Inherited structures in deformations of Poisson pencils

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Abstract: In this paper with study some properties of bi-Hamiltonian deformations of Poisson pencils of hydrodynamic type. More specifically, we are interested in determining those structures of the fully deformed pencils that are inherited through the interaction between structural properties of the dispersionless pencils (in particular exactness or homogeneity) and suitable finiteness conditions on the central invariants (like polynomiality). This approach enables us to gain some information about each term of the deformation *to all orders in ϵ.*

Concretely, we show that deformations of exact Poisson pencils of hydrodynamic type with polynomial central invariants can be put, via a Miura transformation, in a special form, that provides us with a tool to map a fully deformed Poisson pencil with polynomial central invariants of a given degree to a fully deformed Poisson pencil with constant central invariants to all orders in ϵ. In particular, this construction is applied to the so called r-KdV-CH hierarchy that encompasses all known examples with non-constant central invariants.

As far as homogeneous Poisson pencils of hydrodynamic type is concerned, we prove that they can also be put in a special form, if the central invariants are homogeneous polynomials. Through this we can compute the homogeneity degree about the tensorial component appearing in each order in ϵ, namely the coefficient of the highest order derivative of the δ.

MSC 2010: Primary: 37K10, Secondary: 53D45, 58H15, 53D17.

1 Introduction

Toward the end of the twentieth century, Dubrovin and Zhang constructed a general framework to tackle the classification problem for integrable PDEs, motivated in part by questions arising in the theory of Gromov-Witten invariants and topological field theory. One of the cornerstones of their approach is the analysis and classification of Poisson pencils of the form

$$\Pi_\lambda^{ij} = \omega_2^{ij} + \sum_{k \geq 1} \epsilon^k \sum_{l=0}^{k+1} A_{(2)k,l}^{ij}(q, q_x, \ldots, q_{(l)})\delta^{(k-l+1)}(x-y)$$

$$-\lambda \left(\omega_1^{ij} + \sum_{k \geq 1} \epsilon^k \sum_{l=0}^{k+1} A_{(1)k,l}^{ij}(q, q_x, \ldots, q_{(l)})\delta^{(k-l+1)}(x-y) \right) \qquad (1.1)$$

obtained via a bi-Hamiltonian deformation procedure from the dispersionless limit

$$\omega_2^{ij} - \lambda \omega_1^{ij} = g_{(2)}^{ij}\delta'(x-y) + \Gamma_{(2)k}^{ij} q_x^k \delta(x-y) - \lambda \left(g_{(1)}^{ij}\delta'(x-y) + \Gamma_{(1)k}^{ij} q_x^k \delta(x-y) \right),$$

a so called Poisson pencil of hydrodynamic type. In their classification scheme, PDEs related via Miura transformations are considered equivalent, namely two pencils of the form (1.1) are declared equivalent if they are obtained one from the other via a Miura transformation:

$$\tilde{q}^i = F_0^i(q) + \sum_{k \geq 1} \epsilon^k F_k(q, q_x, \ldots, q_{(k)}), \qquad \det \frac{\partial F_0^i}{\partial q^j} \neq 0, \ \deg F_k^i = k. \qquad (1.2)$$

where, by definition $\deg(f(q)) = 0$ and $\deg(q_{(k)}) = k$. According to the theory, as further developed by Dubrovin, Liu and Zhang, once two compatible Poisson structures of hydrodynamic type ω_1 and ω_2 are chosen, equivalence classes of Poisson pencils are labelled by n functions called *central invariants*. This set of invariants arise in the study of certain cohomology groups, called *bi-Hamiltonian cohomology groups* associated to the Poisson bivectors ω_1 and ω_2. In order to define these cohomology groups one has to consider a double differential complex on the Grasmann algebra of multivector fields on the formal loop space $\mathcal{L}(\mathbb{R}^n)$. The two differentials of the complex, denoted by d_{ω_1} and d_{ω_2}, are defined by

$$d_{\omega_1} := [\omega_1, \cdot] : \Lambda^k \to \Lambda^{k+1} \qquad (1.3)$$

$$d_{\omega_2} := [\omega_2, \cdot] : \Lambda^k \to \Lambda^{k+1}, \qquad (1.4)$$

where the square bracket is the Schouten bracket and Λ^k is the space of k-vector fields. The Jacobi conditions

$$[\omega_1, \omega_1] = 0, \qquad [\omega_2, \omega_2] = 0$$

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in this framework read

$$d_{\omega_1}^2 = 0, \qquad d_{\omega_2}^2 = 0.$$

Moreover due to the compatibility of ω_1 and ω_2:

$$[\omega_1, \omega_2] = 0,$$

that is the differentials d_ω and d_{ω_2} anticommute. For much more information about these constructions see [15].

Since the deformed pencil (1.1) is obtained from the pencil of hydrodynamic type via a complicated recursive procedure, in general it is very difficult to obtain information about the various orders of the deformation, and it is even more challenging to get insight about the complete deformed pencil at all orders in ϵ. For instance, even the very existence of the deformation to all orders in ϵ has not been completely established in all cases (the existence of the deformed hierarchy and of one of the correspoding Poisson brackets has been solved with some additional assumptions coming from Gromov-Witten theory in [4, 5]).

The main contribution of our paper to this area is to look for those properties of the complete deformed pencil that are inherited through the interplay between special structural properties of the dispersionless limit (like being exact or homogeneous) and conditions on the central invariants (for instance a finiteness condition like being polynomial). This will enable us to prove some results for the deformed pencil *to all orders in ϵ* and to get some specific pieces of information about the tensorial component appearing in each order in ϵ (this component is just $A_{(2)k,0}^{ij}(q, q_x, \ldots, q_{(l)})$, namely the coefficient of the highest order derivative of the δ).

The key observation to implement this idea is to notice that, in general, those structural properties of the pencil ω_λ of hydrodynamic type that have a counterpart at the level of the bi-differential complex, combined with suitable conditions on the central invariants, give rise to special structures in the deformed pencil, that are natural to call *inherited structures*. It is not possible to expect that these structures depend only on the dispersionless pencil, because the isomorphism class (with respect to Miura transformations) of the fully deformed pencil depends on the choice of central invariants. For instance, in the situation in which the pencil ω_λ is exact, the choice of *constant* central invariants make the fully deformed pencil *exact* (see [17] and Theorem 6 for a generalization). Without this suitable assumption on the central invariants, exactness does not carry over to the fully deformed pencil.

In the present paper we will focus our attention to two cases, that appear to be particularly relevant due to their appearance in the framework of Frobenius manifolds and we will be mainly concerned with *polynomial* central invariants.

The two cases are as follows:

1. The Poisson pencil $\omega_\lambda = \omega_2 - \lambda\omega_1$ is *exact*. In this case there exists a vector field e (sometimes called Liouville vector field) such that

$$\text{Lie}_e\omega_1 \;=\; 0 \tag{1.5}$$

$$\text{Lie}_e\omega_2 \;=\; \omega_1. \tag{1.6}$$

At the level of the double differential complex the above properties imply

$$\text{Lie}_e\, d_{\omega_1} - d_{\omega_1}\text{Lie}_e \;=\; 0 \tag{1.7}$$

$$\text{Lie}_e\, d_{\omega_2} - d_{\omega_2}\text{Lie}_e \;=\; d_{\omega_1}. \tag{1.8}$$

Combining (1.7) and (1.8) one obtains immediately

$$\text{Lie}_e d_{\omega_1}\, d_{\omega_2} - d_{\omega_1}\, d_{\omega_2}\text{Lie}_e = 0. \tag{1.9}$$

2. The Poisson pencil $\omega_\lambda = \omega_2 - \lambda\omega_1$ is *homogeneous*. In this case there exists a vector field E such that

$$\text{Lie}_E\omega_1 \;=\; (d-2)\omega_1 \tag{1.10}$$

$$\text{Lie}_E\omega_2 \;=\; (d-1)\omega_2. \tag{1.11}$$

The properties (1.10) and (1.11) imply

$$\text{Lie}_E d_{\omega_1} - d_{\omega_1}\text{Lie}_E \;=\; (d-2)d_{\omega_1} \tag{1.12}$$

$$\text{Lie}_E d_{\omega_2} - d_{\omega_2}\text{Lie}_E \;=\; (d-1)d_{\omega_2}. \tag{1.13}$$

Combining (1.12) and (1.13) one gets

$$\text{Lie}_E d_{\omega_1}\, d_{\omega_2} - d_{\omega_1}\, d_{\omega_2}\text{Lie}_E = (2d-3)d_{\omega_1}\, d_{\omega_2}. \tag{1.14}$$

As it was hinted above, part of the importance of these two cases stems from the fact that both instances arise in the theory of Frobenius manifolds. In this framework e is the *unity* vector field, E is the *Euler* vector field and d is the *charge* of the Frobenius manifold. More about this case will be presented in Section 2 and 6.

The paper is organized as follows. In Section 2 we recall the main properties of the pencils of hydrodynamic type we will consider, namely those that are semisimple and bi-Hamiltonian. In particular, we present how the two structural properties of exactness and homogeneity appear at the level of the bi-differential complex and their relation with the theory of Frobenius manifolds.

