstract expression space**, and α is called an expression parameter.
(b) For every $\alpha \in I$, $\mathcal{O}_\alpha$ is open, dense and connected, but it may not be simply connected.
(c) For every $\alpha, \beta \in I$, there is a homeomorphism $\phi_\alpha^\beta : \mathcal{O}_\alpha \to \mathcal{O}_\beta$.
(d) For every $g, h \in G$, there is $\alpha \in I$ and continuous path $g(t), h(t) \in G$, $t \in [0,1]$, such that $g(0) = h(0) = e$, $g(1) = g, h(1) = h$ and $g(t), h(t), g(t)h(t)$ are in $\mathcal{O}_\alpha$ for every $t \in [0,1]$.

An open covering $\{\mathcal{O}_\alpha; \alpha \in I\}$ is called **natural covering** of G if it satisfies $(a)\sim(d)$. The condition (c) shows that there is an abstract topological space X homeomorphic to every $\mathcal{O}_\alpha$. We consider a connected covering space $\pi : \tilde{X} \to X$. This is same to say we consider a connected covering $\pi_\alpha : \widetilde{\mathcal{O}}_\alpha \to \mathcal{O}_\alpha$ for each α. It is easy see that $\pi_\alpha^{-1}(e)$ is a group given as a quotient group of the fundamental group of $\mathcal{O}_\alpha$. As G is locally simply connected, $\pi_\alpha^{-1}(e)$ forms a discrete group, and ϕ_α^β lifts to an isomorphism $\tilde{\phi}_\alpha^\beta : \pi_\alpha^{-1}(e) \to \pi_\beta^{-1}(e)$. We denote $\pi_\alpha^{-1}(e) = \mathit{\Gamma}_\alpha$, and the isomorphism class is denoted by $\mathit{\Gamma}$.

Choose $\tilde{e}_\alpha \in \pi_\alpha^{-1}(e)$ and call $\tilde{e}_\alpha$ a tentative identity. For any continuous path $g(t)$ in $\mathcal{O}_\alpha$ such that $g(0) = g(1) = e$, the continuous tracing among the set $\pi^{-1}(g(t))$ starting at $\tilde{e}_\alpha$ gives a group element $\gamma \in \mathit{\Gamma}_\alpha$.

By a standard argument, it is easy to make $\widetilde{\mathcal{O}}_\alpha$ a local group such that π_α is a homomorphism: We define first that $\tilde{e}_\alpha \tilde{e}_\alpha = \tilde{e}_\alpha$. For paths $g(t), h(t), g(t)h(t)$ such that they are in $\mathcal{O}_\alpha$ for every $t \in [0,1]$ and $g(0) = h(0) = e$, we define the product by a continuous tracing among the set-to-set mapping

$$\pi_\alpha^{-1}(g(t))\pi_\alpha^{-1}(h(t)) = \pi_\alpha^{-1}(g(t)h(t)).$$

We set $\mathcal{O}_{\alpha\beta} = \mathcal{O}_\alpha \cap \mathcal{O}_\beta, \quad \mathcal{O}_{\alpha\beta\gamma} = \mathcal{O}_\alpha \cap \mathcal{O}_\beta \cap \mathcal{O}_\gamma$ for simplicity.

As G is locally simply connected, the full inverse $\pi_\alpha^{-1} V$ of a simply connected neighborhood $V \subset \mathcal{O}_\alpha$ of the identity $e \in G$ is the disjoint union $\coprod_\lambda \tilde{V}_\lambda$, each member $\tilde{X}_\lambda$ of which is homeomorphic to V. Moreover $\pi_\alpha^{-1}\mathcal{O}_{\alpha\beta}$ is also a local group for every β.

Isomorphisms modulo $\mathit{\Gamma}$, Controlled discontinuity

For every α, β, we define the notion of "isomorphism" I_α^β of local groups, which corresponds to the notion of intertwiners in the previous section:

$$\begin{array}{ccccccc}
\widetilde{\mathcal{O}}_\alpha & \supset & \pi_\alpha^{-1}\mathcal{O}_{\alpha\beta} & \xrightarrow{I_\alpha^\beta} & \pi_\beta^{-1}\mathcal{O}_{\beta\alpha} & \subset & \widetilde{\mathcal{O}}_\alpha \\
\downarrow \pi_\alpha & & & & & & \downarrow \pi_\beta \\
\mathcal{O}_\alpha & \supset & \mathcal{O}_{\alpha\beta} & = & \mathcal{O}_{\beta\alpha} & \subset & \mathcal{O}_\beta
\end{array}$$

such that $I_\beta^\alpha = (I_\alpha^\beta)^{-1}$, but the cocycle condition $I_\alpha^\beta I_\beta^\gamma I_\gamma^\alpha = 1$ is not required for $\mathcal{O}_{\alpha\beta\gamma}$.

Since the correspondence I_α^β does not make sense as a point set mapping, we should be careful for the definition.

Note that I_α^β is a collection of 1-to-1 mapping $I_\alpha^\beta(g) : \pi_\alpha^{-1}(g) \to \pi_\beta^{-1}(g)$ for every $g \in \mathcal{O}_{\alpha\beta} = \mathcal{O}_{\beta\alpha}$, which may not be continuous in g.

For each g there is a neighborhood V_g of the identity e such that $V_g g \subset \mathcal{O}_{\alpha\beta}$ and the local trivialization $\pi_\alpha^{-1}(V_g g) = V_g g \times \pi_\alpha^{-1}(g)$. Thus $I_\alpha^\beta(g)$ extends to the correspondence

$$\tilde{I}_\alpha^\beta(h, g) : \pi_\alpha^{-1}(hg) \to \pi_\beta^{-1}(hg), \quad h \in V_g$$

which commutes with the local deck transformations.

Definition 2.1 *The collection* $I_\alpha^\beta = \{I_\alpha^\beta(g); g \in \mathcal{O}_{\alpha\beta}\}$ *is called an isomorphism modulo* Γ*, if the product* $I_\beta^\alpha(hg)\tilde{I}_\alpha^\beta(h,g)$ *is in the group* Γ *for every* $g \in \mathcal{O}_{\alpha\beta}$ *and* $h \in V_g$. *(It follows the continuity of* $I_\alpha^\beta(hg)$ *w.r.t.* h.*)*

The condition given by this definition means roughly that $I_\alpha^\beta(g)$ has discontinuity in g only in the group Γ.

$\widetilde{G} = \{\widetilde{\mathcal{O}}_\alpha, \pi_\alpha, I_\alpha^\beta; \alpha, \beta \in I\}$ is called a **blurred covering group** of G if each $\widetilde{\mathcal{O}}_\alpha$ is a covering local group of $\mathcal{O}_\alpha$, where $\{\mathcal{O}_\alpha; \alpha \in I\}$ is a natural open covering of a locally simply arcwise connected topological group G and I_α^β are isomorphisms modulo Γ.

Because of the failure of the cocycle condition, this object does neither form a covering group, nor a topological point set. However, this object looks like a covering group.

For g, let I_g be the set of expression parameters involving g; $I_g = \{\alpha \in I; \mathcal{O}_\alpha \ni g\}$. For every $\alpha \in I(g, h, gh) = I_g \cap I_h \cap I_{gh}$, we easily see that $\pi_\alpha^{-1}(g)\pi_\alpha^{-1}(h) = \pi_\alpha^{-1}(gh)$. In general, this is viewed as set-to-set correspondence, but if g or h is in a small neighborhood of the identity, we can make these correspondence a genuine point set mapping. Hence, we have the notion of indefinite small action or "infinitesimal left/right action" of small elements to the object. This corresponds to the infinitesimal action $w_*^2 *$ or $* w_*^2$ in the previous section.

Next, we choose an element $\tilde{e}_\alpha \in \pi_\alpha^{-1}(e)$, and call it a local identity. On the other hand, $\pi_\alpha^{-1}(e)$ is called *the set of local identities* of $\widetilde{G}$. The failure of the cocycle condition gives that $\mathfrak{M}_\alpha \tilde{e}_\alpha$ may not be a single point set, but forms a discrete abelian group. Hence an identity of our object is always a *local identity.*

Since G is a locally simply connected, there is an open simply connected neighborhood V_β of e contained in $\mathcal{O}_\beta$. Hence, there is the unique lift $\tilde{V}_\beta$ through $\tilde{e}_\beta$. Setting $\tilde{V}_{\beta\gamma} = \tilde{V}_\beta \cap \tilde{V}_\gamma$ e.t.c., we see easily $I_\beta^\gamma(\tilde{V}_{\beta\gamma}) = \tilde{V}_{\gamma\beta}$.

The $\{\tilde{g}_\alpha \in \tilde{\mathcal{O}}_\alpha; \alpha \in I\}$ may be viewed as an element of $\widetilde{G}$ if $I_\alpha^\beta \tilde{g}_\alpha = \tilde{g}_\beta$, but this is not a single point set by the same reason. In spite of this, one can distinguish individual points within a small local area.

3 Star-exponential functions of quadratic forms

For an element H_* of the algebra, we define the $*$-exponential function $e_*^{tH_*}$ as the real analytic solution of

$$\frac{d}{dt} f_*(t) = H_* * f_*(t), \quad f(0)=1, \tag{3.1}$$

provided the solution exists. More precisely, we define $e_*^{tH_*}$ as the family $\{f_t(K)\}$ of univalent solutions of the evolution equation

$$\frac{d}{dt}f_t(K) = {:}H_*{:}_K *_K f_t(K), \tag{3.2}$$

with the initial condition $f_0(K)=1$. We think of $f_t(K)$ as the K-ordered expression of $e_*^{tH_*}$, and denote it by ${:}e_*^{tH_*}{:}_K = f_t(K)$. Uniqueness is ensured if we consider only real analytic solutions. (3.1) is called the *left evolution equation*. The *right evolution equation* is defined similarly, but this is not used except when otherwise mentioned.

If H_* is a $*$-polynomial, (3.1) can be rewritten as a partial differential evolution equation. If the equation $\frac{d}{dt}{:}f_*(t){:}_K = {:}H_*{:}{*}f_*(t){:}_K$ has a unique solution for the initial element $f_*(0){=}g_*$, then the solution will be denoted by ${:}e_*^{tH_*}{*}g_*{:}_K$.

As it was seen in § 1.1, a star exponential function $e_*^{\frac{1}{i\hbar}\langle\boldsymbol{\xi},\boldsymbol{u}\rangle}$ of a linear form $\langle\boldsymbol{\xi},\boldsymbol{u}\rangle$, was welldefined as the family $\{e^{\frac{1}{4i\hbar}\langle\boldsymbol{\xi}K,\boldsymbol{\xi}\rangle}e^{\frac{1}{i\hbar}\langle\boldsymbol{\xi},\boldsymbol{u}\rangle}\}$ for all $K{\in}\mathfrak{S}(n)$. Provided ${:}e_*^{sH_*}{:}_K$ exists for every $s{\in}\mathbb{C}$, they form a complex one parameter subgroup, for the exponential law holds by the uniqueness of real analytic solutions.

Here we give several general remarks on $*$-exponential functions of quadratic forms.

(1) If H_* is a quadratic form, ${:}e_*^{sH_*}{:}_K$ is defined with double branched singularities on a discrete set (c.f. (3.10)). Thus, we have to prepare two sheets to consider ${:}e_*^{sH_*}{:}_K$ for $s\in\mathbb{C}$. But, the origin 0 of
another sheet does not correspond to 1, but -1.

(1.1) In general, there is no reflection symmetry in s for the domain of existence of the solution of (3.2). That is, the existence of ${:}e_*^{sH_*}{:}_K$ does not necessarily imply that ${:}e_*^{-sH_*}{:}_K$ exists: e.g.

$${:}e_*^{t\frac{1}{i\hbar}(u^2+v^2)}{:}_I = \frac{1}{\cos t - \sin t}e^{\frac{1}{i\hbar}\frac{\sin t}{\cos t-\sin t}(u^2+v^2)} \quad cf.(3.17).$$

(1.2) Moreover ${:}e_*^{sH_*}{:}_K$ is double-valued holomorphic function in K on an open connected dense domain, i.e. double-valued holomorphic parallel section.

(2) If H_*, G_* are quadratic forms, then the product ${:}e_*^{tH_*}{*}e_*^{G_*}{:}_K$ is defined as a double-valued holomorphic function of (t,K) defined on an open connected dense domain containing $(0,0)$.

For a given K, suppose that (3.2) has real analytic solutions in t on some domain $D(K)$ including 0 for the initial functions 1 and g. We denote the solution of (3.2) with initial function g by

$${:}e_*^{tH_*}{:}_K *_K g, \quad t{\in}D(K). \tag{3.3}$$

Proposition 3.1 *If H_* is a polynomial and ${:}e_*^{tH_*}{:}_K$ is defined on a domain $D(K)$, then ${:}e_*^{tH_*}{:}_K *_K p(\boldsymbol{u})$ is defined for every polynomial $p(\boldsymbol{u})$ on the same domain $D(K)$.*

If $p(\boldsymbol{u})=\sum A_\alpha(s)\boldsymbol{u}^\alpha$ is a polynomial whose coefficients depend smoothly on s, then the formula

$$\partial_s^\ell{:}e_*^{tH_*}{:}_K *_K p(\boldsymbol{u}) = {:}e_*^{tH_*}{:}_K *_K \partial_s^\ell p(\boldsymbol{u})$$

holds for every ℓ.

Proof Multiplying the defining equation (3.2) by $*p(\boldsymbol{u})$ and applying the associativity in Proposition 1.1, we have

$$\frac{d}{dt}f_t(K)*p(\boldsymbol{u}) = {:}H_*{:}_K *_K (f_t(K)*p(\boldsymbol{u})), \quad f_0(K)=1. \tag{3.4}$$

Since $f_t(K)*p(\boldsymbol{u})$ is a real analytic solution, this is written in our notation as $e_*^{tH_*}*p(\boldsymbol{u})$. Applying ∂_s^ℓ to (3.4) gives the second assertion by a similar argument. □

For a quadratic form $\langle \boldsymbol{u}A,\boldsymbol{u}\rangle_*$, the $*$-exponential function $e_*^{\frac{t}{i\hbar}\langle \boldsymbol{u}A,\boldsymbol{u}\rangle_*}$ is given in a concrete form. For every $\alpha\in\mathfrak{sp}(m,\mathbb{C})$, we consider first the one parameter subgroup $e^{-2t\alpha}$ of $Sp(m,\mathbb{C})$, and consider the inverse image of twisted Cayley transform $C_\kappa^{-1}(e^{-2t\alpha})$: We set

$$C_\kappa^{-1}(e^{-2t\alpha}) = \frac{1}{(I-\kappa)+e^{-2t\alpha}(I+\kappa)}(I-e^{-2t\alpha}) = \frac{1}{\cosh t\alpha-(\sinh t\alpha)\kappa}\sinh t\alpha. \tag{3.5}$$

where $\frac{1}{X}$ stands for X^{-1}.

The exponential function must lie on the integral manifold $\widetilde{\mathcal{O}}_\kappa$ through $(1;0)$, and the point of the integral manifold is determined by its phase part. Hence we have

$$\exp_{*_\kappa}\frac{1}{i\hbar}t\alpha = \Big(\big(\det(I+C_\kappa^{-1}(e^{-2t\alpha})(I+\kappa))\big)^{\frac{1}{2}}; C_\kappa^{-1}(e^{-2t\alpha})\Big). \tag{3.6}$$

In the original notation, we see $e_*^{s\frac{1}{i\hbar}\langle \boldsymbol{u}(\alpha J),\boldsymbol{u}\rangle_*}$ as follows by setting $\kappa{=}JK$:

$$ {:}e_*^{s\frac{1}{i\hbar}\langle \boldsymbol{u}(\alpha J),\boldsymbol{u}\rangle_*}{:}_\kappa = \big(\det(I+C_\kappa^{-1}(e^{-2s\alpha})(I+\kappa))\big)^{\frac{1}{2}} e^{\frac{1}{i\hbar}\langle \boldsymbol{u}(C_\kappa^{-1}(e^{-2s\alpha})J),\boldsymbol{u}\rangle}.$$

More precisely, for every $\alpha\in\mathfrak{sp}(m,\mathbb{C})$, the K-ordered expression of the $*$-exponential function is given as follows: (Cf.[9] [11] [10] for special cases.)

$$ {:}e_*^{\frac{t}{i\hbar}\langle \boldsymbol{u}(\alpha J),\boldsymbol{u}\rangle_*}{:}_K = \frac{2^m}{\sqrt{\det(I-\kappa+e^{-2t\alpha}(I+\kappa))}} e^{\frac{1}{i\hbar}\langle \boldsymbol{u}\frac{1}{I-\kappa+e^{-2t\alpha}(I+\kappa)}(I-e^{-2t\alpha})J,\boldsymbol{u}\rangle} \tag{3.7}$$

where $\kappa{=}JK$. It is not hard to see that this is the real analytic solution of (3.2). By this concrete form we see this is an element of $\mathcal{E}_{2+}(\mathbb{C}^{2m})$ whenever this is defined. But it is remarkable that ${:}e_*^{\frac{t}{i\hbar}\langle \boldsymbol{u}(\alpha J),\boldsymbol{u}\rangle_*}{:}_K$ remains in the space $\mathbb{C}e^{Q(u,v)}$ given in Theorem 3.3.

