DEFORMATION OF ALGEBRAS ASSOCIATED TO GROUP COCYCLES

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Abstract. We define a deformation of algebras endowed with coaction of the reduced group algebras. The deformation parameter is given by a 2-cocycle over the group. We prove K-theory isomorphisms for the cocycles which can be perturbed to the trivial one.

1. Introduction

Deformation of algebras has been an important principle in the study of operator algebras and noncommutative geometry. The noncommutative torus, whose 'function algebra' is generated by two unitaries u, v satisfying $uv = e^{i\theta} vu$ for $\theta \in \mathbb{R}$, is one of the most famous examples which lead to many interesting ideas in the early studies of noncommutative geometry by Connes [Con85] and others. The relation between the generators of the algebra of noncommutative torus indicates that it can be thought as a deformation of the algebra of functions on the 2-torus.

It turned out that the noncommutative torus is an example of a more general deformation procedure called the θ-deformation due to Rieffel [Rie89b]. It takes any C^*-algebra admitting an action of a torus $\mathbb{T}^n$ as the original algebra, and the deformation parameter is given by a skewsymmetric form on $\mathbb{R}^n$. He showed that the θ-deformations have the same K-groups as the original algebras [Rie93], extending the case of the noncommutative torus by Pimsner–Voiculescu [PV80].

The noncommutative torus can be also thought as the twisted group algebra for the case of the discrete group $\mathbb{Z}^2$. Recently, a K-theory isomorphism result of the reduced twisted group algebras $C^*_{r,\omega}(\Gamma)$ was proved for any discrete group Γ satisfying the Baum–Connes conjecture with the compact operator algebra coefficient by Echterhoff et al. [ELPW10]. They showed that the K-groups of the twisted algebras $C^*_{r,\omega}(\Gamma)$ do not change if ω is given by a real 2-cocycle on Γ, which can be thought as a continuous perturbation of the trivial cocycle. We note that Mathai [Mat06] also proved the K-theory invariance under twisting by such cocycles for a slightly different class of groups, building on Lafforgue's Banach algebraic approach [Laf02] to the Baum–Connes conjecture.

In this paper we unify the two frameworks mentioned above. We thus define a way to deform the C^*-algebras admitting coactions of the compact quantum group $C^*_r(\Gamma)$ (in other words, cross sectional algebras of Fell bundles over Γ [Exe97]), and the deformation parameter is given by a U(1)-valued 2-cocycle on Γ. Our main result (Theorem 1) is that, when Γ satisfies the Baum–Connes conjecture with coefficients and the cocycle comes from an $\mathbb{R}$-valued 2-cocycle, the K-groups of the deformed algebra are isomorphic to those of the original algebra.

We also remark that there are several similar formalisms of deformation of operator algebras which do not fall into our approach. The deformation of Fell bundles due to Abadie–Exel [AE01] seems to be closest to ours. The deformation of

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function algebras of compact groups which appeared in the study of ergodic actions with full multiplicity by Wassermann [Was88b] is a close analogue of the twisted group algebra. Finally, there is a similar K-theoretic invariance result by Neshveyev–Tuset [NT11b] for certain C^*-algebraic compact quantum groups and its homogeneous spaces, for the q-deformations of simply connected simple compact Lie groups.

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2. Preliminaries

When A and B are C^*-algebras, $A \otimes B$ denotes their minimal tensor product unless otherwise specified. Likewise when H and K are Hilbert spaces, $H \otimes K$ denotes their tensor product as a Hilbert space. When H is a Hilbert space and X is a Hilbert C^*-module over A, we let $H \otimes X$ denote the tensor product Hilbert C^*-module over A. We let $\mathcal{L}(X)$ denote the algebra of the endomorphisms of a Hilbert C^*-module X.

When A is a C^*-algebra, we let $M(A)$ denote the multiplier algebra.

The coactions of locally compact quantum groups on C^*-algebras are assumed to be the continuous ones. The crossed products with respect to (co)actions of locally compact quantum groups on C^*-algebras are understood to be the reduced ones unless otherwise specified. Our convention is that, when α is an action of a discrete group Γ on a C^*-algebra $A \subset B(H)$, the reduced crossed product is the C^*-algebra generated by the operators $\lambda_g \otimes \mathrm{Id}_H$ for $g \in \Gamma$ and the ones

$$\tilde{\alpha}(a) \colon \delta_g \otimes \xi \mapsto \delta_g \otimes \alpha_g(a)\xi \quad (g \in \Gamma, \xi \in H, a \in A)$$

on $\ell^2(\Gamma) \otimes H$.

2.1. **Group cocycles.** Let Γ be a discrete group. When $(G, +)$ is a commutative group, a *G-valued 2-cocycle* ω on Γ is a map $\omega \colon \Gamma \times \Gamma \to G$ satisfying the cocycle identity

$$\omega(g_0, g_1) + \omega(g_0 g_1, g_2) = \omega(g_1, g_2) + \omega(g_0, g_1 g_2). \tag{1}$$

We always assume that ω satisfies the normalization condition

$$\omega(g, e) = \omega(e, g) = 1 \quad (g \in \Gamma).$$

In this paper we consider the cases $G = \mathbb{R}$ and $G = \mathrm{U}(1)$ as the target group of cocycles. Note that when ω is an $\mathbb{R}$-valued 2-cocycle, we obtain a $\mathrm{U}(1)$-valued cocycle $e^{i\omega}$ by putting $e^{i\omega}(g, h) = e^{i\omega(g,h)}$.

When ω is a $\mathrm{U}(1)$-valued 2-cocycle on Γ, *the twisted reduced group C^*-algebra* $C^*_{r,\omega}(\Gamma)$ is defined to be the C^*-algebraic span of the operators $\lambda^{(\omega)}_g \in B(\ell^2\Gamma)$ for $g \in \Gamma$ defined by

$$\lambda^{(\omega)}_g \delta_h = \omega(g, h)\delta_{gh}.$$

Given Γ and ω, we can consider the fundamental unitary $W = \sum_g \delta_g \otimes \lambda_g$ and another unitary operator $\sum_{g,h} \omega(g, h)\delta_g \otimes \delta_h$ representing ω, both on $\ell^2(\Gamma)^{\otimes 2}$. Then the unitary operator

$$W^{(\omega)} = W\omega \colon \delta_h \otimes \delta_k \mapsto \beta(h, k)\delta_h \otimes \delta_{hk} \tag{2}$$

in the von Neumann algebra $\ell^\infty(\Gamma) \bar{\otimes} B(\ell^2(\Gamma))$ is called the *regular ω-representation* unitary. The algebra $C^*_{r,\omega}(\Gamma)$ can be also defined as the C^*-algebraic span of the operators $\phi \otimes \iota(W^{(\omega)})$ for $\phi \in \ell^1(\Gamma) = \ell^\infty(\Gamma)_*$.

The generators $(\lambda_g^{(\omega)})_{g\in\Gamma}$ satisfy the relations

$$\lambda_g^{(\omega)}\lambda_h^{(\omega)} = \omega(g,h)\lambda_{gh}^{(\omega)}, \qquad (\lambda_g^{(\omega)})^* = \overline{\omega(g,g^{-1})}\lambda_{g^{-1}}^{(\omega)}.$$

From this we see that the vector state for δ_e is tracial. This trace is called the *standard trace* τ on $C^*_{r,\omega}(\Gamma)$.

Two cocycles ω and ω' are said to be *cohomologous* when there exists a map $\psi\colon \Gamma \to \mathrm{U}(1)$ satisfying

$$\psi(g)\psi(h)\omega(g,h)\overline{\psi(gh)} = \omega'(g,h).$$

If this is the case, the algebras $C^*_{r,\omega}(\Gamma)$ and $C^*_{r,\omega'}(\Gamma)$ are isomorphic via the map $\lambda_g^{(\omega)} \mapsto \overline{\psi(g)}\lambda_g^{(\omega')}$.