In Section 3 the definition of central invariants is recalled and worked out in some examples. In particular, we recall the case of the r-KdV-CH hierarchy since it includes all known cases with non constant central invariants, and we are going

to apply some of our results to it. In Section 4 we briefly recall the notion of bi-Hamiltonian cohomology group, their significance in controlling the deformations of Poisson pencils and the fact that for a pencil like (1.1) the deformation of ω_1 can always be eliminated.

In Section 5, we study those properties of the deformed pencil that are inherited from the exactness of the dispersionless limit, we work out some examples and we show how to apply our results to the case of the r-KdV-CH hierarchy with polynomial central invariants. We also show that deformations of exact Poisson pencils of hydrodynamic type with polynomial central invariants can be put, via a Miura transformation, in a special form, that we call normal form. In particular, this provides us with a tool to map a Poisson pencil with polynomial central invariants of a given degree to a Poisson pencil with constant central invariants to *all orders in ϵ*.

In Section 6 we study deformations of homogeneous Poisson pencils of hydrodynamic type. We call homogenous a deformation P_λ of a homogenous Poisson pencil of hydrodynamic type if its central invariants are homogenous functions of the *same* degree D in the canonical coordinates. In particular we prove that such a homogenous Poisson pencil P_λ can be always reduced by a Miura transformation to a Poisson pencil Q_λ of the form

$$Q_\lambda = \omega_2 + \sum_{k=1}^{\infty} \epsilon^{2k} Q_2^{(2k)} - \lambda \omega_1$$

such that

$$\mathrm{Lie}_E Q_2^{(2k)} = [(k+1)(d-1) + kD] Q_2^{(2k)}, \qquad k = 1, 2, \ldots$$

where E is the Euler vector field. In particular, this allows us to predict exactly the homogeneity degree of the tensorial component in each term in ϵ, prediction that we test on a nontrivial example.

The short Section 7 provides some perspectives for future work.

2 The dispersionless case

Consider a semisimple bi-Hamiltonian structure of hydrodynamic type $\omega_\lambda^{ij} := \omega_2^{ij} - \lambda \omega_1^{ij}$, where

$$\omega_A^{ij} = g^{ij}(q)_A \delta'(x - y) + \Gamma_{A,k}^{ij}(q) q_x^k \delta(x - u), \quad A = 1, 2.$$

This means that

1. Each ω_A^{ij} defines a Poisson bivector, which on the other hand, under the assumption $\det(g_A^{ij}) \neq 0$ is equivalent to g_A^{ij} being flat and $\Gamma_{A,k}^{ij} = -g_A^{il} \Gamma_{A,lk}^j$, where $\Gamma_{A,lk}^j$ are the Christoffel symbols of $g_{A,ij}$ (see [14]).

2. The Riemann tensor of the pencil $g_\lambda^{ij} = g_2^{ij} - \lambda g_1^{ij}$ vanishes for any value of λ and the Christoffel symbols $\Gamma_{\lambda,k}^{ij}$ of the pencil g_λ^{ij} are $\Gamma_{(2),k}^{ij} - \lambda \Gamma_{(1),k}^{ij}$ (this is equivalent to the two Poisson structures ω_1 and ω_2 being compatible and thus defining a bi-Hamiltonian structure, see [11])

3. The roots $u^1(q), \ldots, u^n(q)$ of the characteristic equation $\det g_\lambda = \det(g_2 - \lambda g_1) = 0$ are functionally independent (this is the condition of g_λ being semisimple).

Due to the fact that the bi-Hamiltonian structures we are dealing with are semisimple, we can re-express the quantities involved in term of the functions $u^i(q)$ which are called *canonical coordinates*. It can be proved that both metrics g_1 and g_2 are in diagonal form using canonical coordinates [18]:

$$g_1^{ij} = f^i(u)\delta_{ij}, \quad g_2^{ij} = u^i f^i(u)\delta_{ij}. \tag{2.1}$$

A special class of Poisson pencils ω_λ is given by *exact Poisson pencil*. In this case we have the following theorems

Theorem 1. *[17] A semisimple bi-Hamiltonian structure of hydrodynamic type is exact if and only if the functions $f^i(u)$ in (2.1) satisfy the condition*

$$\sum_{k=1}^{n} \frac{\partial f^i(u)}{\partial u^k} = 0, \quad i = 1, \ldots, n \tag{2.2}$$

Moreover, in canonical coordinates all the components of the vector field e are equal to 1.

How the fact that the Poisson pencil is exact translates at the level of cohomology operators is spelled out in the following:

Theorem 2. *Let d_{ω_1} and d_{ω_2} be the Poisson cohomology differentials associated to a semisiple exact Poisson pencil ω_λ. Then the relations (1.7) and (1.8) are satisfied.*

Proof: Let Λ be a k-multivector. Then $(\mathrm{Lie}_e \circ d_{\omega_1})(\Lambda) = [e, [\omega_1, \Lambda]]$, where $[\cdot, \cdot]$ denotes the Schouten-Nijenhuis bracket. By the graded Jacobi identity, the following identity holds:

$$(-1)^{k-1}[e, [\omega_1, \Lambda]] + [\omega_1, [\Lambda, e]] + (-1)^k[\Lambda, [e, \omega_1]] = 0.$$

Since $\mathrm{Lie}_e \omega_1 = 0$, and $[\Lambda, e] = (-1)^k[e, \Lambda]$, the graded Jacobi identity reduces to $(-1)^{k-1}[e, [\omega_1, \Lambda]] + (-1)^k[\omega_1, [e, \Lambda]] = 0$, which is exactly (1.7). For (1.8) the proof is entirely analogous, just observe that in this case the term $[e, \omega_2] = \omega_1$ and $[\Lambda, \omega_1] = (-1)^{2k}[\omega_1, \Lambda]$. $\blacksquare$

Relation (1.7) means that Lie_e is a d_{ω_1}-chain map, so it descends to a map in the d_{ω_1} Poisson cohomology, while equation (1.8) intertwines the action of d_{ω_1} and d_{ω_2} via Lie_e. In particular, it is immediate to see that (1.8) implies that Lie_e descends to a map at the level of the bi-Hamiltonian cohomology groups. It is also true that Lie_e induces maps $H^*(d_{\omega_2})$ to $H^*(d_{\omega_1})$, however as such this is not interesting, since it is already known that $H^*(d_{\omega_i}) = 0$ $i = 1, 2$.

An other case we are going to deal with is a special case of homogeneous Poisson pencils defined by (1.10) and (1.11). We will additionally assume that the pencil g_λ is semisimple and that, in canonical coordinates, the Euler vector field E is given by the formula

$$E = \sum_{i=1}^{n} u^i \frac{\partial}{\partial u^i}. \tag{2.3}$$

The motivation for this assumption comes from the theory of Frobenius manifolds. In such a context we have two flat metrics g_1 and g_2 satisfying the properties

$$\mathrm{Lie}_E g_1^{ij} = (d-2)g_1^{ij} \tag{2.4}$$

$$\mathrm{Lie}_E g_2^{ij} = (d-1)g_2^{ij}. \tag{2.5}$$

Moreover the contravariant components of the second metric, the so-called *intersection form*, are given by

$$g_2^{ij} = g_1^{il} c_{lk}^j E^k \tag{2.6}$$

where c_{lk}^j are the structure constants defining the Frobenius algebra on the tangent spaces. The special form of the Euler vector field in canonical coordinates follows immediately from the formula (2.6) and the semisimplicity assumption. From (2.4) and (2.5) it follows that in canonical coordinates the contravariant components of the metric g_1 are homogeneous functions of degree d and the contravariant components of the metric g_2 are homogenous functions of degree $d+1$. We show now that the Poisson pencil associated to such metrics is homogeneous. Indeed:

$$\mathrm{Lie}_E \omega_1^{ij} =$$

$$\sum_{k,s} \left(\partial_x^s E^k(u(x), \dots) \frac{\partial \omega_1^{ij}}{\partial u_{(s)}^k(x)} - \frac{\partial E^i(u(x), \dots)}{\partial u_{(s)}^k(x)} \partial_x^s \omega_1^{kj} - \frac{\partial E^j(u(y), \dots)}{\partial u_{(s)}^k(y)} \partial_y^s \omega_1^{ik} \right) =$$

$$\sum_k u^k \frac{\partial \omega_1^{ij}}{\partial u^k} + \sum_k u_x^k \frac{\partial \omega_1^{ij}}{\partial u_x^k} - 2\omega_1^{ij} =$$

$$(d-2)g_1^{ij}\,\delta'(x-y) + \sum_k u^k \frac{\partial \Gamma_{(1)l}^{ij}}{\partial u^k} u_x^l\,\delta(x-y) - \sum_k \Gamma_{(1)k}^{ij} u_x^k\,\delta(x-y) =$$

$$(d-2)\left[g_1^{ij}\,\delta'(x-y) + \sum_k \Gamma_{(1)k}^{ij} u_x^k\,\delta(x-y) \right] = (d-2)\omega^{ij}$$

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where, in the last identity, we used the fact that the Christoffel symbols $\Gamma^{ij}_{(1)k}$ are homogeneous functions of degree $d-1$. Similarly using the fact that the Christoffel symbols $\Gamma^{ij}_{(2)k}$ are homogeneous functions of degree d one obtains (1.11). The identities (1.12) and (1.13) can be easily proved using graded Jacobi identity as in Theorem 2.