3.1 Adjoint action to V_{2m}.

$Sp_{\mathbb{C}}^{(\frac{1}{2})}(m)$ is not a genuine Lie group, as elements have double-valued nature in general, and it looks something like a double covering group of $Sp(m,\mathbb{C})$. But, because of this reason, $Sp_{\mathbb{C}}^{(\frac{1}{2})}(m)$ contains several genuine groups such as the metaplectic group which is not contained in $Sp(m,\mathbb{C})$. Moreover, $Sp_{\mathbb{C}}^{(\frac{1}{2})}(m)$ contains $Spin(m)$ under the special ordered expression K_s (cf. [13].) In the case $m=1$, we have seen in [12] some basic properties of Jacobi's θ-functions by means of $*$-exponential functions of quadratic forms.

To avoid the vague issue of sign ambiguity, we first consider adjoint representations of $Sp^{(\frac{1}{2})}_{\mathbb{C}}(m)$ on the linear space of generators, for the sign ambiguity disappears in adjoint representations, and it is independent of the expression parameter K.

For $\alpha\in\mathfrak{sp}(m;\mathbb{C})$, the quadratic form $\langle \boldsymbol{u}(\alpha J),\boldsymbol{u}\rangle$ acts on the space of linear functions:

$$[\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle, \langle \boldsymbol{a},\boldsymbol{u}\rangle]=-\langle \boldsymbol{a}\alpha,\boldsymbol{u}\rangle.$$

Hence, the Lie algebra $\mathfrak{sp}(m;\mathbb{C})$ is obtained by the adjoint representation of quadratic forms

$$\mathrm{ad}(\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle)=-\alpha\in\mathfrak{sp}(m;\mathbb{C}).$$

It follows that for every $*$-function such as $*$-polynomials or $f_*(\boldsymbol{u})=\int_{\mathbb{R}^n}\hat{f}(\xi)e_*^{\frac{1}{i\hbar}\langle\xi,\boldsymbol{u}\rangle}\,\bar{d}\xi$,

$$e^{t\mathrm{ad}(\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle)}f_*(\boldsymbol{u})=f_*(e^{-t\alpha}\boldsymbol{u}),$$

where $e^{-t\alpha}$ is a linear transformation $e^{-t\alpha}\in Sp(m;\mathbb{C})$.

A concrete form for the case $m=1$ is given by using the transposed matrices as follows,

$$\mathrm{ad}(\frac{i}{2\hbar}(au^2+bv^2+2cuv))\begin{bmatrix}u\\ v\end{bmatrix}=\begin{bmatrix}-c & -b\\ a & c\end{bmatrix}\begin{bmatrix}u\\ v\end{bmatrix}\tag{3.8}$$

Let $V_{2m}=\{\langle\boldsymbol{\xi},\boldsymbol{u}\rangle;\boldsymbol{\xi}\in\mathbb{C}^{2m}\}$. For every quadratic form $\frac{1}{2i\hbar}\langle\boldsymbol{u}A,\boldsymbol{u}\rangle_*$, $\mathrm{ad}(\frac{1}{2i\hbar}\langle\boldsymbol{u}A,\boldsymbol{u}\rangle_*)$ is welldefined as a linear mapping independent of expression parameters.

$$\mathrm{ad}(\frac{1}{2i\hbar}\langle\boldsymbol{u}A,\boldsymbol{u}\rangle_*):V_{2m}\to V_{2m},\quad \mathrm{ad}(\frac{1}{2i\hbar}\langle\boldsymbol{u}A,\boldsymbol{u}\rangle_*):Hol(\mathbb{C}^{2m})\to Hol(\mathbb{C}^{2m})$$

It is easy to see that $\mathrm{ad}(\langle\boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle)=-\alpha\in\mathfrak{sp}(m,\mathbb{C})$, hence it extends as a $*$-derivation

$$\mathrm{ad}(\frac{1}{2i\hbar}\langle\boldsymbol{u}A,\boldsymbol{u}\rangle_*):(\mathcal{E}_2(\mathbb{C}^{2m}),*)\to(\mathcal{E}_2(\mathbb{C}^{2m}),*).$$

Linear algebra on finite dimensional vector space gives linear isomorphisms

$$e^{\mathrm{ad}(\frac{1}{2i\hbar}\langle\boldsymbol{u}A,\boldsymbol{u}\rangle_*)}:V_{2m}\to V_{2m},\quad e^{\mathrm{ad}(\frac{1}{2i\hbar}\langle\boldsymbol{u}A,\boldsymbol{u}\rangle_*)}:Hol(\mathbb{C}^{2m})\to Hol(\mathbb{C}^{2m})$$

and a $*$-isomorphism

$$e^{\mathrm{ad}(\frac{1}{2i\hbar}\langle\boldsymbol{u}A,\boldsymbol{u}\rangle_*)}:(\mathcal{E}_2(\mathbb{C}^{2m}),*)\to(\mathcal{E}_2(\mathbb{C}^{2m}),*).$$

Set $A=\alpha J$. Since $:e_*^{\frac{t}{2i\hbar}\langle\boldsymbol{u}A,\boldsymbol{u}\rangle_*}:_K$ is defined as a multi-valued holomorphic mapping from an open connected dense domain D containing the origin into $\mathcal{E}_{2+}(\mathbb{C}^{2m})$. and the first associativity Theorem 1.3 applied to $t=\hbar s$ shows the following:

Lemma 3.1 *Both sides are well-defined and associativity*

$$:(e_*^{\frac{t}{2i\hbar}\langle\boldsymbol{u}A,\boldsymbol{u}\rangle_*}*\langle\boldsymbol{\xi},\boldsymbol{u}\rangle)*e_*^{-\frac{t}{2i\hbar}\langle\boldsymbol{u}A,\boldsymbol{u}\rangle_*}:_K=:e_*^{\frac{t}{2i\hbar}\langle\boldsymbol{u}A,\boldsymbol{u}\rangle_*}*(\langle\boldsymbol{\xi},\boldsymbol{u}\rangle*e_*^{-\frac{t}{2i\hbar}\langle\boldsymbol{u}A,\boldsymbol{u}\rangle_*}):_K$$

holds for every $t\in D$.

Differentiating the identity of Lemma 3.1 by using Theorem 1.3 several times, gives that

$$\frac{d}{dt}{:}e_*^{\frac{t}{2i\hbar}\langle \boldsymbol{u}A,\boldsymbol{u}\rangle_*}*\langle\boldsymbol{\xi},\boldsymbol{u}\rangle*e_*^{-\frac{t}{2i\hbar}\langle \boldsymbol{u}A,\boldsymbol{u}\rangle_*}{:}_K={:}\mathrm{ad}(\frac{1}{2i\hbar}\langle \boldsymbol{u}A,\boldsymbol{u}\rangle_*)(e_*^{\frac{t}{2i\hbar}\langle \boldsymbol{u}A,\boldsymbol{u}\rangle_*}*\langle\boldsymbol{\xi},\boldsymbol{u}\rangle*e_*^{-\frac{t}{2i\hbar}\langle \boldsymbol{u}A,\boldsymbol{u}\rangle_*}){:}_K.$$

Uniqueness of the real analytic solution gives that the matrix obtained is independent of expression parameters:

$${:}e_*^{\frac{t}{2i\hbar}\langle \boldsymbol{u}A,\boldsymbol{u}\rangle_*}*\langle\boldsymbol{\xi},\boldsymbol{u}\rangle*e_*^{-\frac{t}{2i\hbar}\langle \boldsymbol{u}A,\boldsymbol{u}\rangle_*}{:}_K=e^{t\mathrm{ad}(\frac{1}{2i\hbar}\langle \boldsymbol{u}A,\boldsymbol{u}\rangle_*)}\langle\boldsymbol{\xi},\boldsymbol{u}\rangle=\langle\boldsymbol{\xi}e^{-t\alpha},\boldsymbol{u}\rangle\ \ (=\langle\boldsymbol{\xi},e^{-t\alpha}\boldsymbol{u}\rangle),$$

where $A=\alpha J, \alpha\in sp(m,\mathbb{C})$.

Theorem 3.1 *If* ${:}e_*^{\frac{t}{2i\hbar}\langle \boldsymbol{u}A,\boldsymbol{u}\rangle_*}{:}_K$ *is defined, then*

$$e_*^{t\langle\boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle}*\langle\boldsymbol{a},\boldsymbol{u}\rangle*e_*^{-t\langle\boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle}=\langle\boldsymbol{a}e^{-t\alpha},\boldsymbol{u}\rangle.$$

The proof is based on the fact that $e_*^{\frac{t}{2i\hbar}\langle \boldsymbol{u}A,\boldsymbol{u}\rangle_*}*\langle\boldsymbol{a},\boldsymbol{u}\rangle*e_*^{-t\langle\boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle}$ is defined and real analytic on an open dense connected domain of t containing 0. Hence, one may replace $\langle\boldsymbol{a},\boldsymbol{u}\rangle$ by any polynomial.

Since $\{e^{\alpha},\alpha\in\mathfrak{sp}(m,\mathbb{C})\}$ generates $Sp(m,\mathbb{C})$, the following is easy to see:

Proposition 3.2 *As linear transformation of* V_{2m}*, we have* $\mathrm{Ad}(e_*^{t\langle\boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle})=e^{t\mathrm{ad}(\langle\boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle)}$. *Hence,* $\mathrm{Ad}(e_*^{t\langle\boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle})$ *has no singular point and generates the group* $Sp(m,\mathbb{C})$.

This identity holds in spite of the ambiguity of the amplitude of $e_*^{t\langle\boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle}$, because the ambiguity of amplitude disappears in the adjoint formula. Hence,

$\mathrm{Ad}(e_*^{t\langle\boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle})$ generates the group $Sp(m,\mathbb{C})$.
$\mathrm{Ad}:Sp_{\mathbb{C}}^{(\frac{1}{2})}(m)\to Sp(m,\mathbb{C})$ is a 2-to-1 "surjective homomorphism".

The blurred Lie group $Sp_{\mathbb{C}}^{(\frac{1}{2})}(m)$ generated by $e_*^{t\langle\boldsymbol{u}(\frac{1}{i\hbar}\alpha J),\boldsymbol{u}\rangle}$ looks like a double covering group of $Sp(m,\mathbb{C})$ which is known to be simply connected.

3.2 Several point set pictures for blurred subgroups

Recall the surjective "homomorphism"

$$\mathrm{Ad}:Sp_{\mathbb{C}}^{(\frac{1}{2})}(m)\to Sp(m,\mathbb{C}).$$

For every subgroup G of $Sp(m,\mathbb{C})$, the full inverse $\mathrm{Ad}^{-1}G$ may be viewed as a *blurred* covering of G. However, it is often possible that $\mathrm{Ad}^{-1}G$ is a genuine Lie group under a suitable expression parameter.

Suppose we have a subgroup G of $Sp(m,\mathbb{C})$. Take a simple open covering $\{V_\alpha\}_\alpha$ of $Sp(m,\mathbb{C})$ such that $\{V_\alpha\cap G\}_\alpha$ is also a simple open covering of G, and each V_α is contained in some $C_\kappa(\mathcal{O}_\kappa)$. (Cf. Lemma 2.4.) For every α,β,γ we denote

$$V_{\alpha\beta}\cap G=V_\alpha\cap V_\beta\cap G,\quad V_{\alpha\beta\gamma}\cap G=V_\alpha\cap V_\beta\cap V_\gamma\cap G,\quad e.t.c.$$

Although $Sp^{(\frac{1}{2})}_{\mathbb{C}}(m)$ is a blurred double covering, the simplicity of $V_\alpha \cap G$ ensures that $\mathrm{Ad}^{-1}(V_\alpha \cap G) = (V_\alpha \cap G, \mathbb{Z}_2)$, and the patching diffeomorphisms $\phi_{\alpha\beta} : V_{\alpha\beta} \cap G \to \mathbb{Z}_2$ satisfies the cocycle condition

$$\phi_{\alpha\beta}\phi_{\beta\gamma}\phi_{\gamma\alpha} = \pm 1$$

as 2-to-2 mappings. These 2-to-2 patching diffeomorphisms give on each (α, β) two choices of patching diffeomorphisms, say $\pm\phi_{\alpha\beta}$. In a certain case, we can select one of these sign to clear the cocycle condition to obtain a genuine subset.

Theorem 3.2 *For a connected subgroup G of $Sp(m, \mathbb{C})$, if we can select patching diffeomorphisms so that they satisfies the cocycle condition, then there is a group $\tilde{G}$ contained in $\mathrm{Ad}^{-1}(G)$ such that $\mathrm{Ad} : \tilde{G} \to G$ is a surjective homomorphism.*

Proof. Since patching diffeomorphisms are so adjusted that the cocycle condition is satisfied, we have a genuine point set. But it is easy to see that these satisfies the condition of covering group of G. Note that such a point set picture may not be unique. □

We have already in [13] an example that $Sp^{(\frac{1}{2})}_{\mathbb{C}}(m)$ contains $Spin(m)$ under a special ordered expression K_s. Here, we give a simplest example. Note that

$$\frac{i}{2\hbar}[\sum(u_i^2 + v_i^2), \begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{v}\end{pmatrix}] = \begin{bmatrix} 0 & -I_m \\ I_m & 0 \end{bmatrix}\begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{v}\end{pmatrix}.$$

We see that $Sp(m, \mathbb{C})$ contains $U(1)$ in the form

$$U(1) = \left\{ \begin{bmatrix} \cos\theta I_m & -\sin\theta I_m \\ \sin\theta I_m & \cos\theta I_m \end{bmatrix}; \theta \in \mathbb{R} \right\}. \tag{3.9}$$

Hence we see that $\{\mathrm{Ad}(e_*^{\frac{i\theta}{2\hbar}\sum_m (u_i^2+v_i^2)})\} = U(1)$ and the full inverse $\tilde{U}(1) = \mathrm{Ad}^{-1}(U(1))$ is a double covering group of $U(1) \subset Sp(m, \mathbb{C})$. In the next section, we see that there are open subsets $\mathfrak{K}_1$, $\mathfrak{K}_2$ of expression parameters such that

$$:\tilde{U}(1):_K = \begin{cases} U(1) \times \mathbb{Z}_2 & K \in \mathfrak{K}_1 \\ \text{the connected double cover of } U(1) & K \in \mathfrak{K}_2 \end{cases}$$

Then, noting that $Sp(m, \mathbb{R}) \supset U(1)$, the full inverse $\mathrm{Ad}^{-1}(Sp(m, \mathbb{R}))$ is a genuine connected double covering group of $Sp(m, \mathbb{R})$ under the K-ordered expression such that $K \in \mathfrak{K}_2$. This is called the **metaplectic group** and denoted by $Mp(m)$. The metaplectic group is the connected double covering group of $Sp(m, \mathbb{R})$, which appears naturally as patching diffeomorphisms of the symbols of the group of invertible Fourier integral operators. It is known that $Mp(m)$ has no complexification as Lie groups. Thus $Sp^{(\frac{1}{2})}_{\mathbb{C}}(m)$ is viewed as its complexification as blurred Lie groups.

