Asymptotic Analysis of Non-self-adjoint Hill Operators

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Abstract

In this article we obtain asymptotic formulas, uniform with respect to $t \in [0, 2\pi)$, for eigenvalues and eigenfunctions of the Sturm-Liouville operators $L_t(q)$ with potential $q \in L_1[0,1]$ and $t-$periodic boundary conditions. Using these formulas, we find some conditions on q such that the number of spectral singularities in the spectrum of the Hill operator $L(q)$ in $L_2(-\infty,\infty)$ with $q(x)$ periodic is finite. Then we prove that $L(q)$ is, in some sense, asymptotically spectral operator if q satisfies these conditions.

Key Words: Asymptotic formulas for eigenvalues and eigenfunctions, Hill operator, Spectral singularities, Spectral operator.
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1 Introduction and Preliminary Facts

Let $L(q)$ be the Hill operator generated in $L_2(-\infty,\infty)$ by the expression

$$-y^{''}+q(x)y, \tag{1}$$

where $q(x)$ is a complex-valued summable function on $[0,1]$ and $q(x+1)=q(x)$ for a.e. $x \in (-\infty,\infty)$. It is well-known that (see [7], [23] for real and [5], [16-18] for complex-valued q) the spectrum $S(L(q))$ of the operator $L(q)$ is the union of the spectra $S(L_t(q))$ of the Sturm-Liouville operators $L_t(q)$ for $t \in [0,2\pi)$, where $L_t(q)$ is the operator generated in $L_2[0,1]$ by (1) and the boundary conditions

$$y(1)=e^{it}y(0),\ y^{'}(1)=e^{it}y^{'}(0). \tag{2}$$

In this article we obtain asymptotic formulas, uniform with respect to $t \in [0,2\pi)$, for the eigenvalues and eigenfunctions of the operator $L_t(q)$. Note that, the formula

$f(k,t)=O(h(k))$ is said to be uniform with respect to t in a set S if there exists a positive constants M and N, independent of t, such that $\mid f(k,t)) \mid < M \mid h(k) \mid$ for all $t \in S$ and $\mid k \mid \geq N$. Then using these asymptotic formulas, we find some conditions on the potential q such that the number of the spectral singularities in $S(L(q))$ is finite and $L(q)$ is, in some sense, asymptotically spectral operator.

The spectral expansion for the self-adjoint operator $L(q)$ is constructed by Gelfand [7] and Titchmarsh [23]. Tkachenko [24] proved that the non-self-adjoint operator $L(q)$ can be reduced to triangular form if all eigenvalues of the operators $L_t(q)$ for all $t \in [0,2\pi)$ are simple. McGarvey [16-18] proved that $L(q)$ is a spectral operator if and only if the projections of the operator $L(q)$ are uniformly bounded. However, in general, the eigenvalue of the operator $L_t(q)$ are not simple and the projections of the operator $L(q)$ are not uniformly bounded. In fact, Gasymov [6] investigated the operator $L(q)$ with the potentials q which can be continued analytically onto the upper half plane and proved that this operator, in particular $L(q)$ with the simple potential $q(x) = e^{i2\pi x}$, has infinitely many spectral singularities. Note that the spectral singularities of the operator $L(q)$ are the points of its spectrum in neighborhoods of which the projections of the operator $L(q)$ are not uniformly bounded. Veliev [26] proved that a number $\lambda = \lambda_n(t) \in S(L)$ is a spectral singularity of $L(q)$ if and only if the operator $L_t(q)$ has an associated function at the point $\lambda_n(t)$. In the paper [25] (see also [27]) we constructed the spectral expansion for the operator $L(q)$ with a continuous and complex-valued potential. In the paper [28], we obtained the asymptotic formulas formulas of order $O(n^{-l})$ (for all $l > 0$) for the eigenvalue $\lambda_n(t)$ and eigenfunction $\Psi_{n,t}(x)$ of $L_t(q)$ for $t \neq 0, \pi$ with $q \in L_1[0,1]$. Then using these formulas, we proved that the eigenfunctions and associated functions of L_t form a Riesz basis in $L_2[0,1]$ for $t \neq 0, \pi$ and constructed the spectral expansion for the operator $L(q)$ (see also [13,29,30] for the spectral expansion of the differential operators with periodic coefficients). Recently, Gesztezy and Tkachenko [8,9] proved two versions of a criterion for the Hill operator $L(q)$ with $q \in L_2[0,1]$ to be a spectral operator of scalar type, one analytic and one geometric. The analytic version is stated in term of the solutions of Hill's equation. The geometric version of the criterion uses algebraic and geometric properties of the spectra of periodic/antiperiodic and Dirichlet boundary value problems. In this paper we find conditions on the potential q such that the Hill operator $L(q)$ is, in some sense, asymptotically spectral operator of scalar type.

Since the spectral property of $L(q)$ is strongly connected with the operators $L_t(q)$ for $t \in [0, 2\pi)$, let us discuss briefly the papers devoted to $L_t(q)$. It is known [14] that the operator $L_t(q)$ is Birkhoff regular. In the case $t \neq 0, \pi$ it is strongly regular and the root functions of the operator $L_t(q)$ form a Riesz basis (this result is proved independently in [4,12,19]). In the cases $t = 0$ and $t = \pi$ the operator $L_t(q)$ is not strongly regular. In the case when an operator is regular but not strongly regular the root functions, generally, do not form even usual basis. However, it is known [20, 21] that they can be combined in pairs, so that the corresponding 2-dimensional subspaces form a Riesz basis of subspaces.

We note that, last times, necessary and sufficient conditions have been established it order the root functions of periodic and antiperiodic problems to form a Riesz basis. For brevity, we discuss only the periodic problem. The antiperiodic problem is similar to the periodic problem. In 1996 at a seminar in MSU Shkalikov formulated the following result. Assume that $q(x)$ is a smooth potential,

$$q^{(k)}(0) = q^{(k)}(1), \quad \forall\, k = 0, 1, ..., s-1 \tag{3}$$

and $q^{(s)}(0) \neq q^{(s)}(1)$. Then the root functions of the operator $L_0(q)$ form a Riesz basis in $L_2[0,1]$. Kerimov and Mamedov [11] obtained the rigorous proof of this result in the case $q \in C^4[0,1]$, $q(1) \neq q(0)$. Actually, this results remains valid for an arbitrary $s \geq 0$. It is obtained in Corollary 2 of [22].

Another approach is due to Dernek and Veliev [1]. The result was obtained in terms of the Fourier coefficients of the potential q. Namely, we proved that if conditions

$$\lim_{n\to\infty} \frac{\ln|n|}{nq_{2n}} = 0, \tag{4}$$

$$q_{2n} \sim q_{-2n} \tag{5}$$

hold, then the root functions of $L_0(q)$ form a Riesz basis in $L_2[0,1]$, where $q_n = (q, e^{i2\pi nx})$ is the Fourier coefficient of q and everywhere, without loss of generality, it is assumed that $q_0 = 0$. Here $(.,.)$ denotes inner product in $L_2[0,1]$ and $a_n \sim b_n$ means that $a_n = O(b_n)$ and $b_n = O(a_n)$ as $n \to \infty$. Makin [15] improved this result. Using another method he proved that the assertion on the Riesz basis property remains valid if condition (5) holds, but condition (4) is replaced by a less restrictive one: $q \in W_1^s[0,1]$, (3) holds and $| q_{2n} |> c_0 n^{-s-1}$ for $n \gg 1$ with some $c_0 > 0$, where s is a nonnegative integer. Besides, some conditions which imply the absence of the Riesz basis property were presented in [15]. The results which we obtained in [22] are more general and cover all the results discussed above.

Some sharp results on the absence of the Riesz basis property were obtained by Djakov and Mitjagin [2]. Moreover, recently, Djakov and Mitjagin [3] obtained some interesting results about Riesz basis property of the root functions of the operators $L_0(q)$ with trigonometric polynomial potentials. I do not formulate precisely the results of [2,3], since their formulation takes some additional pages which is not related to this paper. Very recently Gesztezy and Tkachenko [10] proved a criterion for the root functions of $L_0(q)$ to form a Riesz basis in term of the spectra of periodic and Dirichlet boundary value problems.

The next we present some preliminary facts, from [22, 28, 1], we need in this paper.

Result 1 (see [22]). *Let $p \geq 0$ be an arbitrary integer, $q \in W_1^p[0,1]$ and (3) holds with some $s \leq p$. Suppose there is a number $\varepsilon > 0$ such that either the estimate*

$$|q_{2n} - S_{2n} + 2Q_0 Q_{2n}| \geq \varepsilon n^{-s-2} \tag{6}$$

or the estimate

$$|q_{-2n} - S_{-2n} + 2Q_0 Q_{-2n}| \geq \varepsilon n^{-s-2} \tag{7}$$

hold, where $Q_k = (Q(x), e^{2\pi ikx})$ and $S_k = (S(x), e^{2\pi ikx})$ are the Fourier coefficients of

$$Q(x) = \int_0^x q(t)\,dt, \text{ and } \quad S(x) = Q^2(x).$$

Then the condition

$$q_{2n} - S_{2n} + 2Q_0 Q_{2n} \sim q_{-2n} - S_{-2n} + 2Q_0 Q_{-2n} \tag{8}$$

is necessary and sufficient for the root functions of $L_0(q)$ to form a Riesz basis. Moreover if (6) and (8) hold then all large eigenvalues of $L_0(q)$ are simple.

Result 2 (see [28]). *The eigenvalue $\lambda_n(t)$ and eigenfunction $\Psi_{n,t}(x)$ of the operator $L_t(q)$ for $t \neq 0, \pi$, satisfy the following asymptotic formulas*

$$\lambda_n(t) = (2\pi n + t)^2 + O(\frac{ln\,|n|}{n}),\ \Psi_{n,t}(x) = e^{i(2\pi n+t)x} + O(\frac{1}{n}). \tag{9}$$

These asymptotic formulas are uniform with respect to t in $[\rho, \pi-\rho]$,where ρ is a sufficiently small fixed number ($\rho \ll 1$). In the other word, there exist positive numbers $N(\rho)$ and $M(\rho)$, independent of t, such that the eigenvalues $\lambda_n(t)$ for $t \in [\rho, \pi-\rho]$ and$| n |> N(\rho)$ are simple and the terms $O(\frac{1}{n})$, $O(\frac{ln|n|}{n})$ in (9) do not depend on t.

Result 3 (see [1]). *Let the conditions (4) and (5) hold. Then:*

(a) All sufficiently large eigenvalues of the operator $L_0(q)$ are simple. They consists of two sequences $\{\lambda_{n,1} : n > N_0\}$ and $\{\lambda_{n,2} : n > N_0\}$ satisfying

$$\lambda_{n,j} = (2\pi n)^2 + (-1)^j p_{2n} + O\left(\frac{ln\,|n|}{n}\right) \tag{10}$$

for $j = 1, 2$, where $p_n = (q_n q_{-n})^{\frac{1}{2}}$. The corresponding eigenfunction $\varphi_{n,j}(x)$ satisfies

$$\varphi_{n,j}(x) = e^{i2\pi nx} + \alpha_{n,j} e^{-i2\pi nx} + O(\frac{1}{n}), \tag{11}$$

where $\alpha_{n,j} \sim 1$, $\alpha_{n,j} = \frac{(-1)^j p_{2n}}{q_{2n}} + O\left(\frac{ln|n|}{nq_{2n}}\right)$, $j = 1, 2$.

(*b*) *The root functions of* $L_0(q)$ *form a Riesz basis in* $L_2(0,1)$.

