Generic matrix superpotentials [1]

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Abstract

A simple and algorithmic description of matrix shape invariant potentials is presented. The complete lists of generic matrix superpotentials of dimension 2×2 and of special superpotentials of dimension 3×3 are given explicitly. In addition, a constructive description of superpotentials realized by matrices of arbitrary dimension is presented. In this way an extended class of integrable systems of coupled Schrödinger equation is classified. Examples of such systems are considered in detail. New integrable multidimensional models which are reduced to shape invariant systems via separation of variables are presented also.

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1 Introduction

Supersymmetry presents effective and elegant tools to solve quantum mechanical problems described by integrable Schrödinger equations. Unfortunately the class of known problems which can be solved using their supersymmetry is rather restricted since they should have the additional quality called shape invariance [1], and this feature appears to be rather rare. The classification of shape invariant (scalar) potentials is believed to be completed at least in the case when they include an additive variable parameter [2].

However there exist an important class of shape invariant potentials which is not classified yet, and they are matrix valued potentials. Such potentials appear naturally in models using systems of Schrödinger-Pauli equations. A famous example of such model was proposed by Pron'ko and Stroganov (PS) [3], its supersymmetric aspects were discovered in papers [4] and [5]. We note that there exist a relativistic version of the PS problem which is shape invariant too [6].

Examples of matrix superpotentials including shape invariant ones were discussed in [7], [8], [9] [10], [11]. A rather general approach to matrix superpotentials was proposed in paper [12], which, however, was restricted to their linear dependence on the variable parameter.

A systematic study of matrix superpotentials was started in recent paper [13] where we presented the complete description of a special class of irreducible matrix superpotentials. These superpotentials include terms linear and inverse in variable parameter, moreover, the linear terms where supposed to be proportional to the unit matrix. In this way we formulated five problems for systems of Schrödinger equations which are exactly solvable thanks to their shape invariance. Three of these problems are shape invariant with respect of shifts of two parameters, i.e., posses the dual shape invariance [13].

The present paper is a continuation and in some sense the completion of the previous one. We classify all irreducible matrix superpotentials realized by matrices of dimension 2×2 with linear and inverse dependence on variable parameter. As a result we find 17 matrix potentials which are shape invariant and give rise to exactly solvable problems described by Schrödinger-Pauli equation. These potentials are defined up to sets of arbitrary parameters thus the number of non-equivalent integrable models which are presented here is rather large. They include as particular cases all superpotentials discussed in [5], [9], [12] and [13], but also a number of new ones. Moreover, the list of found shape invariant potentials is complete, i.e., it includes all such potentials realized by 2×2 matrices.

In addition, we present a constructive description of superpotentials realized by matrices of arbitrary dimension. The case of matrix superpotentials of dimension 3×3 is considered in more detail. A simple algorithm for construction of all non-equivalent 3×3 matrix superpotentials is presented. A certain subclass of such superpotentials is given explicitly.

The found superpotentials give rise to one dimensional integrable models described by systems of coupled Schrödinger equations. However, some of these systems are nothing but reduced versions of multidimensional models, which appear as a result of the separation of variables. Examples of such multidimensional systems are presented in section 8.

These systems are integrable and most of them are new. In particular, we show that the superintegrable model for vector particles, proposed in [21], possesses supersymmetry with shape invariance, and so its solutions can be easily found using tools of SUSY quantum mechanics. The same is true for the arbitrary spin models considered in paper [21], but we do not discuss them here.

We also analyze five of found integrable systems in detail and calculate their spectrum and the related eigenfunctions. In particular we give new examples of matrix oscillator models.

The paper is organized as follows. In section 2 we discuss restrictions imposed on superpotentials by the shape invariance condition and present the determining equations which should be solved to classify these superpotentials. In sections 3 and 4 the 2×2 matrix superpotentials are described and the complete list of them is presented. The case of arbitrary dimensional matrices is considered in section 5, the list of 3×3 matrix superpotentials can be found in section 6.

Sections 7 and 8 are devoted to discussion of new integrable systems of Schrödinger equations which has been effectively classified in the previous sections. In addition, in section 8 we discuss SUSY aspects of superintegrable models for arbitrary spin s proposed in [21].

2 Shape invariance condition

Let us consider a Schrödinger-Pauli type equation

$$H_\kappa \psi = E_\kappa \psi \tag{2.1}$$

where

$$H_k = -\frac{\partial^2}{\partial x^2} + V_k(x), \tag{2.2}$$

and $V_k(x)$ is a matrix-valued potential depending on variable x and parameter k.

We suppose that $V_k(x)$ is an $n \times n$ dimensional hermitian matrix, and that Hamiltonian H_k admits a factorization

$$H_\kappa = a_\kappa^+ a_\kappa^- + c_\kappa \tag{2.3}$$

where

$$a_\kappa^- = \frac{\partial}{\partial x} + W_\kappa, \quad a_\kappa^+ = -\frac{\partial}{\partial x} + W_\kappa,$$

c_k is a constant and $W_k(x)$ is a superpotential.

Let us search for superpotentials which generate shape invariant potentials $V_k(x)$. It means that W_k should satisfy the following condition

$$W_k^2 + W_k' = W_{k+\alpha}^2 - W_{k+\alpha}' + C_k \tag{2.4}$$

were C_k and α are constants.

In the following sections we classify shape invariant superpotentials, i.e., find matrices W_k depending on of x, k and satisfying conditions (2.4). More exactly, we find indecomposable hermitian matrices whose dependence on k is is specified by terms proportional to k and $\frac{1}{k}$.

Let us consider superpotentials of the following generic form

$$W_k = kQ + \frac{1}{k}R + P \tag{2.5}$$

where P, R and Q are $n \times n$ Hermitian matrices depending on x.

Superpotentials of generic form (2.5) were discussed in paper [13] were we considered the case of arbitrary dimension matrices but restrict ourselves to the case when $Q = Q(x)$ is proportional to the unit matrix. Rather surprisingly this supposition enables to make a completed classification of superpotentials (2.5). All such (irreducible) superpotentials include known scalar potentials listed in [2] and five 2×2 matrix superpotentials found in [13].

To complete the classification presented in [13] let us consider generic superpotentials (2.5) with arbitrary hermitian matrices Q, P and R. We suppose W_k be irreducible, i.e., matrices R, P and Q cannot be simultaneously transformed to a block diagonal form.

Substituting (2.5) into (2.4), multiplying the obtained expression by $k^2(k+\alpha)^2$ and equating the multipliers for same powers of k we obtain the following determining equations:

$$Q' = \alpha(Q^2 + \nu I), \tag{2.6}$$

$$P' - \frac{\alpha}{2}\{Q, P\} + \varkappa I = 0, \tag{2.7}$$

$$\{R, P\} + \lambda I = 0 \tag{2.8}$$

$$R^2 = \omega^2 I \tag{2.9}$$

where $Q = \frac{1}{\alpha}\tilde{Q}$, $Q' = \frac{\partial Q}{\partial x}$, $\{Q, P\} = QP + PQ$ is an anticommutator of matrices Q and P, I is the unit matrix and $\varkappa$, λ, ω are constants.

Equations (2.6)–(2.9) have been deduced in [13] where the anticommutator $\{Q, P\}$ was reduced to doubled product of Q with P since Q was considered to be proportional to the unit matrix and so be commuting with P.

The system (2.6)–(2.9) for generic matrices Q, P and R is much more complicated then in the case of diagonal Q. However it is possible to find its exact solutions for matrices of arbitrary dimension.

3 Determining equations for 2×2 matrix superpotentials

At the first step we restrict ourselves to the complete description of superpotentials (2.5) which are matrices of dimension 2×2. In this case it is convenient to represent Q as a linear combination of Pauli matrices

$$Q = q_0\sigma_0 + q_1\sigma_1 + q_2\sigma_2 + q_3\sigma_3 \tag{3.1}$$

where $\sigma_0 = I$ is the unit matrix,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.2}$$

Let is show that up to the unitary transformation realized by a constant matrix the matrix Q can be transformed to a diagonal form. Substituting (3.1) into (2.6) and equating coefficients for the linearly independent Pauli matrices $\sigma_a, a = 1, 2, 3$ we obtain the following system:

$$q'_a = 2\alpha q_0 q_a, \quad a = 1, 2, 3, \tag{3.3}$$

It follows from (3.3) that

$$q_a = c_a F(x), \quad F(x) = \exp\left(2\alpha \int q_0 dx\right) \tag{3.4}$$

where c_a are integration constants. Since all q_a are expressed via the same functions of x multiplied by constants, we can transform Q to the diagonal form:

$$Q \to UQU^\dagger = \begin{pmatrix} q_+ & 0 \\ 0 & q_- \end{pmatrix} \tag{3.5}$$

where $q_\pm = q_0 \pm cF(x)$, $c = \sqrt{c_1^2 + c_2^2 + c_3^2}$, and U is the constant matrix:

$$U = \frac{c + c_3 - \mathrm{i}\sigma_2 c_1 + \mathrm{i}\sigma_1 c_2}{2\sqrt{c(c + c_3)}}$$

In accordance with (3.5) equation (2.6) is reduced to the decoupled system of Riccati equations for $q_\pm$:

$$q'_\pm = \alpha(q_\pm^2 + \nu) \tag{3.6}$$

which is easily integrable. The corresponding matrices P can be found from equation (2.7):

$$P = \begin{pmatrix} p_+ & p \\ p^* & p_- \end{pmatrix} \tag{3.7}$$

with $p_\pm$ being solutions of the following equation

$$p'_\pm = \alpha p_\pm q_\pm + \varkappa, \tag{3.8}$$

and

$$p = \mu \exp\left(\frac{1}{2}\alpha \int (q_+ + q_-)dx\right) \tag{3.9}$$

where μ is an integration constant and the asterisk denotes the complex conjugation. Moreover, up to unitary transformations realized by matrices commuting with Q (3.5) the constant μ can chosen be real and so we can restrict ourselves to p satisfying $p^* = p$.