For concrete computation, note that the adjoint mapping Ad gives

$$e_*^{\frac{r}{\hbar}u^2} \to \begin{bmatrix} 1, & 0 \\ -ri, & 1 \end{bmatrix}, \quad e_*^{\frac{s}{\hbar}iuv} \to \begin{bmatrix} e^{-s}, & 0 \\ 0, & e^{s} \end{bmatrix}, \quad e_*^{\frac{t}{\hbar}v^2} \to \begin{bmatrix} 1, & ti \\ 0, & 1 \end{bmatrix}$$

$$e_*^{\frac{\theta}{2\hbar}(u^2+v^2)} \to \begin{bmatrix} \cosh\theta, & i\sinh\theta \\ -i\sinh\theta, & \cosh\theta \end{bmatrix}, \quad e_*^{\frac{s}{2\hbar}(u^2-v^2)} \to \begin{bmatrix} \cos s, & i\sin s \\ i\sin s, & \cos s \end{bmatrix}$$

In particular, $Sp(1,\mathbb{C}) = SL(2,\mathbb{C})$ contains $SL(2,\mathbb{R})$ and

$$SU(2) = \Big\{ \begin{bmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{bmatrix}; |\alpha|^2+|\beta|^2 = 1 \Big\} \cong S^3,$$
$$SU(1,1) = \Big\{ \begin{bmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{bmatrix}; |\alpha|^2-|\beta|^2 = 1 \Big\}.$$

Through these subgroups we take the full inverse $\mathrm{Ad}^{-1}(G)$. Hence, for $SL(2,\mathbb{R})$ we see

$$\begin{aligned} &\{\mathrm{Ad}(e_*^{\frac{r}{2\hbar}i(u^2+v^2)} * e_*^{\frac{s}{\hbar}iuv} * e_*^{\frac{ti}{\hbar}u^2}); r,s,t \in \mathbb{R}\} \\ &= \Big\{ \begin{bmatrix} \cos r, & -\sin r \\ \sin r, & \cos r \end{bmatrix} \begin{bmatrix} e^{-s}, & 0 \\ 0, & e^{s} \end{bmatrix} \begin{bmatrix} 1, & 0 \\ t, & 1 \end{bmatrix}; r,s,t \in \mathbb{R} \Big\} \end{aligned}$$

Hence, $\mathrm{Ad}^{-1}(SL(2,\mathbb{R}))$ is the connected double covering of $SL(2,\mathbb{R})$ under the K-ordered expression such that $K{\in}\mathfrak{K}_2$.

Similarly, under the K-ordered expression such that $K{\in}\mathfrak{K}_2$, we see $\widetilde{SU}(1,1) = \mathrm{Ad}^{-1}(SU(1,1))$ is the connected double covering group of $SU(1,1)$.

Next, consider
$$\widetilde{SU}(2) = \mathrm{Ad}^{-1}(SU(2)).$$

Indeed, this is the simplest toy model of blurred covering group. More precisely, decompose $SU(2)$ as
$$\Big\{ \begin{bmatrix} \cos\theta, & -\sin\theta e^{i\psi} \\ \sin\theta e^{-i\psi}, & \cos\theta \end{bmatrix} \begin{bmatrix} e^{i\rho}, & 0 \\ 0, & e^{-i\rho} \end{bmatrix}; \theta,\psi,\rho \in \mathbb{R},\ |\theta| < \frac{\pi}{2} \Big\}$$
with singular points at $\cos\theta = 0$, where θ, ψ may be viewed as the latitude and longitude respectively. Under a suitable expression parameter, we have a double covering group of the group $\{e^{i\rho}\}$. Hence, we have a covering space by replacing ρ by $\rho/2$ for each decomposition.

By this observation we see also

Proposition 3.3 *There is no expression parameter K under which all one parameter subgroup are not 2π-periodic but 4π-periodic.*

3.3 Several remarks on $*$-exponential functions

By noting that $\det(e^{t\alpha}){=}1$ for every $\alpha{\in}\mathfrak{sp}(m,\mathbb{C})$, (3.7) is rewritten as

$$:e_*^{\frac{t}{i\hbar}\langle \boldsymbol{u}(\alpha J),\boldsymbol{u}\rangle_*}:_K = \frac{2^m}{\sqrt{\det(e^{t\alpha}(I{-}\kappa){+}e^{-t\alpha}(I{+}\kappa))}} e^{\frac{1}{i\hbar}\langle \boldsymbol{u}\frac{1}{e^{t\alpha}(I-\kappa)+e^{-t\alpha}(I+\kappa)}(e^{t\alpha}-e^{-t\alpha})J,\boldsymbol{u}\rangle} \tag{3.10}$$

In spite of the sign ambiguity of $\sqrt{\ }$, the exponential law

$$\begin{aligned} &:e_*^{(s+t)\frac{1}{i\hbar}\langle \boldsymbol{u}(\alpha J),\boldsymbol{u}\rangle_*}:_K = :e_*^{s\frac{1}{i\hbar}\langle \boldsymbol{u}(\alpha J),\boldsymbol{u}\rangle_*}:_K *_K :e_*^{t\frac{1}{i\hbar}\langle \boldsymbol{u}(\alpha J),\boldsymbol{u}\rangle_*}:_K \\ &:e_*^{s(a+\frac{1}{i\hbar}\langle \boldsymbol{u}(\alpha J),\boldsymbol{u}\rangle_*)}:_K = :e^{as}e_*^{s\frac{1}{i\hbar}\langle \boldsymbol{u}(\alpha J),\boldsymbol{u}\rangle_*}:_K \end{aligned} \tag{3.11}$$

holds under computations such as $\sqrt{a}\sqrt{b}{=}\sqrt{ab}$. This is because that the exponential law and associativity holds on the group $Sp(m,\mathbb{C})$. Note however that $\sqrt{1}{=}\pm 1$.

By this observation we have the following:

Proposition 3.4 *For every fixed α, κ, a suitable choice of angle θ gives various real one parameter subgroups* $:e_*^{se^{i\theta}\frac{1}{i\hbar}\langle\boldsymbol{u}(\alpha J),\boldsymbol{u}\rangle_*}:_K$, $s{\in}\mathbb{R}$. *Moreover, we can find many complex semi-groups on various sectors.*

By the concrete formula (3.10), we have also the following:

Proposition 3.5 *Replacing t by $t\hbar$,* $:e_*^{t\langle\boldsymbol{u}(\alpha J),\boldsymbol{u}\rangle_*}:_K$ *is real analytic in $\hbar$ in an open connected domain containing $\hbar = 0$.*

As (3.10) has double branched singular points, we have to use two sheets by setting *slits* in the complex plane to treat $:e_*^{tH_*}:_K$ univalent way. Although there is no general rule to set the slits, it is natural to set **the slits periodically, since the singular points are distributed periodically**. We adopt this rule throughout this series.

Note that $J \in \mathfrak{sp}(m,\mathbb{C})$ and also $J \in Sp(m,\mathbb{C}) = \{g \in GL(2m,\mathbb{C}); gJ\,{}^tg = J\}$. For every $g \in Sp(m,\mathbb{C})$, $\tilde{J} = gJg^{-1}$ is both an element of Lie algebra and a group element satisfies $\tilde{J}^2{=}-I$ and $e^{t\tilde{J}}{=}\cos tI{+}(\sin t)\tilde{J}$. Recall the formula (3.7), which is rewritten as

$$:e_*^{\frac{t}{i\hbar}\langle\boldsymbol{u}(\alpha J),\boldsymbol{u}\rangle_*}:_{(-J\kappa)}=\frac{2^m}{\sqrt{\det(I{-}\kappa{+}e^{-2t\alpha}(I{+}\kappa))}}e^{\frac{1}{i\hbar}\langle\boldsymbol{u}\frac{1}{I-\kappa+e^{-2t\alpha}(I+\kappa)}(I-e^{-2t\alpha})J,\boldsymbol{u}\rangle},\quad \kappa{=}JK.$$

Setting $\alpha = \tilde{J}$ and noting $\alpha J = gJg^{-1}J = -g\,{}^tg$, we see first

$$\begin{aligned}I{-}\kappa{+}e^{-2t\alpha}(I{+}\kappa) &= I{-}\kappa{+}(\cos 2tI{-}(\sin 2t)\tilde{J})(I{+}\kappa)\\ &= 2(\cos tI{-}(\sin t)\tilde{J})(\cos tI{-}(\sin t)\tilde{J}\kappa)\\ &= 2g(\cos tI{-}(\sin t)J)(\cos tI{-}(\sin t)J\tilde{\kappa})g^{-1},\quad (\tilde{\kappa} = g^{-1}\kappa g).\end{aligned}$$

We have also that

$$(I - e^{-2t\alpha})J{=}J{-}(\cos 2tI{-}(\sin 2t)\tilde{J})J{=}-2g\sin t(\cos tI - (\sin t)J)^t g.$$

Since $\det(\cos tI{-}(\sin t)J) = 1$, it follows

$$\det(I{-}\kappa{+}e^{-2t\alpha}(I{+}\kappa)) = 2^{2m}\det(\cos tI{-}(\sin t)J\tilde{\kappa}).$$

Recalling that $K{=}-J\kappa$, $\kappa{=}g\tilde{\kappa}g^{-1}$, and plugging these, we have

$$:e_*^{-\frac{t}{i\hbar}\langle\boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_{(-J\kappa)}{=}:e_*^{\frac{t}{i\hbar}u\langle\boldsymbol{u}(\alpha J),\boldsymbol{u}\rangle_*}:_{(-J\kappa)}{=}\frac{1}{\sqrt{\det(\cos tI{-}(\sin t)J\tilde{\kappa})}}e^{\frac{1}{i\hbar}\langle\boldsymbol{u}g\frac{-\sin t}{\cos tI-(\sin t)J\tilde{\kappa}},\boldsymbol{u}g\rangle}\tag{3.12}$$

where $\cos tI{-}(\sin t)J\tilde{\kappa}$ is a symmetric matrix.

Now, one may assume in generic ordered expressions, $-J\tilde{\kappa}$ has disjoint $2m$ simple eigenvalues. Considering the diagonalization of $J\tilde{\kappa}$ in (3.12), we easily see that

Lemma 3.2 *In a generic (open dense) ordered expression, the singular points distributed π-periodically along $2m$ lines parallel to the real axis, and the singular points are all simple double branched singular points. Moreover,* $:e_*^{-\frac{t}{i\hbar}\langle \boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_{(-J\kappa)}$ *is rapidly decreasing along lines parallel to the pure imaginary axis of the growth order $e^{-|t|^m}$, where $2m = n$.*

Generic assumption Throughout this series, we suppose above properties for generic ordered expressions except otherwise stated.

In addition to generic assumption, we may suppose the following:

Proposition 3.6 *In generic ordered expression K, one may assume that* $:e_*^{\frac{t}{i\hbar}\langle \boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_{(-J\kappa)}$ *has no singular point on the real line. Hence, the exponential law proved by the uniqueness in the left evolution equation gives that* $:e_*^{\frac{t}{i\hbar}\langle \boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_{(-J\kappa)}$*, $t \in \mathbb{R}$ forms a one parameter subgroup of period π, or 2π depending on the expression parameter K.*

One of the remarkable feature of this concrete formula (3.12) is that it shows several extraordinary properties of $*$-exponential functions. For instance, we will see in the next section the following (cf. (3.15), Lemma 3.2):

Proposition 3.7 *If $\alpha = gJg^{-1}$ for some $g \in Sp(m,\mathbb{C})$, then the $*$-exponential function of quadratic form $\frac{1}{i\hbar}\langle \boldsymbol{u}(\alpha J), \boldsymbol{u}\rangle_*$ in a* **generic** *(open dense) ordered expression κ is 2π-periodic along real line (in precise, π-periodic or alternating π-periodic), and rapidly decreasing in both sides along the imaginary axis $i\mathbb{R}$ in the growth order $e^{-|t|^m}$. Hence such a $*$-exponential function must have singular points by Liouville's theorem.*

By (3.10), we see also that $:e_*^{s\frac{1}{i\hbar}\langle \boldsymbol{u}(\alpha J),\boldsymbol{u}\rangle_*}:_\kappa$ has in general discrete branched singularities on the complex space $s{\in}\mathbb{C}$ with some periodicity depending on the parameter $\kappa{=}JK$. To obtain the value without sign ambiguity, we have to fix the path from 0. To stress this, we use sometimes the notation

$$:e_*^{[0\sim s]\frac{1}{i\hbar}\langle \boldsymbol{u}(\alpha J),\boldsymbol{u}\rangle_*}:_\kappa \tag{3.13}$$

where $[0 \sim s]$ indicates a path joining 0 to s avoiding singular points.

Replacing $-J\kappa$ by K in (3.12), we have $J\tilde{\kappa} = -{}^t gKg$, and replacing t by $-t$ we have the formula (1.10) again:

$$:e_*^{\frac{t}{i\hbar}\langle \boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K = \frac{1}{\sqrt{\det(\cos tI - (\sin t){}^t gKg)}} e^{\frac{1}{i\hbar}\langle \boldsymbol{u}g\frac{\sin t}{\cos tI-\sin t\,{}^t gKg},\boldsymbol{u}g\rangle} \tag{3.14}$$

By requiring 1 at $t = 0$ and by using $\det g = 1$, we have by setting $t = \pm\pi$, and $t = \pm\frac{\pi}{2}$, that

$$:e_*^{\pi\frac{\pm 1}{i\hbar}\langle \boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K = \sqrt{(-1)^{2m}} = \sqrt{1}, \quad :e_*^{\pi\frac{\pm 1}{2i\hbar}\langle \boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K = \frac{1}{\sqrt{\det K}} e^{-\frac{1}{i\hbar}\langle \boldsymbol{u}\frac{1}{K},\boldsymbol{u}\rangle}. \tag{3.15}$$

Note that the r.h.s. of the first equality looks independent of g and the expression parameters, and that the r.h.s. of the second equality looks independent of g. Since $Sp(m,\mathbb{C})$ is connected, it looks the sign of $\sqrt{1}$ and $\sqrt{\det K}$ can be fixed. However, the sign of $\sqrt{1}$ depends both on the expression

K and on the path from 0 to π by which we choose the sign of $:e_*^{\pi\frac{1}{i\hbar}\langle \boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*}:_\kappa$ under the condition $e_*^{0\langle \boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*}:_\kappa=1$, where the path should be so chosen that there is no singular point on the path.

In the case $K{=}0$ (the Weyl ordered expression), the r.h.s. of the second identity diverges and the first identity gives

$$:e_*^{[0\sim t]\frac{1}{i\hbar}\langle \boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_0=\sqrt{(\cos t)^{2m}}e^{\frac{1}{i\hbar}\langle \boldsymbol{u}g\frac{\sin t}{\cos t}I,\boldsymbol{u}g\rangle}=(\cos t)^m e^{\frac{1}{i\hbar}\langle \boldsymbol{u}g\frac{\sin t}{\cos t}I,\boldsymbol{u}g\rangle}$$

by requesting 1 at $t=0$. Hence $:e_*^{[0\sim\pi]\frac{1}{i\hbar}\langle \boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_0=(-1)^m$.

In general the $\pm$-sign depends on the path from 0 to π or $\pi/2$. It depends on which sheet the end point of the path is sitting. By this observation, we see that

$$:e_*^{[0\sim\pi]\frac{\pm 1}{i\hbar}\langle \boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K=(-1)^m,\ \ (\text{resp.}-(-1)^m) \tag{3.16}$$

if $:e_*^{\pi\frac{\pm 1}{i\hbar}\langle \boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K$ is sitting in the same (resp. opposite) sheet as in $:e_*^{0\frac{\pm 1}{i\hbar}\langle \boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K$.

On the other hand, for a fixed K, the r.h.s. of the second equality (3.15) is independent of g. Since $Sp(m,\mathbb{C})$ is connected, it looks that one can fix the sign of $\sqrt{\det K}$ in the r.h.s. of the second equality. Here, we meet the strange phenomenon that we have already met in [13]. We call $e_*^{\pi\frac{1}{2i\hbar}\langle \boldsymbol{u}g,\boldsymbol{u}g\rangle_*}$ the (total) **polar element** and denote this by ε_{00}. The polar element will be discussed in the next section more closely.