We let $\overline{\omega}$ denote the complex conjugate cocycle $\overline{\omega}(g,h) = \overline{\omega(g,h)}$. Then the twisted algebra $C^*_{r,\overline{\omega}}(\Gamma)$ is antiisomorphic to $C^*_{r,\omega}(\Gamma)$ as follows. The formula

$$\tilde{\omega}(g,h) = \omega(h^{-1},g^{-1}) \tag{3}$$

defines another cocycle on Γ. On one hand, the map $\lambda_g^{(\omega)} \mapsto \lambda_{g^{-1}}^{(\tilde{\omega})}$ defines an antiisomorphism from $C^*_{r,\omega}(\Gamma)$ to $C^*_{r,\tilde{\omega}}(\Gamma)$. On the other hand, the equality

$$\begin{aligned}\omega(h^{-1},g^{-1})\omega(g,h)\omega(gh,h^{-1}g^{-1}) &= \omega(h^{-1},g^{-1})\omega(h,h^{-1}g^{-1})\omega(g,g^{-1})\\ &= \omega(h,h^{-1})\omega(g,g^{-1})\end{aligned}$$

shows that $\overline{\omega}$ and $\tilde{\omega}$ are cohomologous with respect to the map $g \mapsto \omega(g,g^{-1})$.

2.2. **Crossed product presentation of twisted group algebras.** The reduced group algebra $C^*_r(\Gamma)$ admits the structure of (the function algebra of) a compact quantum group by the coproduct map $\delta(\lambda_g) = \lambda_g \otimes \lambda_g$.

Suppose that α and β are U(1)-valued 2-cocycles on Γ. Then, with the unitary regular β-representation unitary (2), we have

$$W^{(\beta)}(\lambda_g^{(\alpha\cdot\beta)} \otimes \mathrm{Id}_{\ell^2(\Gamma)})(W^{(\beta)})^* = \lambda_g^{(\alpha)} \otimes \lambda_g^{(\beta)}$$

for any $g \in \Gamma$. This way we obtain a C^*-algebra homomorphism

$$C^*_{r,\alpha\cdot\beta}(\Gamma) \to C^*_{r,\alpha}(\Gamma) \otimes C^*_{r,\beta}(\Gamma), \quad \lambda_g^{(\alpha\cdot\beta)} \mapsto \lambda_g^{(\alpha)} \otimes \lambda_g^{(\beta)}.$$

When either of α or β is trivial, we obtain the coactions

$$\delta_l^{(\omega)}\colon C^*_{r,\omega}(\Gamma) \to C^*_r(\Gamma) \otimes C^*_{r,\omega}(\Gamma), \qquad \delta_r^{(\omega)}\colon C^*_{r,\omega}(\Gamma) \to C^*_{r,\omega}(\Gamma) \otimes C^*_r(\Gamma)$$

of $C^*_r(\Gamma)$ on the twisted group algebra $C^*_{r,\omega}(\Gamma)$. Note that these two carry the same data because $C^*_r(\Gamma)$ is cocommutative.

The crossed product algebra $C^*_{r,\omega}(\Gamma) \rtimes_{\delta_r} C_0(\Gamma)$ with respect to the coaction $\delta_r^{(\omega)}$ is the C^*-algebra generated by $\delta_r^{(\omega)}(C^*_{r,\omega}(\Gamma))$ and $1 \otimes C_0(\Gamma)$ in $B(\ell^2(\Gamma) \otimes \ell^2(\Gamma))$.

This crossed product is actually isomorphic to the compact operator algebra

$$\mathcal{K}(\ell^2(\Gamma)) \simeq \Gamma \ltimes_\lambda C_0(\Gamma) \simeq C^*_r(\Gamma) \rtimes_{\delta_r} C_0(\Gamma),$$

where λ in the middle denotes the left translation action of Γ on $C_0(\Gamma)$. This isomorphism is given by the map

$$C^*_{r,\omega}(\Gamma) \rtimes_{\delta_r^{(\omega)}} C_0(\Gamma) \to C^*_r(\Gamma) \rtimes_{\delta_r} C_0(\Gamma), \quad \lambda_g^{(\omega)}\delta_h \mapsto \omega(g,h)\lambda_g\delta_h. \tag{4}$$

The crossed product $C^*_{r,\omega}(\Gamma) \rtimes_{\delta_r} C_0(\Gamma)$ admits the dual action $\hat{\delta}_r^{(\omega)}$ of Γ defined by

$$(\hat{\delta}_r^{(\omega)})_k(\lambda_g^{(\omega)}\delta_h) = \lambda_g^{(\omega)}\delta_{hk^{-1}}.$$

If we regard $\hat{\delta}_r^{(\omega)}$ as an action of Γ on $C^*_r(\Gamma) \rtimes_{\delta_r} C_0(\Gamma)$ via the isomorphism (4), the dual coaction can be expressed as

$$(\hat{\delta}_r^{(\omega)})_k(\lambda_g\delta_h) = \overline{\omega(g,h)}\omega(g,hk^{-1})\lambda_g\delta_{hk^{-1}}. \tag{5}$$

By the Takesaki–Takai duality, the crossed product $\mathcal{K}(\ell^2(\Gamma)) \rtimes_{\hat{\delta}_r^{(\omega)}} \Gamma$ is strongly Morita equivalent to $C^*_{r,\omega}(\Gamma)$.

2.3. **Coaction of quantum groups and braided tensor products.** Suppose that A is a $C^*_r(\Gamma)$-C^*-algebra. Thus, A admits a coaction α of $C^*_r(\Gamma)$ given by a homomorphism

$$\alpha\colon A \to C^*_r(\Gamma) \otimes A$$

which satisfies the multiplicativity $\iota \otimes \alpha \circ \alpha = \delta \otimes \iota \circ \alpha$ and the condition that $C^*_r(\Gamma)_1 \alpha(A)$ is dense $C^*_r(\Gamma) \otimes A$, called *the cancellation property* or *continuity* of α. We write the coaction as $\alpha(x) = \sum_g \lambda_g \otimes \alpha^{(g)}(x)$. Then $x = \alpha^{(g)}(x)$ is equivalent to $\alpha(x) = \lambda_g \otimes x$. Note that linear span A_{fin} of such elements, the *elements of finite spectrum*, are dense in A. This fact will be frequently utilized later to verify the images of various homomorphisms.

Suppose that A is represented on a Hilbert space H. Then a unitary $X \in M(C^*_r(\Gamma) \otimes \mathcal{K}(H))$ is said to be a *covariant representation* for α if it satisfies $\delta \otimes \iota(X) = X_{13}X_{23}$ and $X^*(1 \otimes a)X = \alpha(a)$.

By analogy with the case of $\Gamma = \mathbb{Z}^2$ [Yam10, Section 3], we would like to consider 'the diagonal coaction' $\alpha \otimes \delta_l^{(\omega)}$ of $C^*_r(\Gamma)$ on $A \otimes C^*_{r,\omega}(\Gamma)$. Nonetheless, a naive attempt

$$A \otimes C^*_{r,\omega}(\Gamma) \to C^*_r(\Gamma) \otimes A \otimes C^*_{r,\omega}(\Gamma), \quad a \otimes x \mapsto \alpha(a)_{12} \delta_l^{(\omega)}(x)_{13}$$

does not define an algebra homomorphism unless Γ is commutative. To remedy this we use the notion of braided tensor product instead.

We consider an action $\mathrm{Ad}^{(\omega)}$ of Γ on $C^*_{r,\omega}(\Gamma)$ given by

$$\mathrm{Ad}_g^{(\omega)}(\lambda_h^{(\omega)}) = \lambda_g^{(\omega)} \lambda_h^{(\omega)} (\lambda_g^{(\omega)})^* = \omega(g,h)\omega(gh, g^{-1})\overline{\omega(g, g^{-1})} \lambda_{ghg^{-1}}^{(\omega)}.$$

Let $\widetilde{\mathrm{Ad}}^{(\omega)}$ denote the algebra homomorphism

$$C^*_{r,\omega}(\Gamma) \to M(C_0(\Gamma) \otimes C^*_{r,\omega}(\Gamma)), \quad x \mapsto \sum_h \delta_h \otimes \mathrm{Ad}_{h^{-1}}^{(\omega)}(x).$$

This is implemented as the adjoint by the ω-representation unitary $W^{(\omega)}$, and satisfies $\iota \otimes \widetilde{\mathrm{Ad}}^{(\omega)} \circ \widetilde{\mathrm{Ad}}^{(\omega)} = \hat{\delta} \otimes \iota \circ \widetilde{\mathrm{Ad}}^{(\omega)}$. Hence it defines a coaction of the dual quantum group $(C_0(\Gamma), \hat{\delta})$.