3 Central invariants

In the semisimple case [22] (that is if ω_λ is semisimple) equivalence classes of equivalent Poisson pencil of the form (1.1) are labelled by n functional parameters called *central invariants*. More precisely two pencils having the same leading order are Miura equivalent if and only if they have the same central invariants. The problem of costructing a pencil for a given choice of the leading term ω_λ and of the central invariants has been solved only in certain cases. In general, as observed in the Introduction, even to prove the existence of the pencil is a non trivial problem. The central invariants are defined as

$$c_i = -\frac{1}{3f^i}\operatorname{Res}_{\lambda=u^i}\operatorname{Tr} g_\lambda^{-1}A_\lambda \tag{3.1}$$

where the tensor A^{ij} is defined by

$$A_\lambda^{ij} = A_{(\lambda)2,0}^{ij} + (g_\lambda^{-1})_{lk}A_{(\lambda)1,0}^{li}A_{(\lambda)1,0}^{kj}.$$

with

$$A_{(\lambda)2,0}^{ij} = A_{(2)2,0}^{ij} - \lambda A_{(1)2,0}^{ij}, \qquad A_{(\lambda)1,0}^{ij} = A_{(2)1,0}^{ij} - \lambda A_{(1)1,0}^{ij}.$$

It turns out [22, 12] that the central invariants c_i depend only on the canonical coordinates u^i and are given by the following expression:

$$c_i(u^i) = \frac{1}{3(f^i)^2}\left(Q_2^{ii} - u^i\,Q_1^{ii} + \sum_{k\neq i}\frac{(P_2^{ki} - u^i\,P_1^{ki})^2}{f^k(u^k - u^i)}\right), \quad i = 1,\ldots,n. \tag{3.2}$$

where P_1^{ij}, P_2^{ij}, Q_1^{ij}, Q_2^{ij} are the components of the tensor fields $A_{2,0}^{(1)ij}$, $A_{2,0}^{(2)ij}$, $A_{3,0}^{(1)ij}$, $A_{3,0}^{(2)ij}$ in canonical coordinates. In particular, each central invariant c_i is a scalar function of only the canonical coordinate u^i.

Now we show how the definition of central invariants works in a couple of examples:

AKNS. Let us consider the Poisson pencil $\omega_2 + \epsilon P_2^{(1)} - \lambda\omega_1$ with

$$\omega_2 + \epsilon P_2^{(1)} - \lambda\omega_1 = \begin{pmatrix} (2u\partial_x + u_x)\delta & v\delta' \\ \partial_x(v\delta) & -2\delta' \end{pmatrix} + \epsilon\begin{pmatrix} 0 & -\delta'' \\ \delta'' & 0 \end{pmatrix} - \lambda\begin{pmatrix} 0 & \delta' \\ \delta' & 0 \end{pmatrix} \tag{3.3}$$

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where, to keep formulas short, we write δ instead of $\delta(x - y)$. This is the Poisson pencil of the so-called AKNS (or two-boson) hierarchy.

In this case

$$g_\lambda = \begin{pmatrix} 2u & v - \lambda \\ v - \lambda & -2 \end{pmatrix}.$$

After some computations we get $A_\lambda = \frac{g_\lambda}{\det g_\lambda}$ and therefore, taking into account that

$$u^1 = v + \sqrt{-4u}, \ u^2 = v - \sqrt{-4u}. \qquad f^1 = \frac{8}{u_2 - u_1}, \ f^2 = \frac{8}{u_1 - u_2},$$

using formula (3.1) we obtain

$$
\begin{aligned}
c_1 &= -\frac{1}{3f^1} \mathrm{Res}_{\lambda = u^1} \mathrm{Tr}\, g_\lambda^{-1} A_\lambda = -\frac{1}{3f^1} \mathrm{Res}_{\lambda = u^1} \frac{2}{\det g_\lambda} = -\frac{1}{12} \\
c_2 &= -\frac{1}{3f^2} \mathrm{Res}_{\lambda = u^2} \mathrm{Tr}\, g_\lambda^{-1} A_\lambda = -\frac{1}{3f^2} \mathrm{Res}_{\lambda = u^2} \frac{2}{\det g_\lambda} = -\frac{1}{12}
\end{aligned}
$$

Two component CH. Moving $P_2^{(1)}$ from P_2 to P_1 in the Poisson pencil of the AKNS hierarchy one obtains the following Poisson pencil [16, 22]

$$P_\lambda = \begin{pmatrix} (2u\partial_x + u_x)\delta & v\delta' \\ \partial_x(v\delta) & -2\delta' \end{pmatrix} - \lambda \begin{pmatrix} 0 & \delta' - \epsilon\delta'' \\ \delta' + \epsilon\delta'' & 0 \end{pmatrix} \qquad (3.4)$$

which is the Poisson pencil that identifies the so called CH_2 hierarchy. The pencil g_λ and the canonical coordinates are the same as in the previous example, while $A_\lambda = \frac{\lambda^2 g_\lambda}{\det g_\lambda}$. Using formula (3.1) we obtain

$$
\begin{aligned}
c_1 &= -\frac{1}{3f^1} \mathrm{Res}_{\lambda = u^1} \mathrm{Tr}\, g_\lambda^{-1} A_\lambda = -\frac{1}{3f^1} \mathrm{Res}_{\lambda = u^1} \frac{2\lambda^2}{\det g_\lambda} = -\frac{(u^1)^2}{12} \\
c_2 &= -\frac{1}{3f^2} \mathrm{Res}_{\lambda = u^2} \mathrm{Tr}\, g_\lambda^{-1} A_\lambda = -\frac{1}{3f^2} \mathrm{Res}_{\lambda = u^2} \frac{2\lambda^2}{\det g_\lambda} = -\frac{(u^2)^2}{12}.
\end{aligned}
$$

Both the above examples are special cases of the r-KdV-CH hierarchy.

r-KdV-CH hierarchy. The r-KdV-CH hierarchy is an encompassing generalization of the Kortweg-de-Vries and Camassa-Holm hierarchies parameterized by $r + 1$ constants. It has been introduced by Antonowicz and Fordy in [1] and [2] (see also [24]) and further studied by Chen, Liu and Zhang ([8]). It appears that the only known bi-Hamiltonian hierarchies with non-constant central invariants are special cases of the r-KdV-CH hierarchy. For details about the r-KdV-CH hierarchy we refer to [8], here we just focus our attention on its bi-Hamiltonian structure (it is actually multi-Hamiltonian) and how our results can be applied in this case.

Fix $r + 1$ constants $(a_0, a_1, \ldots, a_r) \in (\mathbb{R}^{r+1})^*$, so $(a_0, \ldots, a_r) \neq (0, \ldots, 0)$, and coordinates $w^0, \ldots, w^{r-1}$ on a manifold M, which is considered the target manifold for the loop space. The r-KdV-CH hierarchy is endowed with $r + 1$ mutually compatible Hamiltonian structures

$$(P_m)^{ij} = f_m^{ij} \mathcal{D}_{i+j+1-m}, \quad i, j = 0, \ldots, r-1, \quad m = 0, \ldots, r \qquad (3.5)$$

where

$$\mathcal{D}_i = 2w^i \partial_x + w_x^i - \frac{\epsilon^2}{2} a_i \partial_x^3, \quad w^r := 1, \quad w^i = a_i := 0, \quad \text{for } i < 0, \ i > r$$

and where furthermore

$$f_m^{kl} := \begin{cases} 1 & l < m \text{ and } k < m \\ -1 & l \geq m \text{ and } k \geq m \\ 0 & \text{in all other cases} \end{cases}$$

Among the mutually compatible Hamiltonian structures P_m, $m = 0, \ldots, r$, we focus our attention on two of them, say P_k and P_l. Their dispersionless limit is given by

$$Q_b^{ij} = P_{b,|\epsilon=0}^{ij} = f_b^{ij}(2w^{i+j+1-b}\partial_x + w_x^{i+j+1-b}), \quad b = k, l.$$

To study the dispersionless bi-Hamiltonian structure (Q_k, Q_l) one introduces the following coordinates. Consider the polynomial $P(\lambda) = \lambda^r + w^{r-1}\lambda^{r-1} + \cdots + w^0$. If its roots λ_i are pairwise distinct, which is equivalent to $P'(\lambda_i) \neq 0$, then they can be used as a local system of coordinates, in place of w^i. Let us remind that Chen, Liu and Zhang (see [8]) proved that in this case the bi-Hamiltonian structure (Q_k, Q_l) is semisimple and has canonical coordinates $u^i := (\lambda_i)^{l-k}$, $i = 1, \ldots, r$ and that in these coordinates, the associated metrics g_k g_l have non-zero components given by

$$g_k^{ii} = -2(k-l)^2 \frac{\lambda_i^{2l-k-2}}{P'(\lambda_i)}, \quad g_m^{ii} = -2(k-l)^2 \frac{\lambda_i^{3l-2k-2}}{P'(\lambda_i)}, \quad i = 1, \ldots, r.$$