3.3.1 The case $m=1$

In this section, we treat the case of two variables u,v (i.e. the case $m{=}1$). Note first that $\{\langle \boldsymbol{u}g,\boldsymbol{u}g\rangle_*; g{\in}SL(2,\mathbb{C})\}$ is spanned by quadratic forms given by

$$\begin{bmatrix}\cosh r & \sinh r\\ \sinh r & \cosh r\end{bmatrix},\quad \begin{bmatrix}\cos r & i\sin r\\ i\sin r & \cos r\end{bmatrix},\quad \begin{bmatrix}e^{is} & 0\\ 0 & e^{-is}\end{bmatrix},\quad \begin{bmatrix}e^{s} & 0\\ 0 & e^{-s}\end{bmatrix},\quad r,s\in\mathbb{R}.$$

In particular, we treat $*$-exponential functions $e_*^{t\frac{1}{i\hbar}(u_*^2+v_*^2)}$, $e_*^{t\frac{1}{i\hbar}2u\circ v}$ more closely.

In (3.7), we set the expression parameter $K=-J\kappa{=}\left[\begin{smallmatrix}a & c\\ c & b\end{smallmatrix}\right]$, and we set the amplitude part of (3.7) $\frac{1}{\sqrt{\Delta_K(t)}}$ where

$$\Delta_K(t){=}\det((\cos t)I{+}(\sin t)K){=}\cos^2 t{-}(a{+}b)\sin t\cos t{+}(ab{-}c^2)\sin^2 t \tag{3.17}$$

Note that $a{+}b$ and $ab{-}c^2$ can be arbitrary complex numbers.

$\Delta_K(t)$ and the phase part of (3.7) are both π-periodic, but the sign of $\sqrt{\Delta_K(t)}$ depends on the expression parameter K and the path from 0 to t in the complex plane. The sign ambiguity is removed by putting the initial condition $e_*^{0\frac{1}{i\hbar}H_*}{=}1$ at $t{=}0$ only in the case that $a=-b$ and $c^2{+}a^2=1$, i.e. $\Delta_K(t)=1$, or the case that $(a{-}b)^2{+}4c^2{=}0$, i.e. $\Delta_K(t){=}\frac{1}{4}(2\cos t{+}(a{+}b)\sin t)^2$.

Moreover, singular points depend on expression parameters (cf.[13]). The case $c{=}0$ where a,b are arbitrary in $\mathbb{C}$ gives an overview how the singular points are moving:

$$:e_*^{t\frac{1}{i\hbar}(u_*^2+v_*^2)}:_K{=}\frac{1}{\sqrt{(\cos t{-}a\sin t)(\cos t{-}b\sin t)}}\exp\frac{1}{i\hbar}\Big(\frac{\sin t}{\cos t{-}b\sin t}u^2{+}\frac{\sin t}{\cos t{-}a\sin t}v^2\Big).$$

By these observations, we see that the singular points appear π-periodically in general on two lines parallel to the real axis and the $*$-exponential functions have $e^{-|t|}$-growth with the exponential decay on the line parallel to the pure imaginary axis when these do not hit singular points.

The observation here gives in addition the following:

Lemma 3.3 *Choosing the expression parameter K, we can make both $:e_*^{[0\to\pi]\frac{1}{i\hbar}(u_*^2+v_*^2)}:_K = 1$, and -1. Moreover, multiplying e^t, we have an extremal point, called vacuum,*

$$\lim_{t\to\infty} e^t{:}e_*^{it\frac{1}{i\hbar}(u_*^2+v_*^2)}{:}_K = \frac{2}{\sqrt{(1-a)(1-b)}}\exp\frac{1}{i\hbar}\Big(\frac{1}{1-b}u^2+\frac{1}{1-a}v^2\Big).$$

depending on the expression parameters.

We fix the expression parameter K as follows:

$$K_{re}=\begin{bmatrix}\rho & ic'\\ ic' & \rho\end{bmatrix},\ \ or\ \ K_{im}=\begin{bmatrix}i\rho & c\\ c & i\rho\end{bmatrix},\ \rho,\ c,\ c'\in\mathbb{R}.$$

The formula (3.17) is rewritten in this case as

$$\begin{aligned}\Delta_{K_{re}}(t)&=\det((\cos t)I+(\sin t)K_{re})=\cos^2 t+2\rho\sin t\cos t+(\rho^2+c'^2)\sin^2 t,\\ \Delta_{K_{im}}(t)&=\det((\cos t)I+(\sin t)K_{im})=\cos^2 t+2i\rho\sin t\cos t-(\rho^2+c^2)\sin^2 t.\end{aligned}$$

The first one is obviously positive definite if $c'\neq 0$ (i.e. Siegel ordered expression in the case $m=1$) and hence $\sqrt{\Delta_{K_{re}}(t)}$ does not change sign when t moves 0 to π along real line.

On the other hand,

$$\sqrt{\Delta_{K_{im}}(t)}=\frac{1}{2}e^{-it}\sqrt{(1+\rho)^2+c^2}\sqrt{(e^{2it}+\alpha)(e^{2it}+\bar{\alpha})},\quad \alpha=\frac{1-(\rho+ic)}{1+(\rho+ic)}.$$

One may assume generically that $|\alpha|\neq 1$. Hence, $\sqrt{\Delta_{K_{im}}(t)}$ changes sign when t moves from 0 to π. Thus, we have

Lemma 3.4 *$:e_*^{t\frac{1}{i\hbar}(u^2+v^2)}:_{K_{re}}$ is π-periodic, and the two lines of singular points are sitting in both upper and lower half plane. The real line is between these.*

On the other hand, $:e_*^{t\frac{1}{i\hbar}(u^2+v^2)}:_{K_{im}}$ is alternating π-periodic, and the two lines of singular points are sitting in upper or lower half plane depending on the sign of ρ.

The expression parameter K_{im} is the case $m=1$ of the special expression parameter K_s used in [13].

Next, we take our attention to the quadratic form $2u{\circ}v$, but we take a general expression parameter $K=\begin{bmatrix}\delta & c\\ c & \delta'\end{bmatrix}$. A little complicated calculation via intertwiner $I_{K_0}^{K}$ from the normal ordered expression gives by setting $\Delta=e^t+e^{-t}-c(e^t-e^{-t})$ that

$$:e_*^{t\frac{1}{i\hbar}2u\circ v}:_K=\frac{2}{\sqrt{\Delta^2-(e^t-e^{-t})^2\delta\delta'}}\ e^{\frac{1}{i\hbar}\frac{e^t-e^{-t}}{\Delta^2-(e^t-e^{-t})^2\delta\delta'}\Big((e^t-e^{-t})(\delta'u^2+\delta v^2)+2\Delta uv\Big)}. \tag{3.18}$$

The q-scalar and the polar element are obtained by setting $t=\pm\pi i$ and $t=\pm\frac{\pi i}{2}$ respectively.

For the simplest case in (3.18), that the case $c=\delta=\delta'=0$ is the Weyl ordered expression. This is not a generic ordered expression having singular points on the imaginary axis, and this is πi-alternating periodic.

On the contrary, the unit ordered expression is given by $K=I$, i.e. $\delta=1, \delta'=1$, $c=0$. By (3.18), we have

$$:e_*^{t\frac{1}{i\hbar}2u{\circ}v}:_{_I}=\frac{2}{\sqrt{4}}e^{\frac{1}{4i\hbar}(e^{2t}+e^{-2t}+2)(u^2+v^2)+2(e^{2t}-e^{-2t})uv}.$$

This is πi-periodic and there is no singular point.

For the case $\delta=\delta'=0$ but $c\neq 0$ which involves the normal ordered expression, we see that

$$:e_*^{t\frac{1}{i\hbar}2u{\circ}v}:_{_K}=\frac{2}{\sqrt{\Delta^2}}\ e^{\frac{1}{i\hbar}\frac{e^t-e^{-t}}{\Delta^2}\left(2\Delta uv\right)}=\frac{2}{\Delta}\ e^{\frac{1}{i\hbar}\frac{e^t-e^{-t}}{\Delta}2uv}. \tag{3.19}$$

This is the case where the singular points are not branching ones and they are sitting πi periodically on a single line parallel to the imaginary axis whose real part are given by $\log\left|\frac{c+1}{c-1}\right|$. We see also that $:e_*^{\frac{t}{i\hbar}2u{\circ}v}:_{_K}$ is alternating πi-periodic along the imaginary axis.

Suppose in (3.18) that $K{=}K_{re}$;

$$\delta{=}\delta'{=}\rho,\ c=ic' \quad \rho,\ c'\in\mathbb{R},\ c'{\neq}0. \tag{3.20}$$

By setting $\beta{=}\rho+ic'$, we have that

$$\frac{1}{2}\sqrt{\Delta^2{-}(e^t{-}e^{-t})^2\delta\delta'}{=}e^{-t}\sqrt{(1{-}\beta)(1{+}\bar\beta)}\sqrt{e^{2t}{-}\frac{1{+}\beta}{1{-}\beta}}\sqrt{e^{2t}{-}\frac{1{-}\bar\beta}{1{+}\bar\beta}}. \tag{3.21}$$

Obviously $|\frac{1+\beta}{1-\beta}||\frac{1-\bar\beta}{1+\bar\beta}|=1$, but one may assume in generic ordered expression that $|\frac{1+\beta}{1-\beta}|{\neq}1$. Hence, $\sqrt{e^{2t}{-}\frac{1+\beta}{1-\beta}}\sqrt{e^{2t}{-}\frac{1-\bar\beta}{1+\bar\beta}}$ changes the sign when t moves 0 to πi. Thus we see $\frac{1}{2}\sqrt{\Delta^2{-}(e^t{-}e^{-t})^2\delta\delta'}$ does not change the sign on the interval $[0,\pi i]$. Hence $:e_*^{t\frac{1}{i\hbar}2u{\circ}v}:_{_{K_{re}}}$ is πi-periodic. Remark now this is the case in $:e_*^{\frac{it}{i\hbar}\langle \boldsymbol{u}g,\boldsymbol{u}g\rangle}:_{_K}$ where

$$g=\frac{1}{\sqrt{2}}\begin{bmatrix}1&i\\ i&1\end{bmatrix},\quad K_{re}=\begin{bmatrix}\rho&ic'\\ ic'&\rho\end{bmatrix}.$$

The concrete expression of polar element is

$$:\varepsilon_{00}:_{_{K_{re}}}=:e_*^{\pi i\frac{1}{i\hbar}u{\circ}v}:_{_{K_{re}}}{=}\frac{1}{\sqrt{(\rho^2{+}c'^2)}}e^{-\frac{1}{i\hbar}\frac{1}{\rho^2+c'^2}\rho(u^2+v^2)-2c'iuv}. \tag{3.22}$$

Note that the quadratic form $u_*^2{+}v_*^2$ is a representative of general quadratic forms $au_*^2{+}bv_*^2{+}2cu{\circ}v$ with the discriminant $c^2{-}ab=-1$ via $SL(2;\mathbb{C})$-linear change of generators.

Since $SL(2;\mathbb{C})=Sp(1;\mathbb{C})$, such a linear change is covered by a change of expression parameters by the formula (1.10). Thus, even if an expression parameter K is fixed generically, these patterns for the quadratic form $u_*^2{+}v_*^2$ must appear for $au_*^2{+}bv_*^2{+}2cu{\circ}v$ via changing coefficients. We shall show that this appears slightly different, more delicate shape.

In general, we set

$$e_*^{itH_*}{=}e_*^{\frac{it}{i\hbar}(au_*^2+bv_*^2+2cu\circ v)}, \quad c^2{-}ab=1.$$

We have then three patterns as follows:

$(Q(1))$: $e_*^{itH_*}$ is alternating π-periodic and the 2 lines of singular points are in the upper half-plane.

$(Q(2))$: $e_*^{itH_*}$ is alternating π-periodic and the 2 lines of singular points are in the lower half-plane.

$(Q(3))$: $e_*^{itH_*}$ is π-periodic and the real line are between 2 lines of singular points.

$(Q(k))$ are open subsets of $\{(a,b,c); c^2{-}ab{=}1\}$ such that $(Q(1))\cup(Q(2))\cup(Q(3))$ is dense. Since the time reversing sends the line of singularities to the opposite side, we see that $(Q(k))$ has the property

$$(Q(1))^{-1}=(Q(2)), \quad (Q(3))^{-1}=(Q(3)).$$

This means that if $e_*^{itH_*}\in(Q(1))$, then $e_*^{it(-H_*)}\in(Q(2))$.

Remark Alternating π-periodicity appears when no sheet changing occurs. Thus,

$$:e_*^{\pi\frac{1}{i\hbar}(u^2+v^2)}:_{_K}=-1$$

always on the positive sheet, as far as requesting $e_*^{0\frac{1}{i\hbar}(u^2+v^2)}=1$. On the other hand ,

$$:e_*^{\pi\frac{1}{i\hbar}(u^2+v^2)}:_{_K}=1,$$

when the sheet changing occurs on a path from 0 to π. It is very easy to make a mistake.

Recall first the anomalous phenomena mentioned in [13] that a polar element is obtained not only by $e_*^{\frac{\pi i}{i\hbar}u\circ v}$ but also by $e_*^{\frac{\pi i}{2i\hbar}(au_*^2+bv_*^2+c2u\circ v)}$, $c^2{-}ab=1$. This shows that a polar element is sitting on various one parameter subgroups. This is just like the longitude lines starting at the north pole meet again at the South Pole. We show in the next section this is a generic phenomena of $e_*^{\frac{\pi i}{2i\hbar}(au_*^2+bv_*^2+2cu\circ v)}$, $c^2{-}ab=1$. Thus, a polar element has infinitely many *square roots* sitting on the equator.

Beyond the south pole the longitude lines come back again to the north pole, where we give the initial value 1 to every one parameter subgroup parameterized by longitude. However, it is a little surprising that the periodicity of these periodic movement depends on expression parameters.

3.3.2 Product structure

The product formula (2.13) shows that the space $\mathbb{C}e^{Q(u,v)}$ of exponential functions of polynomial of degree 2 forms a very special subclass in the space $\mathcal{E}_{2+}(\mathbb{C}^{2m})$. It is useful to memorize the next theorem:

Theorem 3.3 *In a generic ordered expression K, the $*_{_K}$-product*

$$\pi_{_K}:\mathbb{C}e^{Q(u,v)}\times\mathbb{C}e^{Q(u,v)}\to\mathbb{C}e^{Q(u,v)}$$

is a mapping given in the form $\pi_{_K}(ae^Q,be^R)=ab\sqrt{f(Q,R,K)}\,e^{g(Q,R,K)}$ where f and g are meromorphic functions of Q,R,K. Hence the continuity

$$\lim_{(k,\ell)}\pi_{_K}(a_ke^{Q_k},b_\ell e^{R_\ell})=\pi_{_K}(\lim_k a_ke^{Q_k},\lim_\ell b_\ell e^{R_\ell})$$

holds whenever $\lim_k a_k e^{Q_k}$*,* $\lim_\ell b_\ell e^{R_\ell}$ *are defined in the space* $\mathbb{C}e^{Q(u,v)}$*, and*

$$\pi_K(a_k e^{Q_k}, b_\ell e^{R_\ell}), \quad \pi_K(\lim_k a_k e^{Q_k}, \lim_\ell b_\ell e^{R_\ell})$$

are defined.

As for products, we know already that associativity holds always with sign ambiguity. However the following theorem is useful as a corollary of Proposition 3.5 and the formal associativity theorem (cf. Theorem 1.3),

Theorem 3.4 *For quadratic forms* K_**,* L_**,* M_**, associativity*

$$(e_*^{[0\sim r]K_*}*e_*^{[0\sim s]L_*})*e_*^{[0\sim t]M_*}=e_*^{[0\sim r]K_*}*(e_*^{[0\sim s]L_*}*e_*^{[0\sim t]M_*})$$

holds without sign ambiguity whenever both sides are defined, where paths in both left/right hand sides with same symbol should be same path (synchronized path selecting).