Consider the remaining equations (2.8) and (2.9). In accordance with (2.9) R should be a constant matrix whose eigenvalues are $\pm\omega$. Thus it can be represented as $R = r_1\sigma_1 + r_2\sigma_2 + r_3\sigma_3$ where r_1, r_2 and r_3 are constants satisfying $r_1^2 + r_2^2 + r_3^2 = \omega^2$, or alternatively, $R = \pm\omega I$. Let $\omega \neq 0$ then, in order equation (2.8) to be satisfied we have to exclude the second possibility and to set $r_1 = \varkappa = p_\pm = 0$. As a result we obtain the general solution of the determining equations (2.6)–(2.9) with $\omega \neq 0$ in the following form:

$$P = \sigma_1 p, \quad R = r_3\sigma_3 + r_2\sigma_2, \quad Q = q_+\sigma_+ + q_-\sigma_- \tag{3.10}$$

where $\sigma_\pm = (1 \pm \sigma_3)/2$, $q_\pm$ are solutions of Riccati equation (3.6), p is the function defined by (3.9) and r_a are constants satisfying $r_2^2 + r_3^2 = \omega^2$.

If $\omega = 0$ then conditions (2.8) and (2.9) became trivial. The corresponding matrices Q and P are given by equations (3.5) and (3.7).

4 Complete list of 2×2 matrix superpotentials

Let us write the found matrix superpotentials explicitly and find the corresponding shape invariant potentials.

All nonequivalent solutions of equations (3.6) are enumerated in the following formulae:

$$q_\sigma = 0, \ \nu = 0, \tag{4.1}$$

$$q_\sigma = -\frac{1}{\alpha x + c_\sigma}, \ \nu = 0,$$

$$q_\sigma = \frac{\lambda}{\alpha}\tan(\lambda x + c_\sigma), \quad \nu = \frac{\lambda^2}{\alpha^2} > 0, \tag{4.2}$$

$$\begin{aligned} q_\sigma &= -\frac{\lambda}{\alpha}\tanh(\lambda x + c_\sigma), \quad \nu = -\frac{\lambda^2}{\alpha^2} < 0, \\ q_\sigma &= -\frac{\lambda}{\alpha}\coth(\lambda x + c_\sigma), \quad \nu = -\frac{\lambda^2}{\alpha^2} < 0, \\ q_\sigma &= -\frac{\lambda}{\alpha}, \quad \nu = -\frac{\lambda^2}{\alpha^2} < 0 \end{aligned} \tag{4.3}$$

where $\sigma = \pm, c_\sigma = \pm c$ and c is an integration constant.

Going over solutions (4.1)–(4.3) corresponding to the same values of parameter ν it is not difficult to find the related entries of matrix P (3.7) defined by equations (3.8) and

(3.9). As a result we obtain the following list of superpotentials:

$$W_\kappa^{(1)} = \lambda \left(\kappa \left(\sigma_+ \tan(\lambda x + c) + \sigma_- \tan(\lambda x - c)\right) + \mu\sigma_1 \sqrt{\sec(\lambda x - c)\sec(\lambda x + c)} + \frac{1}{\kappa} R\right), \tag{4.4}$$

$$W_\kappa^{(2)} = \lambda \left(-\kappa(\sigma_+ \coth(\lambda x + c) + \sigma_- \coth(\lambda x - c)) + \mu\sigma_1 \sqrt{\operatorname{csch}(\lambda x - c)\operatorname{csch}(\lambda x + c)} + \frac{1}{\kappa} R\right), \tag{4.5}$$

$$W_\kappa^{(3)} = \lambda \left(-\kappa(\sigma_+ \tanh(\lambda x + c) + \sigma_- \tanh(\lambda x - c)) + \mu\sigma_1 \sqrt{\operatorname{sech}(\lambda x - c)\operatorname{sech}(\lambda x + c)} + \frac{1}{\kappa} R\right), \tag{4.6}$$

$$W_\kappa^{(4)} = \lambda \left(-\kappa(\sigma_+ \tanh(\lambda x + c) + \sigma_+ \coth(\lambda x - c)) + \mu\sigma_1 \sqrt{\operatorname{sech}(\lambda x + c)\operatorname{csch}(\lambda x - c)} + \frac{1}{\kappa} R\right), \tag{4.7}$$

$$W_\kappa^{(5)} = \lambda \left(-\kappa(\sigma_+ \tanh(\lambda x) + \sigma_-) + \mu\sigma_1 \sqrt{\operatorname{sech}(\lambda x)\exp(-\lambda x)} + \frac{1}{\kappa} R\right), \tag{4.8}$$

$$W_\kappa^{(6)} = \lambda \left(-\kappa(\sigma_+ \coth(\lambda x) + \sigma_-) + \mu\sigma_1 \sqrt{\operatorname{csch}(\lambda x)\exp(-\lambda x)} + \frac{1}{\kappa} R\right), \tag{4.9}$$

$$W_\kappa^{(7)} = -\kappa \left(\frac{\sigma_+}{x + c} + \frac{\sigma_-}{x - c}\right) + \frac{\mu\sigma_1}{\sqrt{x^2 - c^2}} + \frac{1}{\kappa} R, \tag{4.10}$$

$$W_\kappa^{(8)} = -\kappa \frac{\sigma_+}{x} + \mu\sigma_1 \frac{1}{\sqrt{x}} + \frac{1}{\kappa} R \tag{4.11}$$

$$W_\kappa^{(9)} = \lambda \left(-\kappa I + \mu \exp(-\lambda x)\sigma_1 - \frac{\omega}{\kappa}\sigma_3\right). \tag{4.12}$$

Here $\sigma_\pm = \frac{1}{2}(\sigma_0 \pm \sigma_3)$, R is the numeric matrix given by equation (3.10), κ, μ and λ are arbitrary parameters.

Formulae (4.4)–(4.11) give the complete list of superpotentials corresponding to non-trivial matrices R. In particular this list includes superpotentials with Q being proportional to the unit matrix which has been discussed in paper [13]. These cases are specified by equation (4.12) and equations (4.4), (4.5), (4.6), (4.10) with $c = 0$ and $R = \omega\sigma_1$.

Finally, let us add the list (4.4)–(4.12) by superpotentials with $R \equiv 0$. Using equations (3.5), (4.1)–(4.3) and (3.7), (3.9) we obtain the following expressions for operators (2.5):

$$W_\kappa^{(10)} = \lambda \left(\sigma_+(\kappa \tan(\lambda x + c) + \nu \sec(\lambda x + c)) + \sigma_-(\kappa \tan(\lambda x - c) + \tau \sec(\lambda x - c)) + \mu\sigma_1 \sqrt{\sec(\lambda x - c)\sec(\lambda x + c)}\right), \tag{4.13}$$

$$W_\kappa^{(11)} = -\lambda \left(\sigma_+(\kappa \coth(\lambda x + c) + \nu \operatorname{csch}(\lambda x + c)) + \sigma_-(\kappa \coth(\lambda x - c) + \tau \operatorname{csch}(\lambda x - c)) + \mu\sigma_1 \sqrt{\operatorname{csch}(\lambda x - c)\operatorname{csch}(\lambda x + c)}\right), \tag{4.14}$$

$$W_\kappa^{(12)} = -\lambda\left(\sigma_+(\kappa\tanh(\lambda x+c)+\nu\,\mathrm{sech}(\lambda x+c))\right.$$
$$\left.+\sigma_-(\kappa\coth(\lambda x-c)+\tau\,\mathrm{csch}(\lambda x-c))+\mu\sigma_1\sqrt{\mathrm{sech}(\lambda x-c)\,\mathrm{csch}(\lambda x+c)}\right), \qquad (4.15)$$

$$W_\kappa^{(13)} = -\lambda\left(\sigma_+(\kappa\tanh(\lambda x+c)+\nu\,\mathrm{sech}(\lambda x+c))\right.$$
$$\left.+\sigma_-(\kappa\tanh(\lambda x-c)+\tau\,\mathrm{sech}(\lambda x-c))+\mu\sigma_1\sqrt{\mathrm{sech}(\lambda x-c)\,\mathrm{sech}(\lambda x+c)}\right), \qquad (4.16)$$