Combined with the coaction $\delta_l^{(\omega)}$ of $C^*_r(\Gamma)$, the algebra $C^*_{r,\omega}(\Gamma)$ becomes a Γ-Yetter–Drinfeld-C^*-algebra [NV10]. Indeed, it amounts to verifying the commutativity of the diagram [NV10, Definition 3.1]

(6)

$$\begin{array}{ccccc} C^*_{r,\omega}(\Gamma) & \xrightarrow{\delta_l^{(\omega)}} & \hat{S} \otimes C^*_{r,\omega}(\Gamma) & \xrightarrow{\iota \otimes \widetilde{\mathrm{Ad}}^{(\omega)}} & M(\hat{S} \otimes S \otimes C^*_{r,\omega}(\Gamma)) \\ \big\downarrow \widetilde{\mathrm{Ad}}^{(\omega)} & & & & \big\downarrow \Sigma_{12} \\ M(S \otimes C^*_{r,\omega}(\Gamma)) & \xrightarrow{\iota \otimes \delta_l^{(\omega)}} & M(S \otimes \hat{S} \otimes C^*_{r,\omega}(\Gamma)) & \xrightarrow{\mathrm{Ad}_W} & M(S \otimes \hat{S} \otimes C^*_{r,\omega}(\Gamma)) \end{array},$$

where $\hat{S} = C^*_r(\Gamma)$, $S = C_0(\Gamma)$, W is the fundamental unitary $\sum_h \delta_h \otimes \lambda_h$ in $M(C_0(\Gamma) \otimes C^*_r(\Gamma))$, and Σ is the transposition of tensors. If we track the image of $\lambda_g^{(\omega)} \in C^*_{r,\omega}(\Gamma)$ along the top-right arrows, we obtain

$$\lambda_g^{(\omega)} \mapsto \lambda_g \otimes \lambda_g^{(\omega)} \mapsto \sum_h \lambda_g \otimes \delta_h \otimes (\lambda_h^{(\omega)})^* \lambda_g^{(\omega)} \lambda_h^{(\omega)} \mapsto \sum_h \delta_h \otimes \lambda_g \otimes (\lambda_h^{(\omega)})^* \lambda_g^{(\omega)} \lambda_h^{(\omega)}$$

Similarly, if we go along the left-bottom arrows, we obtain

$$\lambda_g^{(\omega)} \mapsto \sum_h \delta_h \otimes (\lambda_h^{(\omega)})^* \lambda_g^{(\omega)} \lambda_h^{(\omega)} \mapsto \sum_h \delta_h \otimes \lambda_{hgh^{-1}} \otimes (\lambda_h^{(\omega)})^* \lambda_g^{(\omega)} \lambda_h^{(\omega)}$$
$$\mapsto \sum_h \delta_h \otimes \lambda_g \otimes (\lambda_h^{(\omega)})^* \lambda_g^{(\omega)} \lambda_h^{(\omega)},$$

where we used

$$\begin{aligned}\delta_l^{(\omega)}((\lambda_h^{(\omega)})^* \lambda_g^{(\omega)} \lambda_h^{(\omega)}) &= \overline{\omega(h, h^{-1})} \omega(h^{-1}, g) \omega(h^{-1}g, h) \lambda_{h^{-1}gh} \otimes \lambda_{h^{-1}gh}^{(\omega)} \\ &= \lambda_{h^{-1}gh} \otimes (\lambda_h^{(\omega)})^* \lambda_g^{(\omega)} \lambda_h^{(\omega)}.\end{aligned}$$

Combining these, we conclude that the diagram (6) is commutative.

As proved in [NV10, Theorem 3.2], a Yetter–Drinfeld algebra is the same thing as an algebra endowed with a coaction of the Drinfeld dual. In our setting the Drinfeld dual $\mathsf{D}(\Gamma)$ of Γ is represented by the algebra $C_0(\mathsf{D}(\Gamma)) = C_0(\Gamma) \otimes C_r^*(\Gamma)$ endowed with the coproduct

$$\Delta = (\Sigma \circ \mathrm{Ad}_W)_{23} \circ \hat{\delta} \otimes \delta \colon \delta_h \otimes \lambda_g \mapsto \sum_{h'h''=h} (\delta_{h'} \otimes \lambda_{h''gh''^{-1}}) \otimes (\delta_{h''} \otimes \lambda_g).$$

The above Yetter–Drinfeld algebra structure on $C_{r,\omega}^*(\Gamma)$ corresponds to the coaction

$$C_{r,\omega}^*(\Gamma) \to M(C_0(\mathsf{D}(\Gamma)) \otimes C_{r,\omega}^*(\Gamma)), \quad \lambda_g^{(\omega)} \mapsto \sum_h \delta_h \otimes \lambda_{h^{-1}gh} \otimes \mathrm{Ad}_{h^{-1}}^{(\omega)}(\lambda_g^{(\omega)}).$$

Let A be a $C_r^*(\Gamma)$-C^*-algebra and ω be a U(1)-valued 2-cocycle on Γ. The *braided tensor product* $A \boxtimes C_{r,\omega}^*(\Gamma)$ of A and $C_{r,\omega}^*(\Gamma)$ [NV10, Definition 3.3] is the C^*-algebra of operators on the Hilbert C^*-module $\ell^2(\Gamma) \otimes A \otimes C_{r,\omega}^*(\Gamma)$ generated by the operators of the form $\alpha(a)_{12} \widetilde{\mathrm{Ad}}^{(\omega)}(x)_{13}$ for $a \in A$ and $x \in C_{r,\omega}^*(\Gamma)$. By means of the conditional expectation $\iota \otimes \tau$ from $A \otimes C_{r,\omega}^*(\Gamma)$ onto A, we may regard $A \boxtimes C_{r,\omega}^*(\Gamma)$ as a subalgebra of $\mathcal{L}\left(\ell^2(\Gamma) \otimes A \otimes \ell^2(\Gamma)\right)$. Note that our convention (the Yetter–Drinfeld algebra being the second component in the braided tensor product) is different from that of [NV10, Definition 3.3].

By [Vae05, Proposition 8.3], we have

$$A \boxtimes C_{r,\omega}^*(\Gamma) = \overline{\alpha(A)_{12} \widetilde{\mathrm{Ad}}^{(\omega)}(C_{r,\omega}^*(\Gamma))_{13}}$$

as linear spaces of $\mathcal{L}\left(\ell^2(\Gamma) \otimes A \otimes \ell^2(\Gamma)\right)$.

By [NV10, the remark after Definition 3.3], the braided tensor product $A \boxtimes C_{r,\omega}^*(\Gamma)$ admits a coaction $\alpha \otimes \delta_l^{(\omega)}$ of $C_r^*(\Gamma)$ which we shall call the diagonal coaction. It is given by

$$\alpha \otimes \delta_l^{(\omega)}(\alpha(a)_{12} \widetilde{\mathrm{Ad}}^{(\omega)}(x)_{13}) = \delta \otimes \iota(\alpha(a))_{123} \iota \otimes \widetilde{\mathrm{Ad}}^{(\omega)}(\delta_l^{(\omega)}(x))_{124}.$$

2.4. **Exterior equivalence of actions.** Let us briefly recall the notion of exterior equivalence between the (co)actions on C^*-algebras by the locally compact quantum groups of our interest.

Let α and β be actions of Γ on a C^*-algebra A. These two actions are said to be *exterior equivalent* when there exists a family $(u_g)_{g \in \Gamma}$ of unitaries in $M(A)$ satisfying $u_g \alpha_g(u_h) = u_{gh}$ and $\beta_g = \mathrm{Ad}_{u_g} \circ \alpha_g$ for any $g, h \in \Gamma$. Two actions α and β of Γ on different algebras A and B are said to be *outer conjugate* if there is an isomorphism $\phi \colon B \to A$ such that the action $(\phi \beta_g \phi^{-1})_g$ on A is exterior equivalent to α.

Outer conjugate actions define isomorphic crossed products with conjugate dual (co)actions. An action is exterior equivalent to the trivial one if and only if it is the conjugation action with respect to a group homomorphism from Γ into the unitary group of $M(A)$.

Similarly, two coactions α and β of $C^*_r(\Gamma)$ on a C^*-algebra A is said to be exterior equivalent when there is a unitary element X in $C^*_r(\Gamma) \otimes A$ satisfying $X_{23}\iota \otimes \alpha(X) = \delta \otimes \iota(X)$ and $X\alpha(x)X^* = \beta(x)$ for $x \in A$. Such X is called an α-cocycle.

3. Deformation of algebras

Definition 1. Let A be a C^*-algebra with a coaction α of $C^*_r(\Gamma)$, and ω be a U(1)-valued 2-cocycle on Γ. We define the *deformation $A_{\alpha,\omega}$ of A with respect to α and ω* (*the ω-deformation* of A) to be the fixed point algebra $(A \boxtimes C^*_{r,\overline{\omega}}(\Gamma))^{C^*_r(\Gamma)}$ under the diagonal coaction $\alpha \otimes \delta^{(\overline{\omega})}_l$. When there is no source of confusion for α we write A_ω instead of $A_{\alpha,\omega}$.