We next consider the product $e_*^{sH_*}*e_*^{tK_*}$ for two quadratic forms H_*, K_* such that $[H_*, K_*]=0$. First of all, we show the following

Proposition 3.8 *If* $e_*^{sH_*}*e_*^{tK_*}$ *are defined on* $(s,t)\in[0,a]^2$*, then* $e_*^{sH_*}*e_*^{tK_*}=e_*^{tK_*}*e_*^{sH_*}$*.*

Proof Since $K_**e_*^{sH_*}$ and $e_*^{sH_*}*K_*$ satisfies the equation $\frac{d}{ds}f_s = H_**f_s$ with $f_0 = K_*$, we have $K_**e_*^{sH_*} = e_*^{sH_*}*K_*$. Hence, we have

$$\frac{d}{dt}e_*^{sH_*}*e_*^{tK_*}=K_**e_*^{sH_*}*e_*^{tK_*}, \quad \frac{d}{dt}e_*^{tK_*}*e_*^{tH_*}=K_**e_*^{tK_*}*e_*^{sH_*}$$

with the same initial condition $e_*^{sH_*}$. The uniqueness gives the proof. □

If $(s,t)\in\mathbb{C}^2$, then we have in general $e_*^{sH_*}*e_*^{tK_*} = \pm e_*^{tK_*}*e_*^{sH_*}$ with the sign ambiguity by the product formula (2.11). This means in particular and the phase parts of both sides coincides (the sign ambiguity appears only in the amplitude parts). In general, $e_*^{sH_*}*e_*^{tK_*}$ has a singular set S of complex codimension 1. We see that the origin $(0,0)$ is not contained in S. Since S is a branched singularity, we have to prepare two sheets $\mathbb{C}^2_+$, $\mathbb{C}^2_-$ and "slit" Σ of real codimension 1 to connect these two sheets. Σ is set so that $\mathbb{C}^2\setminus\Sigma$ is locally simply connected and there is no singular point.

Now, restrict the parameter $(s,t)\in\mathbb{R}^2$ in $e_*^{sH_*}*e_*^{tK_*}$ and suppose $e_*^{sH_*}*e_*^{tK_*}$ has a singular point in $(s,t)\in(0,a)\times(0,b)$. One may assume that $\mathbb{R}^2$ is transversal to S in generic ordered expression. Hence if $S\cap\mathbb{R}^2\neq\emptyset$, then this is a discrete set and $\Sigma\cap\mathbb{R}^2$ is a collection of (real one dimensional) curves starting at a singular point ending another singular point or ∞. Hence one may assume that the boundary $\partial([0,a]\times[0,b])$ cuts the slit just once for all.

Proposition 3.9 *Under the assumption as above, we have* $e_*^{sH_*}*e_*^{tK_*} = -e_*^{sK_*}*e_*^{tH_*}$

Proof As we have two sheets, there are two "origin", $(0,0){\in}\mathbb{C}^2_+$ and $(0,0){\in}\mathbb{C}^2_-$. Since $e_*^{0H_*}{*}e_*^{0K_*}$ is 1 in the positive sheet $\mathbb{C}^2_+$, the origin in the negative sheet must be treated as -1. Now, consider $e_*^{aH_*}{*}e_*^{bK_*}$ and $e_*^{bK_*}{*}e_*^{aH_*}$. The first one is defined by the solution of the evolution equation

$$\frac{d}{dt}f_t = H_*{*}f_t, \quad f_0 = e_*^{bK_*}.$$

We indicate this by the notation $e_*^{[0\to a]H_*}{*}e_*^{bK_*}$. This is the clockwise tracing from the origin. On the contrary, $e_*^{[0\to b]K_*}{*}e_*^{aH_*}$ means the anti-clockwise tracing from the origin. Now suppose there is a singular point (s_0,t_0), then one of the paths $e_*^{[0\to a]H_*}{*}e_*^{bK_*}$ and $e_*^{[0\to b]K_*}{*}e_*^{aH_*}$ is crossing the slit hence they are sitting mutually in the opposite sheet. By this way, the sign changes around a singular point. $\square$

4 Rule of setting slits and polar elements

If it is an absolute scalar, then $(\sqrt{1})^2{=}1$ is trivial. Recall first

Proposition 4.1 *If $e^{2\pi\alpha}{=}I$ such as $\alpha{=}J$ (e.g. $\alpha = gJg^{-1}$, $\forall g{\in}Sp(m,\mathbb{C})$), then $:e_*^{\pi\frac{1}{i\hbar}\langle\boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*}:_\kappa = \sqrt{1}$ independent of K.*

Note that l.h.s. is not a *classic* element, for this identity does not hold for $\hbar = 0$.

Hence, the strict exponential law might be failed, that is, $:e_*^{2\pi\frac{1}{i\hbar}\langle\boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*}:_\kappa = 1$ or

$$:e_*^{\pi\frac{1}{i\hbar}\langle\boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*}:_\kappa{*_\kappa}:e_*^{\pi\frac{1}{i\hbar}\langle\boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*}:_\kappa = 1$$

may not hold automatically. If $:e_*^{t\frac{1}{i\hbar}\langle\boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*}:_\kappa$ has a singular point on the interval $[0,2\pi]$, then it may occur $(e_*^{\pi\frac{1}{i\hbar}\langle\boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*})^2{\neq}e_*^{2\pi\frac{1}{i\hbar}\langle\boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*}$, although the equality holds modulo $\pm$ sign.

To avoid such a strange nature, we give a general rule to set *slits*. Because of the double branching singular points, we have to use two sheets by setting *slits* in the complex plane to treat these $*$-exponential functions $:e_*^{tH_*}:_K$ univalent way.

♣ As it is discussed already, it is natural to set the slits periodically, since the singular points are distributed periodically.

By virtue of this rule, we have

Proposition 4.2 *If $e^{2\pi\alpha}{=}I$ (e.g. $\alpha = gJg^{-1}$, $\forall g{\in}Sp(m,\mathbb{C})$), then $:(e_*^{[0\to\pi]\frac{1}{i\hbar}\langle\boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*})^2:_\kappa{=}1$ for every κ-ordered expression such that $:e_*^{t\frac{1}{i\hbar}\langle\boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*}:_\kappa$ has no singular point on the interval $[0,\pi]$. Moreover, we have*

$$:e_*^{2\pi\frac{1}{i\hbar}\langle\boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*}:_\kappa{=}:(e_*^{\pi\frac{1}{i\hbar}\langle\boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*})^2:_\kappa.$$

Proof. Note first that this is by no means trivial. It is crucial that the assumption and the π-periodicity of singular points shows that $:e_*^{t\frac{1}{i\hbar}\langle\boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*}:_\kappa$ has no singular point on the interval $[\pi,2\pi]$, but if there is no rule to set the slit, it may happen that path $[0{\to}2\pi]$ cross the slit only once.

By the rule of setting slits ♣, we see that the slits are set π-periodically. Thus, the line segment $[0, 2\pi]$ must cross the slits even (possibly 0) times. It follows $:e_*^{2\pi\frac{1}{i\hbar}\langle \boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*}:_\kappa{=}1$, since this is sitting in the positive sheet.

To confirm $:(e_*^{\pi\frac{1}{i\hbar}\langle \boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*})^2:_\kappa = :e_*^{2\pi\frac{1}{i\hbar}\langle \boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*}:_\kappa$, we have to recall how the $*$-product $e_*^{\pi\frac{1}{i\hbar}\langle \boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*}{*}g$ is defined. We use the definition which is given by the evolution equation (3.2).

Since $:e_*^{\pi\frac{1}{i\hbar}\langle \boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*}:_\kappa = \pm 1$, one can define

$$:e_*^{t\frac{1}{i\hbar}\langle \boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*}:_\kappa {*_\kappa} :1:_\kappa, \quad \text{or} \quad :e_*^{t\frac{1}{i\hbar}\langle \boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*}:_\kappa {*_\kappa} :(-1):_\kappa$$

by the solution of the evolution equation (3.2) with the initial condition ± 1. By Proposition 3.1, the solution is $:e_*^{t\frac{1}{i\hbar}\langle \boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*}:_\kappa$ or $-:e_*^{t\frac{1}{i\hbar}\langle \boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*}:_\kappa$ respectively. This gives the result. $\square$

4.1 General polar element as q-scalars

It is interesting that polar element ε_{00} behaves just like a scalar, but it behaves various ways. Sometimes, it behaves as if it were -1, and sometimes it looks as if i depending on K. We call such elements q-scalars. But, to treat this as a univalent element, we have to distinguish more strictly.

The strange double-valued nature of the polar element ε_{00} is caused by that $e_*^{\pi\frac{1}{2i\hbar}\langle \boldsymbol{u}g,\boldsymbol{u}g\rangle_*}$ is moving discontinuously in both positive and negative sheets when g moves in $Sp(m,\mathbb{C})$.

In this section, we analyze this phenomenon more clearly. In particular, we investigate the generic patterns of periodicity and singularities of $*$-exponential functions of quadratic forms under the assumption ♣. In particular, we are interested the behaviour of polar element. In what follows, we use several notions for the path as follows:

$[0{\to}a]$: the path starting from the origin 0 ending at a along the line segment, but the $*$-exponential is evaluated at $t{=}a$ by the continuous chase from 0 to a along the path $[0{\to}a]$.

$[0{\sim}a]$: a path starting from the origin 0 ending at a avoiding singular points, but evaluated at a.

$[0{\approx}a]$: a path starting from the origin 0 ending at a avoiding singular points and slits so that the end point is sitting in the same sheet as the origin.

For a fixed g, $\varepsilon_{00} = :e_*^{\pi\frac{1}{2i\hbar}\langle \boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K$ is always viewed as a double-valued single parallel section. If K is fixed, ε_{00} looks independent of g with $\pm$ ambiguity. To distinguish the sign, we use the notation

$$:\varepsilon_{00}[g]:_K = :e_*^{[0\to\pi]\frac{1}{2i\hbar}\langle \boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K = \frac{1}{\sqrt{\det(\cos([0{\to}1]\frac{\pi}{2})I - (\sin([0{\to}1]\frac{\pi}{2})\,{}^t\!gKg)}}e^{-\frac{1}{i\hbar}\langle \boldsymbol{u}\frac{1}{K},\boldsymbol{u}\rangle} \tag{4.1}$$

to fix the sign of ε_{00}, where $[0{\to}a]$ is the path along the straight line segment. Note that $:\varepsilon_{00}[g]:_K$ may not be defined at some g, when a singular point appears in the interval $(0, \pi/2]$. Although $\varepsilon_{00} = \pm\varepsilon_{00}[g]$ and ε_{00} is independent of g, $\varepsilon_{00}[g]$ may not be continuous w.r.t. g. The sign changes discontinuously at some g. For a generic K, there is $g{\in}Sp(m,\mathbb{C})$ such that ${}^t\!gKg$ is a real diagonal matrix. Hence $:e_*^{t\frac{1}{2i\hbar}\langle \boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K$ has a singular point.

Note that for every $g{\in}Sp(m,\mathbb{C})$ there is $k{\in}Sp(m,\mathbb{C})$ such that $-\langle\boldsymbol{u}g,\boldsymbol{u}g\rangle_*{=}\langle\boldsymbol{u}k,\boldsymbol{u}k\rangle_*$. This is shown for instance

$$g\begin{bmatrix}iI & 0\\ 0 & -iI\end{bmatrix}\begin{bmatrix}iI & 0\\ 0 & -iI\end{bmatrix}{}^t\!g = -g\,{}^t\!g. \tag{4.2}$$

Recall the rule ♣ of setting slits. As sheets are set π-periodically we see the next result:

Lemma 4.1 *In generic K-expression,* $:e_*^{[0\to\pi]\frac{1}{i\hbar}\langle\boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K$ *and* $:e_*^{[0\to\pi]\frac{-1}{i\hbar}\langle\boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K$ *belong to the same sheet, and*

$$:e_*^{[0\to\pi]\frac{1}{i\hbar}\langle\boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K = :e_*^{[0\to\pi]\frac{-1}{i\hbar}\langle\boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K = 1 \quad or \quad -1.$$

However, this may not belong to the same (positive) sheet as $:e_*^{0\frac{1}{i\hbar}\langle\boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K$.

Proof If the path $[0{\to}\pi]$ crosses the slit ℓ-times, then the end point $:e_*^{[0\to\pi]\frac{1}{i\hbar}\langle\boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K$ is sitting on the $(-1)^\ell$-sheet. Since sheets are set π-periodically, the path $[0{\to}\pi]$ for $:e_*^{[0\to\pi]\frac{-1}{i\hbar}\langle\boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K$ also crosses the slit ℓ-times. □

On the other hand, the second equality of (3.15) does not necessarily imply that

$$:e_*^{[0\to\pi]\frac{1}{2i\hbar}\langle\boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K = :e_*^{[0\to\pi]\frac{-1}{2i\hbar}\langle\boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K.$$

The sheet change may occur in the continuous tracing of $\sqrt{\det(\cos tI{-}(\sin t){}^t\!gKg)}$ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ if the path from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ crosses the slit odd-times. (3.15) shows

Lemma 4.2 $:\varepsilon_{00}[g]:_K = :\varepsilon_{00}[g]^{-1}:_K$ *if and only if*

$$\sqrt{\det(\cos([0{\to}\pi])I{-}(\sin([0{\to}\pi]){}^t\!gKg)} = 1.$$

If there is no singular point on $:e_*^{t\frac{1}{2i\hbar}\langle\boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K$, $t\in\mathbb{R}$, then this forms a one parameter group, and thus the equality above is equivalent with $:e_*^{[0\to\pi]\frac{1}{i\hbar}\langle\boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K = 1$ by the exponential law.

Lemma 3.3 in the this section shows that for a certain K there are g, g' such that $\sqrt{(-1)^{2m}} = 1$ and -1 respectively. Thus, even if K is fixed, the sign may depend on g and the path from 0 to π. Since $Sp(m,\mathbb{C})$ is connected, the sign changes discontinuously when the path from 0 to π hits a singular point. The sign changes by the changing sheet caused when the path crossing the slit drawn from the set of the singular points.

In the argument above, paths were restricted in line segment to fix the ambiguous sign. In fact, we can relax this condition. The next lemma shows that the sign-changing is caused only when the path moves across the set S of singular points. Take an open connected subset U of $Sp(m,\mathbb{C})$ which may be $U\neq -U$. Suppose we can fix path $:e_*^{[0\sim\pi]\frac{1}{i\hbar}\langle\boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K$ from $t{=}0$ to $t{=}\pi$ avoiding singular points but depending continuously in $g\in U$. By setting $t=\pi$, and $t=\frac{\pi}{2}$, we have the following :

Lemma 4.3 *Under the assumption for U mentioned above, the $*$-exponential function*

$$:e_*^{[0\sim t]\frac{1}{i\hbar}\langle\boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K$$

is defined uniquely without sign ambiguity by the continuous tracing from the identity, and we see

$$:e_*^{[0\sim\pi]\frac{1}{i\hbar}\langle\boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K = \sqrt{(-1)^{2m}}, \tag{4.3}$$

where $\sqrt{(-1)^{2m}} = (-1)^m$, *when the end point of path is sitting in the same (positive) sheet as* 0, *and* $-(-1)^m$, *when the end point of path is sitting in the opposite (negative) sheet.*

On the other hand for the polar element, we have

$$:e_*^{[0\sim\pi]\frac{1}{2i\hbar}\langle\boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K = \frac{1}{\sqrt{\det K}}e^{-\frac{1}{i\hbar}\langle\boldsymbol{u}K^{-1},\boldsymbol{u}\rangle}. \tag{4.4}$$

The sign of $\sqrt{\det K}$ *is determined by the sheet on which the end point of the path* $[0\sim\pi]$ *is sitting.*

Note that $(-1)^m$ in Lemma 4.3 is -1 if m =odd, and 1 if m =even. Thus, the mathematical context depends on $(-1)^m = \pm 1$ in the next Proposition.

Proposition 4.3 *Suppose there is* $g \in Sp(m,\mathbb{C})$ *such that* $:e_*^{[0\to\pi]\frac{1}{i\hbar}\langle\boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K = -1$. *Then, there must exist* $h \in Sp(m,\mathbb{C})$ *such that* $:e_*^{[0\to\pi]\frac{1}{i\hbar}\langle\boldsymbol{u}h,\boldsymbol{u}h\rangle_*}:_K = 1$, *and* $\hat{h} \in Sp(m,\mathbb{C})$ *such that the path* $:e_*^{[0\to\pi]\frac{1}{2i\hbar}\langle\boldsymbol{u}\hat{h},\boldsymbol{u}\hat{h}\rangle_*}:_K$ *must hit a singular points.*

Proof Suppose $:e_*^{[0\to\pi]\frac{1}{i\hbar}\langle\boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K = -1$ for every $g\in Sp(m,\mathbb{C})$ and suppose there is no singular point on the path $[0\to\pi]$.