Thus, in the papers [28] and [1] we obtained the asymptotic formulas for the operator $L_t(q)$, uniform with respect to $t \in [\rho, \pi - \rho]$, and for the operator $L_0(q)$ respectively, where $\rho \ll 1$. In this paper we obtain the uniform asymptotic formulas in the more complicated case $t \in [0, \rho] \cup [\pi - \rho, \pi]$, when the potential q satisfies some conditions (see Theorem 2 and 3). Some estimations and formulas of the Section 2 are similar to the estimations and formulas that were done in [1], [22] and [28]. However, because of the uniformity, with respect to $t \in [0, \rho]$, we search for we can not cite [1, 22, 28] for the related facts. In the other words, in this paper we take a closer look the uniformity, with respect to $t \in [0, \rho]$, of the formulas and estimations that were not done in those papers and that is very important in this paper. Note that the case $t \in [\pi - \rho, \pi]$ is similar to the case $t \in [0, \rho]$ and the eigenvalues of $L_{-t}(q)$ coincide with the eigenvalues of $L_t(q)$. As a result we get the uniform, with respect to t in $[0, 2\pi)$, asymptotic formulas for the operator $L_t(q)$. These formulas imply that if the potential q satisfies some conditions, then there exists a positive constant R, independent of t, such that all eigenvalues of $L_t(q)$ lying outside of the disk $\{\lambda \in \mathbb{C} : |\lambda| \le R\}$ are simple for all value of t in $[0, 2\pi)$. Since the spectral singularity of the operator $L(q)$ is contained in the set of multiply eigenvalues of $L_t(q)$ for $t \in [0, 2\pi)$, we obtain sufficient conditions on q such that the Hill operator $L(q)$ has at most finitely many spectral singularities. Then we prove that the projections $P(\gamma)$ of the operator $L(q)$ for arcs γ lying outside of the disk $\{\lambda \in \mathbb{C} : |\lambda| \le R\}$ are uniformly bounded if q satisfies these conditions, which mains that $L(q)$ is, in some sense, asymptotically spectral operator.

2 Uniform Asymptotic Formulas for $L_t(q)$

It is well-known that the eigenvalues of $L_t(q)$ are the squares of the roots of the equation

$$F(\xi) = 2\cos t, \tag{12}$$

where $F(\xi) = \varphi'(1, \xi) + \theta(1, \xi)$, and $\varphi(x, \xi)$ and $\theta(x, \xi)$ are the solutions of the equation

$$-y'' + q(x)y = \xi^2 y$$

satisfying the initial conditions $\theta(0, \xi) = \varphi'(0, \xi) = 1, \quad \theta'(0, \xi) = \varphi(0, \xi) = 0$. In [14] (see chapter 1, sec. 3) it is proved that

$$F(\xi) - 2\cos\xi = e^{|Im\xi|}\varepsilon(\xi), \quad \lim_{|\xi|\to\infty} \varepsilon(\xi) = 0. \tag{13}$$

Let us consider the functions $F(\xi) - 2\cos\xi$ and $2\cos\xi - \cos t$ on the circle

$$C(n, t, \rho) =: \{\xi \in \mathbb{C} : |\xi - (2\pi n + t)| = 3\rho\}, \tag{14}$$

where $t \in [0, \rho]$ and $\rho \ll 1$. By (13) there exists a positive number $N(0, \rho)$ such that

$$|F(\xi) - 2\cos\xi| < \rho^2 \tag{15}$$

for $\xi \in C(n, t, \rho)$ whenever $n > N(0, \rho)$ and $t \in [0, \rho]$. On the other hand, using the Taylor formula of $\cos\xi$ at the points $2\pi n + t$ for $\xi = 2\pi n + t + 3\rho e^{i\alpha}$, where $\alpha \in [0, 2\pi)$, and taking into account the inequalities $|\sin t| \le \rho$ and $|\cos t| > \frac{9}{10}$ for $t \in [0, \rho]$, $\rho \ll 1$, we obtain

$$|\ 2\cos\xi - 2\cos t\ | = 2\ |\ -3\rho e^{i\alpha}\sin t + \frac{9}{2}\rho^2 e^{2i\alpha}\cos t + O(\rho^4)\ | > 2\rho^2. \tag{16}$$

By Rouche's theorem, it follows from (15) and (16) that the equation (12) has the same number of zeros in $C(n,t,\rho)$, where $n > N(0,\rho)$, as the equation

$$\cos\xi - \cos t = 0. \tag{17}$$

Since the equation (17) has 2 roots inside the circle $C(n,t,\rho)$, the equation (12) has also 2 roots (counting multiplicity) inside this circle for $n > N(0,\rho)$. On the other hand, it is proved in [14] (see chapter 1, sec. 3) that the estimation

$$F(\xi) - 2\cos\xi = o(\cos\xi - \cos t)$$

holds on the boundaries of the admissible strip $K_n =: \{\xi : |\mathrm{Re}\,\xi| < (2n+1)\pi\}$ for $t \in [0,\rho]$, $\rho \ll 1$. Hence the number of roots of the equations (12) and (17) are the same in the strip K_n and in the sets $K_{n+1} \backslash K_n$ for large n. The following remark follows from these arguments.

Remark 1 *There exists a large number $N(0,\rho)$ such that the number of the roots of the equations (12) lying inside of the strip K_N is $2N+1$. Denote these roots by $\xi_n(t)$ for $n = 0, \pm1, \pm2, ..., \pm N$. The roots of the equations (12) lying outside of the strip K_N consist of the roots lying in the contours $C(n,t,\rho)$ for $n > N(0,\rho)$. The roots of (12) lying in $C(n,t,\rho)$ for $n > N(0,\rho)$ consist only of two roots denoted by $\xi_{n,1}(t)$ and $\xi_{n,2}(t)$. Hence*

$$|\xi_{n,j}(t) - (2\pi n + t)| < 3\rho, \ \forall |n| > N(0,\rho), \ t \in [0,\rho], \ j = 1,2. \tag{18}$$

Since the entire function $\frac{dF}{d\xi}$ has a finite number of zeros inside the circle $\{\xi \in \mathbb{C} : |\xi - 2\pi n| = 4\rho\}$ and this circle encloses $C(n,t,\rho)$ for all $t \in [0,\rho]$, there exists only finite $t_1, t_2, ..., t_k$ from $(0,\rho)$ for which $\xi_n(t_k)$ is a double root of (12). Let $0 < t_1 < t_2 < ... < t_k < \rho$. By implicit function theorem the function $\xi_{n,1}(t)$ and $\xi_{n,2}(t)$ can be chosen as analytic in intervals $(0,t_1)$, (t_k,ρ) and (t_s,t_{s+1}) for $s = 1,2,...,k-1$. Let ξ be any limit point of $\xi_{n,1}(t)$ or $\xi_{n,2}(t)$ as $t \to t_s$. Since $F(\xi_{n,j}(t)) = 2\cos t$ for $j = 1,2$ and F is continuous, we have $F(\xi) = 2\cos t_s$. However, this equation has only one double root $\xi_{n,1}(t_s) = \xi_{n,2}(t_s)$ inside $C(n,t_s,\rho)$. Thus

$$\lim_{t\to t_s^-} \xi_{n,1}(t) = \lim_{t\to t_s^+} \xi_{n,1}(t) = \lim_{t\to t_s^-} \xi_{n,2}(t) = \lim_{t\to t_s^+} \xi_{n,1}(t) = \xi_{n,1}(t_s) = \xi_{n,2}(t_s)$$

for $s = 1,2,...,k$. This implies that the eigenvalues $\lambda_{n,1}(t) = \xi_{n,1}^2(t)$ and $\lambda_{n,2}(t) = \xi_{n,2}^2(t)$ of $L_t(q)$ can be chosen as continuous function on $(0,\rho)$. By the result of [28] (see introduction) $\lambda_{n,1}(\rho)$ and $\lambda_{n,2}(\rho)$ are simple eigenvalues of $L_\rho(q)$ for $n > N(\rho)$. Moreover, if $q \in L_1[0,1]$, and (4), (5) hold then by the result of [1], similarly if $q \in W_1^p[0,1]$, and (3), (6), (8) hold then by the result of [22] $\lambda_{n,1}(0)$ and $\lambda_{n,2}(0)$ are simple eigenvalues of L_0 for $n > N_0$. These arguments imply the continuity of the functions $\lambda_{n,1}(t)$, $\lambda_{n,2}(t)$ and

$$d_n(t) =: |\lambda_{n,1}(t) - \lambda_{n,2}(t)| \tag{19}$$

on $[0,\rho]$ for $n > N =: \max\{N(0,\rho), N(\rho), N_0\}$. By (18) we have

$$\left|\lambda_{n,j}(t) - (2\pi n + t)^2\right| < 15\pi n\rho \tag{20}$$

for $t \in [0,\rho]$, $n > N$ and $j = 1,2$. Thus for $t \in [0,\rho]$ and $n > N$ the disk

$$D(n,t,\rho) =: \{\lambda \in \mathbb{C} : \left|\lambda - (2\pi n + t)^2\right| < 15\pi n\rho\} \tag{21}$$

contains two eigenvalues (counting multiplicity) $\lambda_{n,1}(t)$ and $\lambda_{n,2}(t)$ that are continuous function on the interval $[0,\rho]$. In addition to these eigenvalues, the operator $L_t(q)$ for $t \in [0,\rho]$

has only $2N+1$ *eigenvalues.*

Using (20), one can readily see that

$$\left|\lambda_{n,j}(t)-(2\pi(n-k)+t)^2\right|>|k|\,|2n-k| \tag{22}$$

for $k\neq 0,2n$ and $t\in[0,\rho]$, where $n>N$ and $j=1,2$. To obtain the uniform asymptotic formula for eigenvalues $\lambda_{n,j}(t)$ and corresponding normalized eigenfunctions $\Psi_{n,j,t}(x)$ for $t\in[0,\rho]$, we use (22) and the iteration of the formula

$$(\lambda_{n,j}(t)-(2\pi(n-k)+t)^2)(\Psi_{n,j,t},e^{i(2\pi(n-k)+t)x})=(q\Psi_{n,j,t},e^{i(2\pi(n-k)+t)x}). \tag{23}$$

To iterate (23) we use the following lemma.