Although the computation of the central invariants $c^i(u^i)$ for the pair (P_k, P_l) has appeared in Chen, Liu and Zhang, for convenience of the reader and since this example will be worked out later we report here the details of such a computation. Given $P_b = Q_b(w) + \epsilon^2(E(w)_b \partial_x^3 + \ldots)$, $b = k, l$ as above, the i-th central invariant c^i which is just a function of the canonical coordinate u^i is given by

$$c^i(u^i) = \frac{E^{ii}(u)_l - u^i E^{ii}(u)_k}{3(g^{ii}(u)_k)^2}. \qquad (3.6)$$

Equivalently, since $u^i = (\lambda_i)^{l-k}$, we can express (3.6) as a function of λ_i. Now $E_m^{ij}(w) = -\frac{1}{2} f_m^{ij} a_{i+j+1-m}$ and therefore

$$E_m^{ii}(\lambda) = -\frac{1}{2} \sum_{k,l=0}^{r-1} f_m^{kl} a_{k+l+1-m} \frac{\partial \lambda_i}{\partial w^k} \frac{\partial \lambda_i}{\partial w^l},$$

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and using the definition of f_m^{ij} this is equal to

$$E_m^{ii}(\lambda) = -\frac{1}{2}\sum_{k,l=0}^{m-1} a_{k+l+1-m}\frac{\partial\lambda_i}{\partial w^k}\frac{\partial\lambda_i}{\partial w^l} + \frac{1}{2}\sum_{k,l=m}^{r-1} a_{k+l+1-m}\frac{\partial\lambda_i}{\partial w^k}\frac{\partial\lambda_i}{\partial w^l}. \qquad (3.7)$$

Moreover, the following relation holds:

$$\frac{\partial\lambda_i}{\partial w^k} = -\frac{\lambda_i^k}{P'(\lambda_i)}. \qquad (3.8)$$

Indeed, since $P(\lambda) = \lambda^r + w^{r-1}\lambda^{r-1} + \cdots + w^0 = \prod_{i=1}^r(\lambda - \lambda_i)$, we have

$$\frac{\partial P(\lambda)}{\partial w_k} = \lambda^k = -\sum_l\left(\prod_{j\neq l}(\lambda - \lambda^j)\frac{\partial\lambda_l}{\partial w_k}\right).$$

Evaluating this last expression at $\lambda = \lambda_i$ we get immediately $\lambda_i^k = -\prod_{j\neq i}(\lambda_i-\lambda_j)\frac{\partial\lambda_i}{\partial w_k}$ and so

$$\frac{\partial\lambda_i}{\partial w_k} = -\frac{\lambda_i^k}{\prod_{j\neq i}(\lambda_i - \lambda_j)}.$$

Since $P'(\lambda) = \sum_l\left(\prod_{j\neq l}(\lambda - \lambda_j)\right)$, we arrive at equation (3.8). Substituting (3.8) into (3.7) we arrive at the expression

$$E_m^{ii}(\lambda) = -\frac{1}{2}\frac{1}{(P'(\lambda_i))^2}\left(\sum_{k,l=0}^{m-1}\lambda_i^{k+l}a_{k+l+1-m} - \sum_{k,l=m}^{r-1}\lambda_i^{k+l}a_{k+l+1-m}\right) \qquad (3.9)$$

Defining the polynomial $\mathfrak{a}(\lambda) := a_0 + a_1\lambda + \cdots + a_r\lambda^r$, rearranging the sums in the right hand side of (3.9), it is not difficult to see that

$$E_m^{ii}(\lambda) = \frac{\lambda_i^m\mathfrak{a}'(\lambda_i) - m\lambda_i^{m-1}\mathfrak{a}(\lambda_i)}{2(P'(\lambda_i))^2}. \qquad (3.10)$$

Now observe that

$$E_m^{ii}(u) = \left(\frac{\partial u^i}{\partial\lambda_i}\right)^2 E_m^{ii}(\lambda),$$

so since $u^i = (\lambda_i)^{l-k}$ we get

$$E_m^{ii}(\lambda_i(u^i)) = (l - k)^2\lambda^{2(l-k-1)}\frac{\lambda_i^m\mathfrak{a}'(\lambda_i) - m\lambda_i^{m-1}\mathfrak{a}(\lambda_i)}{2(P'(\lambda_i))^2}. \qquad (3.11)$$

To focus on the central invariants for the pair (P_k, P_l) so let us assume $l > k$ and use formula (3.6) to compute the central invariants; using (3.11) and the fact that $u^i = (\lambda_i)^{l-k}$ we get:

$$c^i(u^i) = \frac{E_l^{ii} - \lambda_i^{l-k}E_k^{ii}}{3(g_k^{ii})^2} = \frac{\lambda_i^{1-l}\mathfrak{a}(\lambda_i)}{24(k-l)} = \frac{(u^i)^{\frac{1-l}{l-k}}\mathfrak{a}((u^i)^{\frac{1}{l-k}})}{24(k-l)}.$$

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Let us remark that the central invariants $c^i(u^i)$ for the compatible pair (P_k, P_l) of the r-Kdv-CH hierarchy are rational functions of the canonical coordinates if and only if $l = k + 1$. It is immediate to check that the condition is sufficient. To show that it is necessary, observe that $c^i(u^i)$ are rational if and only if $\frac{1-l}{(l-k)^{r+1}} \in \mathbb{Z}$. If $l = 1$ then the condition is satisfied and $k = 0$. Otherwise, call $q = l - k$, $1 \leq q \leq r$. Now if $q \geq 2$, then $q^{r+1} > |1-r| > |1-l|$ for $2 \leq l \leq r$, and this shows that the ratio we are dealing with can be integer if and only if $q = 1$, which means $l = k + 1$. Moreover, $c^i(u^i)$ are polynomials in the canonical coordinates if and only if $k = 0, l = 1$. In the next few sections we will use the r-KdV-CH hierarchy as an example on which to illustrate some of our results.

4   Bi-Hamiltonian cohomology

For the sake of being reasonably self-contained and to fix notations, in this section we collect some definitions and results about (bi)-Hamiltonian cohomologies and the Dubrovin-Zhang complex (see [15] for full details and proofs). Let g be a flat metric on $\mathbb{R}^n$ and ω be the associated Poisson bivector of hydrodynamic type. In analogy with the case of finite dimensional Poisson manifolds [21] one defines Poisson cohomology groups in the following way:

$$H^j(\mathcal{L}(\mathbb{R}^n), \omega) := \frac{\ker\{d_\omega : \Lambda^j_{\text{loc}} \to \Lambda^{j+1}_{\text{loc}}\}}{\text{im}\{d_\omega : \Lambda^{j-1}_{\text{loc}} \to \Lambda^j_{\text{loc}}\}} \tag{4.1}$$

where $d_\omega := [\omega, \cdot]$ (the square brackets denote the Schouten brackets) and Λ^j_{loc} is the space of local j-multivectors on the loop space $\mathcal{L}(\mathbb{R}^n)$ (see [15] for more details on the definition of this complex). Since Λ^j_{loc} has a natural decomposition in homogenous components which is preserved by d_ω, we have

$$H^j(\mathcal{L}(\mathbb{R}^n), \omega) = \oplus_k H^j_k(\mathcal{L}(\mathbb{R}^n), \omega). \tag{4.2}$$

For Poisson structures of hydrodynamic type, it has been proved in [20] (see also [10] for an independent proof of the cases $n = 1, 2$) that $H^k(\mathcal{L}(\mathbb{R}^n), \omega) = 0$ for $k = 1, 2, \ldots$. The vanishing of these cohomology groups implies that any deformation of a Poisson bivector of hydrodynamic type

$$P^\epsilon = \omega + \sum_{n=1}^\infty \epsilon^n P_n, \tag{4.3}$$

where $P_k \in \Lambda^2_{k+2,\text{loc}}$ can be obtained from ω by performing a Miura transformation.

In order to study deformations of Poisson pencils of hydrodynamic type it is necessary to introduce bi-Hamiltonian cohomology groups [19, 15, 22] They are defined

as

$$H_k^2(\mathcal{L}(\mathbb{R}^n), \omega_1, \omega_2) = \frac{\mathrm{Ker}\left(d_{\omega_1} d_{\omega_2}\big|_{\Lambda_{k,\mathrm{loc}}^1}\right)}{\mathrm{Im}\left(d_{\omega_1}\big|_{\Lambda_{k-2,\mathrm{loc}}^0}\right) \oplus \mathrm{Im}\left(d_{\omega_1}\big|_{\Lambda_{k-2,\mathrm{loc}}^0}\right)}.$$

Liu and Zhang showed that, in the semisimple case,

$$H_k^2(\mathcal{L}(\mathbb{R}^n), \omega_1, \omega_2) = 0 \quad \forall k \neq 2,$$

and that the elements of

$$H_2^2(\mathcal{L}(\mathbb{R}^n), \omega_1, \omega_2)$$

have the form

$$d_2\left(\sum_{i=1}^n \int c^i(u^i) u_x^i \mathrm{log} u_x^i \, dx\right) - d_1\left(\sum_{i=1}^n \int u^i c^i(u^i) u_x^i \mathrm{log} u_x^i \, dx\right) \qquad (4.4)$$

where $c^i(u^i)$ are the central invariants introduced in the previous section. More explicitly, the components of these vector fields, in canonical coordinates, are given by

$$X^i = \sum_{j=1}^n \left[\left(\frac{1}{2}\delta_{ij}\partial_x f^i + A^{ij}\right) c^j u_x^j + (2\delta_{ij}f^i - L^{ij})\partial_x(c^j u_x^j)\right], \ i = 1, \ldots, n. \qquad (4.5)$$

with

$$A^{ij} = \frac{1}{2}\left(\frac{f^i}{f^j}\frac{\partial f^j}{\partial u^i}u_x^j - \frac{f^j}{f^i}\frac{\partial f^i}{\partial u^j}u_x^i\right) \qquad (4.6)$$

$$L^{ij} = \frac{1}{2}\delta_{ij}f^i + \frac{(u^i - u^j)f^i}{2f^j}\frac{\partial f^j}{\partial u^i}. \qquad (4.7)$$

We will use these facts later.

