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DEPTH AND MINIMAL NUMBER OF GENERATORS OF SQUARE FREE MONOMIAL IDEALS

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Abstract. Let I be an ideal of a polynomial algebra S over a field generated by square free monomials of degree $\geq d$. If I contains more monomials of degree d than $(d+1)/d$ of the total number of square free monomials of S of degree $d+1$ then $\operatorname{depth}_S I \leq d$, in particular Stanley's Conjecture holds in this case.

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Let $S = K[x_1, \ldots, x_n]$ be the polynomial algebra in n-variables over a field K and $I \subset S$ a square free monomial ideal. Let d be a positive integer and $\rho_d(I)$ be the number of all square free monomials of degree d of I.

Proposition 1. *If I is generated by square free monomials of degree $\geq d$ and $\rho_d(I) > ((d+1)/d)\binom{n}{d+1}$ then* $\operatorname{depth}_S I \leq d$.

Proof. Apply induction on n. If $n = d$ then there exists nothing to show. Suppose that $n > d$. Let ν_i be the number of the square free monomials of degree d from $I \cap (x_i)$. We may consider two cases renumbering the variables if necessary.

Case 1 $\nu_1 > \binom{n-1}{d}$.

Let $S' := K[x_2, \ldots, x_n]$ and $x_1c_1, \ldots, x_1c_{\nu_1}$, $c_i \in S'$ be the square free monomials of degree d from $I \cap (x_1)$. Then $J = (I : x_1) \cap S'$ contains $(c_1, \ldots, c_{\nu_1})$ and so $\rho_{d-1}(J) \geq \nu_1 > \binom{n-1}{d}$. By induction hypothesis, we get $\operatorname{depth}_{S'} J \leq d-1$. It follows $\operatorname{depth}_S JS \leq d$ by [1, Lemma 3.6] and so $\operatorname{depth}_S I \leq d$ by [6, Proposition 1.2].

Case 2 $\nu_i \leq \binom{n-1}{d}$ for all $i \in [n]$.

We get $\sum_{i=1}^n \nu_i \leq n\binom{n-1}{d}$. Let A_i be the set of the square free monomials of degree d from $I \cap (x_i)$. A square free monomial from I of degree d will be present in d-sets A_i and it follows

$$\rho_d(I) = |\cup_{i=1}^n A_i| \leq (n/d)\binom{n-1}{d} = ((d+1)/d)\binom{n}{d+1}$$

if $n \geq d+1$. Contradiction! □

Remark 2. If I is generated by square free monomials of degree $\geq d$, then $\operatorname{depth}_S I \geq d$. Indeed, since I has a square free resolution the last shift in the resolution of I is at most n. Thus if I is generated in degree $\geq d$, then the resolution can have length at most $n-d$, which means that the depth of I is greater than or equal to d (this

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argument belongs to J. Herzog). Hence in the setting of the above proposition we get $\operatorname{depth}_S I = d$.

Corollary 3. *Let I be an ideal generated by $\mu(I)$ square free monomials of degree d. If $\mu(I) > ((d+1)/d)\binom{n}{d+1}$ then* $\operatorname{depth}_S I = d$.

Example 4. Let $I = (x_1x_2, x_2x_3) \subset S := K[x_1, x_2, x_3]$. Then $d = 2$ and $\mu(I) = 2 > (3/2)\binom{3}{2+1}$. It follows that $\operatorname{depth}_S I = 2$ by the above corollary.

The condition $\mu(I) > ((d+1)/d)\binom{n}{d+1}$ is not necessary in the above corollary as shows the following:

Example 5. Let $I = (x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_5, x_3x_4, x_3x_5, x_4x_5) \subset S := K[x_1, \dots, x_5]$. Then $d = 2$ and $\mu(I) = 8 < 15 = (3/2)\binom{5}{2+1}$ but $\operatorname{depth}_S I = 2$ because $I = (x_1, x_2, x_4, x_5) \cap (x_1, x_3, x_5) \cap (x_2, x_3, x_4)$.

Now, let I be an arbitrary square free monomial ideal and P_I the poset given by all square free monomials of I (a finite set) with the order given by the divisibility. Let $\mathcal{P}$ be a partition of P_I in intervals $[u, v] = \{w \in P_I : u|w, w|v\}$, let us say $P_I = \cup_i [u_i, v_i]$, the union being disjoint. Define $\operatorname{sdepth} \mathcal{P} = \min_i \deg v_i$ and $\operatorname{sdepth}_S I = \max_{\mathcal{P}} \operatorname{sdepth} \mathcal{P}$, where $\mathcal{P}$ runs in the set of all partitions of P_I. This is the so called the Stanley depth of I, in fact this is an equivalent definition given in a general form by [1].