$$W_\kappa^{(14)} = -\lambda\left(\sigma_+(\kappa\tanh\lambda x+\nu\,\mathrm{sech}\,\lambda x)+\sigma_-\kappa)+\mu\sigma_1\sqrt{\mathrm{sech}\,\lambda x\exp(-\lambda x)}\right), \qquad (4.17)$$

$$W_\kappa^{(15)} = -\lambda\left(\kappa(\sigma_+(\kappa\coth\lambda x+\nu\,\mathrm{csch}\,\lambda x+\sigma_-\kappa)+\mu\sigma_1\sqrt{\mathrm{csch}\,\lambda x\exp(-\lambda x)}\right), \qquad (4.18)$$

$$W_\kappa^{(16)} = -\sigma_+\left(\frac{\kappa+\delta}{x+c}+\frac{\omega}{2}(x+c)\right)-\sigma_-\left(\frac{\kappa-\delta}{x-c}+\frac{\omega}{2}(x-c)\right)+\frac{\mu\sigma_1}{\sqrt{x^2-c^2}}, \qquad (4.19)$$

$$W_\kappa^{(17)} = -\sigma_+\left(\frac{2\kappa+1}{2x}-\frac{\omega x}{4}\right)+\sigma_-\left(\frac{\omega x}{2}+c\right)-\mu\sigma_1\frac{1}{\sqrt{x}}. \qquad (4.20)$$

Formulae (4.4)–(4.20) give the complete description of matrix superpotentials realized by matrices of dimension 2×2. These superpotentials are defined up to translations $x\to x+c$, $\kappa\to\kappa+\gamma$, and up to equivalence transformations realized by unitary matrices. In (4.4)–(4.20) we introduce the rescalled parameter $\kappa=\frac{k}{\alpha}$ such that the transformations $k\to k'=k+\alpha$ is reduced to:

$$\kappa\to\kappa'=\kappa+1. \qquad (4.21)$$

The list (4.4)–(4.20) includes all superpotentials obtained earlier in [12] and [13], but also a number of new matrix superpotentials. The corresponding shape invariant potentials are easily calculated starting with superpotentials (4.4)–(4.20) and using the following definition:

$$V_\kappa^{(i)} = W_\kappa^{(i)2}-W_\kappa^{(i)\prime}, \quad i=1,2,...,17. \qquad (4.22)$$

To save a room we will not present all potentials (4.22) explicitly but restrict ourselves to discussions of particular examples of them, see section 7.

5 Matrix superpotentials of arbitrary dimension

Let us consider generic superpotentials (2.5) with arbitrary dimension hermitian matrices Q, P and R. In this case we again come to the determining equations (2.6)–(2.9) where Q,P and R are now hermitian matrices of dimension $K\times K$ with arbitrary integer K.

In accordance with (2.9) R is a constant matrix whose eigenvalues are $\pm\omega$. Thus up to unitary equivalence it can be chosen in the form

$$R=\omega\begin{pmatrix} I_n & 0\\ 0 & -I_m\end{pmatrix}, \quad n+m=K \qquad (5.1)$$

where I_n and I_m are the unit matrices of dimension $n \times n$ and $m \times m$ correspondingly, n and m are the numbers of positive and negative eigenvalues of R.

It is convenient to represent matrix Q in a block form:

$$Q = \begin{pmatrix} A & B \\ B^\dagger & C \end{pmatrix} \tag{5.2}$$

where A, B and C are matrices of dimension $n \times n$, $n \times m$ and $m \times m$. Using the analogous representation for P and taking into account relations (2.8) we right it as

$$P = \begin{pmatrix} 0 & \hat{P} \\ \hat{P}^\dagger & 0 \end{pmatrix} + \tau R \tag{5.3}$$

where $\tau = -\frac{\lambda}{2\omega}$.

Substituting (5.1)–(5.3) into (2.6) and (2.7) we obtain the following equations for the block matrices:

$$A' = \alpha(A^2 + BB^\dagger + \nu I_n), \tag{5.4}$$

$$C' = \alpha(C^2 + B^\dagger B + \nu I_m), \tag{5.5}$$

$$B' = \alpha(AB + BC), \tag{5.6}$$

$$\hat{P}' = \frac{\alpha}{2}(A\hat{P} + \hat{P}C), \tag{5.7}$$

$$2\tau A + B\hat{P}^\dagger + \hat{P}B^\dagger = 2\overline{\mu} I_n, \tag{5.8}$$

$$-2\tau C + B^\dagger \hat{P} + \hat{P}^\dagger B = 2\overline{\mu} I_m \tag{5.9}$$

where $\overline{\mu} = \frac{\mu}{\alpha}$.

Thus the problem of description of matrix valued superpotentials which generate shape invariant potentials is reduced to finding the general solution of equations (5.4)–(5.9) for irreducible sets of square matrices A, C and rectangular matrices B and P. Moreover, A and C are hermitian matrices whose dimension is $n \times n$ and $m \times m$ respectively while dimension of B and P is $(n \times m)$. Without loss of generality we suppose that $n \leq m$.

The system (5.4)–(5.9) is rather complicated, nevertheless it can be solved explicitly. To save a room we shell not present its cumbersome general solution but restrict ourselves to the special subclass of solutions with trivial matrices B. In this case $\overline{\mu} = \tau = 0$ (otherwise the corresponding superpotentials are reduced to a direct sum of 2×2 matrices considered in the above), and the system is reduced to the following equations:

$$A' = \alpha(A^2 + \nu I_n), \quad C' = \alpha(C^2 + \nu I_m), \tag{5.10}$$

$$\hat{P}' = \frac{\alpha}{2}(A\hat{P} + \hat{P}C). \tag{5.11}$$

Without loss of generality the hermitian matrices A and C which solve equations (5.10) can be chosen as diagonal ones, see Appendix. In other words their entries A_{ab} and C_{ab} can be represented as:

$$A_{ab} = \delta_{ab} q_b, \quad C_{ab} = \delta_{ab} q_{n+b} \tag{5.12}$$

where δ_{ab} is the Kronecker symbol and q_σ ($\sigma = b$ or $\sigma = n + b$) are solutions of the scalar Riccati equation

$$q'_\sigma = \alpha(q_\sigma^2 + \nu) \tag{5.13}$$

which is a direct consequence of (5.10) and (5.12). Solutions of equations (5.13) are given by equations (4.1)–(4.3).

Thus matrices A, C and R are defined explicitly by relations (5.10), (4.1)–(4.3) and (5.1) while matrices B are trivial. The remaining components of superpotentials (2.5) are matrices P whose entries $\hat{P}_{ab}$ are easy calculated integrating equations (5.7):

$$\hat{P}_{ab} = \mu_{ab} \exp\left(\frac{1}{2}\alpha \int (q_a + q_{n+b}) dx\right) \tag{5.14}$$

where μ_{ab} are integration constants satisfying $\mu_{ab} = (\mu_{ba})^*$, and q_σ with $\sigma = a, n + b$ are functions (4.1)–(4.3) corresponding to the same value of parameter ν/α.

In analogous way we can describe a special subclass of matrix superpotentials (2.5) with trivial matrices R [14]. In this case it is convenient to start with diagonalization of matrix Q and write its entries as

$$Q_{\alpha\sigma} = \delta_{\alpha\sigma} q_\sigma, \quad \alpha, \sigma = 1, 2, \ldots, K \tag{5.15}$$

where q_σ are functions satisfying equation (5.13). Then the corresponding entries of matrix P satisfying (5.7) are defined as:

$$P_{\alpha\sigma} = \mu_{\alpha\sigma} \exp\left(\frac{1}{2}\alpha \int (q_\alpha + q_\sigma) dx\right). \tag{5.16}$$

Functions q_α and q_σ included into (5.16) have to satisfy equation (5.13) with the same value of parameter ν. In addition, the matrix whose entries are integration constants $\mu_{\alpha\sigma}$ should be hermitian.

6 Superpotentials realized by 3×3 matrices

Let us search for superpotentials (2.5) realized by matrices of dimension 3×3. We will restrict ourselves to the case when parameter ν in the determining equation (2.6) is equal to zero and find the complete list of the related superpotentials. Like (4.10), (4.11), (4.19) and (4.20) they are liner combinations of power functions of $x + c_i$ with some constant c_i.

There are three versions of the related matrices R whose general form is given in (5.1):

$$R = \begin{pmatrix} I_2 & 0 \\ 0 & 1 \end{pmatrix}, \tag{6.1}$$

$$R = I_3, \tag{6.2}$$

$$R = 0_3. \tag{6.3}$$

Let us start with the case presented in (6.1). The corresponding matrices Q and P are given by formulae (5.2) and (5.3) with

$$A = \begin{pmatrix} a_1 & a_2 \\ a_2^* & a_3 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad \hat{P} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \tag{6.4}$$

where $a_1, a_2, a_3, c, b_1, b_2, p_1$ and p_2 are unknown scalar functions. Moreover, a_1, a_3 and c should be real, otherwise Q is not hermitian. Without loss of generality we suppose that p_1 and p_2 be imaginary, since applying a unitary transformation to Q, P and R these functions always can be reduced to a purely imaginary form. Moreover, the corresponding transformation matrix is diagonal.