5 Deformations of exact Poisson pencil of hydrodynamic type

In this section we study deformations of exact Poisson pencil of hydrodynamic type. In the case of costant central invariants we have the following theorem [17].

Theorem 3. *Suppose we are given a Poisson pencil*

$$\Pi_\lambda = P_2 - \lambda P_1 = \omega_2 + \sum_{k=1}^\infty \epsilon^{2k} P_2^{(2k)} - \lambda\left(\omega_1 + \sum_{k=1}^\infty \epsilon^{2k} P_1^{(2k)}\right), \qquad (5.1)$$

whose $\epsilon = 0$ limit $\omega_\lambda = \omega_2 - \lambda\omega_1$ is an exact semisimple bi-Hamiltonian structure of hydrodynamic type. Then the central invariants of Π_λ are constant functions of the canonical coordinates if and only if Π_λ is an exact Poisson pencil.

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In particular in the proof of Theorem 3, it is shown that there exists a Miura transformation reducing the Poisson pencil Π_λ (5.1) to the form

$$\Pi_\lambda = \omega_\lambda + \sum_{k=1}^{\infty} \epsilon^{2k} P_2^{(2k)} \tag{5.2}$$

and there exists a Miura transformation preserving ω_1 and reducing the vector field Z involved in the exactness of Π_λ to $e = \sum_{k=1}^{n} \frac{\partial}{\partial u^k}$. In this system of coordinates, the exactness of the pencil ω_λ is expressed as $\mathrm{Lie}_e \omega_2 = \omega_1$ and $\mathrm{Lie}_e \omega_1 = 0$.

Let us also recall that all the terms $P_2^{(2k)}$ in (5.2) are co-boundaries of ω_1, so they can be written as Lie derivatives of ω_1 with respect to certain vector fields. In particular, the first term in the deformation of Π_λ, namely $P_2^{(2)}$ is equal to

$$P_2^{(2)} = \mathrm{Lie}_X \omega_1 \tag{5.3}$$

where the vector field X has components X^i in canonical coordinates given by (4.5)

Moreover, the vector field X explicitly written as a vector field on the loop manifold $\mathcal{L}(\mathbb{R}^n)$ is given by

$$X = \sum_{i=1}^{n} \sum_{s \geq 0} \partial_x^s X^i \frac{\partial}{\partial u_{(s)}^i}.$$

At this point we can prove the following, which generalizes the conditions about the constancy of the central invariants proved in [17]:

Proposition 4. *Let Π_λ be a bi-Hamiltonian structure as in (5.2) whose dispersionless limit ω_λ is an exact semisimple bi-Hamiltonian structure of hydrodynamic type. Then*

$$\max_{i=1,\dots,n} \deg c_i(u) = l \tag{5.4}$$

if and only if the following conditions are satisfied:

$$\mathrm{Lie}_e^{l+1} P_2^{(2)} = 0, \quad \mathrm{Lie}_e^l P_2^{(2)} \neq 0. \tag{5.5}$$

Proof: First we prove that (5.4) implies (5.5). Using the formulas provided in [DZ] to compute $\mathrm{Lie}_e X_{c_1,\dots,c_n}$, where $X_{c_1,\dots,c_n}$ is given in (4.5), we find

$$(\mathrm{Lie}_e X_{c_1,\dots,c_n})^i = [e, X_{c_1,\dots,c_n}]^i = \sum_{h=1}^{n} \frac{\partial X_{c_1,\dots,c_n}^i}{\partial u^h}.$$

Call D_e the differential operator $\sum_{h=1}^{n} \frac{\partial \cdot}{\partial u^h}$ acting on the components X^i given by the formula (4.5). Since by the exactness of the pencil ω_λ we have that $D_e(f^i) = 0$ ((2.2)), it is immediate to check that similarly $D_e(A^{ij}) = D_e(L^{ij}) = 0$. Moreover, D_e and ∂_x commute, therefore when D_e acts on X^i is just acting on the central invariants $c^i(u)$. This reasoning can be repeated for any iteration of the Lie derivative Lie_e; in

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particular the i-th component of the k-th iterated Lie derivative $[\mathrm{Lie}_e^k(X)]^i$ is just given by the action of the differential operator D_e^k on X^i. So using the notation just introduced, we have that

$$[\mathrm{Lie}_e^k X_{c_1,\ldots,c_n}]^i = X^i_{D_e^j(c_1),\ldots,D_e^j(c_n)}.$$

Therefore, using (1.7), we obtain

$$\mathrm{Lie}_e^l P_2^{(2)} = \mathrm{Lie}_e^l d_{\omega_1} X_{c_1,\ldots,c_n} = d_{\omega_1} \mathrm{Lie}_e^l X_{c_1,\ldots,c_n} = d_{\omega_1} X_{D_e^l(c_1),\ldots,D_e^l(c_n)} \neq 0$$
$$\mathrm{Lie}_e^{l+1} P_2^{(2)} = \mathrm{Lie}_e^{l+1} d_{\omega_1} X_{c_1,\ldots,c_n} = d_{\omega_1} \mathrm{Lie}_e^{l+1} X_{c_1,\ldots,c_n} = d_{\omega_1} X_{D_e^{l+1}(c_1),\ldots,D_e^{l+1}(c_n)} = 0.$$

Now if (5.4) is satisfied, then $D_e^{l+1}(c_i) = 0$, $\forall i$ and at least one of the functions $D_e^l(c_i)$ does not vanish. The theorem follows from the observation that kernel of the map $(f_1,\ldots,f_n) \to X_{f_1,\ldots,f_n}$ is trivial. ∎

The next Theorem provides a first example on how it is possible to transfer information from $P_2^{(2)}$ to all orders in ϵ, if certain conditions are fulfilled.

Theorem 5. *Let* $\Pi_\lambda = \omega_\lambda + \sum_{k=1}^\infty \epsilon^{2k} P_2^{(2k)}$ *be as in (5.2), where* ω_λ *is an exact semisimple bi-Hamiltonian structure of hydrodynamic type. Call* $P_2 := \omega_2 + \sum_{k=1}^\infty \epsilon^{2k} P_2^{(2k)}$. *If* $\mathrm{Lie}_e^n P_2^{(2)} = 0$ *and if* $\mathrm{Lie}_e^{n-1} P_2$ *is a Poisson tensor, then there exists a Miura transformation*

$$\Pi_\lambda = \omega_\lambda + \sum_{k=1}^\infty \epsilon^{2k} P_2^{(2k)} \to \tilde{\Pi}_\lambda = \omega_\lambda + \sum_{k=1}^\infty \epsilon^{2k} \tilde{P}_2^{(2k)}$$

with

$$\mathrm{Lie}_e^n \tilde{P}_2^{(2k)} = 0, \qquad k = 2,3,\ldots$$

Before proving Theorem 5, let us observe the following. In Theorem 5, for $n = 1$ the extra requirement about $\mathrm{Lie}_e^{n-1} P_2$ being Poisson is automatically satisfied (P_2 is indeed a Poisson tensor). In this form the Theorem has been proved in [17] and it states that if the central invariants are constants, then the *entire* pencil Π_λ is exact, and not just its dispersionless limit. For $n = 2$, the Theorem requires the additional assumption that $\mathrm{Lie}_e P_2$ is Poisson. This is in general difficult to satisfy, although it is exactly what happens in the case of the single component Camassa-Holm equation (see next example). In general, the case for $n \geq 3$ requires the additional condition on $\mathrm{Lie}_e^{n-1} P_2$ being Poisson due to the growth of the coefficients in $P_2^{(2k)}$. As we will see in Lemma 6, it is possible to drop this additional condition increasing as a function of k the number of iterations of Lie_e acting on $P_2^{(2k)}$.