Proposition 2. *Let Γ, ω, and A be as above. Then the deformed algebra A_ω is isomorphic to the subalgebra A'_ω of $C^*_{r,\omega}(\Gamma) \otimes A$ consisting of the elements x satisfying $\iota \otimes \alpha(x)_{213} = \delta^{(\omega)}_l \otimes \iota(x)$.*

Proof. Note that the C^*-algebras $A \boxtimes C^*_{r,\overline{\omega}}(\Gamma)$ and $C^*_{r,\omega}(\Gamma) \otimes A \otimes C^*_{r,\overline{\omega}}(\Gamma)$ are represented on $\ell^2(\Gamma) \otimes A \otimes C^*_{r,\overline{\omega}}(\Gamma)$. We have a homomorphism Φ from the former to the latter by $x \mapsto W^{(\overline{\omega})}_{13} x (W^{(\overline{\omega})})^*_{13}$. The effect of Φ on the generators of $A \boxtimes C^*_{r,\overline{\omega}}(\Gamma)$ is described by

$$\alpha(x)_{12} \mapsto \sum_g \lambda^{(\omega)}_g \otimes \alpha^{(g)}(x) \otimes \lambda^{(\overline{\omega})}_g, \qquad \widetilde{\mathrm{Ad}}^{(\overline{\omega})}(y)_{13} \mapsto y_3.$$

Thus the image of Φ is $A'_\omega \otimes C^*_{r,\omega}(\Gamma)$, and the corresponding coaction of $C^*_r(\Gamma)$ is simply given by $(\iota \otimes \delta^{(\overline{\omega})}_l)_{213}$. Hence the fixed point algebra is given by A'_ω. □

Corollary 3. *When the $C^*_r(\Gamma)$-C^*-algebra (A, α) is given by the pair $(C^*_r(\Gamma), \delta)$, the deformed algebra A_ω is isomorphic to $C^*_{r,\omega}(\Gamma)$.*

Proof. By Proposition 2, we may identify the braided tensor product with the subalgebra of $C^*_{r,\omega}(\Gamma) \otimes C^*_r(\Gamma)$ spanned by $\lambda^{(\omega)}_g \otimes \lambda_g$ for $g \in \Gamma$. As this is equal to the image of $\delta^{(\omega)}_l$, we obtain the assertion. □

Corollary 4. *Let A be a C^*-algebra with a coaction α of $C^*_r(\Gamma)$. When the cocycle ω is trivial, the deformed algebra A_ω is isomorphic to A.*

Proof. In this case the algebra A'_ω in Proposition 2 is the image of α. Hence we obtain $A_\omega \simeq A$. □

Corollary 5. *When the coaction α is trivial, A_ω is isomorphic to A for any 2-cocycle ω.*

Remark 6. When $a \in A_{\mathrm{fin}}$, we can consider an element $\sum_g \lambda^{(\omega)}_g \otimes \alpha^{(g)}(a)$ in A'_ω. We let $a^{(\omega)}$ denote the corresponding element in A_ω. The ω-deformation A_ω can be regarded as a certain C^*-algebraic completion of the vector space $\{a^{(\omega)} \mid a \in A_{\mathrm{fin}}\} \simeq A_{\mathrm{fin}}$ endowed with the twisted $*$-algebra structure

$$a^{(\omega)} b^{(\omega)} = \sum_{g,h} \omega(g,h) (\alpha^{(g)}(a) \alpha^{(h)}(b))^{(\omega)}, \quad (a^{(\omega)})^* = \sum_g \overline{\omega(g, g^{-1})} (\alpha^{(g)}(a)^*)^{(\omega)}.$$

Example 7. Let A be a $\mathbb{T}^n$-C^*-algebra for some n, and $(\theta_{jk})_{jk}$ be a skewsymmetric real matrix of size n. Then the *θ-deformation* A_θ of A is given by $(A \otimes C(\mathbb{T}^n)_\theta)^{\mathbb{T}^n}$, where $C(\mathbb{T}^n)_\theta$ is the universal C^*-algebra generated by n unitaries $u_1, \ldots, u_n$ satisfying $u_j u_k = e^{i\theta_{jk}} u_k u_j$, and $\mathbb{T}^n$ acts on $A \otimes C(\mathbb{T}^n)_\theta$ by the diagonal action.

The algebra $C(\mathbb{T}^n)_\theta$ can be regarded as the twisted group algebra of $\mathbb{Z}^n$ with the 2-cocycle $\omega(x, y) = e^{i(\theta x, y)}$. By Proposition 2, A_θ can be identified with A_ω.

Example 8. Let B be a Γ-C^*-algebra. Then the reduced crossed product $\Gamma \ltimes B$ is a $C^*_r(\Gamma)$-C^*-algebra by the dual coaction. If ω is a 2-cocycle on Γ, the deformed algebra $(\Gamma \ltimes B)_\omega$ can be identified with the twisted reduced crossed product $\Gamma \ltimes_{\alpha,\omega} B$ [ZM68].

There is another coaction of $C^*_r(\Gamma)$ on $A \boxtimes C^*_{r,\omega}(\Gamma)$, given by

$$\alpha(x)_{12}\widetilde{\mathrm{Ad}}^{(\overline{\omega})}(y)_{13} \mapsto \iota \otimes \alpha \circ \alpha(x)_{123}\widetilde{\mathrm{Ad}}^{(\overline{\omega})}(y)_{24}.$$

We denote this coaction by α_ω. It is implemented as the adjoint with the dual fundamental unitary $\hat{W} = \sum_g \lambda_g \otimes \delta_g$. It can be easily seen from the definitions that the two coactions α_ω and $\alpha \otimes \delta_l^{(\overline{\omega})}$ of $C^*_r(\Gamma)$ commute with each other. Hence α_ω restricts to the fixed point subalgebra A_ω of $\alpha \otimes \delta_l^{(\overline{\omega})}$.

Remark 9. When ω and η are U(1)-valued 2-cocycles on Γ, we have $(A_\omega)_\eta = A_{\omega\cdot\eta}$ for any $C^*_r(\Gamma)$-C^*-algebra A.

We have the following generalization of the isomorphism (4).

Proposition 10. *The crossed product algebra $C_0(\Gamma) \ltimes_\alpha A_\omega$ is isomorphic to the corresponding algebra $C_0(\Gamma) \ltimes_\alpha A$ of the untwisted case.*

Proof. We identify A_ω with the algebra A'_ω of Proposition 2. Thus, the crossed product $C_0(\Gamma) \ltimes_{\alpha_\omega} A_\omega$ is represented by the C^*-algebra of operators generated by $(\delta_h)_1$ and $\sum_g \lambda_g \otimes \lambda_g^{(\omega)} \otimes \alpha^{(g)}(x)$ on $\ell^2(\Gamma)^{\otimes 2} \otimes A$.

Let V be the unitary operator $\delta_k \otimes \delta_{k'} \mapsto \overline{\omega}(k^{-1}, k)\omega(k^{-1}, k')\delta_k \otimes \delta_{k'}$. The assertion follows once we prove that the image of $\Phi = \mathrm{Ad}_{V_{12}} \colon A'_\omega \to B(\ell^2(\Gamma)^{\otimes 2} \otimes A)$ is equal to

$$C_0(\Gamma) \ltimes A' = \vee \left\{ (\delta_h)_1, \sum_g \lambda_g \otimes \lambda_g \otimes \alpha^{(g)}(x) \mid h \in \Gamma, x \in A \right\}.$$

If $h \in \Gamma$ and $x \in A'_\omega$ has finite spectrum, the action of $\Phi(\alpha_\omega(x)(\delta_h)_1)$ on the vector $\delta_k \otimes \delta_{k'} \otimes b$ is given by

$$\sum_g \delta_{h,k}\omega(k^{-1}, k)\overline{\omega(k^{-1}, k')\omega(k^{-1}g^{-1}, gk)}\omega(k^{-1}g^{-1}, gk')\delta_{gh} \otimes \delta_{gk'} \otimes \alpha^{(g)}(x)b.$$

Using the cocycle identity for ω, we see that this is equal to

$$\sum_g \omega(g, h)\delta_{h,k}\delta_{gh} \otimes \delta_{gk'} \otimes \alpha^{(g)}(x)b,$$

which is equal to the action of $\sum_g \omega(g, h)\lambda_g \otimes \lambda_g \otimes \alpha^{(g)}(x)(\delta_h)_1$. This operator is indeed in $C_0(\Gamma) \ltimes A'$. ∎

We have the following expression of $\hat{\alpha}_\omega$

$$(\hat{\alpha}_\omega)_k((\sum_g \lambda_g \otimes \lambda_g \otimes \alpha^{(g)}(x))(\delta_h)_1) = \omega(g, hk^{-1})\overline{\omega(g, h)}(\sum_g \lambda_g \otimes \lambda_g \otimes \alpha^{(g)}(x))(\delta_{hk^{-1}})_1), \tag{7}$$

regarded as an action on $C_0(\Gamma) \ltimes_\alpha A$ via the isomorphism Φ in the proof of Proposition 10. By the Takesaki–Takai duality, A_ω is strongly Morita equivalent to the crossed product $\Gamma \ltimes_{\hat{\alpha}_\omega} C_0(\Gamma) \ltimes_{\alpha_\omega} A_\omega$ with respect to the dual action $\hat{\alpha}_\omega$ of α_ω by Γ.