Lemma 1 *For the right-hand side of (23) the following equality*

$$(q\Psi_{n,j,t},e^{i(2\pi(n-k)+t)x})=\sum_{m=-\infty}^{\infty}q_m(\Psi_{n,j,t},e^{i(2\pi(n-k-m)+t)x}) \tag{24}$$

and inequality

$$\left|(q\Psi_{n,j,t},e^{i(2\pi(n-k)+t)x})\right|<3M \tag{25}$$

holds, for all $n>N$, $k\in\mathbb{Z}$, $j=1,2$ *and* $t\in[0,\rho]$, *where* $M=\sup_{n\in\mathbb{Z}}|q_n|$, *and* N *is defined in Remark 1. The eigenfunction* $\Psi_{n,j,t}(x)$ *satisfies the following, uniform with respect to* $t\in[0,\rho]$, *asymptotic formulas*

$$\Psi_{n,j,t}(x)=u_{n,j,t}e^{i(2\pi n+t)x}+v_{n,j}(t)e^{i(-2\pi n+t)x}+h_{n,j,t}(x), \tag{26}$$

where $u_{n,j}(t)=(\Psi_{n,j,t}(x),e^{i(2\pi n+t)x})$, $v_{n,j}(t)=(\Psi_{n,j,t}(x),e^{i(-2\pi n+t)x})$,

$$(h_{n,j,t},e^{i(\pm 2\pi n+t)x})=0,\ \ \|h_{n,j,t}\|=O(\frac{1}{n}),\ \ \sup_{\substack{x\in[0,1],\\ t\in[0,\rho]}}|\,h_{n,j,t}(x)\,|=O\left(\frac{ln\,|n|}{n}\right), \tag{27}$$

$$|u_{n,j}(t)|^2+|v_{n,j}(t)|^2=1+O(\frac{1}{n^2}). \tag{28}$$

Proof. The equality (24) is obvious for $q\in L_2(0,1)$. For $q\in L_1(0,1)$ see Lemma 1 of [28]. Since $q\Psi_{n,j,t}\in L_1[0,1]$, we have

$$\lim_{|m|\to\infty}(q\Psi_{n,j,t},e^{i(2\pi(n-k-m)+t)x})=0.$$

Therefore there exists $C(t)$ and $k_0(t)$ such that

$$\max_{s\in\mathbb{Z}}\left|(q\Psi_{n,j,t},e^{i(2\pi s+t)x})\right|=\left|(q\Psi_{n,j,t},e^{i(2\pi(n-k_0)+t)x})\right|=C(t).$$

Now, using (22)-(24) and the obvious relations

$$|q_m|\le M,\ \ \sum_{k\neq 0,2n}\frac{1}{|\,k(2n-k)\,|}=O\left(\frac{\ln n}{n}\right) \tag{29}$$

for $m \in \mathbb{Z}$, we obtain

$$C(t)=\left|(q\Psi_{n,j,t},e^{i(2\pi(n-k_0)+t)x})\right|=\mid\sum_{m=-\infty}^{\infty}q_m(\Psi_{n,j,t},e^{i(2\pi(n-k_0-m)+t)x})\mid=$$

$$\mid q_{-k_0}(\Psi_{n,j,t},e^{i(2\pi n+t)x})+q_{2n-k_0}(\Psi_{n,j,t},e^{i(-2\pi n+t)x})\mid+$$

$$\mid\sum_{m\neq -k_0,2n-k_0}q_m\frac{(q\Psi_{n,j,t},e^{i(2\pi(n-k_0-m)+t)x})}{\lambda_{n,j}(t)-(2\pi(n-k_0-m)+t)^2}\mid\leq 2M+$$
$$\sum_{m\neq -k_0,2n-k_0}\frac{MC(t)}{\mid\lambda_{n,j}(t)-(2\pi(n-k_0-m)+t)^2\mid}=2M+C(t)O\left(\frac{\ln n}{n}\right)$$

which imply that $C(t)<3M$ for all $t\in[0,\rho]$. The inequality (25) is proved. This with (23), (22) and (29) yields

$$\sum_{k\neq\pm n}\left|(\Psi_{n,j,t},e^{i(2\pi k+t)x})\right|=O\left(\frac{\ln n}{n}\right),\ \sum_{k\neq\pm n}\left|(\Psi_{n,j,t},e^{i(2\pi k+t)x})\right|^2=O\left(\frac{1}{n^2}\right).$$

Now decomposing $\Psi_{n,j,t}$ by basis $\{e^{i(2\pi k+t)x}:k\in\mathbb{Z}\}$ and using these equalities we get (26) and (27). The normalization condition $\|\Psi_{n,j,t}\|=1$ with (26) and (27) imply (28) ■

Using (24) in (23), replacing k and m by 0 and n_1 respectively, and then isolating the term containing the multiplicand $(\Psi_{n,j,t},e^{i(-2\pi n+t)x})$ we obtain

$$(\lambda_{n,j}(t)-(2\pi n+t)^2)(\Psi_{n,j,t},e^{i(2\pi n+t)x})-q_{2n}(\Psi_{n,j,t},e^{i(-2\pi n+t)x})= \tag{30}$$
$$\sum_{n_1\neq 0,2n;n_1=-\infty}^{\infty}q_{n_1}(\Psi_{n,j,t},e^{i(2\pi(n-n_1)+t)x}).$$

Note that if $n_1\neq 0,2n$ then it follows from (23) and (24) that

$$(\Psi_{n,j,t},e^{i(2\pi(n-n_1)+t)x})=\sum_{n_2=-\infty}^{\infty}\frac{q_{n_2}(\Psi_{n,j,t},e^{i(2\pi(n-n_1-n_2)+t)x})}{\lambda_{n,j}(t)-(2\pi(n-n_1)+t)^2}. \tag{31}$$

We use this formula only for $n_1\neq 0,2n$, since the denominator of the fraction for $n_1=0,2n$ may be equals to 0, but it is a large number for $n_1\neq 0,2n$ due to (22). Therefore in (30) the terms with $n_1=0,2n$ are isolated. Now we iterate (30) as follows. Use (31) for the terms in (30) with $n_1\neq 0,2n$ and then again isolate the term containing one of the multiplicands $(\Psi_{n,j,t},e^{i(2\pi n+t)x})$, $(\Psi_{n,j,t},e^{i(-2\pi n+t)x})$ (i.e., terms with $n_1+n_2=0,2n$) to get

$$(\lambda_{n,j}(t)-(2\pi n+t)^2)(\Psi_{n,j,t},e^{i(2\pi n+t)x})=$$
$$q_{2n}(\Psi_{n,j,t},e^{i(-2\pi n+t)x})+\sum_{\substack{n_1=-\infty\\ n_1\neq 0,2n}}^{\infty}\frac{q_{n_1}q_{-n_1}(\Psi_{n,j,t},e^{i(2\pi n+t)x})}{\lambda_{n,j}(t)-(2\pi(n-n_1)+t)^2}+ \tag{32}$$
$$\sum_{\substack{n_1=-\infty\\ n_1\neq 0,2n}}^{\infty}\frac{q_{n_1}q_{2n-n_1}(\Psi_{n,j,t},e^{i(-2\pi n+t)x})}{\lambda_{n,j}(t)-(2\pi(n-n_1)+t)^2}+\sum_{\substack{n_1,n_2=-\infty\\ n_1\neq 0,2n,\ n_1+n_2\neq 0,2n}}^{\infty}\frac{q_{n_1}q_{n_2}(\Psi_{n,j,t},e^{i(2\pi(n-n_1-n_2)+t)x})}{\lambda_{n,j}(t)-(2\pi(n-n_1)+t)^2}.$$

Now again using (31) for the terms with $n_1+n_2\neq 0,2n$ in the last summation of (32) and repeating this process $m-$times (i.e., m times isolating the terms containing one of the multiplicands $(\Psi_{n,j,t},e^{i(2\pi n+t)x})$, $(\Psi_{n,j,t},e^{i(-2\pi n+t)x})$ and using (31) for the others) we get

$$(\lambda_{n,j}(t)-(2\pi n+t)^2-A_m(\lambda_{n,j}(t),t))u_{n,j}(t)=(q_{2n}+B_m(\lambda_{n,j}(t),t))v_{n,j}(t)+R_m, \quad (33)$$

where

$$A_m(\lambda_{n,j}(t),t)=\sum_{k=1}^{m}a_k(\lambda_{n,j}(t),t),\ B_m(\lambda_{n,j}(t),t)=\sum_{k=1}^{m}b_k(\lambda_{n,j}(t),t),$$

$$a_k(\lambda_{n,j}(t),t)=\sum_{n_1,n_2,...,n_k}\frac{q_{n_1}q_{n_2}...q_{n_k}q_{-n_1-n_2-...-n_k}}{[\lambda_{n,j}-(2\pi(n-n_1)+t)^2]...[\lambda_{n,j}-(2\pi(n-n_1-...-n_k)+t)^2]},$$

$$b_k(\lambda_{n,j}(t),t)=\sum_{n_1,n_2,...,n_k}\frac{q_{n_1}q_{n_2}...q_{n_k}q_{2n-n_1-n_2-...-n_k}}{[\lambda_{n,j}-(2\pi(n-n_1)+t)^2]...[\lambda_{n,j}-(2\pi(n-n_1-..-n_k)+t)^2]},$$

$$R_m=\sum_{n_1,n_2,...,n_{m+1}}\frac{q_{n_1}q_{n_2}...q_{n_m}q_{n_{m+1}}(q\Psi_{n,j,t},e^{i(2\pi(n-n_1-...-n_{m+1})+t)x})}{[\lambda_{n,j}-(2\pi(n-n_1)+t)^2]...[\lambda_{n,j}-(2\pi(n-n_1-...-n_{m+1})+t)^2]}.$$

Note that, here the sums are taken under conditions $n_1,n_2,...,\neq 0$ and $n_1+n_2+...+n_s\neq 0,2n$ for $s=1,2,....$ Using (22), (25) and (29) one can easily verify that the equalities

$$a_k=O\left((\frac{ln\,|n|}{n})^k\right),\ b_k=O\left((\frac{ln\,|n|}{n})^k\right),\ R_m=O\left((\frac{ln\,|n|}{n})^{m+1}\right) \quad (34)$$

hold uniformly with respect to t in $[0,\rho]$. In the same way the relation

$$(\lambda_{n,j}(t)-(-2\pi n+t)^2-A_m^{'}(\lambda_{n,j}(t),t))v_{n,j}(t)=(q_{-2n}+B_m^{'}(\lambda_{n,j}(t),t))u_{n,j}(t)=R_m^{'} \quad (35)$$

can be obtained, where

$$A_m^{'}(\lambda_{n,j}(t),t)=\sum_{k=1}^{m}a_k^{\prime}(\lambda_{n,j}(t),t),\ B_m^{'}(\lambda_{n,j}(t))=\sum_{k=1}^{m}b_k^{'}(\lambda_{n,j}(t),t),$$

$$a_k^{'}(\lambda_{n,j}(t),t)=\sum_{n_1,n_2,...,n_k}\frac{q_{n_1}q_{n_2}...q_{n_k}q_{-n_1-n_2-...-n_k}}{[\lambda_{n,j}-(2\pi(n+n_1)-t)^2]...[\lambda_{n,j}-(2\pi(n+n_1+...+n_k)-t)^2]},$$

$$b_k^{'}(\lambda_{n,j}(t),t)=\sum_{n_1,n_2,...,n_k}\frac{q_{n_1}q_{n_2}...q_{n_k}q_{-2n-n_1-n_2-...-n_k}}{[\lambda_{n,j}-(2\pi(n+n_1)-t)^2]...[\lambda_{n,j}-(2\pi(n+n_1+...+n_k-t))^2]},$$

$$a_k^{'}=O\left((\frac{ln\,|n|}{n})^k\right),\ b_k^{'}=O\left((\frac{ln\,|n|}{n})^k\right),\ R_m^{'}=O\left((\frac{ln\,|n|}{n})^{m+1}\right). \quad (36)$$

Here the sums are taken under the conditions $n_s\neq 0,\ n_1+n_2+...+n_s\neq 0,-2n$ for $s=1,2,...,k$.

Now in (33) and (35) letting m tend to infinity, using (34) and (36) we obtain

$$(\lambda_{n,j}(t)-(2\pi n+t)^2-A(\lambda_{n,j}(t),t))u_{n,j}(t)=(q_{2n}+B(\lambda_{n,j}(t),t))v_{n,j}(t) \quad (37)$$

and

$$(\lambda_{n,j}(t)-(-2\pi n+t)^2-A^{'}(\lambda_{n,j}(t),t))v_{n,j}(t)=(q_{-2n}+B^{'}(\lambda_{n,j}(t),t))u_{n,j}(t), \quad (38)$$

where

$$A(\lambda,t)=\sum_{k=1}^{\infty}a_k(\lambda,t),\ B=\sum_{k=1}^{\infty}b_k,\ A^{'}=\sum_{k=1}^{\infty}a_k',\ B^{'}=\sum_{k=1}^{\infty}b_k^{'}. \tag{39}$$

The main results of this section are obtained from these formulas. For this we use the following lemmas.

