A SUPPORT THEOREM FOR HILBERT SCHEMES OF PLANAR CURVES

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ABSTRACT. Consider a family of integral complex locally planar curves whose relative Hilbert scheme of points is smooth. The decomposition theorem of Beilinson, Bernstein, and Deligne asserts that the pushforward of the constant sheaf on the relative Hilbert scheme splits as a direct sum of shifted semisimple perverse sheaves. We will show that no summand is supported in positive codimension. It follows that the perverse filtration on the cohomology of the compactified Jacobian of an integral plane curve encodes the cohomology of *all* Hilbert schemes of points on the curve. Globally, it follows that a family of such curves with smooth relative compactified Jacobian ("moduli space of D-branes") in an irreducible curve class on a Calabi-Yau threefold will contribute equally to the BPS invariants in the formulation of Pandharipande and Thomas, and in the formulation of Hosono, Saito, and Takahashi.

1. INTRODUCTION

In this note a *curve* will always be *integral, complete, locally planar, and defined over* $\mathbb{C}$.[1]

Let C be a curve of arithmetic genus g. The Hilbert scheme of points $C^{[d]}$ parameterizes length d subschemes of C; it is complete, integral, d-dimensional, and l.c.i. [AIK, BGS]. If $\pi : \mathcal{C} \to B$ is a family of curves, there is a relative Hilbert scheme $\pi^{[d]} : \mathcal{C}^{[d]} \to B$ with fibres $(\mathcal{C}^{[d]})_b = (\mathcal{C}_b)^{[d]}$. Planarity of the curves ensures the existence of families in which the total space of $\mathcal{C}^{[d]}$ is smooth [S]; ultimately this is a consequence of the smoothness of Hilbert scheme of points on a surface. As the map $\pi^{[d]} : \mathcal{C}^{[d]} \to B$ is proper, the decomposition theorem of Beilinson, Bernstein, and Deligne [BBD] applies and $\mathrm{R}\pi^{[d]}_*\mathbb{C}$ decomposes as a direct sum of shifted intersection complexes associated to local systems on constructible subsets of the base.

Let $\widetilde{\pi} : \widetilde{\mathcal{C}} \to \widetilde{B}$ denote the restriction of π to the smooth locus. The Hilbert schemes of a smooth curve are its symmetric products, and in particular the map $\widetilde{\pi}^{[d]}$ is smooth. Thus the summand of $\mathrm{R}\pi^{[d]}_*\mathbb{C}[d + \dim B]$ with support equal to B is $\bigoplus \mathrm{IC}(B, \mathrm{R}^{d+i}\widetilde{\pi}^{[d]}_*\mathbb{C})[-i]$. As pointed out by Macdonald [M], the cohomology of the symmetric products is expressed in terms of the cohomology of the curves by the formula

$$\mathrm{R}^i\widetilde{\pi}^{[d]}_*\mathbb{C} = \bigoplus_{k=0}^{\lfloor i/2 \rfloor} \left(\bigwedge\nolimits^{i-2k}\mathrm{R}^1\widetilde{\pi}_*\mathbb{C}\right)(-k) = (\mathrm{R}^{2d-i}\widetilde{\pi}^{[d]}_*\mathbb{C})(d-i) \quad \text{for } i \le d \tag{1}$$

Even given this expression, computing $\mathrm{IC}(B, \mathrm{R}^i\widetilde{\pi}^{[d]}_*\mathbb{C})$ is a nontrivial matter, about which we say nothing here. But at least $\mathrm{R}\pi^{[d]}_*\mathbb{C}[d + \dim B]$ contains no other summands:

Theorem 1. *Let $\pi : \mathcal{C} \to B$ be a family of integral plane curves, and let $\widetilde{\pi} : \widetilde{\mathcal{C}} \to \widetilde{B}$ its restriction to the smooth locus. If $\mathcal{C}^{[d]}$ is smooth, then*

$$\mathrm{R}\pi^{[d]}_*\mathbb{C}[d + \dim B] = \bigoplus_{i=-d}^{d} \mathrm{IC}(B, \mathrm{R}^{d+i}\widetilde{\pi}^{[d]}_*\mathbb{C})[-i].$$

[1]This reflects a limitation of the authors rather than a certainty that the methods do not work in characteristic p.

From now on we will use the notation

$$ {}^p\mathrm{R}^i\pi_*^{[d]}\mathbb{C}[d+\dim B] := {}^p\mathcal{H}^i\left(\mathrm{R}\pi_*^{[d]}\mathbb{C}[d+\dim B]\right) $$

for the perverse cohomology sheaves of $\mathrm{R}\pi_*^{[d]}\mathbb{C}[d+\dim B]$.

The central term of Equation 1 can be reinterpreted in terms of the family of Jacobians of the curves. Indeed, taking $\widetilde{\pi}^J : J(\mathcal{C}) \to B$ to be the family of Jacobians over the smooth locus, then there is a (canonical) identification of local systems

$$ \mathrm{R}^i\widetilde{\pi}_*^J\mathbb{C} = \bigwedge\nolimits^i(\mathrm{R}^1\widetilde{\pi}_*\mathbb{C}) \tag{2} $$

Consequently,

$$ \mathrm{R}^i\widetilde{\pi}_*^{[d]}\mathbb{C} = \bigoplus_k(\mathrm{R}^{i-2k}\widetilde{\pi}_*^J\mathbb{C})(-k) = (\mathrm{R}^{2d-i}\widetilde{\pi}_*^{[d]}\mathbb{C})(d-i) \quad \text{for } i \leq d \tag{3} $$

It can be convenient to express Equations (1), (2), and (3) in the following formula:

$$ \sum_{d=0}^{\infty}\sum_{i=0}^{2d} q^d\mathrm{R}^i\widetilde{\pi}_*^{[d]}\mathbb{C} = \frac{\sum_{i=0}^{2g} q^i\bigwedge^i(\mathrm{R}^1\widetilde{\pi}_*\mathbb{C})}{(1-q\mathbb{C})(1-q\mathbb{C}(-1))} = \frac{\sum_{i=0}^{2g} q^i\mathrm{R}^i\pi_*^J\mathbb{C}}{(1-q\mathbb{C})(1-q\mathbb{C}(-1))} \tag{4} $$