As $Sp(m,\mathbb{C})$ is connected, (4.2) and the second equality of (3.15) give that for the mid-point

$$:e_*^{[0\to\pi]\frac{1}{2i\hbar}\langle\boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K=:e_*^{[0\to\pi]\frac{1}{2i\hbar}\langle\boldsymbol{u}k,\boldsymbol{u}k\rangle_*}:_K=:e_*^{[0\to\pi]\frac{-1}{2i\hbar}\langle\boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K. \tag{4.5}$$

The exponential law gives

$$:e_*^{[0\to\pi]\frac{-1}{2i\hbar}\langle\boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K = :\big(e_*^{[0\to\pi]\frac{1}{2i\hbar}\langle\boldsymbol{u}g,\boldsymbol{u}g\rangle_*}\big)^{-1}:_K$$

and therefor multiplying $e_*^{[0\to\pi]\frac{1}{2i\hbar}\langle\boldsymbol{u}g,\boldsymbol{u}g\rangle_*}$ to both sides of (4.5), we have the contradiction

$$-1 = e_*^{[0\to\pi]\frac{1}{i\hbar}\langle\boldsymbol{u}g,\boldsymbol{u}g\rangle_*} = e_*^{[0\to\pi]\frac{1}{2i\hbar}\langle\boldsymbol{u}g,\boldsymbol{u}g\rangle_*}*\big(e_*^{[0\to\pi]\frac{1}{2i\hbar}\langle\boldsymbol{u}g,\boldsymbol{u}g\rangle_*}\big)^{-1} = 1.$$

As a result $Sp(m,\mathbb{C})$ is divided into three parts D_+, D_-, D_{sing} such that

$$:e_*^{[0\to\pi]\frac{1}{i\hbar}\langle\boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K = \begin{cases} -1 & g\in D_+ \\ :e_*^{t\frac{1}{i\hbar}\langle\boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K \text{has a singular point on } (0,\pi) & g\in D_{sing} \\ 1 & g\in D_- \end{cases}$$

and $D_+ \subsetneqq Sp(m,\mathbb{C})$. In particular this yields $D_{sing} \neq \emptyset$.

Now, we show that $D_- \neq \emptyset$. Since the points of D_{sing} are branched singular points, the value of $:e_*^{t\frac{1}{i\hbar}\langle\boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K$ changes sign around branched singular point. Since we assumed as a generic assumption that the singular points distributed π-periodically along $2m$ lines parallel to the real line, there is at most one singular point on $(0,\pi)$. Thus, $:e_*^{[0\to 1]\frac{\pi}{i\hbar}\langle \boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K$ must change sign at $g{\in}D_{sing}$. Hence we see $D_-{\neq}\emptyset$. $\square$

We note that $Sp(k,\mathbb{C})$ is naturally included in $Sp(m,\mathbb{C})$ for $m{>}k$. Apparently, the result mentioned in [13] is a special case for $m=1$, $g=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & i\\ i & 1\end{bmatrix}$ and $K{=}K_0$ (normal ordered expression).

Proposition 4.3 gives in particular that if $D_+{\neq}\emptyset$, then $D_-{\neq}\emptyset$ and $D_{sing}{\neq}\emptyset$.

Consider now whether it is possible $D_+=\emptyset$ in Lemma 4.3. First we note the following:

Lemma 4.4 *If $D_{sing}{\neq}\emptyset$, then $D_{\pm}{\neq}\emptyset$.*

Proof For $(t,g){\in}\mathbb{C}{\times}Sp(m,\mathbb{C})$, the set S of singular points of $:e_*^{t\frac{1}{i\hbar}\langle \boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K$ is a closed subset of complex codimension 1. The slit Σ is set so that $(\mathbb{C}{\times}Sp(m,\mathbb{C}))\backslash\sigma$ is locally simply connected. Hence, if $:e_*^{[0\to\pi]\frac{1}{i\hbar}\langle \boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K$ hits a singular point for some g, then there are $h,h'\in Sp(m,\mathbb{C})$ in a neighborhood of g such that $:e_*^{[0\to\pi]\frac{1}{i\hbar}\langle \boldsymbol{u}h,\boldsymbol{u}h\rangle_*}:_K$ hits Σ, but $:e_*^{[0\to\pi]\frac{1}{i\hbar}\langle \boldsymbol{u}h',\boldsymbol{u}h'\rangle_*}:_K$ does not. Hence, these two must have different sign. $\square$.

Now note that the comment following (4.1) shows that $D_{sing}{\neq}\emptyset$. Thus, we have

Theorem 4.1 *Suppose K is a generic expression parameter. Then, $Sp(m,\mathbb{C})$ is divided into three non empty subsets D_+, D_-, D_{sing}.*

Remark 1 As singular points are distributed π- periodically, if $g\in D_{sing}$, then $:e_*^{t\frac{1}{i\hbar}\langle \boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K$ has singular points not only in the interval $(0,\frac{\pi}{2}]$ but also in the interval $(-\pi,-\frac{\pi}{2}]$.

Theorem 4.1 shows a polar element ε_{00} is a member of various one parameter subgroups with different periodicity $\varepsilon_{00}^2=1$, and $\varepsilon_{00}^2=-1$.

Note Sometimes, D_- contains a compact subgroup of $Sp(m,\mathbb{C})$. Indeed, we will show in the next section that such a case exists. That is, in the case $m=1$ there is a class K_{re} of expression parameters such that

$$:e_*^{[0\to\pi]\frac{1}{i\hbar}\langle \boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K=1 \quad \text{for every} \quad g{\in}SU(2) \quad \text{cf. Proposition 4.5.}$$

Remark 2 A polar element is a double-valued single element. Thus even though $:e_*^{\frac{\pi}{2i\hbar}\langle \boldsymbol{u}g,\boldsymbol{u}g\rangle_*}:_K = :e_*^{\frac{\pi}{2i\hbar}\langle \boldsymbol{u}h,\boldsymbol{u}h\rangle_*}:_K$, square of these may be different

$$:(e_*^{\pm\frac{\pi}{2i\hbar}\langle \boldsymbol{u}g,\boldsymbol{u}g\rangle_*})^2:_K{\neq}:(e_*^{\pm\frac{\pi}{2i\hbar}\langle \boldsymbol{u}h,\boldsymbol{u}h\rangle_*})^2:_K$$

if the paths from 0 to π have different numbers of crossing slits.

It is quite difficult to control the $\pm$ sign in the product formula. We have always to chase continuously from the identity. Even though $:\varepsilon_{00}[g]:_K=\pm{:}\varepsilon_{00}[h]:_K$, it does not necessarily imply $:\varepsilon_{00}[g]^2:_K = :\varepsilon_{00}[h]^2:_K$. Furthermore, we do not have enough information in order to determine $:\varepsilon_{00}[g]{*}\varepsilon_{00}[h]:_K$, though this is $\{\pm 1\}$ by the product formula with sign ambiguity. In such a situation, we cannot use $\varepsilon_{00}[g],\varepsilon_{00}[h]$ as elements of a system with binary operations.

By these observation, it seems to be better to treat every element always together with a path from the origin, and products are defined always by path connecting. However, this is sometimes too much to treat the detail, for the object turns out to be a groupoid. We have to seek an amenable object to treat which gives informations what we want to know.

Strict polar element

Let $[0\approx\pi]$ be a path from 0 to π avoiding singular points and slits so that $e_*^{\pi\frac{1}{2i\hbar}\langle \boldsymbol{u}g,\boldsymbol{u}g\rangle_*}$ is sitting in the same sheet as in $e_*^{0\frac{1}{2i\hbar}\langle \boldsymbol{u}g,\boldsymbol{u}g\rangle_*}$. Then $e_*^{[0\approx\pi]\frac{1}{2i\hbar}\langle \boldsymbol{u}g,\boldsymbol{u}g\rangle_*}$ is determined without sign ambiguity.

$e_*^{[0\approx\pi]\frac{1}{2i\hbar}\langle \boldsymbol{u}g,\boldsymbol{u}g\rangle_*}$ is called the **strict polar element** by requesting that the path is so chosen that $e_*^{\pi\frac{1}{2i\hbar}\langle \boldsymbol{u}g,\boldsymbol{u}g\rangle_*}$ is sitting in the same sheet as in $e_*^{0\frac{1}{2i\hbar}\langle \boldsymbol{u}g,\boldsymbol{u}g\rangle_*}$, and it will be denoted by $\hat\varepsilon_{00}$. In precise,

$$(4.6)\qquad \hat\varepsilon_{00}=e_*^{[0\approx\pi]\frac{1}{2i\hbar}\langle \boldsymbol{u}g,\boldsymbol{u}g\rangle_*},\quad {:}\hat\varepsilon_{00}{:}_K=\frac{1}{\sqrt{\det(\cos([0{\approx}1]\frac{\pi}{2})I-(\sin([0{\approx}1]\frac{\pi}{2})^t gKg)}}e^{-\frac{1}{i\hbar}\langle \boldsymbol{u}\frac{1}{K},\boldsymbol{u}\rangle}$$

but a little care is required for the r.h.s., for the sheet is not distinguished by the notation itself.

Since singular points and slits are not sitting $\pi/2$-periodically but only π-periodically, the square $\hat\varepsilon_{00}^2=\hat\varepsilon_{00}{*}\hat\varepsilon_{00}$ is defined only with sign ambiguity (cf.(3.2)). That is, $\hat\varepsilon_{00}^2=\pm 1$ and the sign depends on g and K discontinuously, while $\hat\varepsilon_{00}^4=1$ (cf. Proposition 4.2).

But recall here that change of generators is covered by change of expression parameters. Hence the same phenomenon must occur in the change of expression parameters even when g is fixed.

4.2 Sign-changing by the order of continuous tracing

Recall § 3.3.2. We have discussed the product formula $e_*^{sH_*}{*}e_*^{tK_*}$ for the case $[H_*,K_*]=0$ in Propositions 3.8, 3.9. In this section, we consider the the case $[H_*,K_*]\neq 0$ and we give the product formula corresponding to Propositions 3.8, 3.9.

As it is mentioned before, the product

$${:}e_*^{t\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle}{*}f{:}_K,\qquad {:}f{*}e_*^{-t\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle}{:}_K$$

are defined by the left/right evolution equations

$$\frac{d}{dt}f_t=\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle{*}f_t,\quad \frac{d}{dt}f_t=f_t{*}\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle,$$

with initial data f Suppose f is another $*$-exponential function ${:}e_*^{t\langle \boldsymbol{u}(\frac{1}{2i\hbar}\beta J),\boldsymbol{u}\rangle}{:}_K$.

$${:}e_*^{t\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle}{*}(f{*}e_*^{-t\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle}){:}_K\quad {:}(e_*^{t\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle}{*}f){*}e_*^{-t\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle}{:}_K\quad \text{e.t.c.}$$

are defined holomorphically, but multi-valued in t on an open connected domain D containing the origin $0\in\mathbb{C}$.

Even in such a case, we can fix the value by tracing along a real analytic path from 0. We have then the following synchronized associativity:

Theorem 4.2 *Whenever the same path is used to fix the value in both sides, associativity*

$${:}e_*^{t\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle}{*}(f{*}e_*^{-t\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle}){:}_K={:}(e_*^{t\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle}{*}f){*}e_*^{-t\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle}{:}_K$$

holds and differentiating this gives as in § 3.1

$$:e_*^{t\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle}*f*e_*^{-t\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle}:_K = :e^{\mathrm{tad}(\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle)}f:_K$$

Using Theorem 1.3 and Theorem 4.2, we see also the following:

Corollary 4.1 *Suppose* $e_*^{s\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle}$ *and* $e_*^{[0\sim t]\langle \boldsymbol{u}(\frac{1}{2i\hbar}\beta J),\boldsymbol{u}\rangle}$ *are defined, where* $[0\sim\pi]$ *is a real analytic curve in* $\mathbb{C}$ *joining* 0 *to* π *avoiding singular points.*

Since for every fixed s*,*

$$e_*^{s\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle}*e_*^{t\langle \boldsymbol{u}(\frac{1}{2i\hbar}\beta J),\boldsymbol{u}\rangle}*e_*^{-s\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle}$$

are defined as a multi-valued holomorphic element on an open connected neighbourhood of $[0\sim\pi]$*, we see*

$$:\mathrm{Ad}(e_*^{s\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle})e_*^{[0\sim t]\langle \boldsymbol{u}(\frac{1}{2i\hbar}\beta J),\boldsymbol{u}\rangle}:_K=:e_*^{[0\sim t]\langle \boldsymbol{u}(\frac{1}{2i\hbar}\tilde{\beta}(s)J),\boldsymbol{u}\rangle}:_K,\ \ \tilde{\beta}(s)=e^{s\alpha}\beta e^{-s\alpha}$$

hold without sign ambiguity, where $[0\sim t]$ *in the r.h.s. is the path naturally given by the adjoint action for the path of the l.h.s.*

In particular, $:e_*^{[0\to t]\langle \boldsymbol{u}(\frac{1}{2i\hbar}\beta J),\boldsymbol{u}\rangle}:_K$ *and* $:e_*^{[0\to t]\langle \boldsymbol{u}(\frac{1}{2i\hbar}\tilde{\beta}(s)J),\boldsymbol{u}\rangle}:_K$ *must have the same periodicity.*

Here we used the same notation as in previous section to stress that $e_*^{[0\to t]\langle \boldsymbol{u}(\frac{1}{2i\hbar}\tilde{\beta}(s)J),\boldsymbol{u}\rangle}$ is defined by solving the evolution equation along the real segment $[0,t]$:

$$\frac{d}{dt}f_*(t) = \langle \boldsymbol{u}(\frac{1}{2i\hbar}\tilde{\beta}(s)J),\boldsymbol{u}\rangle*f_*(t), \quad f_*(0)=1. \tag{4.7}$$

Consider now $\mathrm{Ad}(e_*^{s\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle})e_*^{t\langle \boldsymbol{u}(\frac{1}{2i\hbar}\beta J),\boldsymbol{u}\rangle}$ of two variables $(s,t)\in[0,\pi]\times[0,\pi]$. Note that Corollary 4.1 holds even if there is a singular point (s_0,t_0) in the open square $(0,\pi)\times(0,\pi)$, but there happens another phenomenon of change sheets depending on the order of continuous tracing of values.

By the observation Proposition 3.9 in the previous section, we see that the singular points in $\mathbb{C}^2$ forms a set S of complex codimension 1, which is transversal to the real plane. One may assume that $S\cap\mathbb{R}^2$ is a discrete set. Suppose now there is a singular point (s_0,t_0) in the open square $(0,\pi)\times(0,\pi)$. Then, there must be a slit starting from (s_0,t_0) going outside the square. In what follows, we see that $*$-exponential function $e_*^{t\langle \boldsymbol{u}(\frac{1}{2i\hbar}\tilde{\beta}(s_0)J),\boldsymbol{u}\rangle}$ is discontinuous at $t=t_0$.

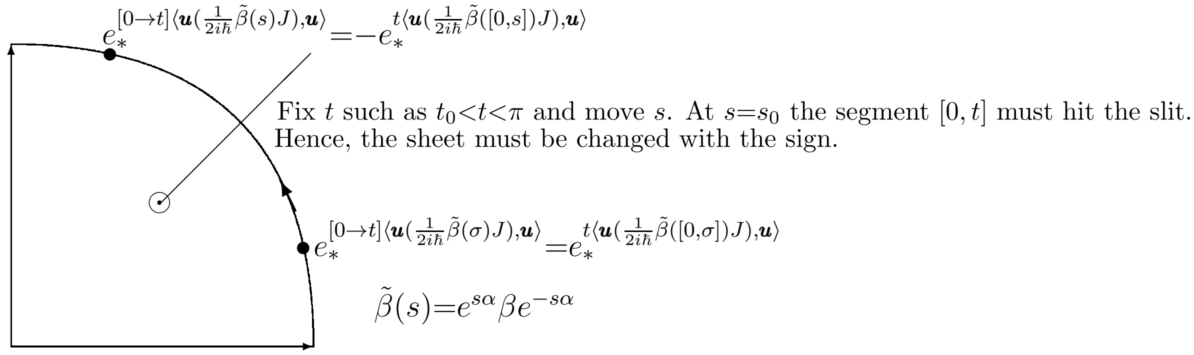
Hence fixing t as $t_0<t<\pi$ and tracing $e_*^{t\langle \boldsymbol{u}(\frac{1}{2i\hbar}\tilde{\beta}(s)J),\boldsymbol{u}\rangle}$ by moving s from $s=0$, the curve must hit the slit and changes the sheet.