For each $g \in \Gamma$, let A_g denote the corresponding spectral subspace consisting of the elements $x \in A$ satisfying $\alpha^{(g)}(x) = x$. Recall that the Fell bundle $(A_g)_{g\in\Gamma}$ associated to A has the *approximation property* [Exe97, Definition 4.4] when there is a sequence a_i of functions from Γ into A_e satisfying

$$\sup_i \left\| \sum_g a_i(g)^* a_i(g) \right\| < \infty \tag{8}$$

and

$$\lim_i \sum_h a_i(gh)^* b a_i(h) = b \quad (g \in \Gamma, b \in A_g). \tag{9}$$

If Γ is amenable, any $C_r^*(\Gamma)$-C^*-algebra has the approximation property. This property also holds when A is given as $\Gamma \ltimes_\beta B$ for some amenable action β of a discrete group Γ on a unital C^*-algebra B.

Lemma 11. *Let Γ, ω, and A be as above. The Fell bundle associated to A has the approximation property if and only if the one associated to A_ω has the approximation property.*

Proof. The algebra $(A_\omega)_e$ is naturally isomorphic to A_e. Hence we may regard a_i as a sequence of functions with values in A_ω. Then the condition (8) is automatic. The other one (9) follows from the equalities

$$\left\| b^{(\omega)} \right\| = \|b\|, \qquad a_i(gh)^* b^{(\omega)} a_i(h) = (a_i(gh)^* b a_i(h))^{(\omega)}$$

for any $g \in \Gamma$ and $b \in A_g$. ☐

We have the following adaptation of [Rie93, Theorem 4.1] in our context.

Proposition 12. *Let Γ, ω, and A be as above, and suppose that the Fell bundle associated to A has the approximation property. Then A_ω is nuclear if and only if A is nuclear.*

Proof. The Fell bundle associated to A_ω also has the approximation property by Lemma 11. By Remark 9, it is enough to prove that A_ω is nuclear when A is nuclear.

By the amenability of the Fell bundle associated to A_ω, the maximal and the reduced crossed product coincide for the dual action of Γ on $C_0(\Gamma) \ltimes_{\alpha_\omega} A_\omega$ [EQ02, Corollary 3.6]. Since $C_0(\Gamma) \ltimes_{\alpha_\omega} A_\omega$ is nuclear by Proposition 10, we conclude that its crossed product by Γ is also nuclear, c.f. [AD02, the proof of Theorem 5.3, (2) $\Rightarrow$ (3)]. ☐

Proposition 13. *Let Γ, ω be as above, and A be a $C_r^*(\Gamma)$-C^*-algebra represented on a Hilbert space H. Suppose that there is a covariant representation $X \in M(C_r^*(\Gamma) \otimes \mathcal{K}(H))$ of $C_r^*(\Gamma)$. Then, the action of $C_{r,\omega}^*(\Gamma) \otimes A$ on $\ell^2(\Gamma) \otimes H$ restricts to the one of the algebra A'_ω of Proposition 2 on $X^*(\delta_e \otimes H)$.*

Proof. Recall that the dual fundamental unitary $\hat{W} = \sum_g \lambda_g^* \otimes \delta_g$ satisfies $\delta(x) = \hat{W}^*(1 \otimes x)\hat{W}$. Hence

$$\delta \otimes \iota(X^*) = \delta \otimes \iota(X^*)_{213} = X_{13}^* X_{23}^*$$

implies

$$X_{13}^* X_{23}^* (\delta_e \otimes \delta_e \otimes \xi) = \mathrm{Ad}_{\hat{W}_{12}^*}(X_{23}^*)(\delta_e \otimes \delta_e \otimes \xi) = \hat{W}_{12}^* X_{23}^* (\delta_e \otimes \delta_e \otimes \xi).$$

for $\xi \in H$. Thus, any $\eta \in X^*(\delta_e \otimes H)$ satisfies $X_{13}^*(\delta_e \otimes \eta) = \hat{W}_{12}^*(\delta_e \otimes \eta)$.

Conversely, if we had $X_{13}^*(\delta_e \otimes \xi) = \hat{W}_{12}^*(\delta_e \otimes \xi)$ for some $\xi \in \ell^2(\Gamma) \otimes H$, we can write ξ as $\sum_g \delta_g \otimes \xi_g$ and conclude that $X^*(\delta_e \otimes \xi_g) = \delta_g \otimes \xi_g$ for any g, i.e., $\xi = X^*(\delta_e \otimes \sum_g \xi_g)$. Hence we can identify $X^*(\delta_e \otimes H)$ with the subspace $\left\{\xi \mid X_{13}^*\xi = \hat{W}_{12}^*\xi\right\}$ of $\delta_e \otimes \ell^2(\Gamma) \otimes H$ via the embedding $\xi \mapsto \delta_e \otimes \xi$.

By the covariance of X, we can characterize A'_ω as the subalgebra of $C^*_{r,\omega}(\Gamma) \otimes A$ satisfying

$$\hat{W}_{12}^*(1 \otimes a)\hat{W}_{12} = X_{13}^*(1 \otimes a)X_{13}.$$

If $\xi \in X^*(\delta_e \otimes H)$ and $a \in A'_\omega$, one has

$$X_{13}^*(1 \otimes a)(\delta_e \otimes \xi) = \hat{W}_{12}^*(1 \otimes a)\hat{W}_{12}X_{13}^*(\delta_e \otimes \xi) = \hat{W}_{12}^*(1 \otimes a)(\delta_e \otimes \xi),$$

which proves the assertion. □

Proposition 14. *Let α and β be exterior equivalent coactions of $C^*_r(\Gamma)$ on A, and ω be a 2-cocycle on Γ. Then the corresponding deformed algebras $A_{\alpha,\omega}$ and $A_{\beta,\omega}$ are strongly Morita equivalent.*

Proof. Let U be an α-cocycle satisfying $U\alpha(x)U^* = \beta(x)$. As in the standard argument, the rank 1 Hilbert A-module A admits a coaction of $C^*_r(\Gamma)$ defined by

$$X_U \colon \xi \otimes x \mapsto \alpha(x)U^*\xi_1 \quad (\xi \in \ell^2(\Gamma), x \in A).$$

This coaction is covariant with respect to the coaction α on A for the left A-module structure and β for the right. Then, as in Proposition 13, we can take the closed subspace $X_U(\delta_e \otimes A)$ in the Hilbert C^*-module $C^*_{r,\omega}(\Gamma) \otimes A$ which is closed under the left action of $A'_{\alpha,\omega}$ and the right action of $A'_{\beta,\omega}$. This bimodule is the imprimitivity bimodule between the two algebras. □

Corollary 15. *Let A be a $C^*_r(\Gamma)$-C^*-algebra and ω be a 2-cocycle on Γ. Then the deformed algebra A_ω is strongly Morita equivalent to the twisted crossed product $\Gamma \ltimes_{\hat{\alpha},\omega} C_0(\Gamma) \ltimes_\alpha A$.*

Proof. The double dual coaction of $C^*_r(\Gamma)$ on the iterated crossed product $C^*_r(\Gamma) \ltimes_{\hat{\alpha}} C_0(\Gamma) \ltimes_\alpha A$ and the amplification of α on $\mathcal{K}(\ell^2(\Gamma)) \otimes A$ are outer conjugate by the Takesaki–Takai duality. The assertion follows from Proposition 14 and the natural identification $(\mathcal{K} \otimes A)_\omega \simeq \mathcal{K} \otimes A_\omega$. □

This corollary shows that the twisted crossed product (Example 8) is the universal example up to the strong Morita equivalence. We can also see that Proposition 10, and the resulting strong Morita equivalence between A_ω and $\Gamma \ltimes_{\hat{\alpha}_\omega} C_0(\Gamma) \ltimes_\alpha A$ is an adaptation of the 'untwisting' of twisted crossed products by Packer–Raeburn [PR89, Corollary 3.7].