Proof: The proof is by induction. Assume $\mathrm{Lie}_e^n P_2^{(2k)} = 0$, for $k = 1,\ldots,N$. Than we show that $\mathrm{Lie}_e^n \tilde{P}_2^{(2N+2)} = 0$, where $\tilde{P}_2^{(2N+2)}$ is obtained from $P_2^{(2N+2)}$ using

a suitable Miura transformation. By assumption $\mathrm{Lie}_e^{n-1} P_2$ is Poisson, which means $[\mathrm{Lie}_e^{n-1} P_2, \mathrm{Lie}_e^{n-1} P_2] = 0$ or equivalently

$$d_{\omega_2} \mathrm{Lie}_e^{n-1} P_2^{(2l+2)} = -\frac{1}{2} \sum_{k=1}^{l} \left[\mathrm{Lie}_e^{n-1} P_2^{(2k)}, \mathrm{Lie}_e^{n-1} P_2^{(2l-2k+2)} \right], \quad l \in \mathbb{N}. \qquad (5.6)$$

Taking (5.6) for $l = N$ and applying Lie_e on both sides, we get

$$\mathrm{Lie}_e d_{\omega_2} \mathrm{Lie}_e^{n-1} P_2^{(2N+2)} = -\frac{1}{2} \sum_{k=1}^{N} \mathrm{Lie}_e \left(\left[\mathrm{Lie}_e^{n-1} P_2^{(2k)}, \mathrm{Lie}_e^{n-1} P_2^{(2N-2k+2)} \right] \right).$$

By (1.7), (1.8) and the fact that $d_{\omega_1} P_2^{(2k)} = 0$ this last expression is equal to

$$d_{\omega_2} \mathrm{Lie}_e^n P_2^{(2N+2)} =$$

$$= -\frac{1}{2} \sum_{k=1}^{N} \left(\left[\mathrm{Lie}_e^n P_2^{(2k)}, \mathrm{Lie}_e^{n-1} P_2^{(2N-2k+2)} \right] + \left[\mathrm{Lie}_e^{n-1} P_2^{(2k)}, \mathrm{Lie}_e^n P_2^{(2N-2k+2)} \right] \right) = 0,$$

by the inductive assumption $\mathrm{Lie}_e^n P_2^{(2k)} = 0$, $k = 1, \ldots, N$. Moreover, by (1.7) it is immediate to see that $d_{\omega_1} \mathrm{Lie}_e^n P_2^{(2N+2)} = 0$. The fact that $\mathrm{Lie}_e^n P_2^{(2N+2)}$ is a cocycle for d_{ω_1} and d_{ω_2} means that $\mathrm{Lie}_e^n P_2^{(2N+2)} = \mathrm{Lie}_{X_2^{(2N+2)}} \omega_1$ for a suitable vector field $X_2^{(2N+2)}$ satisfying $d_{\omega_1} d_{\omega_2} (X_2^{(2N+2)}) = 0$. Due to the triviality of $H_{2N+2}^2(\mathcal{L}(\mathbb{R}^n), \omega_1, \omega_2)$ we have

$$X_2^{(2N+2)} = d_{\omega_1} H_2^{(2N+2)} + d_{\omega_2} K_2^{(2N+2)},$$

for suitable functionals $H_2^{(2N+2)}$ and $K_2^{(2N+2)}$ with densities that are differential polynomials. Now suppose to look for a functional $\tilde{K}_2^{(2N+2)}$ such that $\mathrm{Lie}_e^n \tilde{K}_2^{(2N+2)} = K_2^{(2N+2)}$. This equation with unknown $\tilde{K}_2^{(2N+2)}$ can always be solved and indeed it admits infinitely many solutions (see [17]). Consider now the Miura transformation generated by the vector field $\epsilon^{2N+2} d_{\omega_1} \tilde{K}_2^{(2N+2)}$. This transformation preserves ω_1 since this vector field is a coboundary of ω_1, while P_2 is changed in such a way that:

$$\Pi_\lambda \to \tilde{\Pi}_\lambda = \omega_\lambda + \epsilon^2 P_2^{(2)} + \cdots + \epsilon^{2N+2} \left(P_2^{(2N+2)} + \mathrm{Lie}_{d_{\omega_1} \tilde{K}_2^{(2N+2)}} \omega_2 \right) + O(\epsilon^{2N+4}).$$

Therefore $\tilde{P}_2^{(2N+2)} = \left(P_2^{(2N+2)} + \mathrm{Lie}_{d_{\omega_1} \tilde{K}_2^{(2N+2)}} \omega_2 \right)$. Now it is immediate to check that $\mathrm{Lie}_e^n \tilde{P}_2^{(2N+2)} = 0$. Indeed, we have that

$$\mathrm{Lie}_e^n \tilde{P}_2^{(2N+2)} = \mathrm{Lie}_e^n P_2^{(2N+2)} + \mathrm{Lie}_e^n d_{\omega_2} d_{\omega_1} \tilde{K}_2^{(2N+2)} = \mathrm{Lie}_e^n P_2^{(2N+2)} + d_{\omega_2} d_{\omega_1} K_2^{(2N+2)},$$

due to (1.7), (1.8) and $\mathrm{Lie}_e^n \tilde{K}_2^{(2N+2)} = K_2^{(2N+2)}$. Moreover,

$$d_{\omega_2} d_{\omega_1} K_2^{(2N+2)} = -d_{\omega_1} d_{\omega_2} K_2^{(2N+2)} = -\mathrm{Lie}_{X_2^{(2N+2)}} \omega_1 = -\mathrm{Lie}_e^n P_2^{(2N+2)}$$

and thus $\mathrm{Lie}_e^n \tilde{P}_2^{(2N+2)} = 0$. ∎

Camassa-Holm Consider the Poisson pencil of the Camassa-Holm equation [6]:

$$2m\delta'(x-y) + m_x\delta(x-y) - \lambda(\delta'(x-y) - \delta'''(x-y)).$$

In the coordinate u related to m by the relation $m = u + \epsilon u_x$ this pencil becomes

$$2u\delta'(x-y) + u_x\delta(x-y) - \lambda\delta'(x-y) + P_2$$

with

$$
\begin{aligned}
P_2 &= \sum_{n=1} \epsilon^{2n} \left[\partial_x u\, \delta^{(2n)}(x-y) + \partial_x^{2n} u\, \delta'(x-y) \right] \\
&\quad + \sum_{n=1} \epsilon^{2n+1} \left(\partial_x \left[u\delta^{(2n+1)}(x-y) \right] - \partial_x^{2n+1} \left[u\delta'(x-y) \right] \right).
\end{aligned}
$$

It is easy to check that it satisfy the conditions.

$$\mathrm{Lie}_e\omega_1 = 0 \tag{5.7}$$
$$\mathrm{Lie}_e\omega_2 = \omega_1 \tag{5.8}$$
$$\mathrm{Lie}_e^2 P_2^{(2)} = 0, \tag{5.9}$$

and moreover $\mathrm{Lie}_e P_2$ is Poisson, since $\mathrm{Lie}_e P_2$ is a pencil with constant coefficients which is skew-symmetric. Then by Theorem 5, we conclude that $\mathrm{Lie}_e^2 \tilde{P}_2 = 0$ for a suitable $\tilde{P}_2$ obtained using a Miura transformation. However in this specific case it turns out that $\tilde{P}_2 = P_2$.

Two component CH In general the requirement that $\mathrm{Lie}_e^{n-1} P_2$ is a Poisson tensor seems very restrictive. Let us consider for instance the Poisson pencil (3.4). Performing the Miura transformation

$$u = \xi$$
$$v = \eta + \epsilon\eta_x$$

it becomes

$$
\tilde{P}_\lambda = \begin{pmatrix} (2\xi\partial_x + \xi_x)\delta & (\eta + \epsilon\eta_x)\sum_{k=0}^{\infty} \epsilon^k \delta^{(k+1)} \\ \sum_{k=0}^{\infty}(-1)^k \epsilon^k \partial_x^{k+1}[(\eta + \epsilon\eta_x)\delta] & -2\sum_{k=0}^{\infty} \epsilon^{2k} \delta^{(2k+1)} \end{pmatrix} - \lambda \begin{pmatrix} 0 & \delta' \\ \delta' & 0 \end{pmatrix}.
\tag{5.10}
$$

The first order deformation can be written as

$$
\begin{pmatrix} 0 & \eta\delta'' + \eta_x\delta' \\ -\eta\delta'' - \eta_x\delta' & 0 \end{pmatrix} = -\mathrm{Lie}_X \begin{pmatrix} (2\xi\partial_x + \xi_x)\delta & \eta\delta' \\ \partial_x(\eta\delta) & -2\delta' \end{pmatrix}
$$

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where
$$X = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta \xi} \\ \frac{\delta H}{\delta \eta} \end{pmatrix}, \qquad H = -\int_{S^1} \frac{\eta(x)^3}{12} \, dx.$$

This means that the Miura transformation generated by the vector field X (up to terms of order $\mathcal{O}(\epsilon^3)$) reduces the pencil $\tilde{P}_\lambda$ to the form $\tilde{\tilde{P}}_\lambda =$

$$\begin{pmatrix} (2\xi\partial_x + \xi_x)\delta & \eta\delta' \\ \partial_x(\eta\delta) & -2\delta' \end{pmatrix} - \lambda \begin{pmatrix} 0 & \delta' \\ \delta' & 0 \end{pmatrix} +$$
$$\epsilon^2 \left[\begin{pmatrix} 0 & \eta\delta''' + \eta_x\delta'' \\ \eta\delta''' + 2\eta_x\delta'' + \eta_{xx}\delta' & -2\delta''' \end{pmatrix} + \frac{1}{2}\mathrm{Lie}_X^2 \begin{pmatrix} (2\xi\partial_x + \xi_x)\delta & \eta\delta' \\ \partial_x(\eta\delta) & -2\delta' \end{pmatrix} \right]$$

Since the dispersionless limit of the above pencil coincides with the dispersionless limit of the pencil of the AKNS hierarchy, we have $\mathrm{Lie}_Z \omega_2 = \omega_1$, $\mathrm{Lie}_Z \omega_1 = 0$ with $Z = \frac{\partial}{\partial \eta}$. Moreover it is easy to check that $\mathrm{Lie}_Z^3 P_2^{(2)} = 0$ due to the quadratic dependence of the central invariants on the canonical coordinates. It is also possible to check that the condition $\mathrm{Lie}_Z^3 P_2^{(n)}$ is no longer satisfied by higher order deformations $(n > 2)$ due to the presence of terms as $\eta^n \delta^{(n+1)}$.