As it mentioned in Proposition 3.9, the sheet changing gives

$$e_*^{t\langle \boldsymbol{u}(\frac{1}{2i\hbar}\tilde{\beta}([0,s])J),\boldsymbol{u}\rangle} = -e_*^{[0\to t]\langle \boldsymbol{u}(\frac{1}{2i\hbar}\tilde{\beta}(s)J),\boldsymbol{u}\rangle} \tag{4.8}$$

where the l.h.s. is the element obtained by tracing continuously from $e_*^{t\langle \boldsymbol{u}(\frac{1}{2i\hbar}\tilde{\beta}(0)J),\boldsymbol{u}\rangle} = e_*^{t\langle \boldsymbol{u}(\frac{1}{2i\hbar}\beta J),\boldsymbol{u}\rangle}$ to $e_*^{t\langle \boldsymbol{u}(\frac{1}{2i\hbar}\beta(s)J),\boldsymbol{u}\rangle}$ under a fixed t.

One may understand how the sign changes by noting the difference $([0,t],s)$ and $(t,[0,s])$ in the next picture.



Now the formal associativity Theorem 1.3 gives the translation identity from the right evolution equation into the left evolution equation:

$$:e_*^{s\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle} * e_*^{[0\to t]\langle \boldsymbol{u}(\frac{1}{2i\hbar}\beta J),\boldsymbol{u}\rangle}:_K = :e_*^{[0\to t]\langle \boldsymbol{u}(\frac{1}{2i\hbar}\tilde\beta(s)J),\boldsymbol{u}\rangle} * e_*^{s\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle}:_K. \tag{4.9}$$

On the other hand, note that $e_*^{t\langle \boldsymbol{u}(\frac{1}{2i\hbar}\tilde\beta([0,s])J),\boldsymbol{u}\rangle}$ is the solution of

$$\frac{d}{d\eta}f_*(\eta) = [\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle, f_*(\eta)], \quad f_*(0)=e_*^{t\langle \boldsymbol{u}(\frac{1}{2i\hbar}\beta J),\boldsymbol{u}\rangle}, \quad t \text{ is fixed.} \tag{4.10}$$

Since this is real analytic, the formal associativity theorem gives for fixed t that

$$:e_*^{[0\to s]\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle} * e_*^{t\langle \boldsymbol{u}(\frac{1}{2i\hbar}\beta J),\boldsymbol{u}\rangle}:_K = :e_*^{t\langle \boldsymbol{u}(\frac{1}{2i\hbar}\tilde\beta([0,s])J),\boldsymbol{u}\rangle} * e_*^{[0\to s]\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle}:_K. \tag{4.11}$$

If we use the tracing (4.8), then we have

$$\begin{aligned} :e_*^{[0\to s]\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle} * e_*^{t\langle \boldsymbol{u}(\frac{1}{2i\hbar}\beta J),\boldsymbol{u}\rangle}:_K &= -:e_*^{[0\to t]\langle \boldsymbol{u}(\frac{1}{2i\hbar}\tilde\beta(s)J),\boldsymbol{u}\rangle} * e_*^{[0\to s]\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle}:_K \\ &= -:e_*^{[0\to t]\langle \boldsymbol{u}(\frac{1}{2i\hbar}\tilde\beta(s)J),\boldsymbol{u}\rangle} * e_*^{s\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle}:_K \end{aligned}$$

for $e_*^{[0\to s]\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle}$ on the right hand side can be replaced simply by $e_*^{s\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle}$ without changing meaning. It follows a little tricky result as follows:

Theorem 4.3 *If the square $[0,s]\times[0,t]$ contains no singular point, then the identify*

$$:e_*^{[0\to s]\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle} * e_*^{t\langle \boldsymbol{u}(\frac{1}{2i\hbar}\beta J),\boldsymbol{u}\rangle}:_K. = :e_*^{[0\to t]\langle \boldsymbol{u}(\frac{1}{2i\hbar}\tilde\beta(s)J),\boldsymbol{u}\rangle} * e_*^{s\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle}:_K$$

holds, but if the square $[0,s]\times[0,t]$ contains a singular point (s_0,t_0) in the interior, then

$$:e_*^{[0\to s]\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle} * e_*^{t\langle \boldsymbol{u}(\frac{1}{2i\hbar}\beta J),\boldsymbol{u}\rangle}:_K. = -:e_*^{[0\to t]\langle \boldsymbol{u}(\frac{1}{2i\hbar}\tilde\beta(s)J),\boldsymbol{u}\rangle} * e_*^{s\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle}:_K$$

since the sheet is exchanged.

Corollary 4.2 *Suppose $\tilde\beta(\pi)=e^{\pi\alpha}\beta e^{-\pi\alpha}=-\beta$. If there is no singular point in $(0,\pi)\times(0,\pi)$, then*

$$:e_*^{[0\to \pi]\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle} * e_*^{\pi\langle \boldsymbol{u}(\frac{1}{2i\hbar}\beta J),\boldsymbol{u}\rangle}:_K. = :e_*^{[0\to \pi]\langle \boldsymbol{u}(\frac{-1}{2i\hbar}\beta J),\boldsymbol{u}\rangle} * e_*^{\pi\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle}:_K$$

but if there is a singular point in $(0,\pi)\times(0,\pi)$, then

$$:e_*^{[0\to \pi]\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle} * e_*^{\pi\langle \boldsymbol{u}(\frac{1}{2i\hbar}\beta J),\boldsymbol{u}\rangle}:_K. = -:e_*^{[0\to \pi]\langle \boldsymbol{u}(\frac{-1}{2i\hbar}\beta J),\boldsymbol{u}\rangle} * e_*^{\pi\langle \boldsymbol{u}(\frac{1}{2i\hbar}\alpha J),\boldsymbol{u}\rangle}:_K$$

The relation such as $e^{\pi\alpha}\beta e^{-\pi\alpha}=-\beta$ appears naturally in the next section, but the relation in Corollary 4.2 is *not* a classical relation, for such a relation does not hold in the limit $\hbar\to 0$.

4.2.1 Formula obtained by adjoint relations

In this subsection, we apply these results to the case $m=1$. First of all, we recall

Proposition 4.4 *In a generic ordered expression K, $:e_*^{\frac{t}{i\hbar}(au^2+bv^2+2cu{\circ}v)}:_K$ has no singular point on the real line and the pure imaginary line.*

Providing $c^2-ab=1$, polar element $:e_*^{\frac{\pi i}{2i\hbar}(au^2+bv^2+2cu{\circ}v)}:_K$ depends only on K and the path from 0 to π.

Except otherwise stated, the path is chosen as the segment $[0{\to}\pi]$.

Note first that $\frac{1}{i\hbar}[u{\circ}v, \begin{bmatrix}u\\v\end{bmatrix}]=\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}$. It follows that $e^{\frac{it}{i\hbar}\mathrm{ad}(u\circ v)}\begin{bmatrix}u\\v\end{bmatrix}=\begin{bmatrix}e^{it}&0\\0&e^{-it}\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}$, and hence for any $*$-function such as $f_*(u,v,\hbar)=\frac{1}{2\pi}\int_{\mathbb{R}^2}\hat{f}(s,t,\hbar)e_*^{\frac{1}{i\hbar}(su+tv)}dsdt$ depending real analytically on $\hbar$ in some interval involving $\hbar=0$, we have

$$e_*^{\frac{is}{i\hbar}u\circ v}{*}f_*(u,v,\hbar){*}e_*^{-\frac{is}{i\hbar}u\circ v}{=}f_*(e^{is}u,e^{-is}v,\hbar)$$

by the formal associativity theorem. Furthermore, we have by the same reason that

$$e_*^{\frac{is}{i\hbar}u\circ v}{*}f_*(u,v,\hbar){=}f_*(e^{is}u,e^{-is}v,\hbar){*}e_*^{\frac{is}{i\hbar}u\circ v}. \tag{4.12}$$

$$\frac{1}{2i\hbar}[u^2{-}v^2,\begin{bmatrix}u\\v\end{bmatrix}]=\begin{bmatrix}0&-1\\-1&0\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix},\quad \frac{1}{2i\hbar}[u^2{+}v^2,\begin{bmatrix}u\\v\end{bmatrix}]=\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}.$$

It follows that

$$e^{\frac{it}{2i\hbar}\mathrm{ad}(u^2-v^2)}\begin{bmatrix}u\\v\end{bmatrix}=\begin{bmatrix}\cos t&-i\sin t\\-i\sin t&\cos t\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix},\quad e^{\frac{t}{2i\hbar}\mathrm{ad}(u^2+v^2)}\begin{bmatrix}u\\v\end{bmatrix}=\begin{bmatrix}\cos t&\sin t\\-\sin t&\cos t\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}.$$

Now, even if f_* is a $*$-exponential function of quadratic form $:e_*^{t\langle\boldsymbol{u}(\frac{1}{2i\hbar}\beta J),\boldsymbol{u}\rangle}:_K$, we can make several commutation relations by using the product formula (2.11). But for that purpose, we have to use *synchronized* path in both sides.

Polar elements are splitting

We next compute the case $c{=}0$ and $\delta\delta'{\neq}\pm 1$ in (3.18) i.e. $K=diag\{\delta,\delta'\}$. Then,

$$\begin{aligned}&\sqrt{\Delta^2{-}(e^t{-}e^{-t})^2\delta\delta'}\\&{=}\sqrt{1{-}\delta\delta'}e^{-t}\sqrt{e^{4t}{+}2\frac{1{+}\delta\delta'}{1{-}\delta\delta'}e^{2t}{+}1}{=}\sqrt{1{-}\delta\delta'}e^{-t}\sqrt{(e^{2t}{+}\beta)(e^{2t}{+}\beta^{-1})}\end{aligned}$$

where $\beta=\frac{1+\sqrt{\delta\delta'}}{1-\sqrt{\delta\delta'}}$. If $|\beta|\neq 1$ i.e. $\delta\delta'\not\in\mathbb{R}_{<0}$ then only one of $\sqrt{e^{2t}{+}\beta}$ or $\sqrt{e^{2t}{+}\beta^{-1}}$ changes sign when t moves from 0 to πi. Thus, this is the case where the singular points are distributed πi-periodically along two lines parallel to the imaginary axis both positive and negative real parts, whose real parts are $\pm\log\left|\frac{\sqrt{\delta\delta'}+1}{\sqrt{\delta\delta'}-1}\right|$, and $:e_*^{\frac{t}{i\hbar}2u\circ v}:_K$ is πi-periodic along the imaginary axis.

Lemma 4.5 *Suppose* $K=\begin{bmatrix}\delta & 0\\ 0 & \delta'\end{bmatrix}$ *such that* $\delta\delta'\neq 0,1$, *and* $\delta\delta'$ *is not a negative real. Then,* $:e_*^{\frac{t}{i\hbar}2u{\circ}v}:_K$ *is* πi*-periodic along the pure imaginary axis and singular points distributed* πi*-periodically along two lines parallel to the imaginary axis both positive and negative real parts.*

If $\delta\delta'=1$, *then* $:e_*^{\frac{t}{i\hbar}2u{\circ}v}:_K$ *is* πi*-periodic along the pure imaginary axis and there is no singular point.*

However, if $|\beta|=1$ i.e. $\delta\delta'$ is negative real, then $:e_*^{t\frac{1}{i\hbar}2u{\circ}v}:_K$, $K=diag\{\delta,\delta'\}$ has two branching singular points on the open interval $i(0,\pi)$. Even if this is the case, one may change δ,δ' slightly so that $\delta\delta'$ avoids negative real and 1. By this procedure, we have the same periodical nature and the pattern of singularities as above.

We next change the generator by $(u,v)\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$, and the expression parameters by two different ways:

$$
(4.13)\qquad
\begin{aligned}
K_{re}&=\frac{1}{2}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}\rho{-}ic' & 0\\ 0 & \rho{+}ic'\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}=\begin{bmatrix}\rho & ic'\\ ic' & \rho\end{bmatrix}\\
K_{im}&=\frac{1}{2}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}i\rho{-}c & 0\\ 0 & i\rho{+}c\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}=\begin{bmatrix}i\rho & c\\ c & i\rho\end{bmatrix}.
\end{aligned}
$$

Then for both cases, we see by (1.10) that

$$
:e_*^{\frac{t}{i\hbar}(u^2-v^2)}:_{K_{re}}=:e_*^{\frac{t}{i\hbar}2u'v'}:_{\hat{K}_0},\qquad :e_*^{\frac{t}{i\hbar}(u^2-v^2)}:_{K_{im}}=:e_*^{\frac{t}{i\hbar}2u'v'}:_{K_0'},
$$

where $u'=\frac{1}{\sqrt{2}}(u-v)$, $v'=\frac{1}{\sqrt{2}}(u+v)$, $\hat{K}_0=diag\{\rho{-}ic,\rho{+}ic\}$, $K_0'=diag\{i\rho{-}c,i\rho{+}c\}$.

Note now that

$$
\begin{bmatrix}\cos r & i\sin r\\ i\sin r & \cos r\end{bmatrix}\subset\begin{bmatrix}\rho & ic\\ ic & \rho\end{bmatrix},\quad \rho,c\in\mathbb{R},
$$

and

$$
\begin{bmatrix}\cos r & i\sin r\\ i\sin r & \cos r\end{bmatrix},\quad \begin{bmatrix}\cos s & -\sin s\\ sins & \cos s\end{bmatrix},\quad \begin{bmatrix}e^{i\theta} & 0\\ 0 & e^{-i\theta}\end{bmatrix}
$$

generate $SU(2)$.

By these observation, we have first the following:

Proposition 4.5 *If* $K_{re}=\begin{bmatrix}\rho & ic'\\ ic' & \rho\end{bmatrix}$ *with* $c',\rho\in\mathbb{R}$, *satisfies* $|\frac{1+\rho+ic'}{1-\rho-ic'}|{\neq}1$, *then* K_{re} *ordered expressions of those three* $*$*-exponential functions*

$$
e_*^{\frac{it}{i\hbar}2u{\circ}v},\quad e_*^{\frac{t}{i\hbar}(u^2+v^2)},\quad e_*^{\frac{it}{i\hbar}(u^2-v^2)},
$$

have no singular point on the real axis and π*-periodic, but each of them has singular points sitting* π*-periodically along two lines parallel to the real axis on both upper and lower half plane.*

Hence, the polar element ε_{00} *may be written in the* K_{re}*-expression by*

$$
:\varepsilon_{00}:_{K_{re}}=:e_*^{\frac{\pi i}{i\hbar}u{\circ}v}:_{K_{re}}=:e_*^{\frac{\pi i}{2i\hbar}(u^2-v^2)}:_{K_{re}}=:e_*^{-\frac{\pi}{2i\hbar}(u^2+v^2)}:_{K_{re}}.
$$

and $\varepsilon_{00}^2 = 1$. *Therefore, we have three square roots*

$$e_1 = e_*^{\frac{\pi i}{2i\hbar}u\circ v}, \quad e_2 = e_*^{\frac{\pi}{4i\hbar}(u^2+v^2)}, \quad e_3 = e_*^{\frac{\pi i}{4i\hbar}(u^2-v^2)}$$

such that $e_i^2 = \varepsilon_{00}$.

To avoid possible confusion, we restrict the expression parameter in the class K_{re} in what follows.

For every s, (4.12) and Corollary 4.1 gives in generic ordered expression K that

$$\begin{aligned}
&:e_*^{\frac{\pi i}{4i\hbar}(u\circ v)}*e_*^{[0\sim s]\frac{1}{i\hbar}(u_*^2+v_*^2)}*e_*^{-\frac{\pi i}{4i\hbar}(u\circ v)}:_K = :e_*^{[0\sim s]\frac{i}{i\hbar}(u_*^2-v_*^2)}:_K,\\
&:e_*^{\frac{\pi i}{2i\hbar}(u\circ v)}*e_*^{[0\sim s]\frac{1}{i\hbar}(u_*^2+v_*^2)}*e_*^{-\frac{\pi i}{2i\hbar}(u\circ v)}:_K = :e_*^{[0\sim s]\frac{-1}{i\hbar}(u_*^2+v_*^2)}:_K
\end{aligned} \tag{4.14}$$

without sign ambiguity, where $[0{\sim}s]$ in the l.h.s. is a path from 0 to s in a complex plane on which there is no singular point, and $[0{\sim}s]$ in the r.h.s. is the path given naturally by the adjoint transformation.