The family of Jacobians can be extended over the singular locus of π to the *compactified Jacobian* [AK], $\pi^J : \overline{J}^d(\mathcal{C}) \to B$, whose fibre $\overline{J}^d(\mathcal{C})_b = \overline{J}^d(\mathcal{C}_b)$ parameterizes rank one, degree d torsion free sheaves on $\mathcal{C}$.[2] The map π^J is proper, and for Gorenstein curves there is an Abel-Jacobi map $AJ : \mathcal{C}^{[d]} \to \overline{J}^d(\mathcal{C})$ taking a subscheme to the dual of its ideal sheaf.[3] For $d > 2g-2$, the map AJ is a $\mathbb{P}^{d-g}$ bundle; thus the statement in Theorem 1 is true in this range for the map π^J as well. Over sufficiently small open set, π admits a section σ with image in the smooth locus of the curves; twisting by $\mathcal{O}(\sigma)$ identifies the $\overline{J}^d(\mathcal{C})$ for varying d and so $\pi_*^J\mathbb{C}$ does not depend on d. It can be shown [S, Prop. 14] that smoothness of the relative compactified Jacobian implies smoothness of all relative Hilbert schemes. Therefore taking IC sheaves in Equations (1) and (2) yields the following Corollary.

Corollary 2. *Let $\pi : \mathcal{C} \to B$ be a family of integral plane curves of arithmetic genus g. If the relative compactified Jacobian $\overline{J}(\mathcal{C})$ is smooth, then:*

$$ {}^p\mathrm{R}^{i-d}\pi_*^{[d]}\mathbb{C}[d+\dim B] = \bigoplus_{k=0}^{\lfloor i/2\rfloor} {}^p\mathrm{R}^{i-g-2k}\pi_*^J\mathbb{C}[g+\dim B](-k) \quad \text{for } 0 \leq i \leq d $$

(The ${}^p\mathrm{R}^{i-d}$ for $i > d$ are determined similarly by duality.)

This corollary has a consequence for the enumerative geometry of Calabi-Yau three-folds, which we briefly sketch. Gopakumar and Vafa argued in [GV] that the cohomology of the moduli space of D-branes (roughly speaking, semistable sheaves supported on curves) on a Calabi-Yau Y should give rise to *integer* “BPS” invariants, one for each genus and homology class in $\mathrm{H}_2(Y,\mathbb{Z})$, which encode the Gromov-Witten invariants of Y. Hosono, Saito, and Takahashi [HST] use intersection cohomology and the tools of [BBD] to give a precise formulation; however, their proposal is known *not* to give the desired BPS numbers in general [BP]. A different definition of integer BPS invariants is given by Pandharipande and Thomas [PT] using

[2] It also extends to the *generalized Jacobian* $J(\mathcal{C})$ whose fibre $J(\mathcal{C})_b$ parameterizes line bundles on $\mathcal{C}_b$; this is a commutative group scheme of dimension g of which the affine part is of dimension $\delta(\mathcal{C})_b$. This is a subscheme of the compactified Jacobian, and acts on it. Such actions are central to Ngô's arguments, but play no role here.

[3] In general, it is better to define the Abel-Jacobi map from the Quot scheme of the dualizing sheaf, see [AK].

the closely related spaces of "stable pairs", which for integral planar curves are just the Hilbert schemes of points. By the work of Behrend [B], the BPS invariants are extracted by a weighted Euler characteristic of these spaces, the weighting function depending only on the singularities of the moduli space. For BPS invariants associated to *irreducible* homology classes, it is sensible to discuss the contribution of an individual curve in both theories; if the moduli space of sheaves on Y is smooth along the locus of sheaves supported on a curve C, then the intersection cohomology considerations may be neglected in [HST], and likewise the weighting function of Behrend may be neglected in [PT]. In this case, taking Euler characteristics in the Corollary yields the equality of the contributions of the curve C to these two theories.

Theorem 1 is inspired by the support theorem of B. C. Ngô [N], and is a consequence of it when $d > 2g-2$. Nonetheless our proofs – we give two – do not logically depend on his work.

Acknowledgements. Corollary 2 was conjectured during a discussion between the authors and Lothar Göttsche. We are indebted to Zhiwei Yun for the suggestion that the induction procedure of Section 5 be categorified. We also thank Davesh Maulik, Alexei Oblomkov, and Richard Thomas for enlightening conversations, and Sam Gunningham and Ben Webster for helpful comments (on MathOverflow) on Proposition 15. A different approach to the main theorem can be found in the work of Davesh Maulik and Zhiwei Yun [MY], who deduce it, under additional hypotheses but in arbitrary characteristic, from the support theorem of Ngô.

Conventions. We follow [BBD] in declaring $\mathcal{F} \in \mathrm{D}^b_c(X)$ perverse when $\dim \mathrm{Supp}\mathcal{H}^i(\mathcal{F}) \leq -i$, and the same holds for the Verdier dual. That is, if X is smooth and n dimensional, $\mathbb{C}[n]$ is perverse. In arguments of a topological nature, we omit Tate twists. As mentioned at the outset, all curves are integral and have singularities of embedding dimension 2. All families of curves will enjoy a smooth base. For a curve C, we write $\delta(C)$ for the difference between its arithmetic and geometric genera, which we term the *cogenus*.