The above example shows that in some cases the assumption $\mathrm{Lie}_e^n P_2^{(2)} = 0$ is satisfied while the condition $\mathrm{Lie}_e^{n-1} P_2$ being Poisson is violated. In the remaining part of this section we will discuss what can be said without this last condition.

We have the following

Theorem 6. *Let $\Pi_\lambda = \omega_\lambda + \sum_{k=1}^\infty \epsilon^{2k} P_2^{(2k)}$ be as in (5.2), where ω_λ is an exact semisimple bi-Hamiltonian structure of hydrodynamic type. Call $P_2 := \omega_2 + \sum_{k=1}^\infty \epsilon^{2k} P_2^{(2k)}$. Suppose that $\mathrm{Lie}_e^n P_2^{(2)} = 0$, for some positive integer n. Then there exists a Miura transformation that preserves ω_1 and deforms P_2 to $\tilde{P}_2 = \omega_2 + \sum_{k=1}^\infty \epsilon^{2k} \tilde{P}_2^{(2k)}$ with*
$$\mathrm{Lie}_e^{nk-k+1} \tilde{P}_2^{(2k)} = 0, \quad k = 1, 2, \dots.$$

Proof: For $k = 1$ there is nothing to prove, since this is the hypothesis of the Theorem. By inductive hypothesis we assume to have proved that we can construct at each step Miura transformations such that $\mathrm{Lie}_e^{kn-k+1} P_2^{(2k)} = 0$ for $k = 1, \dots, N$. (We call again the transformed components $P_2^{(2k)}$ although they have been obtained applying Miura transformations.) At the next step, we have $\mathrm{Lie}_e^{(N+1)(n-1)+1} P_2^{(2N+2)} \neq 0$ and we have to show that we can find a Miura transformation preserving ω_1 such that the transformed term $\tilde{P}_2^{(2N+2)}$ satisfies $\mathrm{Lie}_e^{(N+1)(n-1)+1} \tilde{P}_2^{(2N+2)} = 0$.

First we show $d_{\omega_1}(\mathrm{Lie}_e^{(N+1)(n-1)+1} P_2^{(2N+2)}) = 0$. This is immediate since Lie_e commutes with d_{ω_1} and each $P_2^{(2k)}$ is a co-boundary of ω_1 due to compatibility. Next we show $d_{\omega_2}(\mathrm{Lie}_e^{(N+1)(n-1)+1} P_2^{(2N+2)}) = 0$. Using (1.8) and the co-boundary property,

we have that $d_{\omega_2}(\text{Lie}_e^{(N+1)(n-1)+1}P_2^{(2N+2)}) = \text{Lie}_e^{(N+1)(n-1)+1}(d_{\omega_2}P_2^{(2N+2)})$. By the fact that P_2 is Poisson, we have $d_{\omega_2}P_2^{(2N+2)} = -\frac{1}{2}\sum_{k=1}^{N}[P_2^{(2k)}, P_2^{(2(N-k+1))}]$.

It is therefore sufficient to show that $\text{Lie}_e^{(N+1)(n-1)+1}[P_2^{(2k)}, P_2^{(2(N-k+1))}] = 0$. For this, let us observe that

$$\text{Lie}_e^{(N+1)(n-1)+1}[P_2^{(2k)}, P_2^{(2(N-k+1))}] =$$
$$\sum_{j=0}^{(N+1)(n-1)+1}\binom{(N+1)(n-1)+1}{j}[\text{Lie}_e^j P_2^{(2k)}, \text{Lie}_e^{(N+1)(n-1)+1-j}P_2^{(2(N-k+1))}].$$

By inductive hypothesis, $\text{Lie}_e^j P_2^{(2k)} = 0$ for $j \geq kn - k + 1$, while

$$\text{Lie}_e^{(N+1)(n-1)+1-j}P_2^{(2(N-k+1))} = 0$$

for $(N+1)(n-1)+1-j \geq (N-k+1)n-(N-k+1)+1$ or equivalently for $j \leq kn-k$. Therefore each term is zero and we have proved that $d_{\omega_2}(\text{Lie}_e^{(N+1)(n-1)+1}P_2^{(2N+2)}) = 0$.

Let us denote with $\alpha := (N+1)(n-1)+1$. Since $d_{\omega_1}\text{Lie}_e^\alpha P_2^{(2N+2)} = 0$, $\text{Lie}_e^\alpha P_2^{(2N+2)} = d_{\omega_1}X^{(2N+2)}$, for a suitable vector field $X^{(2N+2)}$. Moreover, since $d_{\omega_2}\text{Lie}_e^\alpha P_2^{(2N+2)} = 0$ the vector field $X^{(2N+2)}$ defines a class in the bi-Hamiltonian cohomology group, which in this degree is however trivial. Thus $X^{(2N+2)} = d_{\omega_1}H_2^{(2N+2)} + d_{\omega_2}K_2^{(2N+2)}$, for suitable local functionals $H_2^{(2N+2)}$ and $K_2^{(2N+2)}$. Since $\text{Lie}_e^\alpha P_2^{(2N+2)} = d_{\omega_1}X^{(2N+2)}$, we also have

$$\text{Lie}_e^\alpha P_2^{(2N+2)} = d_{\omega_1}d_{\omega_2}K_2^{(2N+2)}. \tag{5.11}$$

To construct the Miura transformation we are looking for, we proceed as follows. Given the functional $K_2^{(2N+2)}$, we can always find another functional $\tilde{K}_2^{(2N+2)}$ such that $\text{Lie}_e^\alpha \tilde{K}_2^{(2N+2)} = K_2^{(2N+2)}$. Indeed, it has been proved in [17] that the equation $\text{Lie}_e K_1 = K_2^{(2N+2)}$ is always solvable given $K_2^{(2N+2)}$, and therefore, repeating with $\text{Lie}_e K_2 = K_1$, etc. one gets that there exists a local functional $\tilde{K}_2^{(2N+2)}$ with $\text{Lie}_e^\alpha \tilde{K}_2^{(2N+2)} = K_2^{(2N+2)}$.

Now we consider the Miura transformation generated by the vector field $\epsilon^{2N+2}d_{\omega_1}\tilde{K}_2^{(2N+2)}$. Since this is a co-boundary of d_{ω_1}, ω_1 is undeformed, while

$$P_2 \mapsto \tilde{P}_2 = \omega_2 + \sum_{k=1}^{N}\epsilon^{2k}P_2^{(2k)} + \epsilon^{2N+2}(\tilde{P}_2^{(2N+2)}) + O(\epsilon^{(2N+4)}),$$

where

$$\tilde{P}_2^{(2N+2)} = P_2^{(2N+2)} + \text{Lie}_{d_1\tilde{K}_2^{(2N+2)}}\omega_2.$$

Therefore

$$\text{Lie}_e^\alpha \tilde{P}_2^{(2N+2)} = \text{Lie}_e^\alpha P_2^{(2N+2)} + \text{Lie}_e^\alpha(d_{\omega_2}d_{\omega_1}\tilde{K}_2^{(2N+2)}) =$$
$$= d_{\omega_1}d_{\omega_2}K_2^{(2N+2)} + d_{\omega_2}d_{\omega_1}K_2^{(2N+2)} = 0,$$

the first term by equation (5.11) and the second due to the fact that Lie_e^α commutes with $d_{\omega_2}d_{\omega_1}$. ∎

In view of Theorem 6 we introduce the following

Definition 7. *Let ω_1 and ω_2 a pair of exact Poisson structures with $\mathrm{Lie}_e\omega_2 = \omega_1$ and $\mathrm{Lie}_e\omega_1 = 0$. Consider the compatible pencil ω_1 and $P_2 := \omega_2 + \sum_{k=1}^{\infty} \epsilon^{2k} P_2^{2k}$, with $\mathrm{Lie}_e^n P_2^{(2)} = 0$. Then the pair (ω_1, P_2) is said to be in* normal form *if $\mathrm{Lie}_e^{nk-k+1} P_2^{(2k)} = 0$ for all $k \in \mathbb{N}$.*

Theorem 6 combined with Proposition 4 can be reformulated by saying that any Poisson pencil Π_λ with *polynomial* central invariants whose dispersionless limit is an exact Poisson pencil of hydrodynamic type can be put in normal form via a sequence of Miura transformations.