Lemma 2 *(a) The following equalities hold uniformly with respect to t in $[0,\rho]$;*

$$A(\lambda_{n,j}(t),t)=O(n^{-1}),\ A^{'}(\lambda_{n,j}(t),t)=O(n^{-1}). \tag{40}$$

(b) Let $q\in W_1^p[0,1]$, and (3) holds with some $s\leq p$. Then the equalities

$$B(\lambda_{n,j}(t),t)=o\left(n^{-s-1}\right),\ B^{'}(\lambda_{n,j}(t),t)=o\left(n^{-s-1}\right) \tag{41}$$

hold uniformly with respect to t in $[0,\rho]$.

Proof. (a) First, let us prove that

$$a_1(\lambda_{n,j}(t),t)=\frac{1}{4\pi^2}\sum_{k\neq 0,\,2n}\frac{q_k\,q_{-k}}{k(2n-k)}+O\left(\frac{1}{n}\right). \tag{42}$$

Using (20) and taking into account that $t<\rho\ll 1$ one can see that if $|k|\leq 3\,|n|$ then

$$\mid \lambda_{n,j}-(2\pi(n-k)+t)^2-4\pi^2k(2n-k)\mid\leq |n|\,. \tag{43}$$

Conversely, if $|k|>3\,|n|$, then

$$\mid \lambda_{n,j}-(2\pi(n-k)+t)^2\mid > k^2>n^2, \mid 4\pi^2k(2n-k)\mid > k^2>n^2. \tag{44}$$

Therefore, taking into account the inequality in (29), we obtain

$$\sum_{k:|k|>3|n|}\frac{q_k\,q_{-k}}{\lambda_{n,j}-(2\pi(n-k)+t)^2}-\frac{1}{4\pi^2}\sum_{k:|k|>3|n|}\frac{q_k\,q_{-k}}{k(2n-k)}=O\left(\frac{1}{n}\right),$$

$$\mid\sum_{k:|k|\leq 3|n|,k\neq 0,\,2n}\left(\frac{q_k\,q_{-k}}{\lambda_{n,j}-(2\pi(n-k)+t)^2}-\frac{q_k\,q_{-k}}{4\pi^2k(2n-k)}\right)\mid\leq$$

$$\sum_{k:|k|\leq 3|n|}\frac{M^2\,|n|}{k^2(2n-k)^2}\leq\sum_{k:|k|\leq|n|}\frac{M^2\,|n|}{k^2n^2}+\sum_{k:|n|<|k|\leq 3|n|}\frac{M^2\,|n|}{n^2(2n-k)^2}=O(\frac{1}{n}).$$

Thus (42) holds. In (42) grouping the terms $\frac{q_k\,q_{-k}}{k(2n-k)}$ and $\frac{q_{-k}\,q_k}{-k(2n+k)}$ we get

$$a_1(\lambda_{n,j}(t),t)=\frac{1}{4\pi^2}\sum_{k>0,\ k\neq\,2n}\frac{q_k\,q_{-k}}{(2n+k)(2n-k)}+O\left(\frac{1}{n}\right). \tag{45}$$

To estimate the sum in (45) we consider, as in the paper [22], the function

$$G(x,n)=\int_0^x q(t)e^{-2\pi i(2n)t}dt-q_{2n}x.$$

The Fourier coefficients $G_k(n)=:(G(x,n),e^{2\pi ikx})$ of $G(x,n)$ are $G_k(n)=\frac{1}{2\pi ik}q_{2n+k}$ for $k\neq$

0, and hence we have

$$G(x,n)=G_0(n)+\sum_{k\neq 2n}\frac{q_k}{2\pi i(k-2n)}e^{2\pi i(k-2n)x}.$$

Therefore using the integration by parts and taking into account the obvious equalities $G(1,n)=G(0,n)=0$, $G(x,n)-G_0(n)=O(1)$ we obtain

$$\frac{1}{4\pi^2}\sum_{k>0,\ k\neq 2n}\frac{q_k\,q_{-k}}{(2n+k)(2n-k)}=\int_0^1(G(x,n)-G_0(n))^2e^{2\pi i(4n)x}\,dx=$$

$$\frac{-1}{2\pi i(4n)}\int_0^1 2(G(x,n)-G_0(n))(q(x)e^{-2\pi i(2n)x}-q_{2n})e^{2\pi i(4n)x}\,dx=O\left(\frac{1}{n}\right).$$

This with (45) and (34) imply the first equality of (40). In the same way we get the second equality of (40).

(b) If the assumptions of (b) hold, then

$$q_{2n}=o(n^{-s}),\ q_{n_1}q_{n_2}\cdots q_{n_k}q_{\pm 2n-n_1-n_2-\cdots-n_k}=o(n^{-s}) \tag{46}$$

(see p. 655 of [22]). Using this and (22), in a standard way, we get

$$b_k(\lambda_{n,j}(t))=o\left(\frac{\ln^k n}{n^{k+s}}\right)=o(n^{-s-1}),\quad b_k^{'}(\lambda_{n,j}(t))=o\left(\frac{\ln^k n}{n^{k+s}}\right)=o(n^{-s-1}) \tag{47}$$

for $k\geq 2$. It remains to prove that

$$b_1(\lambda_{n,j}(t))=o(n^{-s-1}),\quad b_1^{'}(\lambda_{n,j}(t))=o(n^{-s-1}). \tag{48}$$

Instead of the inequality in (29) using the equality $q_kq_{2n-k}=o(n^{-s})$ (see (46)) and arguing as in the proof of (42) we get

$$b_1(\lambda_{n,j}(t),t)=\frac{1}{4\pi^2}\sum_{k\neq 0,\ 2n}\frac{q_k\,q_{2n-k}}{k(2n-k)}+o(n^{-s-1}) \tag{49}$$

for all $t\in[0,\rho]$. In [22] (see p. 655) the summation in (49) is denoted by S_{2n} and it is proved that $S_{2n}=o(n^{-s-1})$ (see p. 658). Thus from (49) we obtain the first equality of (48). In the same way we get the second equality of (48). ■

Now we consider some properties of the functions $A(\lambda,t),\ B(\lambda,t),\ A^{'}(\lambda,t),\ B^{'}(\lambda,t)$ defined in (39) for $t\in[0,\rho]$ and $\lambda\in D(n,t,\rho)$, where $D(n,t,\rho)$ is the disk defined in (21).

Lemma 3 (a) *There exists a constant K, independent of $n>N$ and $t\in[0,\rho]$, such that*

$$\mid A(\lambda,t)-A(\mu,t)\mid< Kn^{-2}\mid\lambda-\mu\mid,\quad \mid A^{'}(\lambda,t)-A^{'}(\mu,t)\mid< Kn^{-2}\mid\lambda-\mu\mid, \tag{50}$$

$$\mid C(\lambda_{n,j}(t),t)\mid< tKn^{-1},\quad \mid C(\lambda,t))-C(\mu,t))\mid< tKn^{-2}\mid\lambda-\mu\mid \tag{51}$$

for all $\lambda,\mu\in D(n,t,\rho)$, where $C(\lambda,t)=\frac{1}{2}(A(\lambda,t)-A^{'}(\lambda,t))$ and N is defined in Remark 1.

(b) *Let $q\in W_1^p[0,1]$, and (3) holds with some $s\leq p$. Then the functions $b_k(\lambda,t)$, $b_k^{'}(\lambda,t)$, $B(\lambda,t)$, $B^{'}(\lambda,t)$ for $\lambda,\mu\in D(n,t,\rho)$, $k=1,2,...$, satisfy the following, uniform with respect to t in $[0,\rho]$, condition*

$$f(\lambda,t)-f(\mu,t)=(\lambda-\mu)o(n^{-s-2}). \tag{52}$$

Proof. (a) If $\lambda\in D(n,t,\rho)$, then

$$\left|\lambda-(2\pi(n-k)+t)^2\right|>|k|\,|2n-k|\,,\ \forall k\neq 0,2n. \tag{53}$$

To prove the estimations (50) and (51) we use (53) and the following obvious equality

$$\sum_{k\neq 0,-2n}\frac{1}{\mid k^s(2n-k)^m\mid}=O\left(\frac{1}{n^p}\right) \tag{54}$$

if $\max\{s,m\}\geq 2$, where $p=\min\{s,m\}\geq 1$. The inequality (54) with (53) imply that the series in the formula for the functions $a_k(\lambda,t)$, $a_k^{'}(\lambda,t)$, $b_k(\lambda,t)$, $b_k^{'}(\lambda,t)$ converge uniformly in a neighborhood of λ, which yields that these functions are the continuous functions of λ. Moreover, the estimations (34) and (36) hold if we replace $\lambda_{n,j}(t)$ by λ. Therefore the series in the formulas for the functions $A(\lambda,t)$, $B(\lambda,t)$, $A^{'}(\lambda,t)$ and $B^{'}(\lambda,t)$ converge uniformly in a neighborhood of λ. Using (53) and (54) one can easily verify that these series can be differentiated, with respect to λ, term by term. Moreover, taking into account the inequality

$$\mid \frac{d}{d\lambda}(\frac{1}{\lambda-(2\pi(n-k)+t)^2})\mid\leq\frac{1}{k^2(2n-k)^2}$$

and (54), we see that the absolute values of the derivatives of $a_k(\lambda,t)$, $a_k^{'}(\lambda,t)$, $b_k(\lambda,t)$, $b_k^{'}(\lambda,t)$ with respect to λ is $O(n^{-k-1})$. Therefore, these functions satisfy the condition

$$g(\lambda,t)-g(\mu,t)=(\lambda-\mu)O(n^{-k-1}). \tag{55}$$

Now (50) follows from (55).

To prove the first inequality of (51) we use substitutions $-n_1-n_2-\cdots-n_k=j_1,\ n_2=j_k,\ n_3=j_{k-1},\ \ldots,\ n_k=j_2$ in the formula for the expression a_k'. Then the inequalities for the forbidden indices $n_p\neq 0$, $n_1+n_2+\cdots+n_p\neq 0,-2n$ for $1\leq p\leq k$ in the formula for a_k' take the form $j_p\neq 0$, $j_1+j_2+...+j_p\neq 0,2n$ for $1\leq p\leq k$, and

$$a_k^{'}(\lambda_{n,j}(t))=\sum_{n_1,n_2,\ldots,n_k}\frac{q_{n_1}q_{n_2}...q_{n_k}q_{-n_1-n_2-...-n_k}}{[\lambda_{n,j}-(2\pi(n-n_1)-t)^2]...[\lambda_{n,j}-(2\pi(n-n_1-...-n_k)-t)^2]}.$$

Using (22) and (54) one can readily see that

$$\sum_{\substack{k=-\infty,\\ k\neq 0,2n}}^{\infty}\mid\frac{1}{\lambda_{n,j}(t)-(2\pi(n-k)+t)^2}-\frac{1}{\lambda_{n,j}(t)-(2\pi(n-k)-t)^2}\mid=tO(\frac{1}{n}).$$

This with the inequality in (29) imply the first inequality in (51). Now arguing as in the proof of (50), we get the proof of the second inequality of (51).

(b) Using (46) and repeating the proof of (50) we get the proof of (b) ■

Now using Lemma 2 and Lemma 3 we prove the following main result.