In the previous section we *apriori* restrict ourselves to trivial matrices B. Let us show that this restriction is not necessary at least for the considered case $\nu = 0$.

First let us prove that system (5.4)–(5.9) is compatible iff $\overline{\mu} = \tau = 0$, and it is true for all versions of matrix R enumerated in (6.1)–(6.3). Calculating traces of matrices present in (5.8) and (5.9) we obtain the following relation:

$$\tau(\mathtt{Tr}A + \mathtt{Tr}C) + \overline{\mu}(m - n) = 0 \tag{6.5}$$

where $m = 2, 3, 0$ for versions (6.1), (6.2), (6.1) respectively, and $n = 3 - m$.

Differentiating all terms in (6.5) w.r.t. x and using equations (5.4), (5.5) we obtain:

$$\tau(\mathtt{Tr}A^2 + \mathtt{Tr}C^2 + 2\mathtt{Tr}B^\dagger B + \nu(m + n)) = 0. \tag{6.6}$$

Three the first terms in brackets are positive defined and we supposed that $\nu = 0$. If $\tau \neq 0$ all terms in brackets should be zero. If the trace of the square of hermitian matrix is zero, this matrix is zero too, the same is true for matrix $B^\dagger B$. Thus for $\tau \neq 0$ matrix Q (5.2) is trivial. To obtain non-trivial solutions we have to set $\tau = 0$, then from (6.5) we obtain that $\overline{\mu} = 0$ also.

Substituting (6.4) into (5.8) and (5.9) (remember that $\overline{\mu} = \tau = 0$) we obtain the following relations:

$$p_1 b_2^* - b_1 p_2 = 0, \quad p_2 b_1^* - b_2 p_1 = 0, \quad p_1(b_1 - b_1^*) = 0, \quad p_2(b_2 - b_2^*) = 0. \tag{6.7}$$

In accordance with (6.7) there are three qualitatively different possibilities:

$$(a): \ p_1 = p_2 = 0, \quad (b): \ p_1 b_2 = p_2 b_1, \ b_1^* = b_1, \ b_2^* = b_2 \tag{6.8}$$

and

$$(c): \ b_1 = b_2 = 0. \tag{6.9}$$

In the cases (6.8) the corresponding superpotentials are reducible. Indeed, in the case (a) the only condition we need to satisfy is equation (2.6). But matrix Q can be diagonalized (see Appendix) and so the related superpotential can be reduced to the direct sum of three scalar potentials.

The only possibilities to realize the case (b), which differs from cases (a) and (c) is to suppose that

$$p_1 = \alpha p_2 \quad \text{and} \quad b_1 = \alpha b_2 \tag{6.10}$$

or $p_1 = \alpha b_1$ and $p_2 = \alpha b_2$ where α is a constant parameter. The second possibility is excluded since p_α can be proportional to b_α only in the case when these functions are reduced to constants (otherwise p_α and b_α are linearly independent, compare equations (5.6) and (5.7)), and so this possibility is reduced to (6.10) also.

But if conditions (6.10) are realized the corresponding superpotential is reducible too since the transformation $W \to UWU^\dagger$ with

$$U = \frac{1}{\sqrt{1+\alpha^2}} \begin{pmatrix} \alpha & 1 & 0 \\ -1 & \alpha & 0 \\ 0 & 0 & \sqrt{1+\alpha^2} \end{pmatrix}$$

makes it block diagonal.

Thus to obtain an irreducible superpotential (2.5) we should impose the condition (6.9), and our problem is reduced to solving the system (5.10), (5.11) where $\nu = 0$. Like in section 3 the 2×2 matrix A can be chosen diagonal, i.e., we can set $a_2 = 0$ in (6.4) while the remaining (diagonal) entries of matrix Q can be denoted as $a_1 = q_1, a_2 = q_2, C = q_3$, compare with (5.12). In accordance with (5.13) with $\nu = 0$, functions q_i can independently take the following values:

$$\begin{aligned} q_1 &= -\frac{1}{x+c_1} \quad \text{or} \quad q_1 = 0, \\ q_2 &= -\frac{1}{x+c_2} \quad \text{or} \quad q_2 = 0, \\ q_3 &= -\frac{1}{x+c_3} \quad \text{or} \quad q_3 = 0 \end{aligned} \tag{6.11}$$

where c_1, c_2 and c_3 are integration constants. The corresponding values of p_1 and p_2 are easy calculated using equation (5.14). As a result we obtain the following irreducible superpotentials:

$$\begin{aligned} W &= (S_1^2 - 1)\frac{\kappa}{x+c_1} + (S_2^2 - 1)\frac{\kappa}{x+c_2} + (S_3^2 - 1)\frac{\kappa}{x} \\ &+ S_1\frac{\mu_1}{\sqrt{x(x+c_1)}} + S_2\frac{\mu_2}{\sqrt{x(x+c_2)}} + \frac{\omega}{\kappa}(2S_3^2 - 1), \end{aligned} \tag{6.12}$$

$$W = (S_1^2 - 1)\frac{\kappa}{x} + (S_2^2 - 1)\frac{\kappa}{x+c} + S_1\frac{\mu_1}{\sqrt{x}} + S_2\frac{\mu_2}{\sqrt{x+c}} + \frac{\omega}{\kappa}(2S_3^2 - 1), \tag{6.13}$$

$$W = (S_1^2 - 1)\frac{\kappa}{x+c} + (S_3^2 - 1)\frac{\kappa}{x} + S_1\frac{\mu_1}{\sqrt{x}} + S_3\frac{\mu_2}{\sqrt{x(x+c)}} + \frac{\omega}{\kappa}(2S_3^2 - 1), \tag{6.14}$$

$$W = (S_1^2 - 1)\frac{\kappa}{x} + S_1 c + S_2\frac{\mu}{\sqrt{x}} + \frac{\omega}{\kappa}(2S_3^2 - 1) \tag{6.15}$$

where c, c_1, c_2, μ, μ_1 and μ_2 are integration constants, and

$$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6.16}$$

are matrices of spin $s = 1$.

Formulae (6.12)–(6.15) give the complete list of 3×3 matrix superpotentials including matrix R in form (6.1). If this matrix is proportional to the unit one (i.e., if the version (6.2) is realized), the related matrix P should be trivial, see equation (2.8) with $\lambda = 0$. Diagonalizing the corresponding matrix Q we obtain the direct sum of three scalar potentials, i.e., the related superpotentials are reducible.

In the case (6.3) we again can restrict ourselves to diagonal matrices Q whose entries are enumerated in (6.11). The corresponding matrices P can be calculated using equation (5.16). As a result we obtain the following superpotentials:

$$\begin{aligned} W &= (S_1^2 - 1)\frac{\kappa}{x + c_1} + (S_2^2 - 1)\frac{\kappa}{x + c_2} + (S_3^2 - 1)\frac{\kappa}{x} \\ &+ S_1\frac{\mu_1}{\sqrt{x(x + c_1)}} + S_2\frac{\mu_2}{\sqrt{x(x + c_2)}} + S_3\frac{\mu_3}{\sqrt{(x + c_1)(x + c_2)}}, \end{aligned} \tag{6.17}$$

$$W = (S_1^2 - 1)\frac{\kappa}{x} + (S_2^2 - 1)\frac{\kappa}{x + c} + S_1\frac{\mu_1}{\sqrt{x}} + S_2\frac{\mu_2}{\sqrt{x + c}} + S_3\frac{\mu_3}{\sqrt{x(x + c)}}, \tag{6.18}$$

$$W = (S_1^2 - 1)\frac{\kappa}{x} + S_1 c + S_3\frac{\mu_1}{\sqrt{x}} + S_2\frac{\mu_2}{\sqrt{x}}. \tag{6.19}$$

Formulae (6.12)–(6.19) present the complete list of irreducible 3×3 matrix superpotentials corresponding to zero value of parameter ν in (2.6).

In full analogy with the above we can find superpotentials with ν nonzero. In this case the list of solutions (6.11) is changed to solutions (4.1)–(4.3) with $\sigma = 1, 2, 3$ and the same value of ν for all values of σ. The corresponding matrices P again is calculated using equation (5.14) for non-trivial R and equation (2.8) for R trivial.

7 Examples of integrable matrix models

Thus we obtain a collection of integrable models with matrix potentials. The related superpotentials of dimension 2×2 are given by equations (4.4)–(4.20). They are defined up to arbitrary constants $c, \lambda, \mu, \nu, \ldots$. In addition, in section 5 we present the infinite number of superpotentials realized by matrices of arbitrary dimension. So we have a rather large database of shape invariant models, whose potentials have the form indicated in (4.22).