We have also

$$\begin{aligned}
&e_*^{\frac{\pi i}{4i\hbar}u\circ v}*e_*^{\frac{i[0\sim s]}{i\hbar}(u_*^2-v_*^2)}*e_*^{-\frac{\pi i}{4i\hbar}u\circ v} = e_*^{-\frac{[0\sim s]}{i\hbar}(u_*^2+v_*^2)},\\
&e_*^{\frac{\pi i}{4i\hbar}u\circ v}*e_*^{\frac{i[0\sim s]}{i\hbar}(u_*^2-v_*^2)} = e_*^{-\frac{[0\sim s]}{i\hbar}(u_*^2+v_*^2)}*e_*^{\frac{\pi i}{4i\hbar}u\circ v}
\end{aligned} \tag{4.15}$$

Taking the synchronized use of path, we have

$$\begin{aligned}
&e_*^{\frac{\pi}{4i\hbar}(u_*^2+v_*^2)}*e_*^{\frac{i[0\sim s]}{i\hbar}u\circ v} = e_*^{-\frac{i[0\sim s]}{i\hbar}u\circ v}*e_*^{\frac{\pi}{4i\hbar}(u_*^2+v_*^2)},\\
&e_*^{\frac{\pi i}{4i\hbar}(u_*^2-v_*^2)}*e_*^{\frac{i[0\sim s]}{i\hbar}u\circ v} = e_*^{-\frac{i[0\sim s]}{i\hbar}u\circ v}*e_*^{\frac{\pi i}{4i\hbar}(u_*^2-v_*^2)}.
\end{aligned} \tag{4.16}$$

In these notations, we have also

$$e_*^{\frac{\pi i}{4i\hbar}u\circ v}*e_*^{\frac{[0\sim\pi]}{4i\hbar}(u_*^2+v_*^2)}*e_*^{-\frac{\pi i}{4i\hbar}u\circ v} = e_*^{\frac{i[0\sim\pi]}{4i\hbar}(u_*^2-v_*^2)},$$

Applying the second equality of (4.16) to the part $e_*^{\frac{[0\sim\pi]}{4i\hbar}(u^2+v^2)}*e_*^{-\frac{\pi i}{4i\hbar}u\circ v}$, we have

$$e_*^{\frac{\pi i}{4i\hbar}u\circ v}*e_*^{\frac{[0\sim\pi]}{4i\hbar}(u^2+v^2)}*e_*^{-\frac{\pi i}{4i\hbar}u\circ v} = e_*^{\frac{\pi i}{4i\hbar}u\circ v}*e_*^{\frac{\pi i}{4i\hbar}u\circ v}*e_*^{\frac{[0\sim\pi]}{4i\hbar}(u^2+v^2)} = e_*^{\frac{i[0\sim\pi]}{4i\hbar}(u^2-v^2)}$$

hold. This may be written simply by

$$e_*^{\frac{\pi i}{2i\hbar}u\circ v}*e_*^{\frac{\pi}{4i\hbar}(u^2+v^2)} = e_*^{\frac{\pi i}{4i\hbar}(u^2-v^2)} \tag{4.17}$$

Note also that (4.16) yields a tricky result as follows:

Proposition 4.6 *The polar element* $e_*^{\frac{i[0\sim\pi]}{i\hbar}u\circ v}$ *satisfies the equality*

$$e_*^{\frac{\pi}{4i\hbar}(u_*^2+v_*^2)}*e_*^{\frac{i[0\sim\pi]}{i\hbar}u\circ v} = e_*^{-\frac{i[0\sim\pi]}{i\hbar}u\circ v}*e_*^{\frac{\pi}{4i\hbar}(u_*^2+v_*^2)}.$$

Hence, such a polar element commutes with another square root of a polar element, if and only if $e_*^{-\frac{i[0\sim\pi]}{i\hbar}u\circ v} = e_*^{\frac{i[0\sim\pi]}{i\hbar}u\circ v}$.

Recall first (4.17). This gives $e_1{*}e_2 = e_3$ in the K_{re}-ordered expression.

Generally, adjoint relations of quadratic forms give the following master relations for elements of square roots of the polar element.

Lemma 4.6 *Let H_* be a quadratic form with the discriminant 1. Then, $e^{\pi i \mathrm{ad}(H_*)}e_j{=}e_j^{-1}$. This implies that $e_i{*}e_j{*}e_i^{-1}{=}e_j^{-1}$ by Theorem 4.2. These relations hold without sign ambiguity.*

Proof The first equality is easy to see. The second identity is a special case of the identity which is proved by using formal associativity theorem. $\square$

By the master relation, we have in general

$$e_i{*}e_j = e_j^{-1}{*}e_i = \varepsilon_{00}{*}e_j{*}e_i.$$

By the identity $e_3 = e_1{*}e_2$, we have

$$e_2{*}e_3 = e_2{*}e_1{*}e_2 = e_2{*}e_2^{-1}{*}e_1 = e_1.$$

Similarly,

$$e_3{*}e_1 = e_3{*}e_2{*}e_3 = e_3{*}e_3^{-1}{*}e_2 = e_2.$$

Note that all e_i are elements of $\mathcal{E}_{2+}(\mathbb{C}^2)$. Hence, we have

Theorem 4.4 *In the K_{re}-ordered expression such that $|\frac{1+\rho+ic'}{1-\rho-ic'}|{\neq}1$, $\{\varepsilon_{00}, e_1, e_2, e_3\}$ generates an algebra $\mathcal{A}$ where exist two idempotent elements $\frac{1}{2}(1{+}\varepsilon_{00})$, $\frac{1}{2}(1{-}\varepsilon_{00})$ such that*

$$1 = \frac{1}{2}(1{+}\varepsilon_{00}){+}\frac{1}{2}(1-\varepsilon_{00}), \quad \frac{1}{2}(1{+}\varepsilon_{00}){*}\frac{1}{2}(1-\varepsilon_{00}) = 0.$$

The subalgebra $\frac{1}{2}(1{-}\varepsilon_{00}){*}\mathcal{A}$ is naturally isomorphic to the complexification $\mathbb{C}{\otimes}\mathbb{H}$ of the quaternion field $\mathbb{H}$ such that by denoting $\hat{1} = \frac{1}{2}(1{-}\varepsilon_{00})$

$$\hat{\varepsilon}_{00} = \frac{1}{2}(1{-}\varepsilon_{00}){*}\varepsilon_{00} = -\hat{1}, \quad \hat{e}_i^2 = -\hat{1}, \quad \hat{e}_i{*}\hat{e}_j = -\hat{1}{*}\hat{e}_j{*}\hat{e}_i, \quad 1 \le i, j \le 3,$$

where $\hat{e}_i = \frac{1}{2}(1{-}\varepsilon_{00}){*}e_i$, and the subalgebra $\frac{1}{2}(1{+}\varepsilon_{00}){*}\mathcal{A}$ is the group ring over $\mathbb{C}$ of the Klein's four group.

4.3 Independence of ordering principle and its failure

In differential geometry, it is widely accepted that geometrical notion should have coordinate free expression. Obviously, algebraic structure of $(\mathbb{C}[\boldsymbol{u}], *_\Lambda)$ depends only on the skew part of Λ. It seems reasonable to accept the independence of ordering principle as a basic principle that the physical implication should be independent of ordered expressions. Theorem 1.1 supports this principle for elements in a class $\mathcal{E}_2(\mathbb{C}^n)$.

By a direct calculation of intertwiner, we see that

$$I_K^{K'}(e^{\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}) = e^{\frac{1}{4i\hbar}\langle \boldsymbol{a}(K'-K),\boldsymbol{a}\rangle}e^{\frac{1}{i\hbar}\langle \boldsymbol{a},\boldsymbol{u}\rangle}. \tag{4.18}$$

Hence, $\{e^{\frac{1}{4i\hbar}\langle\boldsymbol{a}K,\boldsymbol{a}\rangle}e^{\frac{1}{i\hbar}\langle\boldsymbol{a},\boldsymbol{u}\rangle}; K\in\mathfrak{S}_{\mathbb{C}}(2m)\}$ is a parallel section of $\coprod_{K\in\mathfrak{S}_{\mathbb{C}}(2m)} Hol(\mathbb{C}^{2m})$.

We denoted this collection symbolically by $e_*^{\frac{1}{i\hbar}\langle\boldsymbol{a},\boldsymbol{u}\rangle}$ and we regard each member

$$:e_*^{\frac{1}{i\hbar}\langle\boldsymbol{a},\boldsymbol{u}\rangle}:_K = e^{\frac{1}{4i\hbar}\langle\boldsymbol{a}K,\boldsymbol{a}\rangle}e^{\frac{1}{i\hbar}\langle\boldsymbol{a},\boldsymbol{u}\rangle} = e^{\frac{1}{4i\hbar}\langle\boldsymbol{a}K,\boldsymbol{a}\rangle+\frac{1}{i\hbar}\langle\boldsymbol{a},\boldsymbol{u}\rangle} \tag{4.19}$$

as its K-expression. Furthermore for every K, $e_*^{\frac{s}{i\hbar}\langle\boldsymbol{a},\boldsymbol{u}\rangle}$ is the solution of the evolution equation

$$\frac{d}{dt}{:}e_*^{\frac{s}{i\hbar}\langle\boldsymbol{a},\boldsymbol{u}\rangle}{:}_K = \frac{1}{i\hbar}{:}\langle\boldsymbol{a},\boldsymbol{u}\rangle{:}_K *_K {:}e_*^{\frac{s}{i\hbar}\langle\boldsymbol{a},\boldsymbol{u}\rangle}{:}_K \text{ with initial data } {:}1{:}_K = 1.$$

Note also that ${:}\langle\boldsymbol{a},\boldsymbol{u}\rangle{:}_K = \langle\boldsymbol{a},\boldsymbol{u}\rangle$. $e_*^{s\frac{1}{i\hbar}\langle\boldsymbol{a},\boldsymbol{u}\rangle} = \{e^{s^2\frac{1}{4i\hbar}\langle\boldsymbol{a}K\boldsymbol{a}\rangle}e^{s\frac{1}{i\hbar}\langle\boldsymbol{a},\boldsymbol{u}\rangle}; K\in\mathfrak{S}(2m)\}$ forms a one parameter group of parallel sections. The product formula in K-ordered expression gives the exponential law ${:}e_*^{s\frac{1}{i\hbar}\langle\boldsymbol{a},\boldsymbol{u}\rangle}{:}_K *_K {:}e_*^{t\frac{1}{i\hbar}\langle\boldsymbol{a},\boldsymbol{u}\rangle}{:}_K = {:}e_*^{(s+t)\frac{1}{i\hbar}\langle\boldsymbol{a},\boldsymbol{u}\rangle}{:}_K$ for every $K\in\mathfrak{S}(2m)$. Hence, this may be written by omitting the suffix K as $e_*^{s\frac{1}{i\hbar}\langle\boldsymbol{a},\boldsymbol{u}\rangle} * e_*^{t\frac{1}{i\hbar}\langle\boldsymbol{a},\boldsymbol{u}\rangle} = e_*^{(s+t)\frac{1}{i\hbar}\langle\boldsymbol{a},\boldsymbol{u}\rangle}$. The product formula may be written as

$$e_*^{\frac{1}{i\hbar}\langle\boldsymbol{a},\boldsymbol{u}\rangle} * e_*^{\frac{1}{i\hbar}\langle\boldsymbol{b},\boldsymbol{u}\rangle} = e^{\frac{1}{2i\hbar}\langle\boldsymbol{a}J,\boldsymbol{b}\rangle}e_*^{\frac{1}{i\hbar}\langle(\boldsymbol{a}+\boldsymbol{b}),\boldsymbol{u}\rangle}. \tag{4.20}$$

The main point is that we do not use operator theory, but instead various ordered expressions under the leading principle that a physical/mathematical object should be free from ordered expressions (**the independence of ordering principle**, (IOP) in short), just as geometrical objects are independent of local coordinate expressions.

Recall this principle in geometry forced to accept the absolute abstract notion "underlying topological space" before a collection of local coordinate system. However, we saw in [13] that the topology of a set depends on expression parameters. That is,

$$\mathfrak{P}^{(2)}_{K_0} \cong SO(m,\mathbb{C})\times\mathbb{Z}_2, \quad \text{but} \quad \mathfrak{P}^{(2)}_{K_s} \cong Spin(m)\otimes\mathbb{C}.$$

Furthermore, if we apply this principle to our system, then it becomes a true nature that every linear form has two different inverses, for this holds for generic (open dense) expressions. In general, parallel sections of $\coprod_{K\in\mathfrak{S}(n)}\mathcal{E}_{2+}(\mathbb{C}^n)$ are multi-valued with branched singular points depending on expression parameters. It is difficult to explain multi-valued parallel section in a picture of point set topology. Thus we have to think twice about the role of expression parameters in geometry and physics.

4.3.1 Philosophy of general dynamics

It was widely accepted in classical physics that every dynamical movement must be caused by some Hamiltonian H. Another word, this is the definition of "dynamical movement". (IOP) is also widely accepted together with differential geometrical expressions, e.g. contact geometry, G-structures.

The philosophy was succeeded in non-relativistic quantum dynamics by replacing H by a quantum Hamiltonian. This is given by the evolution equation of every quantum observable $f_t(\boldsymbol{u})$:

$$\frac{d}{dt}f_t(\boldsymbol{u}) = [H, f_t(\boldsymbol{u})]_*$$

and the solution is given by $f_t = e^{t\mathrm{ad}(H)}f_0$, where $e^{t\mathrm{ad}(H)}$ acts on the space of quantum observables.

In the relativity theory, "time" t is never an absolute scalar, but a coordinate function of "space-time". Thus, the Hamiltonian H which governs a relativistic movement is given by $H = H(t, e(t, \boldsymbol{u}))$ involving t and the quantum canonical conjugate e of t. e is called the "energy" variable, relating each other by $e = e(t, \boldsymbol{u})$, or $t = t(e, \boldsymbol{u})$.

The equation of the relativistic movement is written by similar differential equation by using "proper time" τ viewed as the individual time of observer:

$$\frac{d}{d\tau}\phi_\tau(e, t, \boldsymbol{u}) = [H, \phi_\tau(e, t, \boldsymbol{u})], \quad \frac{d}{d\tau}e{*}t = [H, e{*}t] = 0,$$

where ϕ_τ is any quantum observable, and the solution is given by $\phi_\tau = e^{\tau \mathrm{ad}(H)}\phi_0$, where $e^{\tau \mathrm{ad}(H)}$ acts on the space of quantum observables. If one forgets about physics by neglecting the positivity of energy, such equations can be treated as Fourier integral operators, and the principle (IOP) remains safe.

However in the field theory, quantum observables ϕ_0 are regarded as operators acting on some pre-Hilbert space $\mathbb{H}$, and we are requested to have $e_*^{\tau H}$ acting on $\mathbb{H}$ with suitable associativity such that ϕ_τ may be written as

$$\phi_\tau = (e_*^{\tau H}{*}\phi_0){*}e_*^{-\tau H} = e_*^{\tau H}{*}(\phi_0{*}e_*^{-\tau H}).$$

On the other hand, as it is seen throughout this series of papers, $*$-exponential functions such as $e_*^{\tau H}$ often has branched singular point and the periodicity depends on the expression parameters. We have delicate problems of failing associativity related to **moving branched singular points**, which depends on expression parameters. Stone's theorem shows that there is no essential selfadjoint operator H such that $\int_{\mathbb{R}} e^{tH} dt$ is finite.

At a first glance it is natural to replace $e_*^{\tau H}$ by $\mathrm{Ad}(e_*^{\tau H})$ and operator representation of $\mathrm{Ad}(e_*^{\tau H})$. But this can not be the mathematical Messiah, because such strange phenomena are already involved in the transcendently extended algebra of ordinary calculus. Hence phisysists are required always the mathematical consistency. Strictly speaking, this means that physical phenomenon in field theory depends on how the element is expressed. Nature of individual element, in particular the nature of periodicity depends on expression parameters.

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