Theorem 1 *Let $q\in W_1^p[0,1]$ and (3) holds with some $s\leq p$. Suppose (5) holds and*

$$\mid q_{2n}\mid>cn^{-s-1} \tag{56}$$

for some $c>0$. If at least one of the following inequalities

$$\operatorname{Re}q_{2n}q_{-2n}\geq 0, \tag{57}$$

$$| \operatorname{Im} q_{2n}q_{-2n} | \geq \varepsilon \mid q_{2n}q_{-2n} \mid \tag{58}$$

holds for some $\varepsilon > 0$ *and for* $n > N$*, where* N *is defined in Remark 1, then the eigenvalue* $\lambda_{n,j}(t)$ *of* $L_t(q)$ *for* $n > N$*,* $j = 1, 2$ *and* $t \in [0, \rho]$ *is simple.*

Proof. It follows from (56) and (41) that

$$q_{2n} + B(\lambda_{n,j}(t), t) \neq 0, \ q_{-2n} + B^{'}(\lambda_{n,j}(t), t) \neq 0 \tag{59}$$

for $t \in [0, \rho]$. Let us prove that this with the formulas (37), (38) and (28) imply that

$$u_{n,j}(t)v_{n,j}(t) \neq 0. \tag{60}$$

If $u_{n,j}(t) = 0$ then by (28) $v_{n,j}(t) \neq 0$ and by (37) $q_{2n} + B(\lambda_{n,j}(t), t) = 0$ which contradicts (59). Similarly, if $v_{n,j}(t) = 0$ then by (28) and (38) $q_{-2n} + B^{'}(\lambda_{n,j}(t), t) = 0$ which again contradicts (59). Now multiplying (37) and (38) side by side and then canceling $u_{n,j}(t)v_{n,j}(t)$, we get

$$\begin{aligned}&(\lambda_{n,j}(t) - (2\pi n + t)^2 - A(\lambda_{n,j}(t), t))(\lambda_{n,j}(t) - (2\pi n - t)^2 - A^{'}(\lambda_{n,j}(t), t)) \\ &= (q_{2n} + B(\lambda_{n,j}(t), t))(q_{-2n} + B^{'}(\lambda_{n,j}(t), t).\end{aligned} \tag{61}$$

Introduce the notation $x =: \lambda_{n,j}(t) - (2\pi n + t)^2 - A_m(\lambda_{n,j}(t))$. Then

$$\lambda_{n,j}(t) - (2\pi n - t)^2 - A_m^{'}(\lambda_{n,j}(t)) = x + 8\pi nt + A_m(\lambda_{n,j}(t)) - A_m^{'}(\lambda_{n,j}(t)).$$

Using this notation in (61) we get

$$x^2 + (8\pi nt + A_m - A_m^{'})x - (q_{2n} + B)(q_{-2n} + B^{'}) = 0 \tag{62}$$

This means that $\lambda_{n,j}(t)$ satisfies either the equation

$$\lambda = (2\pi n + t)^2 + \frac{1}{2}(A(\lambda, t) + A^{'}(\lambda, t)) - 4\pi nt + \sqrt{D(\lambda, t)} \tag{63}$$

or

$$\lambda = (2\pi n + t)^2 + \frac{1}{2}(A(\lambda, t) + A^{'}(\lambda, t)) - 4\pi nt - \sqrt{D(\lambda, t)}, \tag{64}$$

where $D(\lambda, t)$ is the discriminant of (62), that is,

$$D(\lambda, t) = (4\pi nt)^2 + q_{2n}q_{-2n} + D_1(\lambda, t) + D_2(\lambda, t), \tag{65}$$

$$D_1(\lambda, t) = 8\pi ntC(\lambda, t) + C^2(\lambda, t), \ D_2(\lambda, t) = q_{2n}B^{'}(\lambda, t) + q_{-2n}B(\lambda, t) + B(\lambda, t)B^{'}(\lambda, t)$$

and $C(\lambda, t) = A(\lambda, t) - A^{'}(\lambda, t)$ (see Lemma 3(a)).

Let us prove that

$$|D(\lambda_{n,j}(t), t)| > \frac{\varepsilon}{4}(|q_{-2n}q_{2n}| + (4\pi nt)^2), \tag{66}$$

$$D(\lambda_{n,j}(t), t) = ((4\pi nt)^2 + q_{2n}q_{-2n})(1 + o(1)). \tag{67}$$

If follows from (51) and (41), (56), (5) that

$$D_1(\lambda_{n,j}(t), t) = t^2 O(1), \ D_2(\lambda_{n,j}(t), t) = o(q_{2n}n^{-s-1}) = o(q_{2n}q_{-2n}). \tag{68}$$

Therefore, we have

$$D_1(\lambda_{n,j}(t),t)+\ D_2(\lambda_{n,j}(t),t)=o(|q_{-2n}q_{2n}|+(4\pi nt)^2). \tag{69}$$

Thus to prove (66) and (67) it is enough to show that

$$\left|q_{-2n}q_{2n}+(4\pi nt)^2\right|>\frac{\varepsilon}{3}(|q_{-2n}q_{2n}|+(4\pi nt)^2). \tag{70}$$

For this we consider two cases. First case is $(4\pi nt)^2\leq 2\left|q_{2n}q_{-2n}\right|)$. Then

$$|q_{2n}q_{-2n}|\geq\frac{1}{3}(|q_{2n}q_{-2n}|+(4\pi nt)^2). \tag{71}$$

If the condition (57) holds then $\left|q_{2n}q_{-2n}+(4\pi nt)^2\right|\geq|q_{2n}q_{-2n}|$. Therefore (70) follows from (71). If the condition (58) holds then $\left|q_{2n}q_{-2n}+(4\pi nt)^2\right|\geq|\operatorname{Im}q_{2n}q_{-2n}|\geq\varepsilon\left|q_{2n}q_{-2n}\right|$ and again (70) follows from (71). Now let us consider the second case $(4\pi nt)^2>2\left|q_{2n}q_{-2n}\right|)$. Then

$$\left|q_{2n}q_{-2n}+(4\pi nt)^2\right|>(4\pi nt)^2-|q_{2n}q_{-2n}|>\frac{1}{3}(|q_{2n}q_{-2n}|+(4\pi nt)^2),$$

that is, (70) holds. Thus (66) and (67) are proved.

Now suppose that both eigenvalues $\lambda_{n,1}(t)$ and $\lambda_{n,2}(t)$ of the operator $L_t(q)$ lying in the disk $D(n,t,\rho)$ (see (21)) satisfy the equation (63). Then

$$\lambda_{n,1}(t)-\lambda_{n,2}(t)=[\frac{1}{2}(A(\lambda_{n,1}(t),t)-A(\lambda_{n,2}(t),t))+\frac{1}{2}(A^{'}(\lambda_{n,1}(t),t)-A^{'}(\lambda_{n,2}(t),t)]+ \tag{72}$$

$$\left[\sqrt{D(\lambda_{n,1}(t),t)}-\sqrt{D(\lambda_{n,2}(t),t)}\right].$$

By (50) we have

$$\mid A(\lambda_{n,1},t)-A(\lambda_{n,2},t)+A^{'}(\lambda_{n,1},t)-A^{'}(\lambda_{n,2},t)\mid<2Kn^{-2}\mid\lambda_{n,1}(t)-\lambda_{n,2}(t)\mid. \tag{73}$$

Using (65), (51), (46), (41) and Lemma 3(b) one can easily verify that

$$\mid D(\lambda_{n,1}(t),t)-D(\lambda_{n,2}(t),t)\mid\leq(5\pi t^2Kn^{-1}+n^{-2s-2})\mid\lambda_{n,1}(t)-\lambda_{n,2}(t)\mid. \tag{74}$$

On the other hand it follows from (67), (70) and (56), (5) that

$$\left|\sqrt{D(\lambda_{n,1}(t),t)}+\sqrt{D(\lambda_{n,2}(t),t)}\right|=\left|2\sqrt{D(\lambda_{n,1}(t),t)}(1+o(1))\right|>\gamma(n^{-s-1}+nt), \tag{75}$$

where γ is a positive constant. From (74) and (75) we obtain that

$$\left|\sqrt{D(\lambda_{n,1}(t),t)}-\sqrt{D(\lambda_{n,2}(t),t)}\right|=O(n^{-1})\mid\lambda_{n,1}(t)-\lambda_{n,2}(t)\mid. \tag{76}$$

Now using this and (73) in (72) we get

$$\lambda_{n,1}(t)=\lambda_{n,2}(t). \tag{77}$$

In the same way we prove that if both eigenvalues $\lambda_{n,1}(t)$ and $\lambda_{n,2}(t)$ satisfy the equation (64) then (77) holds.

Now suppose that one of them, say $\lambda_{n,1}(t)$, satisfies (63) and the other $\lambda_{n,2}(t)$ satisfies

(64). Then

$$\lambda_{n,1}(t)-\lambda_{n,2}(t)=[\frac{1}{2}(A(\lambda_{n,1},t)-A(\lambda_{n,2},t))+\frac{1}{2}(A^{'}(\lambda_{n,1},t)-A^{'}(\lambda_{n,2},t)]+ \tag{78}$$

$$\left[\sqrt{D(\lambda_{n,1}(t),t)}+\sqrt{D(\lambda_{n,2}(t),t)}\right].$$

Therefore using (73) and (75) we obtain

$$\mid \lambda_{n,1}(t)-\lambda_{n,2}(t) \mid > \delta(n^{-s-1}+nt), \tag{79}$$

where δ is a positive constant.

Now it follows from (77) and (79) that the value $d_n(t)$ of the function d_n, defined in (19), for $t\in[0,\rho]$ belongs to the union of the disjoint sets $(\delta(n^{-s-1}+nt),\infty)$ and $\{0\}$. Moreover, as it is proved in Remark 1, d_n is a continuous function on $[0,\rho]$ which implies that the set $\{d_n(t):t\in[0,\rho]\}$ is a connected set. Therefore, taking into account that $d_n(\rho)\in(\delta(n^{-s-1}+nt),\infty)$ (see (9)), we get $\{d_n(t):t\in[0,\rho]\}\in(\delta(n^{-s-1}+nt),\infty)$. This mean that $\lambda_{n,1}(t)$ and $\lambda_{n,2}(t)$ are different simple eigenvalues and one of them satisfies (63) and the other satisfies (64). Without loss of generality, it can be assumed that

$$\lambda_{n,j}(t)=(2\pi n+t)^2+\frac{1}{2}(A(\lambda_{n,j}(t),t)+A^{'}(\lambda_{n,j}(t),t))-4\pi nt+(-1)^j\sqrt{D(\lambda_{n,j}(t),t)}, \tag{80}$$

where square root in (81) is taken with positive real part. Note that

$$\operatorname{Re}(\sqrt{D(\lambda_{n,j}(t),t)})\neq 0 \tag{81}$$

due to the following reason. By (56)-(58)) $\left|\arg(q_{-2n}q_{2n}+(4\pi nt)^2)\right|<\pi-\alpha$ for some positive constant α. Thus by (66) and (67) $\arg D(\lambda_{n,j}(t),t)\neq\pi$ and hence (81) holds ■

Lemma 4 *Suppose that all conditions of the Theorem 1 hold. Let $\lambda_{n,j}(t)$ be eigenvalue of $L_t(q)$ satisfying (80), and $\Psi_{n,t,j}(t)$ be the corresponding eigenfunction. Then the relations*

$$v_{n,1}(t)\sim 1,\ u_{n,2}(t)\sim 1 \tag{82}$$

hold uniformly for $t\in[0,\rho]$.