For instance, in Example 4, we have $P_I = \{x_1x_2, x_2x_3, x_1x_2x_3\}$ and we may take $\mathcal{P}: \quad P_I = [x_1x_2, x_1x_2x_3] \cup [x_2x_3, x_2x_3]$ with $\operatorname{sdepth}_S \mathcal{P} = 2$. Moreover, it is clear that $\operatorname{sdepth}_S I = 2$. When I is generated by $\mu(I) > ((d+1)/d)\binom{n}{d+1} > \binom{n}{d+1}$ square free monomials of degree d then $\operatorname{sdepth}_S I = d$. Thus the Proposition 1 says that in this case $\operatorname{depth}_S I \leq \operatorname{sdepth}_S I$, which was in general conjectured by Stanley [7]. Stanley's Conjecture holds for intersections of four monomial prime ideals of S by [2] and [4] and for square free monomial ideals of $K[x_1, \dots, x_5]$ by [3] (a short exposition on this subject is given in [5]).

Remark 6. The hypothesis of Corollary 3 is too strong. If $\mu(I) > \binom{n}{d+1}$ then $\operatorname{sdepth}_S I = d$ and we may get $\operatorname{depth}_S I = d$ if Stanley's Conjecture holds.

In the Example 5 we have $P_I = [x_1x_2, x_1x_2x_4] \cup [x_1x_3, x_1x_3x_5] \cup [x_1x_4, x_1x_4x_5] \cup [x_2x_3, x_1x_2x_3] \cup [x_3x_4, x_1x_3x_4] \cup [x_3x_5, x_3x_4x_5] \cup [x_4x_5, x_2x_4x_5] \cup [x_2x_3x_4, x_2x_3x_4] \cap [x_2x_3x_5, x_2x_3x_5] \cup (\cup_\alpha [\alpha, \alpha])$, where α runs in the set of square free monomials of I of degree $4, 5$. It follows that $\operatorname{sdepth}_S I = 3$. But as we know $\operatorname{depth}_S I = 2$.

Proposition 7. *If I is generated by square free monomials of degree $\geq d$ and $\rho_d(I) \leq \binom{n}{d+1}$ then* $\operatorname{sdepth}_S I \geq d + 1$.

Proof. Apply induction on n. If $n = d + 1$ then there exists nothing to show. Suppose that $n > d + 1$. Let $S' = K[x_2, \dots, x_n]$ and $I' = I \cap S'$. Let $x_1c_1, \dots, x_1c_e$, $c_i \in S'$ be the square free monomials of degree d from $I \cap (x_1)$ and $a_1, \dots, a_s$ be the square free monomials of degree d from $I \setminus (I \cap (x_1))$. We have $\rho_d(I) = e + s$. Set $r = \max\{\rho_d(I) - \binom{n-1}{d}, 0\}$. If $r > 0$ we may suppose that a_i is a multiple of c_i for each $1 \leq i \leq r$, after renumbering of (c_i), (a_j). Certainly, it is possible that several a_j are multiples of the same c_i but we just pick one of them a_i. Set $L = (a_1, \ldots, a_r)S'$. As $r \leq \binom{n}{d+1} - \binom{n-1}{d} = \binom{n-1}{d+1}$ we get $\operatorname{sdepth}_{S'} L \geq d+1$ by induction hypothesis. Then there exists a partition of P_L of the form

$$P_L = (\cup_{i=1}^r [a_i, b_i]) \cup (\cup_t [t, t]),$$

where $b_i, t \in S'$, $\deg b_i = d+1$ and t runs in the set of all square free monomials of L different of (b_i) of degree $> d$. Then there exists a partition $\mathcal{P}$ of P_I of the form

$$P_I = (\cup_{i=1}^r [a_i, b_i]) \cup (\cup_{i=1}^r [x_1 c_i, x_1 a_i]) \cup (\cup_{j>r}^s [a_j, x_1 a_j]) \cup (\cup_p [p, p]),$$

where p runs in the set of all square free monomials of I different of (b_i), $(x_1 a_j)$ of degree $> d$. Thus $\operatorname{sdepth} \mathcal{P} = d+1$ and so $\operatorname{sdepth}_S I \geq d+1$. □

Example 8. Let Δ be the simplicial complex on the vertex set $\{1, \ldots, 6\}$, associated to the canonical triangulation of the real projective plane $\mathbb{P}^2$, whose facets are

$$\mathcal{F}(\Delta) = \{125, 126, 134, 136, 145, 234, 235, 246, 356, 456\}.$$

Then the Stanley-Reisner ideal of Δ is

$$I_\Delta = (x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_1x_4x_6, x_1x_5x_6, x_2x_3x_6, x_2x_4x_5, x_2x_5x_6, x_3x_4x_5, x_3x_4x_6).$$

We have $n = 6$, $d = 3$ and $\rho_3(I_\Delta) \leq \binom{6}{4}$. By the above proposition $\operatorname{sdepth}_S I_\Delta \geq 4$, the inequality being in fact equality. It is known that $\operatorname{depth} I_\Delta = 4$ if $\operatorname{char} K \neq 2$ and $\operatorname{depth} I_\Delta = 3$ if $\operatorname{char} K = 2$. Hence Stanley's Conjecture holds in this case. Now, let $S' := K[x_1, \ldots, x_5]$ and $I' := I_\Delta \cap S'$. Then $\rho_3(I') \leq \binom{5}{4}$ and we get also $\operatorname{sdepth}_S I' = 4$.

Corollary 9. *In the above setting, the following statements are equivalent:*

(1) $\rho_d(I) > \binom{n}{d+1}$
(2) $\operatorname{sdepth}_S I = d$.

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