2. Background on relative Hilbert schemes and versal deformations

The Hilbert schemes of points on integral planar curves are singular, but not hopelessly so:

Theorem 3. [AIK, BGS]. *Let C be a complete integral planar curve. Then $C^{[d]}$ is integral, complete, d-dimensional, and locally a complete intersection.*

We systematically employ versal deformations of curve singularities. We will always mean this in the sense of analytic spaces, see [GLS] for a thorough treatment. The base of a versal deformation of a plane curve singularity is smooth. If $\pi : \mathcal{C} \to B$ is a family of curves, we say it is *locally versal* at b if it induces versal deformations of all the singularities of $\mathcal{C}_b$, or equivalently if the tangent map to the product of the first order deformations of the singularities of B is surjective. Such families have in particular the following properties:

Theorem 4. [DH, T]. *Let $\pi : \mathcal{C} \to B$ be a family of curves. The cogenus is an upper semicontinuous function on B. Local versality is an open condition, and in a locally versal family the locus of curves of cogenus at least δ is equal to the closure of the locus of δ-nodal curves. In particular, the locus of curves of cogenus δ has codimension δ.*

Any curve singularity can be found on a rational curve; for an explicit construction see e.g. [L]. Moreover, if $\mathcal{C} \to B$ is a family of curves, then locally near $b \in B$ one can find a different family $\mathcal{C}' \to B$ such that $\mathcal{C}'_b$ is rational with the same singularities as $\mathcal{C}_b$ and the two families induce the same deformations of the singularities of the central fibre.

Proposition 5. [FGvS]. *The map from the base of a versal deformation of an integral locally planar curve to the product of the versal deformations of its singularities is a smooth surjection.*

Corollary 6. *Let $\pi : \mathcal{C} \to B$ be a family of curves. Fix $b \in B$, and let $\overline{\mathcal{C}_b}$ be the normalization of $\mathcal{C}_b$. Then there exists a neighborhood $b \in U \subset B$ and a family $\pi' : \mathcal{C}' \to U$ such that $\mathcal{C}'_b$ is rational with the same singularities as $\mathcal{C}_b$, and $\mathcal{C}$ and $\mathcal{C}'$ induce the same deformations of these singularities on U, and in particular have the same discriminant locus. Moreover, on U, we have an equality of local systems $\mathrm{R}^1\widetilde{\pi}_*\mathbb{C} = \mathrm{R}^1\widetilde{\pi}'_*\mathbb{C} \bigoplus \mathrm{H}^1(\overline{\mathcal{C}_b})$, the latter summand meaning the constant local system with the specified fibre.*

Proof. Let C' be a rational curve with the same singularities as $\mathcal{C}_b$; let $\mathcal{C}' \to \mathbb{V}(C')$ a versal deformation of C', and let $\overline{\mathbb{V}}(\mathcal{C}_b)$ be the product of the versal deformations of the singularities of $\mathcal{C}_b$. The map $\mathbb{V}(C') \to \overline{\mathbb{V}}(\mathcal{C}_b)$ is a smooth surjection, so we may choose a local section over some neighborhood U. Possibly shrinking U, we compose the maps $U \to \overline{\mathbb{V}}(\mathcal{C}_b) \to \mathbb{V}(C')$ and pull back $\mathcal{C}'$ to obtain a family of rational curves $\pi' : \mathcal{C}'_B \to B$.

Shrink U further so that the inclusion $\mathcal{C}_b \to \mathcal{C}|_U$ is a homotopy equivalence. Let $\mathcal{V}$ be the summand of $R^1\widetilde{\pi}_*\mathbb{C}$ whose fibre at $\tilde{b}$ is the kernel of the composition of the specialization map $\mathrm{H}^1(\mathcal{C}_{\tilde{b}}) \to \mathrm{H}^1(\mathcal{C}_b)$ with the pullback to the normalization $\mathrm{H}^1(\mathcal{C}_b) \to \mathrm{H}^1(\overline{\mathcal{C}_b})$. This is a symplectic summand, let $\mathcal{V}^\perp$ be its orthogonal complement. As $\mathcal{V}$ contains all vanishing cycles, the Picard-Lefschetz formula ensure $\mathcal{V}^\perp$ has trivial monodromy and thus extends extends to a trivial local system over B with fibre $\mathcal{V}_b^\perp = \mathrm{H}^1(\overline{\mathcal{C}}_b)$. On the other hand, $\mathcal{V}$ depends only on the deformation of the singularities, which is the same in $\mathcal{C}$ and $\mathcal{C}'$. □

To make use of such a replacement, it is necessary to know that the relative Hilbert scheme $\mathcal{C}'^{[d]}$ is smooth if $\mathcal{C}^{[d]}$ is. This follows from results of the second author on the smoothness of relative Hilbert schemes [S], which we now review.

Proposition 7. [S, Prop. 14] *Let $\pi : \mathcal{C} \to B$ be a family of curves. If $\mathcal{C}^{[d]}$ is smooth, then $\mathcal{C}^{[n]}$ is smooth for any $n \leq d$.*

Theorem 8. *Let $\mathcal{C} \to B$ be a family of curves. For $b \in B$, let I be the image of T_bB in the product of the first-order deformations of the singularities of $\mathcal{C}_b$. Then:*

1. *The smoothness of $\mathcal{C}^{[d]}$ along $\mathcal{C}_b^{[d]}$ depends only on I.*
2. *If $\mathcal{C}^{[d]}$ is smooth along $\mathcal{C}_b^{[d]}$, then $\dim I \geq \min(d, \delta)$.*
3. *If $\dim I \geq d$ and I is general among such subspaces, $\mathcal{C}^{[d]}$ is smooth along $\mathcal{C}_b^{[d]}$.*
4. [FGvS] *$\mathcal{C}^{[d]}$ is smooth along $\mathcal{C}_b^{[d]}$ for all d if and only if I is transverse to the image of the "equigeneric ideal". It suffices for I to be generic of dimension at least δ.*