Of course it is impossible to present a consistent analysis of all found models in one paper. But we can discuss at least some of them. In this and the following sections we consider some particular examples of found models.

7.1 Matrix Hamiltonians with Hydrogen atom spectra

Let us start with the superpotential given by equation (4.10). In addition to variable parameter κ it includes an arbitrary parameter μ, and two additional parameters, r_2 and r_3, which define matrix R (3.10). Moreover, $\mu^2 + r_2^2 \neq 0$, otherwise operator (4.10) reduces to a direct sum of two scalar superpotentials.

The simplest version of the considered superpotential corresponds to the case $\mu = 0$ and $r_3 = 0, r_2 = \omega$. Then with the unitary transformation $W_\kappa^{(8)} \rightarrow W_\kappa = UW_\kappa^{(8)}U^\dagger, U = (1 + i\sigma_3)/\sqrt{2}$ we transform $W_\kappa^{(8)}$ to the following (real) form:

$$W_\kappa = -\frac{\sigma_+ \kappa}{x} - \frac{\sigma_1 \omega}{\kappa}. \tag{7.1}$$

The corresponding potential (4.22) looks as:

$$V_\kappa = \frac{\kappa(\kappa - 1)\sigma_+}{x^2} + \frac{\omega\sigma_1}{x} + \frac{\omega^2}{\kappa^2}. \tag{7.2}$$

Shape invariance of potential (7.2) is almost evident. Calculating its superpartner $V_\kappa^+ = W_\kappa^2 + W_\kappa'$ we easily find that

$$V_\kappa^+ = V_{\kappa+1} + C_\kappa \tag{7.3}$$

where

$$C_\kappa = \frac{\omega^2}{(\kappa + 1)^2} - \frac{\omega^2}{\kappa^2}.$$

Using this property (which makes our model be in some aspects similar to the non-relativistic Hydrogen atom) we immediately find spectrum of Hamiltonian (2.2) with potential (7.2):

$$E_N = -\frac{\omega^2}{N^2} \tag{7.4}$$

where $N = \kappa + n$, $n = 0, 1, 2, \ldots$

The ground state vector $\psi_0(\kappa, x)$ should solve the equation

$$a_\kappa^- \psi_0(\kappa, x) \equiv \left(\frac{\partial}{\partial x} + W_\kappa\right)\psi_0(\kappa, x) = 0 \tag{7.5}$$

where W_κ is 2×2 matrix superpotential (7.1) and $\psi_0(\kappa, x)$ is a two component function:

$$\psi_0(\kappa, x) = \begin{pmatrix} \varphi \\ \xi \end{pmatrix}. \tag{7.6}$$

Substituting (7.1) and (7.6) into (7.5) we obtain the following system:

$$\varphi' - \frac{\kappa}{x}\varphi - \frac{\omega}{\kappa}\xi = 0, \tag{7.7}$$

$$\kappa\xi' - \omega\varphi = 0. \tag{7.8}$$

Substituting $\varphi = \frac{\kappa}{\omega}\xi'$ obtained from (7.8) into (7.7) we come to the second-order equation for ξ:

$$\xi'' - \frac{\kappa}{x}\xi' - \frac{\omega^2}{\kappa^2}\xi = 0,$$

whose normalizable solutions are:

$$\xi = \omega x^\nu K_\nu\left(\frac{\omega x}{\kappa}\right), \quad \nu = \frac{\kappa+1}{2} \tag{7.9}$$

where $K_\nu(\cdot)$ are modified Bessel functions. The first component of function (7.6) is easy calculated using (7.8):

$$\varphi = \kappa(\kappa+1)x^{\nu-1}K_\nu\left(\frac{\omega x}{\kappa}\right) - \omega x^\nu K_{\nu+1}\left(\frac{\omega x}{\kappa}\right). \tag{7.10}$$

Solutions (7.6), (7.9), (7.10) are square integrable for any positive κ and $0 \leq x \leq \infty$. Solution $\psi_n(\kappa, x)$ corresponding to the n^{th} exited state can be obtained from the ground state vector using the following standard relation of SUSY quantum mechanics (see, e.g. [2]):

$$\psi_n(\kappa, x) = a_\kappa^+ a_{\kappa+1}^+ \cdots a_{\kappa+n-1}^+ \psi_0(\kappa+n, x). \tag{7.11}$$

It is not difficult to show that vectors (7.11) are square integrable too provided κ is positive.

In analogous manner it is possible to handle ground state wave functions corresponding to superpotential (4.10) with other values of parameters r_2, r_3 and μ. In general case (but for $r_2\mu \neq 0$) these wave functions are expressed via products of exponentials $\exp\left(-\frac{\omega x}{\kappa}\right)$, powers of x and Kummer functions $U_{\alpha\nu}\left(\frac{2\omega x}{\kappa}\right)$. We will not present the corresponding cumbersome formulae here.

7.2 Matrix Hamiltonians with oscillator spectra

Consider the next relatively simple model which corresponds to superpotential (4.20) with $c = 0$. Denoting $W_\kappa^{(17)} = W_\kappa$ we obtain the following potential:

$$\begin{aligned} V_\kappa = W_\kappa^2 - W_\kappa' = \sigma_+ \left(\frac{4\kappa^2 - 1}{4x^2} + \frac{\omega^2 x^2}{16} + \frac{\mu^2}{x} - (\kappa+1)\frac{\omega}{2}\right) \\ + \sigma_- \left(\frac{\omega^2 x^2}{4} + \frac{\mu^2}{x} - \frac{\omega}{2}\right) - \sigma_1 \left(\frac{3\omega x}{4} - \frac{2\kappa+1}{2x}\right)\frac{\mu}{\sqrt{x}}. \end{aligned} \tag{7.12}$$

It is easy to verify that the superpartner of V_κ, i.e., $V_\kappa^+ = W_\kappa^2 + W_\kappa'$ is equal to $V_{\kappa+1}$ up to a constant term:

$$V_\kappa^+ = V_{\kappa+1} + \omega. \tag{7.13}$$

In other words, potential (7.12) is shape invariant in accordance with the definition given in [1]. Moreover, like in the case of supersymmetric oscillator, the superpartner potential V_κ^+ differs from $V_{\kappa+1}$ by the constant term ω which does not depend on variable parameter k.

Using standard tools of SUSY quantum mechanics it is possible to find the spectrum of system (2.1), (2.2) with hamiltonian (7.12):

$$E_n = n\omega, \quad n = 0, 1, 2, ... \tag{7.14}$$

which coincides with the spectrum of supersymmetric oscillator.

The ground state vector is defined as a square integrable solution of equation (7.5). Substituting (4.20) and (7.6) into (7.5) we obtain the following system:

$$\varphi' - \left(\frac{2\kappa+1}{2x} - \frac{\omega x}{4}\right)\varphi - \frac{\mu}{\sqrt{x}}\xi = 0 \tag{7.15}$$

$$\xi' + \frac{\omega x}{2}\xi - \frac{\mu}{\sqrt{x}}\varphi = 0. \tag{7.16}$$

Changing in (7.15), (7.16)

$$\varphi = \exp\left(-\frac{\omega x^2}{4}\right)\tilde{\varphi}, \quad \xi = \exp\left(-\frac{\omega x^2}{4}\right)\tilde{\xi}, \tag{7.17}$$

solving equation (7.16) for $\tilde{\xi}$ and substituting the found expression into (7.15) we obtain the second order equation for $\tilde{\varphi}$:

$$\tilde{\varphi}'' - \left(\frac{\omega x}{4} + \frac{\kappa}{x}\right)\tilde{\varphi}' - \frac{\mu^2}{x}\tilde{\varphi} = 0.$$

Its solutions are linear combinations of Heun biconfluent functions: $\tilde{\varphi} = c_1\tilde{\varphi}_1 + c_2\tilde{\varphi}_2$ where

$$\tilde{\varphi}_1 = H_B(-a_+, 0, a_-, b, cx), \quad \tilde{\varphi}_2 = x^{\kappa+1}H_B(a_+, 0, a_-, b, cx) \tag{7.18}$$

where

$$a_\pm = 1 \pm \kappa, \quad b = \frac{4\sqrt{2}\mu^2}{\sqrt{\omega}}, \quad c = \frac{\sqrt{2\omega}}{4}. \tag{7.19}$$

Thus, in accordance with (7.17), (7.18) and (7.15) we have two ground state solutions (7.6) with

$$\begin{aligned}
\varphi = \varphi_1 &= \exp\left(-\frac{\omega x^2}{4}\right) H_B(-a_+, 0, a_-, b, cx),\\
\xi = \xi_1 &= \exp\left(-\frac{\omega x^2}{4}\right)\left(\frac{\sqrt{2\omega x}}{4\mu}H_B'(-a_+, 0, a_-, b, cx)\right.\\
&\left. -\frac{1}{2\mu}\left(\frac{2\kappa+1}{\sqrt{x}} + \frac{\omega x^{\frac{3}{2}}}{2}\right)H_B(-a_+, 0, a_-, b, cx)\right)
\end{aligned} \tag{7.20}$$

and

$$\begin{aligned}
&\varphi = \varphi_2 = x^{\kappa+1} H_B(a_+, 0, a_-, b, cx),\\
&\xi = \xi_2 = \exp\left(-\frac{\omega x^2}{4}\right)\left(\frac{\sqrt{2\omega x}}{4\mu} H_B'(a_+, 0, a_-, b, cx)\right.\\
&\left.-\frac{1}{2\mu}\left(\frac{2\kappa+1}{\sqrt{x}} + \frac{\omega x^{\frac{3}{2}}}{2}\right) H_B(a_+, 0, a_-, b, cx)\right).
\end{aligned} \tag{7.21}$$

Functions (7.6) whose components are defined in (7.20) and (7.21) are square integrable for any real values of parameters κ, μ and positive ω.