3.1. ***K*-theory isomorphism of deformed algebras.** Let ω be a normalized U(1)-valued 2-cocycle on γ. For each $k \in \Gamma$, consider the unitary element

$$v_k = \left(\sum_g \omega(gk, k^{-1})\delta_g\right)\left(\sum_h \lambda_{hk^{-1}h^{-1}}\delta_h\right) \tag{10}$$

in $M(C^*_r(\Gamma) \rtimes_{\delta_r} C_0(\Gamma)) = B(\ell^2(\Gamma))$. The second sum is actually the unitary ρ_k which implements the right translation $\delta_h \mapsto \delta_{hk^{-1}}$. From the relation

$$\lambda_{g'^{-1}}v_k\lambda_{g'} = \sum_g \omega(gk, k^{-1})\delta_{g'^{-1}g}\rho_k = \sum_g \omega(g'gk, k^{-1})\delta_g\rho_k,$$

we conclude

$$v_k\lambda_{g'}v_k^{-1} = \lambda_{g'}\sum_g \omega(g'gk, k^{-1})\overline{\omega(gk, k^{-1})}\delta_g.$$

Combining this and

$$v_k\delta_{h'}v_k^{-1} = \rho_k\delta_{h'}\rho_k^{-1} = \delta_{h'k^{-1}},$$

we obtain

$$v_k \lambda_{g'} \delta_{h'} v_k^{-1} = \omega(g'h', k^{-1}) \overline{\omega(h', k^{-1})} \lambda_{g'} \delta_{h'k^{-1}}.$$

By the cocycle condition for ω and (5), we see that the right hand side above is equal to $(\hat{\delta}_r^{(\omega)})_k(\lambda_{g'} \delta_{h'})$.

The failure of the multiplicativity of $(v_k)_k$ is controlled by the cocycle ω. By

$$\begin{aligned} v_k v_{k'} &= \sum_g \omega(gk, k^{-1}) \delta_g \rho_k \sum_h \omega(hk', k'^{-1}) \delta_h \rho_{k'} \\ &= \sum_{g = hk^{-1}} \omega(gk, k^{-1}) \omega(hk', k'^{-1}) \delta_g \rho_{kk'} \end{aligned}$$

and the cocycle relation (1) for $g_0 = gkk'$, $g_1 = k'^{-1}$, and $g_2 = k^{-1}$, we obtain

$$v_k v_{k'} = \omega(k'^{-1}, k^{-1}) v_{kk'}. \tag{11}$$

Remark 16. Suppose that the cocycle ω above is of the form $e^{i\omega_0}$ for some $\mathbb{R}$-valued 2-cocycle ω_0. Then we obtain its opposite cocycle $\tilde{\omega}_0$ as in (3). When H is a finite subgroup of Γ, the 2-cocycle $\tilde{\omega}_0|_H$ is a coboundary because of $H^2(H, \mathbb{R})$ is trivial. Hence there exists a map ϕ from H into $\mathbb{R}$ satisfying

$$\tilde{\omega}_0(h_0, h_1) = \phi(h_0) - \phi(h_0 h_1) + \phi(h_1) \quad (h_0, h_1 \in H). \tag{12}$$

The normalization condition on ω implies the one $\phi(e) = 0$ for ϕ. The unitaries $(e^{-i\phi(h)} v_h)_{h \in H}$ are multiplicative by (11), and they implement the action $\hat{\delta}_r^{(\omega)}|_H$ on $C_r^* \rtimes C_0(\Gamma)$ modulo the isomorphism (4) by (5).

Now, assume that ω is induced by an $\mathbb{R}$-valued 2-cocycle ω_0 as above. Our goal is to show that the K-groups of A_ω are isomorphic to those of A.

Let I denote the closed unit interval $[0, 1]$. Generalizing the method of [ELPW10, Section 1], we put $\omega_\theta = e^{i\theta\omega_0}$ for $\theta \in I$ and consider the following C^*-$C(I)$-algebra $A_{\omega_\star}$ over I, whose fiber at θ is given by A_{ω_θ}.

First, we consider the Hilbert space $L^2(I; \ell^2(\Gamma)) \simeq L^2(I) \otimes \ell^2(\Gamma)$, and the operators $\lambda_g^{(\omega_\star)}$ for $g \in \Gamma$ defined by

$$(\lambda_g^{(\omega_\star)} \xi)_\theta = \lambda_g^{(\omega_\theta)} \xi_\theta \quad (\xi \in L^2(I; \ell^2(\Gamma)), \theta \in I).$$

Thus we obtain a C^*-$C(I)$-algebra $C^*_{r,\omega_\star}(\Gamma)$, given as the C^*-algebra generated by these operators and the natural action of $C(I)$ on $L^2(I; \ell^2(\Gamma))$.

Next, we see that $C^*_{r,\omega_\star}(\Gamma)$ is a Γ-Yetter–Drinfeld algebra by the coaction

$$\lambda_g^{(\omega_\star)} \mapsto \lambda_g \otimes \lambda_g^{(\omega_\star)}, f \mapsto 1 \otimes f \quad (g \in \Gamma, f \in C(I))$$

of $C_r^*(\Gamma)$ and the one

$$\lambda_g^{(\omega_\star)} \mapsto \delta_h \otimes (\lambda_h^{(\omega_\star)})^* \lambda_g^{(\omega_\star)} \lambda_h^{(\omega_\star)}, f \mapsto 1 \otimes f \quad (g, h \in \Gamma, f \in C(I))$$

of $C_0(\Gamma)$. This $C(I)$-algebra and its Γ-Yetter–Drinfeld algebra structure is induced by the twisted fundamental unitary

$$W^{(\omega_\star)} = W\omega_\star \colon \delta_g \otimes \delta_h \mapsto (e^{\theta i\omega_0(g,h)} \delta_g \otimes \delta_{g,h})_\theta$$

on $L^2(I; \ell^2(\Gamma)^{\otimes 2})$ which commutes with $C(I)$.

Thus we can take the braided tensor product $A \boxtimes C^*_{r,\omega_\star}(\Gamma)$ which is again a $C(I)$-C^*-algebra with a compatible coaction of $C_r^*(\Gamma)$. The algebra $A_{\omega_\star}$ is defined to be the fixed point algebra for this coaction. By an argument analogous to Proposition 2, this is isomorphic to the subalgebra $A'_{\omega_\star}$ of $A \otimes C^*_{r,\omega_\star}(\Gamma)$ consisting of the elements a satisfying $\alpha_{13}(a) = \delta_{12}^{(\omega_\star)}(a)$.

The algebra $A_{\omega_\star}$ admits a coaction $\alpha_{\omega_\star}$ of $C_r^*(\Gamma)$ defined in the obvious way. The crossed product $C_0(\Gamma) \ltimes_{\alpha_{\omega_\star}} A_{\omega_\star}$ is a Γ-C^*-$C(I)$-algebra, and the evaluation at each fiber is a Γ-homomorphism.

Lemma 17. *The C^*-$C(I)$-algebra $C_0(\Gamma) \ltimes_{\alpha_{\omega_\star}} A_{\omega_\star}$ is isomorphic to the constant field with fiber $C_0(\Gamma) \ltimes_\alpha A$.*

Proof. The proof is essentially the same as that of Proposition 10. The formula

$$(V\delta_k \otimes \delta_{k'})_\theta = \overline{\omega_\theta}(k^{-1}, k)\omega_\theta(k^{-1}, k')\delta_k \otimes \delta_{k'}.$$

defines unitary operator V on $L^2(I; \ell^2(\Gamma)^{\otimes 2})$ which commutes with $C(I)$. When $x \in A$ and $h \in \Gamma$, the constant section $(\sum_g \lambda_g \otimes \lambda_g \otimes \alpha^{(g)}(x))(\delta_h)_1$ of $C(I) \otimes C_0(\Gamma) \ltimes_\alpha A'$ is mapped to the element

$$\left(\left(\sum_g \overline{\omega_\theta(g,h)}\lambda_g \otimes \lambda_g^{(\omega_\theta)} \otimes \alpha^{(g)}(x)\right)(\delta_h)_1\right)_\theta$$

of $C_0(\Gamma) \ltimes_{\alpha_{\omega_\star}} A'_{\omega_\star}$. ☐

Thus, $A_{\omega_\star}$ is $\mathcal{R}KK(I, -, -)$-equivalent to the crossed product of $C(I) \otimes C_0(\Gamma) \ltimes_\alpha A$ by an action of Γ corresponding to $\hat{\alpha}_\star$ via the isomorphism of Lemma 17. Using (7), we can express this action as

$$\left((\hat{\alpha}_\star)_k\left(\left(\sum_g \lambda_g \otimes \lambda_g \otimes \alpha^{(g)}(x)\right)(\delta_h)_1\right)\right)_\theta = \omega_\theta(g, hk^{-1})\overline{\omega_\theta(g,h)}\left(\sum_g \lambda_g \otimes \lambda_g \otimes \alpha^{(g)}(x)\right)(\delta_{hk^{-1}})_1 \tag{13}$$

for $x \in A$ and $h, k \in \Gamma$.