Proof. Item (1) holds because, as shown in [S], if $\overline{\pi} : (\mathcal{X}, x) \to (\mathbb{V}, 0)$ is a versal deformation of a curve singularity, then for any subscheme $z \subset \mathcal{X}_0$ set theoretically supported at x, the germ $(\mathcal{X}^{[d]}, [z])$ is smooth. We explain in detail: take z a subscheme of $\mathcal{C}_b^{[d]}$ which decomposes as $z = \coprod z_i$ into subschemes of lengths d_i supported at points c_i. Let $(\overline{\mathcal{C}}_i, c_i) \to (\mathbb{V}_i, 0)$ be miniversal deformations of the curve singularities $(\mathcal{C}_b, c_i)$ and $(B, b) \to \prod(\mathbb{V}_i, 0)$ a map along which the (multi-)germ $\prod(\mathcal{C}_b, c_i)$ pulls back. Then analytically locally the (multi-)germ $\prod(\mathcal{C}_b^{[d_i]}, [z_i])$ pulls back from $\prod(\overline{\mathcal{C}}_i^{[d_i]}, [z_i])$ along the same map. As the fibres of $(\overline{\mathcal{C}}_i^{[d_i]}, [z_i]) \to (\mathbb{V}_i, 0)$ are reduced of dimension d_i and the total space is smooth, the smoothness of the pullback depends only on the image of T_bB in $\prod T_0\mathbb{V}_i$, which is well defined as the $\mathbb{V}_i$ were taken miniversal. The miniversal deformation of the germ of a curve at a smooth point being trivial, only the singularities contribute. To check (2), we may by (1) assume the map $T_bB \to I$ is an isomorphism and then identify locally B with its image in some representative $\overline{B}$ of $\prod(\mathbb{V}_i, 0)$. Shrink $\overline{B}$ until it can be written as $B \times \mathbb{D}$ for some polydisc $\mathbb{D}$; by openness of smoothness we may shrink $\mathbb{D}$ further until $\mathcal{C}^{[d]}|_{B\times\epsilon}$ is smooth for all $\epsilon \in \mathbb{D}$. By Theorem 4, some points in $\overline{B}$ will correspond to curves with $\delta(\mathcal{C}_b)$ nodes; choose ϵ so the slice $B \times \epsilon$ contains such a point p. If $d \leq \delta$, there is a point $z \in \mathcal{C}_p^{[d]}$ be a point naming a subscheme supported at d nodes. The Zariski tangent space $T_z\mathcal{C}_p^{[d]}$ is $2d$ dimensional, so $\mathcal{C}_p^{[d]}$ cannot be smoothed out over a base of dimension less than d. Item (3) appears in [S] as Theorem B. Item (4) is stated in [FGvS] for the compactified Jacobian; it follows for $\mathcal{C}^{[d]}$ for $d \gg 0$ because this fibres smoothly over the Jacobian, and for lower d by Proposition 7. □

Corollary 9. *If $\mathcal{C} \to B$ is a family of curves with $\mathcal{C}^{[d]}$ smooth, then for $\delta \leq d$, the locus of curves with cogenus δ is of codimension at least δ in B.*

Proof. Suppose not; let B' be a generic $\delta - 1$ dimensional smooth subvariety of B, then the restriction $\mathcal{C}^{[d]} \times_B B'$ is smooth and B' intersects the locus of curves of cogenus δ. This contradicts (2) of Theorem 8. □

Remark. Corollary 9 explains why we do not require a "δ-regularity" assumption as in [N] – in the case of Hilbert schemes and Jacobians, it follows from smoothness of the total space.

3. Estimates

The following is a variation on the "Goresky-MacPherson inequality" of [N], Section 7.3.

Lemma 10. *Let $\pi : X \to Y$ be a locally projective morphism of smooth varieties with irreducible fibres of dimension n. Then*

$$\mathcal{H}^i({}^p\mathrm{R}^j\pi_*\mathbb{C}[\dim X]) = 0 \quad \text{for } i \geq n - \dim Y - |j| \text{ and } i > -\dim Y$$

In particular, every summand of $\mathrm{R}\pi_*\mathbb{C}$ is supported on a subvariety of codimension $< n$.

Proof. Since the estimate is symmetric in j and, by relative hard Lefschetz, ${}^p\mathrm{R}^j\pi_*\mathbb{C}[\dim X] \simeq {}^p\mathrm{R}^{-j}\pi_*\mathbb{C}[\dim X]$, we may assume $j \geq 0$. We check at a point $y \in Y$, where by [BBD], $\mathcal{H}^i({}^p\mathrm{R}^j\pi_*\mathbb{C}[\dim X])_y$ is a summand of $\mathrm{H}^{i+j+\dim X}(X_y, \mathbb{C})$. This vanishes for dimension reasons if $i + j + \dim X = i + j + \dim Y + n > 2\dim X_y = 2n$. Finally, as the fibres are irreducible, $R^{2n}\pi_*\mathbb{C} \simeq \mathbb{C}$. This top dimensional cohomology is already accounted for by the summand ${}^p\mathrm{R}^n\pi_*\mathbb{C}[\dim X]$ and thus the vanishing for $j = i$ is ensured. The final statement follows because a summand supported on a subvariety Y' is the IC sheaf associated to some local system on an open subset of Y' and consequently the stalk of the cohomology sheaf in degree $-\dim Y'$ is non zero on a general point of Y'; this is prohibited by the stated estimate when $\dim Y - \dim Y' \geq n$. □

Lemma 11. *Let $\pi : \mathcal{C} \to B$ be a family of curves such that $\mathcal{C}^{[d]}$ is smooth. Then for $i > 0$, the sheaf $\mathcal{H}^i(\mathrm{IC}(B, \mathrm{R}^j\widetilde{\pi}_*^{[d]}\mathbb{C})[-\dim B])$ is supported on the locus of curves of cogenus $> i$.*

Proof. We check at some point $b \in B$ and write δ for the cogenus of $\mathcal{C}_b$. By semicontinuity of cogenus, in some neighborhood all curves have cogenus $\leq \delta$; we shrink B to this neighborhood and show that $\mathcal{H}^i(\mathrm{IC}(B, \mathrm{R}^j\widetilde{\pi}_*^{[d]}\mathbb{C})[-\dim B]) = 0$ for all $i \geq \delta$. Shrinking B further if necessary, let $\pi' : \mathcal{C}' \to B$ be the family of curves constructed in Corollary 6, which we recall has the property that $\mathcal{C}'_b$ is rational, $\mathrm{R}^1\widetilde{\pi}_*\mathbb{C} = \mathrm{R}^1\widetilde{\pi}'_*\mathbb{C} \oplus \mathrm{H}^1(\overline{\mathcal{C}_b})$, and by item (1) of Theorem 8, $\mathcal{C}'^{[d]}$ is smooth. Taking exterior powers and comparing with Equation (1), we see that $\mathrm{R}^j\widetilde{\pi}_*^{[d]}\mathbb{C}$ is a sum of $\mathrm{R}^{\leq j}\widetilde{\pi}'^{[d]}_*\mathbb{C}$; it will therefore suffice to check the assertion for the family $\mathcal{C}'$.