Notice that for integer κ solutions (7.20) and (7.21) are linearly dependent. We will not present here the cumbersome expression of the second solution linearly independent with (7.20) which can be easy found solving system (7.15), (7.16) for κ integer.

One more matrix superpotential generating the spectrum of supersymmetric oscillator is given by equation (4.19). Setting for simplicity $\delta = 0$ we obtain the corresponding Hamiltonian (4.22) in the following form:

$$\begin{aligned}
V_\kappa = &\frac{\omega^2}{4}(x^2 + c^2) + \frac{\mu^2}{x^2 - c^2} + \left(\kappa + \frac{1}{2}\right)\omega - \sigma_1 \mu x \left(\frac{2\kappa - 1}{(x^2 - c^2)^{\frac{3}{2}}} + \frac{\omega}{(x^2 - c^2)^{\frac{1}{2}}}\right)\\
&+ \kappa(\kappa - 1)\left(\sigma_+ \frac{1}{(x + c)^2} + \sigma_- \frac{1}{(x - c)^2}\right).
\end{aligned} \tag{7.22}$$

Like (7.12), potential (7.22) satisfies the form-invariance condition written in the form (7.13). The spectrum of the corresponding Hamiltonian (2.2) is given by equation (7.14). The ground state vectors, i.e., solutions of equation (7.5) where $W_\kappa = W_\kappa^{(16)}$ is superpotential (4.19) with $\delta = 0$, are given by equation (7.6) with components φ and ξ given below:

$$\begin{aligned}
&\varphi = \varphi_1 = \exp\left(-4\omega(x + c)^2\right)(c^2 - x^2)^\kappa (c - x)^{\frac{1}{2}} H_C\left(a, b_-, b_+, d, r; \frac{x + c}{2c}\right),\\
&\xi = \xi_1 = \frac{\mathrm{i}}{cx}\exp\left(-4\omega(x + c)^2\right)(c^2 - x^2)^\kappa \left(\frac{x - c}{2} H_C'\left(a, b_-, b_+, d, r; \frac{x + c}{2c}\right)\right.\\
&\left.+\left(\kappa + \frac{1}{2}\right) H_C\left(a, b_-, b_+, d, r; \frac{x + c}{2c}\right)\right)
\end{aligned} \tag{7.23}$$

where $H_C(\dots)$ is the confluent Heun function,

$$a = -4\omega c^2, \quad b_\pm = \kappa \pm \frac{1}{2}, \quad d = 2\omega c^2, \quad r = 2b_+ c^2 \omega + \frac{1}{2}\kappa^2 + \frac{3}{8} - \mu^2. \tag{7.24}$$

There exist one more ground state vector for Hamiltonian (2.2) with potential (7.22) whose components are

$$\varphi_2 = \exp\left(-4\omega(x+c)^2\right)(c^2-x^2)^{\frac{1}{2}}(c-x)^{\kappa}H_C\left(a,-b_-,b_+,d,r;\frac{x+c}{2c}\right),$$
$$\xi_2 = \frac{\mathrm{i}}{cx}\exp\left(-4\omega(x+c)^2\right)(c-x)^{\kappa}\left(c\,(2c\kappa-x)\,H_C\left(a,-b_-,b_+,d,r;\frac{x+c}{2c}\right)\right. \quad (7.25)$$
$$\left.+\frac{x^2-c^2}{2}H_C'\left(a,-b_-,b_+,d,r;\frac{x+c}{2c}\right)\right)$$

where $a, b_\pm, d$ and r are parameters defined by equation (7.24).

For $\omega > 0$ and $k > 0$ functions (7.23) and (7.25) are square integrable on whole real axis.

7.3 Potentials including hyperbolic functions

An important model of ordinary (scalar) SUSY quantum mechanics is described by Schrödinger equation with the hyperbolic Scarf potential

$$V_\kappa = -\kappa(\kappa-1)\operatorname{sech}^2(x). \quad (7.26)$$

This model possesses a peculiar nature at integer values of the parameter κ, namely, it is a reflectionless (non-periodic finite-gap) system which is isospectral with the free quantum mechanical particle. In addition, this model possesses a hidden (bosonized) nonlinear supersymmetry [15].

Let us consider shape invariant matrix potentials including (7.26) as an entry. The corresponding superpotential can be chosen in the form (4.8) where $\mu = 0$ and $c_3 = 0, c_2 = \omega$. Then with the unitary transformation $W_\kappa^{(7)} \to W_\kappa = UW_\kappa^{(7)}U^\dagger, U = (1+i\sigma_3)/\sqrt{2}$ we transform it to the real form:

$$W_\kappa = \lambda\left(-\kappa(\sigma_+\tanh(\lambda x)+\sigma_-)+\sigma_1\frac{\omega}{\kappa}\right). \quad (7.27)$$

The corresponding potential looks as:

$$V_\kappa = \lambda^2\left(-\sigma_+\kappa(\kappa-1)\operatorname{sech}^2(x)-\sigma_1\omega(\tanh(\lambda x)+1)+\frac{\omega^2}{\kappa^2}+\kappa^2\right). \quad (7.28)$$

Potential (7.28) satisfies the shape invariance condition (7.3) with

$$C_\kappa = \lambda^2\left(\frac{\omega^2}{(\kappa+1)^2}+(\kappa+1)^2-\frac{\omega^2}{\kappa^2}-\kappa^2\right) \quad (7.29)$$

thus the discrete spectrum of the corresponding Hamiltonian (2.2) is given by the following formula:

$$E = -\lambda^2\left(\frac{\omega^2}{(\kappa+n)^2}+(\kappa+n)^2\right)$$

where

$$n = 0, 1, \ldots, \quad \kappa + n < 0, \quad (\kappa + n)^2 > \omega. \tag{7.30}$$

Conditions (7.30) will be justified in what follows.

To find the ground state vector we should solve equation (7.5) with W_κ and $\psi_0(\kappa, x)$ given by formulae (7.27) and (7.6) correspondingly. This equation is easy integrable and has the following normalizable solutions:

$$\begin{aligned} &\varphi = y^{-\frac{\sqrt{\kappa^4+\omega^2}}{\kappa}}(1-y)^{\frac{\omega}{2\kappa}-\frac{\kappa}{2}}\,{}_2F_1(a, b, c; y), \\ &\xi = \frac{\kappa}{\omega}\left(\kappa(2y-1)\varphi + 2y(y-1)\frac{\partial \varphi}{\partial y}\right). \end{aligned} \tag{7.31}$$

Here ${}_2F_1(a, b, c; y)$ is the hypergeometric function,

$$\begin{aligned} &c = 1 - \frac{1}{\kappa}\sqrt{\kappa^4 + \omega^2}, \quad b = c + \frac{\kappa}{2} + \frac{\omega}{2\kappa}, \quad a = b - \kappa - 1, \\ &y = \frac{1}{2}(\tanh \lambda x + 1). \end{aligned} \tag{7.32}$$

Wave functions for exited states can be found starting with (7.32) and using equation (7.11). In order to functions (7.32) and the corresponding functions be square integrable, parameters κ, ω and n should satisfy condition (7.30), see discussion of normalizability of state vectors including the hypergeometric function in section 10 of paper [13].