Proof. Multiplying (37) and (38) by $v_{n,j}(t)$ and by $u_{n,j}(t)$ respectively and then subtracting each other we get

$$(-8\pi nt+A^{'}(t)-A(t))u_{n,j}(t)v_{n,j}(t)=(q_{2n}+B(t))v_{n,j}^2(t)-(q_{-2n}+B^{'}(t))u_{n,j}^2(t), \tag{83}$$

where, for brevity, $A(\lambda_{n,j}(t),t),A^{'}(\lambda_{n,j}(t),t),B(\lambda_{n,j}(t),t)$ and $B^{'}(\lambda_{n,j}(t),t)$ is denoted by $A(t),A^{'}(t),B(t)$ and $B^{'}(t)$ respectively.

First, suppose that $nt\leq|q_{2n}|$. Then it follows from (51) that $A^{'}(t)-A(t)=o(q_{2n})$ and

$$\mid -8\pi nt+A^{'}(t)-A(t)\mid < 9\pi\left|q_{2n}\right|. \tag{84}$$

On the other hand, the relations (5), (41) and (56) imply that

$$q_{2n}+B(\lambda_{n,j}(t),t)\sim q_{-2n}+B^{'}(\lambda_{n,j}(t),t)\sim q_{2n}. \tag{85}$$

Therefore using (83)-(85) and taking into account that, if the relation $u_{n,j}(t) \sim v_{n,j}(t)$ does not hold then $u_{n,j}(t)v_{n,j}(t) = o(1)$ (see (28)), we obtain $u_{n,j}(t) \sim v_{n,j}(t) \sim 1$ for $j = 1, 2.$ Thus (82) holds for the case $nt \le |q_{2n}|$.

Now consider the case $nt > |q_{2n}|$. Using (80) in (37) and (38) we obtain

$$(C(t) - 4\pi nt + (-1)^j \sqrt{D(t)})u_{n,j}(t) = (q_{2n} + B(t))v_{n,j}(t), \tag{86}$$

$$(-C(t) + 4\pi nt + (-1)^j \sqrt{D(t)})v_{n,j}(t) = (q_{-2n} + B^{'}(t))u_{n,j}(t). \tag{87}$$

Since $\mathrm{Re}(\sqrt{D(\lambda_{n,j}(t),t)}) > 0$, it follows from (86) for $j = 1$ and (51) that

$$| C(\lambda_{n,1}(t),t) - 4\pi nt - \sqrt{D(\lambda_{n,1}(t),t)} | \ge \mathrm{Re}(4\pi nt(1 + O(n^{-2})) + \sqrt{D(\lambda_{n,1}(t),t)}) > |q_{2n}|.$$

Using this and (85) in (86) for $j = 1$ we get $v_{n,1}(t) \sim 1$. In the same way we get the second relation of (82) from (87) for $j = 2$. ■

To obtain asymptotic formulas of arbitrary accuracy we define successively the following functions

$$F_{n,j,1}(t) = (2\pi n + t)^2 - 4\pi nt + (-1)^j \sqrt{(4\pi nt)^2 + q_{2n}q_{-2n}},$$
$$F_{n,j,m+1}(t) = (2\pi n + t)^2 + \frac{1}{2}(A(F_{n,j,m},t) + A^{'}(F_{n,j,m},t)) - 4\pi nt + (-1)^j \sqrt{D(F_{n,j,m},t)}$$

for $m = 1, 2, ...$ Moreover we use the functions A^*, B^* which are obtained from A, B respectively by replacing q_{n_1} with $e^{i(2\pi(n-n_1)+t)x}$.

Theorem 2 *(a)If the conditions of Theorem 1 hold, then the eigenvalue $\lambda_{n,j}(t)$ satisfies the following , uniform with respect to $t \in [0,\rho]$, formulas*

$$\lambda_{n,j}(t) = (2\pi n + t)^2 - 4\pi nt + (-1)^j \sqrt{(4\pi nt)^2 + q_{2n}q_{-2n}} + O(\frac{1}{n}), \tag{88}$$

$$\lambda_{n,j}(t) = F_{n,j,m}(t) + O(\frac{1}{n^m}),\ m = 1, 2, ... \tag{89}$$

(b) The normalized eigenfunction $\Psi_{n,j,t}(x)$ corresponding to $\lambda_{n,j}(t)$ is $\frac{\varphi_{n,j,t}(x)}{\|\varphi_{n,j,t}(x)\|}$, where $\varphi_{n,j,t}(x)$ satisfies the following , uniform with respect to $t \in [0,\rho]$, formulas

$$\varphi_{n,1,t}(x) = e^{i(-2\pi n+t)x} + \alpha_{n,1}e^{i(2\pi n+t)x} + A^*(F_{n,1,m},t) + \alpha_{n,1}B^*(F_{n,1,m},t) + O(n^{-m-1}),$$

$$\varphi_{n,2,t}(x) = e^{i(2\pi n+t)x} + \alpha_{n,2}e^{i(-2\pi n+t)x} + A^*(F_{n,2,m},t) + \alpha_{n,2}B^*(F_{n,2,m},t) + O(n^{-m-1}),$$

$$\alpha_{n,1}(t) = \frac{C(F_{n,1,m},t) - 4\pi nt - \sqrt{D(F_{n,1,m},t)}}{q_{-2n} + B'(F_{n,1,m},t)} + O(\frac{1}{q_{2n}n^{m+1}}) = O(1),$$

$$\alpha_{n,2}(t) = \frac{-C(F_{n,2,m},t) - 4\pi nt + \sqrt{D(F_{n,2,m},t)}}{q_{2n} + B(F_{n,2,m},t)} + O(\frac{1}{q_{2n}n^{m+1}}) = O(1).$$

Proof. By (80) and (40) to prove (88) it is enough to show that

$$\sqrt{D(\lambda_{n,j}(t),t)} = \sqrt{(4\pi nt)^2 + q_{2n}q_{-2n}} + O(\frac{1}{n}). \tag{90}$$

Using (67) and (70) one can easily verify that

$$| \sqrt{D(\lambda_{n,j}(t),t)} + \sqrt{(4\pi nt)^2 + q_{2n}q_{-2n}} |=| (2 + o(1))\sqrt{(4\pi nt)^2 + q_{2n}q_{-2n}} |\ge$$

$$\sqrt{\frac{\varepsilon}{6}}(4\pi nt+\mid\sqrt{q_{2n}q_{-2n}}\mid.$$

Therefore we have

$$\mid\sqrt{D(\lambda_{n,j}(t),t)}-\sqrt{(4\pi nt)^2+q_{2n}q_{-2n}}\mid=\mid\frac{D_1(\lambda_{n,j}(t),t)+D_2(\lambda_{n,j}(t),t)}{\sqrt{D(\lambda_{n,j}(t),t)}+\sqrt{(4\pi nt)^2+q_{2n}q_{-2n}}}\mid\leq$$

$$c_1(\mid\frac{D_1(\lambda_{n,j}(t),t)}{4\pi nt}\mid+\mid\frac{D_2(\lambda_{n,j}(t),t)}{\sqrt{q_{2n}q_{-2n}}}\mid), \tag{91}$$

where c_1 is a positive, independent of n and t, constant. Moreover from (68) and (5) we obtain

$$\frac{D_1(\lambda,t)}{4\pi nt}=O(\frac{1}{n}),\ \frac{D_2(\lambda,t)}{\sqrt{q_{2n}q_{-2n}}}=o(n^{-s-1}). \tag{92}$$

Hence (90) follows from (91) and (92). Thus (88) is proved.

It follows from Lemma 3 and from the proof of (76) that the functions $A(\lambda,t)$, $A^{'}(\lambda,t)$, $B(\lambda,t)$, $B^{'}(\lambda,t)$ and $\sqrt{D(\lambda,t)}$ satisfy the equality

$$f(F_{n,j,k}(t)+O(n^{-k}),t)=f(F_{n,j,k}(t),t)+O\left(n^{-k-1}\right). \tag{93}$$

Now we prove (89) by induction. It is proved for $m=1$ (see (88) and the definition of $F_{n,j,1}(t)$). Assume that (89) is true for $m=k$. Substituting the value of $\lambda_{n,j}(t)$ given by (89) for $m=k$, in the right-hand side of (80) and using (93) we get (89) for $m=k+1$.

(b) Writing the decomposition of the normalized eigenfunction $\Psi_{n,j,t}(x)$ corresponding to the eigenvalue $\lambda_{n,j}(t)$ by the basis $\{e^{i(2\pi(n-n_1)+t)x}:n_1\in\mathbb{Z}\}$ we obtain

$$\Psi_{n,j,t}(x)-u_{n,j}(t)e^{i(2\pi n+t)x}-v_{n,j}(t)e^{i(-2\pi n+t)x}= \tag{94}$$

$$\sum_{n_1\neq0,2\pi;n_1=-\infty}^{\infty}(\Psi_{n,j,t}(x),e^{i(2\pi(n-n_1)+t)x})e^{i(2\pi(n-n_1)+t)x}.$$

The right-hand side of (94) can be obtained from the right-hand side of (30) by replacing q_{n_1} with $e^{i2\pi(n-n_1)x}$. Since (37) is obtained from (30) by iteration, doing the same, we obtain

$$\Psi_{n,j,t}(x)=u_{n,j}(t)e^{i(2\pi n+t)x}+v_{n,j}(t)e^{i(-2\pi n+t)x}+u_{n,j}(t)A^*(\lambda_{n,j},t)+v_{n,j}(t)B^*(\lambda_{n,j},t) \tag{95}$$

from (94). First let us consider the case $j=2$. Using (89) and (93) in (37), taking into account (41), (56) we get

$$\frac{v_{n,2}(t)}{u_{n,2}(t)}=\frac{-C(F_{n,2,m}(t),t)-4\pi nt+\sqrt{D(F_{n,2,m}(t),t)}}{q_{2n}+B(F_{n,2,m}(t),t)}+O(\frac{1}{q_{2n}n^{m+1}}), \tag{96}$$

where $m>s$. Now dividing both sides of (95) by $u_{n,2}(t)$, and denoting $\alpha_{n,2}(t)=\frac{v_{n,2}(t)}{u_{n,2}(t)}$, $\varphi_{n,2,t}(x)=\frac{\Psi_{n,2,t}(x)}{u_{n,2}(t)}$ we obtain

$$\varphi_{n,2,t}(x)=e^{i(2\pi n+t)x}+\alpha_{n,2}(t)e^{i(-2\pi n+t)x}+A^*(\lambda_{n,2}(t),t)+\alpha_{n,2}(t)B^*(\lambda_{n,2}(t),t). \tag{97}$$

Here $\alpha_{n,2}(t)=O(1)$ due to (82). On the other hand one can readily see that the functions $A^*(\lambda,t)$ and $B^*(\lambda,t)$ also satisfy (93). Therefore from (97) we get the proof of (b) for $j=2$. In the same way we get the proof of (b) for $j=1$. ■

To obtain the asymptotic formulas for the eigenvalue $\lambda_{n,j}(t)$ for $|n|>N$ and $t\in[\pi-\rho,\pi]$, instead of (30) we use the formula

$$(\lambda_{n,j}(t)-(2\pi n+t)^2)(\Psi_{n,j,t},e^{i(2\pi n+t)x})-q_{2n+1}(\Psi_{n,j,t},e^{i(-2\pi(n+1)+t)x})= \tag{98}$$

$$\sum_{n_1\neq 0,2n+1;n_1=-\infty}^{\infty} q_{n_1}(\Psi_{n,j,t},e^{i(2\pi(n-n_1)+t)x}).$$