Note δ is the common arithmetic genus of the fibres of π'. From Equation (1), all summands of $\mathrm{R}^i\widetilde{\pi}'^{[d]}_*\mathbb{C}$ appear already as summands of $\mathrm{R}^i\widetilde{\pi}'^{[\min(d,\delta)]}_*\mathbb{C}$. As $\mathcal{C}'^{[\min(d,\delta)]}$ is smooth by Proposition 7, we may as well assume $d \leq \delta$. By relative hard Lefschetz, it suffices to check the assertion for $j \leq d$. But now $j \leq d \leq \delta \leq i$, thus by the previous lemma, we are done. ☐

Remark. Being an IC sheaf ensures that the above mentioned cohomology is supported on *some* subspace of codimension $i+1$. The force of the lemma is to show this subspace lies inside the codimension $i+1$ locus of curves of cogenus $i+1$. Experimental evidence suggests that the support is *much* smaller, and it would be interesting to have a precise characterization.

Lemma 12. *Let $\pi : \mathcal{C} \to B$ be a family of curves, $B' \subset B$ a smooth closed subvariety, and $\pi' : \mathcal{C}' \to B'$ the restricted family. Assume $\mathcal{C}^{[d]}$ and $\mathcal{C}'^{[d]}$ are smooth. Denote by $\widetilde{\pi}$ and $\widetilde{\pi}'$ the respective smooth loci of the maps. Then $\mathrm{IC}(B, \mathrm{R}^i\widetilde{\pi}^{[d]}_*\mathbb{C})|_{B'}[\dim B' - \dim B] = \mathrm{IC}(B', \mathrm{R}^i\widetilde{\pi}'^{[d]}_*\mathbb{C})$.*

Proof. By induction on the codimension of B' in B, we are reduced to proving the statement for B' a Cartier divisor in B. By [BBD], Cor. 4.1.12, the complex $K := \mathrm{IC}(B, \mathrm{R}^i\widetilde{\pi}^{[d]}_*\mathbb{C})|_{B'}[-1]$ is a perverse sheaf. By proper base change, K is a summand of $\mathrm{R}\pi'^{[d]}_*\mathbb{C}[d + \dim B']$. As $\mathcal{C}'^{[d]}$ is smooth, K must be the sum of IC complexes, and by Corollary 9 the locus of curves of cogenus $\delta \leq d$ appears in codimension δ in B'. By Lemma 11 and the fact that the fibre is d-dimensional, $\dim \mathrm{Supp}\, \mathcal{H}^i(K) < -i$ for $i \neq -\dim B'$. Therefore no summand of K is an IC complex associated to a local system supported in positive codimension in B', and the claimed isomorphism follows from the obvious fact that, on the smooth locus, K coincides with the (shifted) local system $\mathrm{R}^i\widetilde{\pi}'^{[d]}_*\mathbb{C}[\dim B']$. ☐

Corollary 13. *Let $\pi : \mathcal{C} \to B$ be a family of curves, and $\pi' : \mathcal{C}' \to B'$ its restriction to a smooth subvariety of the base; assume $\mathcal{C}^{[d]}$ and $\mathcal{C}'^{[d]}$ are smooth. Let $\mathcal{F}$ be the summand of $\mathrm{R}\pi^{[d]}_*\mathbb{C}[d + \dim B]$ not supported on all of B, and similarly $\mathcal{F}'$ for B'. If $B' \not\subset \mathrm{Supp}\, \mathcal{F}$, then $\mathcal{F}' = \mathcal{F}|_{B'}[\dim B' - \dim B]$.*

4. Proof via reduction to rational curves

Proposition 14. *Let $\pi : \mathcal{C} \to B$ be a family of curves of cogenus bounded by δ. Then Theorem 1 holds for $d \leq \delta$.*

Proof. Suppose not; let $\mathcal{C} \to B$ be a counterexample over a base of minimal dimension. Let $b \in B$ be any point in the support of a summand $\mathcal{F}$ of $\mathrm{R}\pi^{[d]}_*\mathbb{C}$ not supported on all of B. If $\delta(b)$ is the cogenus of $\mathcal{C}_b$, then by Theorem 8 and Corollary 13, the restriction of the family to a general slice of dimension $\delta(b)$ passing through b remains a counterexample. Therefore we may assume $\delta = \delta(b) = \dim B$. By Lemma 10, the support of $\mathcal{F}$ is of codimension $< d \leq \delta$, thus it intersects a general $\delta - 1$ dimensional slice of B. Again by Corollary 13, the restricted family remains a counterexample, contradicting the assumption of minimal dimensionality. ☐

Now let $\pi : \mathcal{C} \to B$ be a family of curves; shrinking to a neighborhood of some $b \in B$, let $\pi' : \mathcal{C}' \to B$ be the replacement family of Corollary 6. Then from Equation (4), we see

$$\sum_{d=0}^{\infty}\sum_{i=0}^{2d} q^d \mathrm{R}^i\widetilde{\pi}^{[d]}_*\mathbb{C} = \left(\sum_{d=0}^{\infty}\sum_{i=0}^{2d} q^d \mathrm{H}^i(\overline{\mathcal{C}_b}^{[d]})\right) \otimes \left(\sum_{i=0}^{2\delta(\mathcal{C}_b)} q^i {\textstyle\bigwedge}^i \mathrm{R}^1\widetilde{\pi}'_*\mathbb{C}\right) \tag{5}$$

As the final term is manifestly symmetric about q^δ, the series is determined by its first δ terms.