Remark 18. The C^*-$C(I)$-algebra $A_{\omega_\star}$ becomes a continuous field of C^*-algebras when the Fell bundle associated to A has the approximation property [Rie89, Corollary 2.7], see also the proof of Proposition 12.

Proposition 19. *Let H be any finite subgroup of Γ and $\theta \in I$. Then the restriction of $\hat{\alpha}_\star$ to H is outer conjugate to the restriction of the constant field of the action $\hat{\alpha}_{\omega_\theta}$.*

Proof. We first prove the assertion for the case $\theta = 0$. As in Remark 16, we can take a map ϕ from H to $\mathbb{R}$ satisfying (12). Now, consider the unitaries

$$(w_k)_{\theta'} = e^{-i\theta'\phi(k)}\sum_g \omega_{\theta'}(gk, k^{-1})\delta_g \quad (\theta' \in I)$$

in $M(C(I) \otimes C_0(\Gamma))$ for $k \in H$. This is a $\hat{\delta}_r$-cocycle. Indeed, we have

$$\begin{aligned}(14)\quad & \left(w_k(\hat{\delta}_r)_k(w_{k'})\right)_{\theta'} \\ &= e^{-i\theta'\phi(k)}\left(\sum_g \omega_{\theta'}(gk, k^{-1})\delta_g\right)e^{-i\theta'\phi(k')}\left(\sum_{g'} \omega_{\theta'}(g'k', k'^{-1})\delta_{g'k^{-1}}\right) \\ &= e^{-i\theta'(\phi(k)+\phi(k'))}\sum_g \omega_{\theta'}(gk, k^{-1})\omega_{\theta'}(gkk', k'^{-1})\delta_g.\end{aligned}$$

Using (12), one sees that $e^{-i\theta'(\phi(k)+\phi(k'))}$ is equal to $e^{-\theta'\phi(kk')}\overline{\omega_{\theta'}(k'^{-1}, k^{-1})}$. Applying (1) for $g_0 = gk$, $g_1 = k'^{-1}$, and $g_2 = k^{-1}$, we see that the right hand side of (14) is equal to $w_{kk'}$.

If we regard $(w_k)_k$ as elements of $M(C_0(\Gamma) \ltimes_\alpha A)$, they are $\hat{\alpha}_\star$-cocycle by definition of the dual (co)action. We next see that they implement the conjugation between $\hat{\alpha}$ and $\hat{\alpha}_{\omega_\star}$. Indeed, recalling that $\hat{\alpha}$ is the conjugation by $(\rho_g)_g$, we can compute

$$\left(\mathrm{Ad}_{(w_k)_1} \circ \hat{\alpha}_k \left(\left(\sum_g \lambda_g \alpha^{(g)}(x)\right)(\delta_h)_1\right)\right)_{\theta'}$$
$$= \mathrm{Ad}_{(v_k^{(\theta')})_1} \left(\left(\sum_g \lambda_g \alpha^{(g)}(x)\right)(\delta_h)_1\right)$$
$$= \sum_g \omega_{\theta'}(gh, k^{-1})\overline{\omega_{\theta'}(h, k^{-1})}\lambda_g \delta_{hk^{-1}} \lambda_g \alpha^{(g)}(x)(\delta_h)_1$$

using the unitaries $(v_k^{(\theta')})_k$ defined in the same way as (10) but ω being replaced by $\omega_{\theta'}$. By (13) and the cocycle identity for $\omega_{\theta'}$, the right hand side of the above formula is indeed equal to $\hat{\alpha}_\star$. Thus we obtain the outer conjugacy of the actions of H for $\theta = 0$.

For the general value of θ, we can argue in the same way as above that the actions $\hat{\alpha}$ and $\hat{\alpha}_{\omega_\theta}$ are outer conjugate. Thus we can compose the above outer conjugacy with the constant field of conjugacy between $\hat{\alpha}$ and $\hat{\alpha}_{\omega_\theta}$, which implies the assertion for the arbitrary value of θ. ☐

We recall that the 'left hand side' of the Baum–Connes conjecture with coefficients can be computed in the following way.

Proposition 20 ([ELPW10, Proposition 1.6])**.** *Let G be a second countable locally compact group, and A and B be G-C^*-algebras. If $z \in KK^G(A, B)$ induces an isomorphism $K_*^H(A) \to K_*^H(B)$ for any compact subgroup H of G, the Kasparov product with z induces an isomorphism from $K_*^{top}(G; A)$ to $K_*^{top}(G; B)$.*

Theorem 1. *Let Γ be a discrete group satisfying the Baum–Connes conjecture with coefficients, A be a $C_r^*(\Gamma)$-C^*-algebra, and ω_0 be an $\mathbb{R}$-valued 2-cocycle on Γ. Then the K-groups $K_i(A_\omega)$ $(i = 0, 1)$ of the deformed algebra A_ω are isomorphic to $K_i(A)$ for the cocycle $\omega = e^{i\omega_0}$.*

Proof. It is enough to show that the evaluation map ev_θ at $\theta \in I$ for the C^*-$C(I)$-algebra $\Gamma \ltimes_{\hat{\alpha}_{\omega_\star}} C_0(\Gamma) \ltimes_{\alpha_{\omega_\star}} A_{\omega_\star}$ induces an isomorphism in the K-theory for any θ.

Proposition 19 implies that for any finite group H of Γ, the H-homomorphism ev_θ induces an isomorphism of the crossed products by H. By the Green–Julg isomorphism $K_*^H(X) \simeq K_*(H \ltimes X)$ which holds for any H-C^*-algebra X, we obtain that ev_θ induces an isomorphism on the K^H-groups. By Proposition 20, ev_θ induces an isomorphism

$$K_*^{\mathrm{top}}(\Gamma; C_0(\Gamma) \ltimes_{\alpha_{\omega_\star}} A_{\omega_\star}) \simeq K_*^{\mathrm{top}}(\Gamma; C_0(\Gamma) \ltimes_{\alpha_{\omega_\theta}} A_{\omega_\theta}).$$

By the assumption on Γ, the both hand sides are isomorphic to the K-groups of the crossed products by Γ. ☐

We have a slight variation of the above theorem for the groups satisfying the strong Baum–Connes conjecture.

Theorem 2. *Let Γ be a discrete group satisfying the strong Baum–Connes conjecture, A be a $C_r^*(\Gamma)$-C^*-algebra, and ω_0 be an $\mathbb{R}$-valued 2-cocycle on Γ. Then the deformed algebra A_ω is KK-equivalent to A for the cocycle $\omega = e^{i\omega_0}$.*

Proof. Recall the following formulation of the strong Baum–Connes conjecture due to Meyer–Nest [MN06]. The group Γ satisfies the conjecture with arbitrary coefficients [MN06, Definition 9.1] if and only if the descent functor $KK^\Gamma \to KK, A \mapsto \Gamma \ltimes_r A$ maps weak equivalences to isomorphisms [MN06, p. 213].

The evaluation maps for the C^*-$C(I)$-algebra $C_0(\Gamma) \ltimes_{\alpha_{\omega_\star}} A_{\omega_\star}$ are weak equivalences by Proposition 19. Thus, the reduced crossed products by Γ are KK-equivalent if Γ satisfies the strong Baum–Connes conjecture. □

Remark 21. Suppose that A is nuclear, the Fell bundle associated to A has the approximation property, and that Γ satisfies the strong Baum–Connes conjecture. Then the continuous field $A_{\omega_\star}$ is an $\mathcal{R}KK$-fibration in the sense of [ENOO09] by Proposition 12, Theorem 2, and [ENOO09, Corollary 1.6].