Consider also superpotential (4.8) with $\omega = 0$ and $\mu \neq 0$:

$$W_\kappa = -\lambda\left(\kappa(\sigma_+(\tanh \lambda x + \sigma_-) + \sigma_1 \mu \sqrt{\operatorname{sech} \lambda x \exp(-\lambda x)}\right). \tag{7.33}$$

The corresponding potential (4.22) have the following form:

$$\begin{aligned} &V_\kappa = \lambda^2\left(-\sigma_+\kappa(\kappa-1)\operatorname{sech}^2 \lambda x + \kappa^2\right. \\ &\left. +\mu^2 \operatorname{sech} \lambda x \exp(-\lambda x) + \sigma_1 \mu(2\kappa - 1)\exp\frac{\lambda x}{2}\operatorname{sech}^{\frac{3}{2}} \lambda x\right). \end{aligned} \tag{7.34}$$

Solving equation (7.5) with W_κ and $\psi_0(\kappa, x)$ given by formulae (7.27) and (7.6) we find components of the ground state vector:

$$\begin{aligned} &\varphi = \frac{1}{\mu}\sqrt{\frac{1 + \exp(2\lambda x)}{2}}(\kappa\xi - \xi'), \quad \xi = y^\nu (1-y)^{-\frac{\kappa}{2}}\,{}_2F_1(a, b, c; y), \\ &a = \nu - \frac{\kappa}{2}, \quad b = a + \kappa + \frac{1}{2}, \quad c = 1 + 2\nu, \quad \nu = \frac{1}{2}\sqrt{\kappa^2 + 2\mu^2}, \quad y = \frac{1}{2}(\tanh \lambda x + 1) \end{aligned}$$

which are square integrable for $\kappa < 0$. The discrete spectrum of Hamiltonian (2.2) with potential (7.2) is given by the following formula: $E = -\lambda^2(\kappa + n)^2$ where n are natural numbers satisfying the condition $\kappa + n < 0$. If this condition is violated, the related eigenvectors (7.11) are not normalizable.

8 Multidimensional integrable models

The models considered in previous subsections are one dimensional in spatial variable. Of course it is more interesting to search for multidimensional (especially, three dimensional) models which can be reduced to integrable models by separation of variables. Famous examples of such models are the (non-relativistic) Hydrogen atom and the Pron'ko-Stroganov system [3] which can be reduced to a scalar and matrix shape invariant systems correspondingly. A more "fresh" example is the reduction of the AdS/CFT holographythe model to the Poschl-Teller system proposed in [16].

In this section we consider new examples of the three-dimensional Schrödinger-Pauli equations which can be reduced to a shape invariant form by separation of variables. Moreover, the related effective potentials in radial variable belong to the shape invariant potentials deduced above.

8.1 Spinor models

Consider shape invariant potential generated by the following superpotential:

$$W = (\mu\sigma_3 - j - 1)\frac{1}{x} + \frac{\omega}{2(j+1)}\sigma_1. \tag{8.1}$$

This operator belongs to the list of matrix superpotentials presented in section 4, see equation (4.10). More exactly, to obtain (8.1) it is necessary to set $c = 0, \kappa = j+1, r_2 = 0$ and $r_1 = -\frac{\omega}{2}$ in (4.10) and (3.10). Then such specified superpotential $W_\kappa^{(7)}$ appears to be unitary equivalent to W (8.1), namely,

$$W = UW_\kappa^{(7)}U^\dagger \quad \text{with} \quad U = \frac{1}{\sqrt{2}}(1 + \mathrm{i}\sigma_2). \tag{8.2}$$

Calculating potential (4.22) corresponding to superpotential (8.1) with $\mu = \frac{1}{2}$ we obtain:

$$V = W^2 - W' = \left(j(j+1) + \frac{1}{4} - \left(j + \frac{1}{2}\right)\sigma_3\right)\frac{1}{x^2} - \frac{\omega}{x}\sigma_1. \tag{8.3}$$

By construction, potential (8.3) is shape invariant, thus the related eigenvalue problem

$$\left(-\frac{\partial^2}{\partial x^2} + V\right)\psi = E\psi \tag{8.4}$$

can be solved exactly using standard tools of SUSY quantum mechanics. The corresponding ground state vector is a two-component function (7.6) with

$$\varphi = y^{j+\frac{3}{2}}K_1(y), \quad \tilde{\xi} = y^{j+\frac{3}{2}}K_0(y), \tag{8.5}$$

where $y = \frac{\omega x}{2(j+1)}$. Energy spectrum is given by equation (7.4) with $N = 2j + n + 1, \quad n = 0, 1, \ldots$

The eigenvalue problem (8.4) with potential (8.3) includes the only independent variable x. However, it can be treated as a radial equation corresponding to the following three dimensional Hamiltonian with the Pauli type potential:

$$H = -\Delta + \omega \boldsymbol{\sigma} \cdot \mathbf{B}, \quad \mathbf{B} = \frac{\mathbf{x}}{x^2}. \tag{8.6}$$

Here δ is the Laplace operator, $\frac{1}{2}\,\boldsymbol{\sigma}$ is a spin vector whose components are Pauli matrices (3.2), and $\mathbf{B}$ is the coordinate three vector divided by $x^2 = x_1^2 + x_2^2 + x_3^2$.

Of course, $\mathbf{B}$ has nothing to do with the magnetic field since $\nabla \cdot \mathbf{B} \neq 0$. However, it can represent another field, e.g., the axion one [17]. Existence of such solutions for equations of axion electrodynamics was indicated recently [18].

Expanding solutions of the eigenvalue problem for Hamiltonian (8.6) via spherical spinors we obtain exactly equation (8.5) for radial functions. We will not present the corresponding calculation here which can be done using the standard representations for the Laplace operator and matrix $\boldsymbol{\sigma} \cdot \mathbf{x}$ in the spherical spinor basis, which can be found, e.g., in [19].

Let us remind that the Pron'ko-Stroganov model is based on the following (rescalled) Hamiltonian

$$H = -\Delta + \frac{\sigma_1 x_2 - \sigma_2 x_1}{r^2}, \quad r^2 = x_1^2 + x_2^2 \tag{8.7}$$

which is reduced to the following form in cylindrical variables:

$$\hat{H}_m = -\frac{\partial^2}{\partial r^2} + m(m - \sigma_3)\frac{1}{r^2} + \sigma_1 \frac{1}{r} \tag{8.8}$$

(we ignore derivatives w.r.t. x_3).

Hamiltonian (8.8) can be expressed in the form (2.3). Moreover, the corresponding superpotential again can be obtained starting with superpotential (4.10) by setting $\kappa = m + \frac{1}{2}, \mu = \frac{1}{2}, c = r_2 = 0, r_1 = \frac{1}{2}$ and making transformation (8.2).

8.2 Vector models

Hamiltonian (8.7) corresponds to the physically realizable system, i.e., the neutral fermion moving in the field of straight line constant current. A natural desire to generalize this model for particles with spin higher then $\frac{1}{2}$ appears to be hardly satisfied since if we simple change Pauli matrices in (8.7) by matrices, say, of spin one, the resultant model will not be integrable [20].

In paper [21] integrable generalizations of model Hamiltonian (8.7) to the case of arbitrary spin have been formulated. The price paid for this progress was the essential complication of the Pauli interaction term present in (8.7). However, there are rather strong physical arguments for such a complication [21], see also [22] for arguments obtained in frames of the relativistic approach.

In this section we present a new formulation of the spin-one Pron'ko model [21]. Doing this we perform the following goals: to apply our abstract analysis of shape invariant matrix potentials to a physically relevant system and to show that this model is shape invariant and so can be easily solved using SUSY technique.

Let us start with superpotential (6.12) with $c_1 = c_2 = \mu_2 = 0,\ \mu_1 = 1$. Making the unitary (rotation) transformation $W \to UWU^\dagger$ with $U = \exp(\mathrm{i}S_2\pi/4)$ and changing the notations $x \to r, \kappa \to m + \frac{1}{2}$ we reduce it to the following form:

$$W = \frac{1}{r}S_3 - \frac{\omega}{2m+1}\left(2S_1^2 - 1\right) - \frac{2m+1}{2x}. \tag{8.9}$$

In addition, we transform the spin matrices to the Gelfand-Tsetlin form:

$$S_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 = \frac{\mathrm{i}}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{8.10}$$

The corresponding shape-invariant potential looks as

$$V_m = W^2 + W' = \left((m - S_3)^2 - \frac{1}{4}\right)\frac{1}{r^2} + \omega\left(2S_1^2 - 1\right)\frac{1}{r} + \frac{\omega^2}{(2m+1)^2}. \tag{8.11}$$

So far we simple represented one of the numerous supersymmetric toys classified in the above. Now we are ready to formulate a two dimensional model which generates the effective potential (8.11). The corresponding Hamiltonian can be written as:

$$H = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + \omega\frac{2(\mathbf{S}\cdot\mathbf{H})^2 - \mathbf{H}^2}{|\mathbf{H}|}. \tag{8.12}$$

Here $\mathbf{H}$ is the two-dimensional vector of magnetic field generated by an infinite straight current; its components are $H_1 = q\frac{x_2}{r^2}$ and $H_2 = -q\frac{x_1}{r^2}$.

First we note that the last term in (8.12) is a particular case of the interaction term found in [21], see equations (15), (21), (29) here for $s = 1$ and $\beta_1 = \beta_0$. However, we believe that our formulation of this term is more transparent.