To finish the proof of Theorem 1, it would suffice to show that

$$\sum q^d \mathrm{H}^*(\mathcal{C}_b^{[d]}) = \Big(\sum q^d \mathrm{H}^*(\overline{\mathcal{C}_b}^{[d]})\Big)\, \mathcal{Z}_C(q) \tag{6}$$

for a generating polynomial of vector spaces $\mathcal{Z}_C(q)$ of degree 2δ with coefficients symmetric around q^δ. Indeed, then the fibre at b of both sides of the equality asserted in Theorem 1 would be determined in the same way by their values for $C^{[\leq\delta]}$, which by Proposition 14 are equal.

However, we know no direct way to establish Equation 6, although of course it will follow as a consequence of Theorem 1. Instead, we prove the product formula and check the symmetry in the Grothendieck group of varieties, in which we denote by $\mathbb{L}$ the class of the affine line. This is still sufficient, because the *weight polynomial* both factors through the Grothendieck group of varieties and serves to witness the non-existence of summands of $\mathrm{R}\pi_*^{[d]}\mathbb{C}[\dim B]$. For K a complex of vector spaces carrying a weight filtration, we write the weight polynomial $\mathfrak{w}(K) := \sum_{i,j} t^i(-1)^{i+j}\dim \mathrm{Gr}^i_W \mathrm{H}^j(K)$. For a variety Z, we abbreviate $\mathfrak{w}(Z)$ for $\mathfrak{w}(\mathrm{H}^*_c(Z))$.

Proposition 15. *Suppose given a proper map $f : X \to Y$ between smooth varieties, and some summand $\mathcal{F}$ of $\mathrm{R}f_*\mathbb{C}[\dim X]$. If, for all $y \in Y$, we have $\mathfrak{w}(\mathcal{F}_y[-\dim X]) = \mathfrak{w}(X_y)$, then $\mathcal{F} = \mathrm{R}\pi_*\mathbb{C}[\dim X]$.*

Proof. Let $\mathrm{R}f_*\mathbb{C}[\dim X] = \mathcal{F} \bigoplus \mathcal{G}$; we must show that if $\mathfrak{w}(\mathcal{G}_y) = 0$ for all $y \in Y$, then $\mathcal{G} = 0$. $\mathcal{G}$ is a direct sum of shifted complexes of the form $\mathrm{IC}(L_i)$, with L_i a local system supported on a locally closed subset of B underlying a pure variation of Hodge structures. Then for y a general point of the support of one with highest weight, the vanishing of the weight polynomial forces the vanishing of the local system. $\square$

Let C be a curve, C^{sm} its smooth locus, and $\overline{C}$ its normalization. For $p \in C$, we write $(C,p)^{[n]}$ for the subvariety of $C^{[n]}$ parameterizing subschemes set-theoretically supported at p; our notation is meant to recall that it depends only on the germ of C at p. Let $b(p)$ be the number of analytic local branches of C near p. Splitting subschemes according to their support gives the following equality in the Grothendieck group of varieties:

$$\sum q^n[C^{[n]}] = \sum q^n[(C^{sm})^{[n]}] \prod_{p\in C\setminus C^{sm}} \sum q^n[(C,p)^{[n]}] \tag{7}$$

$$= \Big(\sum q^n[\overline{C}^{[n]}]\Big)\left(\prod_{p\in C\setminus C^{sm}} (1-q)^{b(p)} \sum q^n[(C,p)^{[n]}]\right) \tag{8}$$

This is the desired product formula. It remains to show that the final term of Equation 8 is symmetric around q^δ. After passing to Euler characteristics, this is shown in [PT] using Serre duality; the argument below is similar.

Proposition 16. *Let C be a Gorenstein curve of cogenus δ, with smooth locus C^{sm} and $b(p)$ analytic local branches at a point $p \in C$. Define*

$$Z_C(q) := \prod_{p\in C\setminus C^{sm}} (1-q)^{b(p)} \sum q^n[(C,p)^{[n]}]$$

Then $Z_C(q)$ is a polynomial in q of degree 2δ. Moreover, writing $\mathbb{L}$ for the class of the affine line, we have $Z_C(q) = (q^2\mathbb{L})^\delta Z_C(1/q\mathbb{L})$.

Proof. By Equation 8, we may assume C is a rational curve of arithmetic genus g; note in this case $Z(C) = (1-q)(1-q\mathbb{L})\sum q^d[C^{[d]}]$. Fix a degree 1 line bundle $\mathcal{O}(1)$ on C. We map $C^{[d]} \to \overline{J}^0(C)$ by by associating the ideal $I \subset \mathcal{O}_C$ to the sheaf $I^* = \mathcal{H}om(I,\mathcal{O}_C) \otimes \mathcal{O}(-d)$;

the fibre is $\mathbb{P}(H^0(C, I^*))$. For $\mathcal{F}$ a rank one degree zero torsion free sheaf, we write the Hilbert function as $h_{\mathcal{F}}(d) = \dim \mathrm{H}^0(C, \mathcal{F} \otimes \mathcal{O}(d))$. Then since over the strata with constant Hilbert function, the map from the Hilbert schemes to the compactified Jacobian is the projectivization of a vector bundle, we have the equality $\sum q^d [C^{[d]}] = \sum_h [\{\mathcal{F} \,|\, h_{\mathcal{F}} = h\}] \sum q^d [\mathbb{P}^{h(d)-1}]$.

Fix $h = h_{\mathcal{F}}$ for some $\mathcal{F}$. Evidently h is supported in $[0, \infty)$, and by Riemann-Roch and Serre duality is equal to $d + 1 - g$ in $(2g-2, \infty)$. Inside $[0, 2g-2]$, it either increases by 0 or 1 at each step. Let $\phi_{\pm}(h) = \{d \,|\, 2h(d-1) - h(d-2) - h(d) = \pm 1\}$; evidently $\phi_- \subset [0, 2g]$ and $\phi_+ \subset [1, 2g-1]$, and

$$Z_h(q) := (1-q)(1-q\mathbb{L}) \sum q^d [\mathbb{P}^{h(d)-1}] = \sum_{d \in \phi_-(\mathcal{F})} q^d \mathbb{L}^{h(d)-1} - \sum_{d \in \phi_+(\mathcal{F})} q^d \mathbb{L}^{h(d-1)}$$

This is a polynomial in q of degree at most $2g$, hence so is $Z_C(q)$.