3.2. **Deformation of equivariant spectral triples.** We see that the 'equivariant Dirac operators' for a given coaction of $C^*_r(\Gamma)$ give isospectral deformations on the ω-deformations, which induce the same index map modulo the K-theory isomorphism of Theorem 1.

As in Proposition 13, let (A, H, X) be a covariant representation of a $C^*_r(\Gamma)$-C^*-algebra A on H. Suppose that D is a (possibly unbounded) self-adjoint operator on H, and $\mathcal{A}$ is a subalgebra of A such that $a(1 + D^2)^{-1}$ is compact and $[D, a]$ is bounded for any $a \in \mathcal{A}$. Thus, $(\mathcal{A}, D, H)$ is an odd spectral triple. By abuse of notation, we let $\mathrm{Id}_{\ell^2(\Gamma)} \otimes D$ the closure of the operator $\xi \otimes \eta \mapsto \xi \otimes D\eta$ for $\xi \in \ell^2(\Gamma)$ and $\eta \in \mathrm{dom}(D)$.

We assume that $\mathrm{Id}_{\ell^2(\Gamma)} \otimes D$ commutes X (in particular, X preserves the domain of $(\mathrm{Id}_{\ell^2(\Gamma)} \otimes D)$) and one has $\alpha^g(a) \in \mathcal{A}$ for any $a \in \mathcal{A}$ and $g \in \Gamma$. These conditions respectively correspond to the equivariance of the Dirac operator and the smoothness of the action. We shall call such a spectral triple as a $C^*_r(\Gamma)$-*equivariant spectral triple*. By the equivariance of D, the operator $\mathrm{Id}_{\ell^2(\Gamma)} \otimes D$ restricts to $X^*(\delta_e \otimes H)$. This restriction is unitarily equivalent to D.

Let $\mathcal{A}_{\mathrm{fin}}$ denote the subalgebra of $\mathcal{A}$ consisting of the elements with finite α-spectrum. Then the commutators of $\mathrm{Id}_{\ell^2(\Gamma)} \otimes D$ and $\sum_g \lambda_g^{(\omega)} \otimes \alpha^{(g)}(a) \in A'_\omega$ for $a \in \mathcal{A}_{\mathrm{fin}}$ are bounded. Thus, if we let $\mathcal{A}_{\omega,\mathrm{fin}}$ denote the algebra generated by the $a^{(\omega)}$ for $a \in \mathcal{A}_{\mathrm{fin}}$, we obtain a new spectral triple

$$(\mathcal{A}_{\omega,\mathrm{fin}}, \mathrm{Id}_{\ell^2(\Gamma)} \otimes D|_{X^*(\delta_e \otimes H)}, X^*(\delta_e \otimes H)),$$

which is an isospectral deformation of the original triple. By means of the unitary operator X between H and $X^*(\delta_e \otimes H)$, we consider this as a spectral triple represented on H, denoted by

$$(\mathcal{A}_{\omega,\mathrm{fin}}, D, H).$$

If the original spectral triple (A, D, H) is even, the above construction gives an even spectral triple over $\mathcal{A}_{\omega,\mathrm{fin}}$ provided X is compatible with the grading on H, that is $X \in M(C^*_r(\Gamma) \otimes \mathcal{K}(H)^{\mathrm{even}})$.

Assume that (A, D, H) is an even triple, and let $F = D\,|D|^{-1}$ be the phase of D. The above construction of the deformed spectral triple give a Fredholm module (F, H) over A_ω, which is in $KK_0(A_\omega, \mathbb{C})$. The next theorem shows that this element induce the essentially same map on the K-group if ω is a real 2-cocycle.

Theorem 3. *Suppose that Γ satisfies the Baum–Connes conjecture with coefficients and ω_0 be an $\mathbb{R}$-valued 2-cocycle on Γ. Let A be a $C^*_r(\Gamma)$-C^*-algebra admitting an equivariant even spectral triple (H, D). Then the even Fredholm module (F, H) for $F = D\,|D|^{-1}$ induce the same map modulo the isomorphism given in Theorem 1.*

Proof. The isomorphisms of the K-groups are induced by the evaluation maps of the C^*-$C(I)$-algebra $A'_{\omega_\star}$.

The algebra $A'_{\omega_\star}$ acts on the field of Hilbert space $X(\delta_e \otimes H) \otimes L^2(I)$ over I, and its elements have the bounded commutator with the self-adjoint operator

$$\left(\mathrm{Id}_{\ell^2(\Gamma)} \otimes F|_{X(\delta_e \otimes H)}\right) \otimes \mathrm{Id}_{L^2(I)}\,.$$

This operator defines an element of $\mathcal{R}KK(I; A'_{\omega_\star}, C(I))$. It is clear from the construction that, if we specialize this element to a point $\theta \in I$, we obtain the Fredholm module (F, H) on A_{ω_θ}. $\square$

Remark 22. There is a corresponding statement for the odd equivariant spectral triples. It can be proved in the same way, or can be reduced to the even case by taking the graded tensor product with the standard odd spectral triple over $C^\infty(S^1)$.

4. Concluding remarks

Remark 23. Suppose that G is a compact group, ω is a 2-cocycle on the dual $\hat{G}$ of G. Wassermann [Was88b] defined a deformation $C(G)_\omega$ of $C(G)$ as in [Was88b], endowed with the action λ^ω of G. When G is commutative, this construction can be identified with ours. More generally, we can generalize this construction to arbitrary 2-cocycles over discrete quantum groups.

When A is a G-C^*-algebra, we can define its deformation by $A_\omega = (A \otimes C(G)_\omega)^G$. We may expect similar phenomenons in this context too, but we lack nontrivial examples in this context. For example, the U(1)-valued 2-cocycles on the duals of semisimple compact Lie groups which can be perturbed to the trivial one are always induced from the dual of the maximal torus [Was88, NT10]. In general, suppose that H is a subgroup of G and ω is a cocycle in $L(H) \otimes L(H) \subset L(G) \otimes L(G)$. Then we have the natural identification $C(G)_\omega = \operatorname{Ind}_H^G C(H)_\omega$ which leads to $A_\omega \simeq (\operatorname{Res}_H^G A)_\omega$ for any G-C^*-algebra A. Hence we can reduce the computation to $\hat{H}$ which is an ordinary discrete group if H is commutative. We note that a recent work of Kasprzak [Kas10] handles this situation.

Remark 24. The compact quantum groups $C^*_r(\Gamma)$ can be characterized as the commutative ones among the general compact quantum groups. The arguments in Section 3.1 depend on this commutativity in the following way. If G is a compact group as above and A is a C^*-algebra endowed with an action α of G, we can define the deformation of A by taking the fixed point algebra $(A \otimes C(G)_\omega)^{\alpha \otimes \lambda^\omega}$. When G is commutative, this algebra is invariant under α (or λ_ω) by

$$\alpha_g \otimes \iota \circ \alpha_h \otimes \lambda^\omega_h = \alpha_{gh} \otimes \lambda^\omega_h = \alpha_{hg} \otimes \lambda^\omega_h = \alpha_h \otimes \lambda^\omega_h \circ \alpha_g \otimes \iota.$$

Then we can take the crossed product $G \ltimes_\alpha (A \otimes C(G)_\omega)^{\alpha \otimes \lambda^\omega}$, which is isomorphic to the corresponding algebra for the case $\omega = 1$. This way we can reduce the problem of $(A \otimes C(G)_\omega)^{\alpha \otimes \lambda^\omega}$ to the corresponding one for the actions of $\hat{G}$ on $G \ltimes_\alpha (A \otimes C(G))^{\alpha \otimes \lambda}$.

Remark 25. For a noncommutative compact quantum group G, one may consider another form of deformation of the function algebra with respect to a 2-cocycle on the dual discrete quantum group. Namely, if $\hat{\delta}$ is the coproduct of C^*G and ω is a 2-cocycle, $\delta_\omega = \omega \hat{\delta} \omega^{-1}$ defines another coproduct on C^*G. Thus the dual Hopf algebra H_ω of (C^*G, δ_ω) can be regarded as a deformation of $C(G)$. Moreover, the cocycle condition for ω can be relaxed to the twist condition for some associator Φ. A result of Neshveyev–Tuset [NT11] for q-deformations of simply connected simple compact Lie groups suggests that the K-theory of H_ω do not change if ω and the associator Φ_ω vary continuously in an appropriate sense.

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