From (30) we obtained (37), (38). In the same way from (98) we get

$$(\lambda_{n,j}(t)-(2\pi n+t)^2-\widetilde{A}(\lambda_{n,j}(t),t))u_{n,j}(t)=(q_{2n+1}+\widetilde{B}(\lambda_{n,j}(t),t))v_{n,j}(t),$$

$$(\lambda_{n,j}(t)-(-2\pi(n+1)+t)^2-\widetilde{A}'(\lambda_{n,j}(t),t))v_{n,j}(t)=(q_{-2n-1}+\widetilde{B}'(\lambda_{n,j}(t),t))u_{n,j}(t),$$

where

$$\widetilde{A}(\lambda,t)=\sum_{k=1}^{\infty}\widetilde{a}_k(\lambda,t),\ \widetilde{B}=\sum_{k=1}^{\infty}\widetilde{b}_k,\ \widetilde{A}'=\sum_{k=1}^{\infty}\widetilde{a}'_k,\ \widetilde{B}'=\sum_{k=1}^{\infty}\widetilde{b}'_k.$$

Here $\widetilde{a}_k,\widetilde{a}'_k,\widetilde{b}_k,\widetilde{b}'_k$ differ from a_k,a'_k,b_k,b'_k respectively, in the following sense. The sums in $\widetilde{a}_k,\widetilde{a}'_k,\widetilde{b}_k,\widetilde{b}'_k$ are taken under the conditions $n_1+n_2+...+n_s\neq 0,\pm(2n+1)$ instead of the condition $n_1+n_2+...+n_s\neq 0,\pm 2n$ for $s=1,2,...,k$. Besides in $\widetilde{b}_k,\widetilde{b}'_k$ the multiplicand $q_{\pm 2n-n_1-n_2-...-n_k}$ of b_k,b'_k is replaced by $q_{\pm(2n+1)-n_1-n_2-...-n_k}$. Moreover, instead of $F,\alpha_{n,j},\ A^*,\ B^*$ we use $\widetilde{F},\widetilde{\alpha}_{n,j},\ \widetilde{A}^*,\ \widetilde{B}^*$ that are defined in a similar way. Thus instead of (5), (56), (57) and (58) using the relations

$$q_{2n+1}\sim q_{-2n-1},\quad |\,q_{2n+1}\,|>cn^{-s-1}, \tag{99}$$

$$\operatorname{Re} q_{2n+1}q_{-2n-1}\geq 0, \tag{100}$$

$$|\operatorname{Im} q_{2n+1}q_{-2n-1}\,|\geq\varepsilon\,|\,q_{2n+1}q_{-2n-1}\,| \tag{101}$$

respectively and repeating the proof of Theorem 1 and Theorem 2 we get:

Theorem 3 *Let $q\in W_1^p[0,1]$ and (3) holds with some $s\leq p$. Suppose (99) and at least one of the inequalities (100), (101) holds. Then the eigenvalue $\lambda_{n,j}(t)$ for $n>N$ and $t\in[\pi-\rho,\pi]$ is simple and satisfies the formulas*

$$\lambda_{n,j}(t)=(2\pi n+t)^2-2\pi(2n+1)(t-\pi)+(-1)^j\sqrt{(2\pi(2n+1)(t-\pi))^2+q_{2n+1}q_{-2n-1}}+O(\frac{1}{n}), \tag{102}$$

$$\lambda_{n,j}(t)=\widetilde{F}_{n,j,m}(t)+O(n^{-m}),\ m=1,2,.... \tag{103}$$

The normalized eigenfunction $\Psi_{n,j,t}(x)$ corresponding to $\lambda_{n,j}(t)$ is $\frac{\varphi_{n,j,t}(x)}{\|\varphi_{n,j,t}(x)\|}$, where $\varphi_{n,j,t}(x)$ satisfies the following , uniform with respect to $t\in[\pi-\rho,\pi]$, formulas

$$\varphi_{n,1,t}=e^{i(-2\pi(n+1)+t)x}+\widetilde{\alpha}_{n,1}(t)e^{i(2\pi n+t)x}+\widetilde{A}^*(\widetilde{F}_{n,1,m},t)+\widetilde{\alpha}_{n,1}(t)\widetilde{B}^*(\widetilde{F}_{n,1,m},t)+O(n^{-m-1}),$$

$$\varphi_{n,2,t}=e^{i(2\pi n+t)x}+\widetilde{\alpha}_{n,2}(t)e^{i(-2\pi(n+1)+t)x}+\widetilde{A}^*(\widetilde{F}_{n,2,m},t)+\alpha_{n,2}(t)\widetilde{B}^*(\widetilde{F}_{n,2,m},t)+O(n^{-m-1}).$$

The following remark follows from Remark 1 and theorems 1-3

Remark 2 *Suppose the conditions of Theorem 1 and Theorem3 hold. One can readily see that (88) for $t=0$ and $t=\rho$ give the formulas (10) and (9) respectively, if we use the notation: $\lambda_{n,1}(t)=:\lambda_{-n}(t)$ for $n=1,2,...$ and $\lambda_{n,2}(t)=:\lambda_n(t)$ for $n=0,1,2,...$ Similarly (102) for $t=\pi$ and $t=\pi-\rho$ give the formula obtained in [1] for $\lambda_{n,j}(\pi)$ and (9). Moreover* *there is one-to-one correspondence between the eigenvalues (counting with multiplicities) and integers. Indeed by (9), Theorems 1-3 and Remark 1 there exists a number* N *such that for all* $|n| > N$ *and for all* $t \in [0, \pi]$ *the eigenvalues* $\lambda_n(t)$ *and* $\lambda_{-n}(t)$ *are simple and the number of the remaining eigenvalues of* $L_t(q)$ *is equal to* $2N + 1$. *Using the above notation, we see that the spectrum of* $L_t(q)$ *is*

$$S(L_t(q)) = \{ \lambda_n(t) : n \in \mathbb{Z}\} = \{\lambda_{n,1}(t) : n = 1, 2, ...\} \cup \{ \lambda_{n,2}(t) \ n = 0, 1, 2, ...\}. \qquad (104)$$

We use both notation $\lambda_n(t)$ *and* $\lambda_{n,j}(t)$. *Since* $\lambda_n(t)$ *for* $|n| > N$ *is a simple root of*

$$F(\lambda) = 2\cos t, \qquad (105)$$

where $\frac{1}{2}F(\lambda)$ *is the Hill's discriminant, it is an analytic function on neighborhood of* $[0, \pi]$. *Thus we have*

$$F(\lambda_n(t)) = 2\cos t, \ \frac{dF(\lambda_n(t))}{d\lambda} \neq 0, \ \frac{d\lambda_n(t)}{dt} = -(\frac{dF}{d\lambda})^{-1} 2\sin t \qquad (106)$$

for $|n| > N$, *and* $t \in [0, \pi]$. *This implies that*

$$\Gamma_n =: \{\lambda_n(t) : t \in [0, \pi]\} \qquad (107)$$

is a simple (i.e. $\lambda_n : [0, \pi] \to \Gamma_n$ *is injective) analytic arc with endpoints* $\lambda_n(0)$ *and* $\lambda_n(\pi)$.

The eigenvalues of $L_{-t}(q)$ *coincides with the eigenvalues of* $L_t(q)$, *because they are roots of the equation (105) and* $\cos(-t) = \cos t$. *We define the eigenvalue* $\lambda_n(-t)$ *of* $L_{-t}(q)$ *by*

$$\lambda_n(-t) = \lambda_n(t), \ \forall t \in (0, \pi). \qquad (108)$$

Then $\lambda_n(t)$ *is an analytic function on neighborhood of* $(-\pi, \pi]$.

Using Theorems 1-3 and taking into account Remark 2, we get

Theorem 4 *Let* $q \in W_1^p[0, 1]$ *and (3) holds with some* $s \leq p$. *If,* $q_n \sim q_{-n}$, $\mid q_n \mid > cn^{-s-1}$ *and at least one of the following inequalities*

$$\operatorname{Re} q_n q_{-n} \geq 0, \quad \mid \operatorname{Im} q_n q_{-n} \mid \geq \varepsilon \mid q_n q_{-n} \mid$$

holds, where c *and* ε *are positive constants, then the eigenvalues* $\lambda_n(t)$ *of* $L_t(q)$ *for* $|n| > N$ *and* $t \in [0, \pi]$ *are simple. They and the corresponding eigenfunctions* $\Psi_{n,t}(x)$ *satisfy the formulas (9) and the formulas obtained in Theorems 2 and 3.*

3 Asymptotic Analysis of L(q)

Since the spectrum $S(L(q))$ of the operator $L(q)$ is the union of the spectra $S(L_t(q))$ of the operators $L_t(q)$ for $t \in [0, 2\pi)$, it follows from (104), (107) and (108) that

$$S(L(q)) = \cup_{n \in \mathbb{Z}} \Gamma_n.$$

By (106) and (107) the subset $\gamma =: \{\lambda_n(t) : t \in [\alpha, \beta]\}$, where $[\alpha, \beta] \subset [0, \pi]$, of Γ_n for $|n| > N$ is a regular spectral arc of $L(q)$ in sense of [9] (see Definition 2.4 of [9]). Following [24, 26, 9], we define the projection $P(\gamma)$ for the arc γ as follows

$$P(\gamma)f = \frac{1}{2\pi} \int_{\gamma} (\Phi_+(x, \lambda) F_-(\lambda, f) + \Phi_-(x, \lambda) F_+(\lambda, f)) \frac{\varphi(1, \lambda)}{p(\lambda)} d\lambda, \qquad (109)$$

where $p(\lambda)=\sqrt{4-F^{2}(\lambda)},\ F_{\pm}(\lambda,f)=\int_{\mathbb{R}}f(x)\Phi_{\pm}(x,\lambda)dx$ and

$$\Phi_{\pm}(x,\lambda)=:\theta(x,\lambda)+(\varphi(1,\lambda))^{-1}(e^{\pm it}-\theta(1,\lambda))\varphi(x,\lambda)$$

is the Floquet solution. Recall that the spectral singularities of the operator $L(q)$ are the points of $S(L(q))$ in neighborhoods of which the projections of the operator $L(q)$ are not uniformly bounded. To estimate the projections we use the following lemma of the paper[16]:

Lemma 5.12 of [16] Let $A^{'}$ be in $L_{\infty}((0,2\pi);B(L_{2}(0,1)))$. Then for f in $L_{2}(-\infty,\infty)$ the limit in mean

$$Af=\lim_{N_{i}\to\infty}\frac{1}{2\pi}\sum_{-N_{1}}^{N_{2}}\sum_{-N_{3}}^{N_{4}}T_{j}^{*}\int_{0}^{2\pi}e^{it(j-k)}A^{'}(t)T_{k}fdt \tag{110}$$

exists and defines a bounded operator in $L_{2}(-\infty,\infty)$ of norm $\parallel A\parallel\leq\parallel A^{'}\parallel_{\infty}$, where T_{k} is defined by $T_{k}(f(x))=f(x+k)$ for $x\in[0,1)$, $T_{k}(f(x))=0$ for $x\neq[0,1)$ and

$T_{j}^{*}(f(x))=f(x-j)$ for $x\in[j,j+1)$, $T_{j}^{*}(f(x))=0$ for $x\neq[j,j+1)$.