Now let $\mathcal{G} = \mathcal{F}^* \otimes \omega_C \otimes \mathcal{O}(2-2g)$, and $h^\vee = h_{\mathcal{G}}$. By Serre duality and Riemann-Roch, $h^\vee(d) = h(2g-2-d) + d + 1 - g$, so in particular, $d \in \phi_{\pm}(h^\vee) \iff 2g - d \in \phi_{\pm}(h)$. It follows that $q^{2g}\mathbb{L}^g Z_h(1/q\mathbb{L}) = Z_{h^\vee}(q)$. As $Z_C(q) = \sum_h [\{\mathcal{F} \,|\, h_{\mathcal{F}} = h\}] Z_h(q)$, we obtain the final stated equality. $\square$

This completes the (first) proof of Theorem 1.

5. Proof by reduction to nodal curves

Lemma 17. *If Theorem 1 holds for all versal families of curves, then it holds for all families.*

Proof. By Corollary 13, the hypothesis implies that Theorem 1 holds for any subfamily of a versal family. Now let $\pi : \mathcal{C} \to B$ be a family such that the theorem fails; let $\mathcal{F}$ be the summand of $\pi_*^{[d]}\mathbb{C}$ whose support is not all of B, and let $b \in B$ be a point such that $\mathcal{F}_b \neq 0$. Let $\phi : B \to \mathbb{V}(\mathcal{C}_b)$ be a map to the miniversal deformation, and let $B' \subset B$ be a smooth closed subvariety such that $\mathrm{d}\phi_b|_{B'}$ is injective. By item (1) of Theorem 8, $\mathcal{C}^{[d]}|_{B'}$ is still smooth. According to Corollary 13, choosing $B' \not\supset \operatorname{Supp} \mathcal{F}$ ensures that the restricted family still provides a counterexample in any neighborhood of b. Shrinking still further, the map $B' \to \mathbb{V}(\mathcal{C}_b)$ may be taken to be the embedding of a smooth subvariety, giving a contradiction. $\square$

We now prove Theorem 1 for the versal family. The argument is an induction on the cogenus, which depends crucially on the properties of the versal family identified in Theorems 4 and 8. For clarity, we separate topological generalities from the specific properties of the versal family.

Definition 18. *Let X be a smooth complex analytic space with a constructible stratification $X = \coprod X_i$ such that X_i is everywhere of codimension $\geq i$. We write $\mathfrak{N}(\coprod X_i)$ for the full subcategory of $\mathrm{D}^b_c(X)$ whose objects $\mathcal{F}$ have the following property.*

> *For $x \in X_i$, for generic, sufficiently small, polydiscs $X \supset \mathbb{D}^i \times \mathbb{D} \supset \mathbb{D}^i \times 0 \ni x$, for sufficiently small $\epsilon \in \mathbb{D}$, the restriction*
>
> $$\mathcal{F}_x = \mathrm{R}\Gamma(\mathbb{D}^i \times 0, \mathcal{F}|_{\mathbb{D}^i \times 0}) = \mathrm{R}\Gamma(\mathbb{D}^i \times \mathbb{D}, \mathcal{F}|_{\mathbb{D}^i \times \mathbb{D}}) \to \mathrm{R}\Gamma(\mathbb{D}^i \times \epsilon, \mathcal{F})$$
>
> *is an isomorphism.*

Lemma 19. *$\mathfrak{N}(\coprod X_i)$ is a thick triangulated subcategory of $\mathrm{D}^b_c(X)$, i.e., it is closed under shifts, triangles, and taking summands.*

Lemma 20. *Let $X^+ \subset X$ be an open subset such that $X_i \setminus X^+$ is of codimension $> i$. Then the restriction $\mathfrak{N}(\coprod X_i) \to \mathrm{D}^b_c(X^+)$ is faithful.*

Proof. Consider $\mathcal{F} \in \mathfrak{N}(X,\Sigma)$ such that $\mathcal{F}|_{X^+} = 0$. We must show $F_x = 0$ for all $x \in X$. Suppose by induction $F_x = 0$ for $x \in X_{<i}$ and consider $x \in X_i \setminus X^+$. Evidently $(X_i \setminus X^+) \cup X_{>i}$ is of codimension $> i$, so the generic $\mathbb{D}^i \times \epsilon$ from the definition of $\mathfrak{N}(X,\Sigma)$ passing near x misses this locus completely. Thus by assumption and the induction hypothesis, $\mathcal{F}_x = \mathrm{R}\Gamma(\mathbb{D}^i \times \epsilon, \mathcal{F}) = 0$. $\square$

Proposition 21. *Let $\pi : \mathcal{C} \to B$ be a locally versal family of curves. Let B_i be the locus of curves of cogenus i. Then $\mathrm{R}\pi^{[d]}_* \mathbb{C}[\dim B] \in \mathfrak{N}(\coprod B_i)$.*

Proof. We check at some $b \in B_\delta$. The definition of $\mathfrak{N}$ is local on the base; as $\widetilde{\pi}$ is proper, after shrinking B the inclusion $\mathcal{C}_b^{[d]} \hookrightarrow \mathcal{C}^{[d]}$ becomes a homotopy equivalence. Any sufficiently small polydisc $\lambda \in \mathbb{D}^\delta \times \mathbb{D} \subset \Lambda$ will induce homotopy equivalences $\mathcal{C}_b^{[d]} \to \mathcal{C}^{[d]}|_{\mathbb{D}^\delta \times 0} \to \mathcal{C}^{[d]}|_{\mathbb{D}^\delta \times \mathbb{D}}$. By item (3) of Theorem 8, a generic choice ensures that the latter two spaces are smooth, possibly after further shrinking the discs; by openness of smoothness we can shrink $\mathbb{D}$ still further so that the projection $\mathcal{C}^{[d]}_{\mathbb{D}^\delta \times \mathbb{D}} \to \mathbb{D}$ is smooth. It follows that, possibly after shrinking $\mathbb{D}^\delta$ further, that $\mathrm{H}^*(\mathcal{C}^{[d]}_{\mathbb{D}^\delta \times \mathbb{D}}) = \mathrm{R}\Gamma(\mathbb{D}^\delta \times \mathbb{D}, \pi^{[d]}_* \mathbb{C}) \to \mathrm{R}\Gamma(\mathbb{D}^\delta \times \epsilon, \pi^{[d]}_* \mathbb{C}) = \mathrm{H}^*(\mathcal{C}^{[d]}_{\mathbb{D}^\delta \times \epsilon})$ is an isomorphism. $\square$