Secondly, introducing radial and angular variables such that $x_1 = r\cos\theta,\ x_2 = r\sin\theta$, and expanding eigenfunctions of Hamiltonian (8.12) via eigenfunctions ψ_m of the symmetry operator $J_3 = \mathrm{i}\left(x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2}\right) + S_3$ which can be written as:

$$\psi_m = \frac{1}{\sqrt{r}}\begin{pmatrix} \exp(\mathtt{i}(m+1)\theta)\phi_1(r) \\ \exp(\mathtt{i}m\theta)\phi_2(r) \\ \exp(\mathtt{i}(m-1)\theta)\phi_3(r) \end{pmatrix} \tag{8.13}$$

we come to the following hamiltonian in radial variables: $H = -\frac{\partial^2}{\partial r^2} + V_m$ where V_m is the effective potential which coincides with (8.11). Thus Hamiltonian (8.12) is shape invariant and its discrete spectrum and the corresponding eigenvectors are easily calculated using the standard tools of SUSY quantum mechanics.

To end this section we present one more integrable model for vector boson. This model is three dimensional in spatial variables and is characterized by the following Hamiltonian

$$H = -\Delta + \omega \frac{2(\mathbf{S}\cdot\mathbf{B})^2 - \mathbf{B}^2}{|\mathbf{B}|}. \tag{8.14}$$

Here $\mathbf{B}$ is the three dimensional vector defined in (8.6) and $\mathbf{S}$ is the matrix vector whose components are given in equation (8.10).

Like (8.6), Hamiltonian (8.14) corresponds to the shape invariant potential in radial variables, which looks as

$$V = \left(j(j+1) + S_3^2 - (2j+1)S_3\right)\frac{1}{x^2} + (2S_1^2 - 1)\frac{\omega}{x}.$$

The corresponding superpotential can be obtained from (8.9) changing $m \rightarrow j + \frac{1}{2}$.

9 Discussion

In spite of that (scalar) shape invariant potentials had been classified long time ago, there exist a great number of other such potentials which where not known till now, and they belong to the class of matrix potentials. The first attempt to classify these potentials which we made in recent paper [13] enabled to find five types of them which are defined up to arbitrary parameters. They give rise to new integrable systems of Schrödinger equations which can be easily solved within the standard technique of SUSY quantum mechanics.

In the present paper we present an infinite number of such integrable systems. In particular we present the list of superpotentials realized by matrices of dimension 2×2, see equations (4.4)–(4.20). The main value of the list is its completeness, i.e., it includes all superpotentials realized by 2×2 matrices which correspond to Schrödinger-Pauli systems (2.1) which are shape invariant w.r.t. shifts of variable parameters.

In section 5 an extended class of arbitrary dimension matrix superpotentials is described. We do not present the proof that this class includes all irreducible matrix superpotentials. However such assumption looks rather plausible since by a consequent differentiation of equations (5.8) and (5.9) with using conditions (5.4)–(5.7) we can obtain an infinite number of algebraic compatibility conditions for system (5.4)–(5.9) which are nontrivial but can be satisfied asking for B be equal to zero. An alternative solution of these compatibility conditions is $P = 0$, but it leads to reducible superpotentials. Computing experiments with system (5.4)–(5.9) for the cases of $n\times n$ matrix superpotential with $n \leq 5$ also support the assumption $B = 0$, see section 6 where we did not make this assumption *a priori* but prove it. Notice that for $n = 2$ this condition is not necessarily satisfied, but it is seemed to be the only exceptional case.

Nevertheless in section 5 we consider the condition $B = 0$ as an additional requirement, which enables to find superpotentials of arbitrary dimension in a straightforward way.

Thus we obtain an entire collection of integrable systems of Schrödinger equations. Some examples of these models are considered in section 7 where we we find their energy spectra and ground state solutions. Among them there are two oscillator-like matrix models whose spectra are linear in the main quantum number, see section 7.2.

One dimensional integrable models classified in the present paper are especially interesting in the cases when they can be used to construct solutions of two- and three-dimensional systems. A perfect example of such shape invariant system is the radial equation for the Hydrogen atom. Thus an important task is to search for multidimensional (in particular, two- and three-dimensional) models which can be reduced to found shape invariant systems after separation of variables. Some results of our search can be found in section 8 were we present new integrable problems for two- and three-dimensional equations of Schrödinger-Pauli type. In particular we discuss SUSY aspects of the Pron'ko-Stroganov model generalized to the case of vector particles (such generalization was proposed in paper [21]). It happens that the spin-one model is shape invariant and so it can be easily integrated using tools of SUSY quantum mechanics.

Except the case $s = 1$ we did not discuss superintegrable models proposed in [21] for arbitrary spin s. However, it is possible to show that they are supersymmetric too.

Let us note that the results presented in section 8 can be considered only as an advertisement, and we plane to present the detailed discussion of integrable multidimensional models in the following publications.

A Solutions of matrix Riccati equations

The corner stone of our classification of matrix superpotentials is the diagonalization of matrix Riccati equations (2.6) and (5.10). For completeness, we present here a simple and constructive algorithm of such diagonalization for hermitian matrices of any dimension while the case of 2×2 matrices was already discussed in section 3.

Let us start with equation (2.6) and make the following change of the dependent variable Q:

$$Q = M + qI \tag{A1}$$

where I is the unit matrix and $q = q_\sigma$ is one of the particular solutions (4.2) of the scalar Riccati equation (5.13) (the unnecessary subindex σ can be omitted). Thanks to the randomness of integration constants in solutions (4.2) we always can suppose that the unknown matrix M be nondegenerated.

Substituting (A1) into (2.6) and using the identity $(M^{-1})' = -M^{-1}M'M^{-1}$ we obtain the following *linear* equation for M^{-1}:

$$(M^{-1})' = -\alpha(I + 2qM^{-1}). \tag{A2}$$

Its solutions have the following generic form:

$$M^{-1} = \rho(x)I + \theta(x)C \tag{A3}$$

where $\rho(x)$ and $\theta(x)$ are scalar real valued functions whose exact expressions can be easily found for any particular solution (4.2), and C is an arbitrary constant matrix.

In accordance with (A3) matrix M^{-1} is *diagonalizable* (remember that M^{-1} should be hermitian). The same is true for matrix M and so matrix Q (A2) is diagonalizable too.

In complete analogy with the above one can justify the diagonalizability of matrices A and C satisfying equations (5.10).

References

[1] L. Gendenshtein, JETP Lett. 38 (1983) 356.

[2] F. Cooper, A. Khare and U. Sukhatme, Phys. Rep. 251 (1995) 267.

[3] G. P. Pron'ko, Y. G. Stroganov, Sov. Phys. JETP 45 (1977) 1075

[4] A. I. Voronin, Phys. Rev. A 43 (1991) 29.

[5] L. V. Hau, G. A. Golovchenko, and M. M. Burns, Phys. Rev. Lett. 74 (1995) 3138.

[6] E. Ferraro, N. Messina and A.G. Nikitin, Phys. Rev. A 81, 042108 (2010)

[7] A. A. Andrianov and M. V. Ioffe, Phys. Lett. B 255 (1991) 543; A. A. Andrianov, M. V. Ioffe, V. P. Spiridonov and L. Vinet, Phys. Lett. B 272 (1991) 297.

[8] A. A. Andrianov, F. Cannata, M. V. Ioffe and D. N. Nishnianidze, J. Phys. A: Math. Gen. 30 (1997) 5037.

[9] M. V Ioffe, S. Kuru, J. Negro and L. M. Nieto, J. Phys. A: Math. Theor.39 (2006) 6987.

[10] R. de Lima Rodrigues, V. B. Bezerra and A. N. Vaidyac, Phys. Lett. A 287 (2001) 45.

[11] V. M. Tkachuk, P. Roy, Phys. Lett. A 263 (1999) 245;
V. M. Tkachuk, P. Roy, J. Phys. A 33 (2000) 4159.

[12] T. Fukui, Phys. Lett. A 178 (1993) 1.

[13] A. G. Nikitin and Yuri Karadzhov, J. Phys. A: 44 (2011) 305204

[14] Yuri Karadzhov. Matrix superpotentials linear in variable parameter. Arxiv 1106.xxxx (to be published)

[15] F. Correa and M. S. Plyushchay. Ann. Phys. 322 (2007) 2493;

[16] F. Correa, G. V. Dunne and M. S. Plyushchay, Ann. Phys. 324 (2009) 1078.

[17] F. Wilczek, Phys. Rev. Lett. 58 (1987) 1799 .

[18] A. G. Nikitin and Oksana Kuriksha. Group analysis and exact solutions for equations of axion electrodynamics, arXiv:1002.0064

[19] W. I. Fushchich and A. G. Nikitin. Symmetries of equations of quantum mechanics. Allerton Press Inc., New York, 1994.

[20] L.Vestergaard Hau, J.A.Golovchenko and M.M. Burns, Phys. Rev. Lett. 74 (1995); K.Berg-Sorensen, M.M. Burns, J.A.Golovchenko and L.Vestergaard Hau, Phys. Rev. A 53 (1996) 1653.

[21] G. P. Pronko J. Phys. A: Math. Theor. 40 (2007) 13331.

[22] J. Beckers, N. Debergh, and A. G. Nikitin, Fortsch. der Phys. 43 (1995) 81.