Let $\{\chi_{n,t}:n\in\mathbb{Z}\}$ be the system of the eigenfunctions of L_{t}^{*} biorthogonal to $\{\Psi_{n,t}:n\in\mathbb{Z}\}$ and $\Psi_{n,t}^{*}(x)$ be normalized eigenfunction of $(L_{t}(q))^{*}$ corresponding to $\overline{\lambda_{n}(t)}$. Then

$$\chi_{n,t}(x))=\frac{1}{\overline{\alpha_{n}(t)}}\Psi_{n,t}^{*}(x),\ \ \alpha_{n}(t)=(\Psi_{n,t}(x),\Psi_{n,t}^{*}(x))_{(0,1)}, \tag{111}$$

where $(.,.)_{(a,b)}$ denotes the inner product in $L_{2}(a,b)$. One can easily verify that

$$\Psi_{n,t}(x)=\frac{\Phi_{+}(x,\lambda_{n}(t))}{\mid\Phi_{+}(x,\lambda_{n}(t))\mid},\ \ \chi_{n,t}(x))=\frac{1}{\overline{\alpha_{n}(t)}}\frac{\overline{\Phi_{-}(x,\lambda_{n}(t))}}{\mid\overline{\Phi_{-}(x,\lambda_{n}(t))}\mid}, \tag{112}$$

$$\Psi_{n,t}(x+1)=e^{it}\Psi_{n,t}(x),\ \ \chi_{n,t}(x+1)=e^{it}\chi_{n,t}(x). \tag{113}$$

Now we are ready to prove the result of this chapter:

Theorem 5 *If all conditions of Theorem 4 hold, then*

(a) *The spectrum of the operator $L(q)$ in a neighborhood of ∞ consist of separated simple analytic arcs Γ_{n} for $|n|>N$ with endpoints $\lambda_{n}(0)$ and $\lambda_{n}(\pi)$.*

(b) *The operator $L(q)$ has at most finitely many spectral singularities.*

(c) *The projections $P(\gamma)$ of $L(q)$ for all $\gamma\subset\Gamma_{n}$ and $|n|>N$ are uniformly bounded.*

Proof. (a) Due to Remark 2 we need only to note that Γ_{n} for $|n|>N$ are separated, that is, $\Gamma_{n}\cap\Gamma_{k}=\varnothing$ for $k\in\mathbb{Z}\backslash\{n\}$. This is true due to the following reason. The equality $\lambda_{n}(t)=\lambda_{k}(t)$ contradicts the simplicity of $\lambda_{n}(t)$. The equality $\lambda_{n}(t)=\lambda_{k}(t^{'})$ for $t'\neq t$ and $t'\in[0,\pi]$ contradicts the first equality in (106).

(b) By Theorem 4 the equation $\frac{dF(\lambda)}{d\lambda}=0$ has no zeros at Γ_{n} for $|n|>N$. Since $\frac{dF(\lambda)}{d\lambda}$ is an entire function it has at most finite number roots on the compact set $\cup_{|n|\leq N}\Gamma_{n}$. Now the proof of ($b$) follows from the well-known fact that the spectral singularities of $L(q)$ is contained in the set $\{\lambda:\frac{dF(\lambda)}{d\lambda}=0,\ \lambda\in S(L(q))\}$ (see [9,26]).

(c) Changing the variable λ to the variable t in the integral in (109), using

$$d\lambda=-p(\lambda)\left(\frac{dF}{d\lambda}\right)^{-1}dt,\ \ \frac{dF(\lambda_{n}(t))}{d\lambda}=-\varphi(1,\lambda_{n}(t))(\Phi_{+}(x,\lambda_{n}(t)),\overline{\Phi_{-}(x,\lambda_{n}(t))})$$

(see 106) and (2.33) of [9]) and (112) by simple calculations we get

$$P(\gamma)f(x)=\frac{1}{2\pi}\int_{\delta}(f,\chi_{n,t})_{\mathbb{R}}\Psi_{n,t}(x))dt, \tag{114}$$

where $\delta=\{t\in[0,2\pi):\lambda_n(t)\in\gamma\}$. Let $A^{'}(t)$ be operator defined by

$$A^{'}(t)f=(f,\chi_{n,t})_{[0,1]}\Psi_{n,t}(x) \tag{115}$$

for $t\in\delta$ and $A^{'}(t)=0$ for $t\in[0,2\pi)\backslash\delta$. By (111) we have

$$\parallel A^{'}(t)\parallel=\mid\alpha_n(t)\mid^{-1}, \tag{116}$$

where α_n is a continuous function and $\alpha_n(t)\neq 0$, since $\lambda_n(t)$ is a simple eigenvalue for $t\in\delta$. Therefore $A^{'}\in L_\infty((0,2\pi);B(L_2(0,1)))$.

Let $f\in C_0$, where C_0 is the set of all compactly supported continuous function, and A be the operator defined by (110). Then using Lemma 5.12 of [16] and (113)-(115) we get

$$A=\lim_{N_i\to\infty}\frac{1}{2\pi}\sum_{j=-N_1}^{N_2}\sum_{k=-N_3}^{N_4}T_j^*\int_0^{2\pi}e^{it(j-k)}A^{'}(t)T_kf(x)dt=$$

$$\lim_{N_i\to\infty}\frac{1}{2\pi}\sum_{j=-N_1}^{N_2}\sum_{k=-N_3}^{N_4}T_j^*\int_0^{2\pi}e^{itj}(f(x+k)e^{-itk},\chi_{n,t})_{[0,1]}\Psi_{n,t}(x)dt=$$

$$\lim_{N_i\to\infty}\frac{1}{2\pi}\sum_{j=-N_1}^{N_2}\int_0^{2\pi}(f,\chi_{n,t})_{\mathbb{R}}e^{itj}T_j^*\Psi_{n,t}(x)dt=P(\gamma)f(x).$$

Hence $Af=P(\gamma)f$ for all $f\in C_0$, where C_0 is dense in $L_2(-\infty,\infty)$. Moreover A is bounded by Lemma 5.12 of [16] and $P(\gamma)$ is bounded since $\gamma\subset\Gamma_n$ and Γ_n for $|n|>N$ does not contain spectral singularities. Therefore, we have $A=P(\gamma)$. Now Lemma 5.12 of [16] with (116) imply that

$$\parallel P(\gamma)\parallel\leq\sup_{t\in\delta}\mid\alpha_n(t)\mid^{-1}.$$

Therefore the proof of (c) follows from the following lemma. ■

Lemma 5 *If all conditions of Theorem 4 hold, then there exists a positive constant d, independent on n for $\mid n\mid>N$ and $t\in[0,2\pi)$, such that*

$$\mid\alpha_n(t)\mid^{-1}<d. \tag{117}$$

Proof. For $t\in[\rho,\pi-\rho]$ the inequality (117) follows from (9). Now we prove this for $t\in[0,\rho]$. The other cases are similar. Since the boundary condition (2) is self-adjoint we have $(L_t(q))^*=L_t(\overline{q})$. Moreover the Fourier coefficients of $\overline{q}$ has the form

$$(\overline{q},e^{i2\pi nx})=\overline{q_{-n}}.$$

Therefore one can readily verify that if q satisfies the conditions of Theorem 4 then $\overline{q}$ also satisfies these conditions. Thus all formulas and theorems obtained for L_t are true for L_t^* if we replace q_n with $\overline{q_{-n}}$. Since formula (26) holds for the operator L_t^* too, we have

$$\Psi_{n,j,t}^*(x)=u_{n,j}^*(t)e^{i(2\pi n+t)x}+v_{n,j}^*(t)e^{i(-2\pi n+t)x}+h_{n,j,t}^*(x),$$

where $u_{n,j}^*(t)=(\Psi_{n,j,t}^*(x),e^{i(2\pi n+t)x})$, $v_{n,j}^*(t)=(\Psi_{n,j,t}^*(x),e^{i(-2\pi n+t)x})$. Then

$$(\Psi_{n,j,t}(x),\Psi_{n,j,t}^*(x))=u_{n,j}(t)\overline{u_{n,j}^*(t)}+v_{n,j}(t)\overline{v_{n,j}^*(t)}+O(n^{-1}). \tag{118}$$

Since Lemma 4 is also true for L_t^*, we have $v_{n,1}^*\sim 1$, $u_{n,2}^*(t)\sim 1$. Using this and (82) in

(118) for $j = 1$ we get

$$(\Psi_{n,1,t}, \Psi^*_{n,1,t}) = v_{n,1}(t)\overline{v^*_{n,1}(t)}(1 + \frac{u_{n,1}(t)\overline{u^*_{n,1}(t)}}{v_{n,1}(t)\overline{v^*_{n,1}(t)}}) + O(n^{-1}). \tag{119}$$

It follows from (86) and (87) that

$$\frac{u_{n,1}}{v_{n,1}} = \frac{(q_{2n} + B(t))}{(C(t) - 4\pi nt - \sqrt{D(t)})} = \frac{-C(t) + 4\pi nt - \sqrt{D(t)}}{(q_{-2n} + B^{'}(t))}. \tag{120}$$

Then $\frac{u^*_{n,1}(t)}{v^*_{n,1}(t)}$ satisfies the formula obtained from (120) by replacing q_n with $\overline{q_{-n}}$. Hence $\overline{\frac{u^*_{n,1}(t)}{v^*_{n,1}(t)}}$ satisfies the formula obtained from (120) by replacing q_n with q_{-n}. Thus we have

$$\frac{u_{n,1}}{v_{n,1}}\frac{\overline{u^*_{n,1}(t)}}{\overline{v^*_{n,1}(t)}} = \frac{-C(t) + 4\pi nt - \sqrt{D}}{(q_{-2n} + B^{'}(t))}\frac{(q_{-2n} + B^*(t))}{(C^*(t) - 4\pi nt - \sqrt{D^*(t)})},$$

where B^*, C^* and D^* are obtained from B, C and D by replacing q_n with q_{-n}. Since

$$(q_{-2n} + B^{'}(t)) = q_{-2n}(1 + o(1)),\ (q_{-2n} + B^*(t)) = q_{-2n}(1 + o(1))$$

(see (41), (56)) the last equality can be written in the form

$$\frac{u_{n,1}}{v_{n,1}}\frac{\overline{u^*_{n,1}(t)}}{\overline{v^*_{n,1}(t)}} = \frac{-C(t) + 4\pi nt - \sqrt{D(t)}}{C^*(t) - 4\pi nt - \sqrt{D^*(t)}}(1 + o(1)). \tag{121}$$

Using (51) and (67) for L_t and L^*_t one can easily see that

$$| C(t) | + | \sqrt{D(t)} | + | C^*(t) | + | \sqrt{D^*(t)} | = O(f(n,t)),$$

where $f(n,t) = | 4\pi nt | + | \sqrt{q_{2n}q_{-2n}} |$. This, (119), (121) and the relations $v_{n,1} \sim 1$, $v^*_{n,1} \sim 1$ yield

$$\frac{1}{| (\Psi_{n,1,t}, \Psi^*_{n,1,t}) |} < c_2 \left| \frac{C^*(t) - 4\pi nt - \sqrt{D^*(t)}}{C^*(t) - \sqrt{D^*(t)} - C(t) - \sqrt{D(t)} + o(f(n,t))} \right|, \tag{122}$$

where c_2 is a positive constant, independent on n and t. Now let us estimate the nominator and denominator of the fraction in (122). Using (51) and (67) for L^*_t we get

$$| C^*(t) - 4\pi nt - \sqrt{D^*(t)} | < | 9\pi nt | + 2 | \sqrt{q_{2n}q_{-2n}} | < 3f(n,t). \tag{123}$$

Similarly using (51), (67) and (70) we obtain

$$| C^*(t) - \sqrt{D^*(t)} - C(t) - \sqrt{D(t)} + o(f(n,t)) | > c_3 f(n,t), \tag{124}$$

where c_3 is a positive constant, independent of n and t. Thus using (123) and (124) in (122) we get the proof of the lemma ■

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