Proposition 22. *Theorem 1 holds for all locally versal families of curves.*

Proof. Let $\pi : \mathcal{C} \to B$ be a locally versal family of curves, and let B_i be the locus of curves of cogenus i. Let $\mathcal{F}$ be any summand of $\mathrm{R}\pi^{[d]}_* \mathbb{C}$ supported on a proper subvariety of B. Then by Lemma 19 and Proposition 21, $\mathcal{F} \in \mathfrak{N}(\coprod B_i)$. By Theorem 4 [DH, T], the locus of nodal curves is dense in each B_i; thus by Lemma 20 we need only check that the restriction of $\mathcal{F}$ to the locus of nodal curves is zero, i.e., that Theorem 1 holds for families of nodal curves. $\square$

Lemma 23. *Theorem 1 holds for locally versal families of nodal curves.*

Proof. Let $\pi : \mathcal{C} \to B$ be such a family. Let $b \in B$ be the base point, let $\{c_1, \cdots c_\delta\} \subseteq \mathcal{C}_b$ be the nodal set of the central curve $\mathcal{C}_b$, and denote by r its geometric genus. Shrink B if necessary, we can assume:

(1) the discriminant locus is a normal crossing divisor $\Delta = \cup D_i$ with $i = 1, \cdots, \delta$, where D_i is the locus in which the $i-$th node c_i is preserved.
(2) If b_0 is such that $\mathcal{C}_{b_0}$ is nonsingular, the vanishing cycles $\{\zeta_1, \cdots, \zeta_\delta\}$ in $\mathcal{C}_{b_0}$ associated with the nodes of $\mathcal{C}_b$ are disjoint.

As the curve $\mathcal{C}_b$ is irreducible, the cohomology classes in $\mathrm{H}^1(\mathcal{C}_{b_0})$ of these vanishing cycles are linearly independent, and can then be completed to a symplectic basis.

Let T_i be the generators of the (abelian) local fundamental group $\pi_1(B \setminus \Delta, b_0)$ where T_i corresponds to "going around D_i". Then the monodromy defining the local system $\mathrm{R}^1\widetilde{\pi}_*\mathbb{C}$ on $B \setminus \Delta$ is given via the Picard-Lefschetz formula, and, in the symplectic basis above, has a Jordan form consisting of δ Jordan blocks of length 2. From this it is easy to compute the invariants of the local systems obtained applying any linear algebra construction to $\mathrm{R}^1\widetilde{\pi}_*\mathbb{C}$, such as those who appear in $\mathrm{R}^i\widetilde{\pi}^{[d]}_*\mathbb{C}$. Let $SS^{i,[d]}$ be the linear algebra operation, described by Formula 1, such that $\mathrm{R}^i\widetilde{\pi}^{[d]}_*\mathbb{C} = SS^{i,[d]}R^1\pi_*\mathbb{C}$. Denote by $j : B \setminus \Delta \to B$ the open inclusion.

We have a natural isomorphism

$$\left(SS^{i,[d]}\mathrm{H}^1(\mathcal{C}_{b_0})\right)^{\pi_1(B\setminus\Delta, b_0)} = \mathcal{H}^{-\dim B}(\mathrm{IC}(B, \mathrm{R}^i\widetilde{\pi}^{[d]}_*\mathbb{C}))_b$$

between the monodromy invariants on $SS^{i,[d]}\mathrm{H}^1(\mathcal{C}_{b_0})$ and the stalk at b of the first non-vanishing cohomology sheaf of the intersection cohomology complex of $\mathrm{R}^i\widetilde{\pi}^{[d]}_*\mathbb{C}$. The decomposition theorem in [BBD] then implies that $\mathrm{H}^*(\mathcal{C}_b^{[d]})$ contains the Hodge structure

$$\mathbb{H}^{[d]} := \bigoplus_i \left(SS^{i,[d]}\mathrm{H}^1(\mathcal{C}_{b_0})\right)^{\pi_1(B\setminus\Delta, b_0)}$$

as a direct summand, with the weight filtration defined in the standard way by the logarithms of the monodromy operators (see [CK]).

It is easy to compute $\mathbb{H}^{[d]}$ explicitly; presumably $\mathrm{H}^*(\mathcal{C}_b^{[d]})$ can be computed by elementary methods and shown to match; this would complete the proof. In the absence of such a calculation, we use Proposition 15 and instead compare weight polynomials. On the one hand, we compute $\sum q^d \mathfrak{w}(\mathbb{H}^{[d]}) = (1-q+t^2q^2)^\delta(1+tq)^{2r}/(1-q)(1-t^2q)$.

On the other hand, when $C = \mathbb{P}^1_+$ is a rational curve with a single node, Riemann-Roch ensures that the Abel map is a projective bundle for any $d \geq 1$; when $d = 1$ we have $[\overline{J}^0(\mathbb{P}^1_+)] = [\mathbb{P}^1_+] = \mathbb{L}$. Thus we get the formula $\sum q^d[(\mathbb{P}^1_+)^{[d]}] = (1-q+q^2\mathbb{L})/((1-q)(1-q\mathbb{L}))$. Comparison with Equation (8) gives $\sum q^d[\mathcal{C}_b^{[d]}] = \left(\sum q^d[\overline{\mathcal{C}_b}^{[d]}]\right)(1-q+q^2\mathbb{L})^\delta$; taking weight polynomials gives the desired result. $\square$

This completes the (second) proof of Theorem 1.

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