Using these tools we can map a Poisson pencil with central invariants that are polynomials of degree $n-1$ to a pencil with constant central invariants. This map is somehow mysterious since at this point we do not have an interpretation in terms of known transformations, like a combination of Miura and reciprocal transformations.

Theorem 8. *Let (ω_2, ω_1) be an exact bi-Hamiltonian structure with respect to the Liouville vector field e ($\mathrm{Lie}_e\omega_2 = \omega_1$, $\mathrm{Lie}_e\omega_1 = 0$), such that ω_1 and $P_2 = \omega_2 + \sum_{k=1}^{\infty} \epsilon^{2k} P_2^{(2k)}$ are compatible Poisson structures. Furthermore, assume $\mathrm{Lie}_e^n P_2^{(2)} = 0$, $\mathrm{Lie}_e^{n-1} P_2^{(2)} \neq 0$ and that the pair (ω_1, P_2) is in normal form. Then the pair of compatible Poisson structures ω_1 and $Q_2 := \omega_2 + \sum_{k=1}^{n} Q_2^{(2k)}$, where*

$$Q_2^{(2k)} := \frac{\mathrm{Lie}_e^{nk-k} P_2^{(2k)}}{[k(n-1)]!}$$

has constant central invariants and it is also again in normal form.

Proof: First of all, we notice that $\mathrm{Lie}_e Q_2^{(2)} = 0$, so by [17] the pair ω_1 and Q_2 has constant central invariants. Now it remains to prove that the pair ω_1 and Q_2 is a pair of compatible Poisson structures. The compatibility of Q_2 with ω_1 is immediate, we need to prove that Q_2 is Poisson. For this we show that

$$d_{\omega_2} Q_2^{(2N+2)} = -\frac{1}{2} \sum_{k=1}^{N} \left[Q_2^{(2k)}, Q_2^{(2(N-k+1))} \right]. \tag{5.12}$$

Substituting everywhere in (5.12) the expression for $Q_2^{(2l)} = \mathrm{Lie}_e^{nl-l} P_2^{(2l)}$ we find

$$d_{\omega_2} \mathrm{Lie}_e^{(N+1)(n-1)} P_2^{(2(N+1))} = -\frac{1}{2} \sum_{k=1}^{N} \left[\mathrm{Lie}_e^{k(n-1)} P_2^{(2k)}, \mathrm{Lie}_e^{(N-k+1)(n-1)} P_2^{(2(N-k+1))} \right].$$

Since each $P_2^{(2l)}$ is a d_{ω_1} co-boundary, we have we that $d_{\omega_2} \mathrm{Lie}_e^{(N+1)(n-1)} P_2^{(2(N+1))}$ is equal to $\mathrm{Lie}_e^{(N+1)(n-1)} d_{\omega_2} P_2^{(2(N+1))}$. Since P_2 is Poisson, we have

$$\mathrm{Lie}_e^{(N+1)(n-1)} d_{\omega_2} P_2^{(2(N+1))} = \mathrm{Lie}_e^{(N+1)(n-1)} \left(-\frac{1}{2} \sum_{k=1}^{N} \left[P_2^{(2k)}, P_2^{(2(N-k+1))} \right] \right). \tag{5.13}$$

On the other hand

$$\text{Lie}_e^{(N+1)(n-1)} \left[P_2^{(2k)}, P_2^{(2(N-k+1))} \right] =$$

$$\sum_{j=0}^{(N+1)(n-1)} \binom{(N+1)(n-1)}{j} \left[\text{Lie}_e^j P_2^{(2k)}, \text{Lie}_e^{(N+1)(n-1)-j} P_2^{(2(N-k+1))} \right], \quad (5.14)$$

and since (ω_1, P_2) is in normal form, we have that $\text{Lie}_e^j P_2^{(2k)} = 0$ for $j \geq nk - k + 1$ and $\text{Lie}_e^{(N+1)(n-1)-j} P_2^{(2(N-k+1))} = 0$ for $(N+1)(n-1) - j \geq n(N-k+1) - (N-k+1) + 1$ or equivalently for $j \leq nk - k - 1$. Therefore, the only surviving term in the sum on the right hand side of equation (5.14) corresponds to $j = nk - k$. Thus we have

$$\text{Lie}_e^{(N+1)(n-1)} \left[P_2^{(2k)}, P_2^{(2(N-k+1))} \right] =$$

$$\binom{(N+1)(n-1)}{k(n-1)} \left[\text{Lie}_e^{nk-k} P_2^{(2k)}, \text{Lie}_e^{(N-k+1)(n-1)} P_2^{(2(N-k+1))} \right]. \quad (5.15)$$

Substituting (5.15) in the right hand side of (5.13) we get

$$\text{Lie}_e^{(N+1)(n-1)} d_{\omega_2} P_2^{(2(N+1))} =$$

$$-\frac{1}{2} \sum_{k=1}^{N} \binom{(N+1)(n-1)}{k(n-1)} \left[\text{Lie}_e^{nk-k} P_2^{(2k)}, \text{Lie}_e^{(N-k+1)(n-1)} P_2^{(2(N-k+1))} \right], (5.16)$$

or

$$d_{\omega_2} \left(\frac{\text{Lie}_e^{(N+1)(n-1)} P_2^{(2(N+1))}}{[(N+1)(n-1)]!} \right) = -\frac{1}{2} \sum_{k=1}^{N} \left[\frac{\text{Lie}_e^{nk-k} P_2^{(2k)}}{[k(n-1)]!}, \frac{\text{Lie}_e^{(N-k+1)(n-1)} P_2^{(2(N-k+1))}}{[(N+1-k)(n-1)]!} \right],$$

$$(5.17)$$

that is

$$d_{\omega_2} Q_2^{(2(N+1))} = -\frac{1}{2} \sum_{k=1}^{N} \left[Q_2^{(2k)}, Q_2^{(2(N-k+1))} \right].$$

This proves that Q_2 is indeed Poisson and thus we have transformed a pencil in normal form (ω_1, P_2) with central invariants of order $n - 1$ to a pencil (ω_1, Q_2) with constant central invariants. Notice also that (ω_1, Q_2) is still in normal form: in this case $\text{Lie}_e Q_2^{(2)} = 0$ and also $\text{Lie}_e Q_2^{(2k)} = 0$, as it is immediate to check. $\blacksquare$

The case of r-KdV-CH hierarchy As an application of the previous Theorem, let us study how Lie_e^r acts on the r-KdV-CH hierarchy. We will use the notation introduced in Section 3. We will show the following

Proposition 9. *In the case $k = 0$, $l = 1$, i.e. in the case in which the central invariants are polynomials of degree r parametrized by $(a_0, \ldots, a_r)$, Lie_e^r acts on the r-KdV-CH hierarchy, mapping the system with parameters $(a_0, \ldots, a_r)$ to the one with parameters $(a_r, 0, \ldots, 0)$.*

Proof: We fix our attention to the bi-Hamiltonian structure (P_k, P_l), where k and l are general for the moment. First, in λ_i coordinates, we normalize P_k up to the order $O(\epsilon^4)$, eliminating with a Miura transformation the term in ϵ^2. In particular, looking the leading term in derivatives of $\delta(x-y)$ in ϵ^2, one can use as an ansatz for the vector field Z generating the Miura transformation

$$Z^i = \epsilon^2(b^i(\lambda)\lambda_{i,xx} + \dots) + O(\epsilon^4).$$

Now it is immediate to see that

$$\mathrm{Lie}_Z P_k^{ii} = -\epsilon^2 \left[b^i(\lambda) g_k^{ii} \delta^{(3)}(x-y) + \dots \right].$$

Since we want to normalize P_k^{ij} up to $O(\epsilon^4)$, we have that we have to impose

$$\mathrm{Lie}_Z P_k^{ii} = -\epsilon^2 \left(E_k^{ii}(\lambda) + \dots \right) = -\epsilon^2 \left[\frac{\lambda_i^k \mathfrak{a}'(\lambda_i) - k\lambda_i^{k-1}\mathfrak{a}(\lambda_i)}{2(P'(\lambda_i))^2} \right] \delta^{(3)}(x-y) + \dots.$$

Since $g_k^{ii} = -\frac{2\lambda_i^k}{P'(\lambda_i)}$, one gets

$$b_i(\lambda_i) = -\frac{\mathfrak{a}'(\lambda_i) - k\mathfrak{a}(\lambda_i)/\lambda_i}{4P'(\lambda_i)}.$$

In this way, we eliminate the term in ϵ^2 from P_k. Now the same Miura transformation applied to P_l is computed as follows. Using the formula for Lie derivatives one finds:

$$\mathrm{Lie}_Z P_l^{ii} = -\epsilon^2 \left(b_i(\lambda_i) g_l^{ii} \delta^{(3)}(x-y) + \dots \right)$$

and substituting the expression for $b_i(\lambda_i)$ computed above and the expression for g_l^{ii} we find

$$\mathrm{Lie}_Z P_l^{ii} = \epsilon^2 \left[-\frac{\mathfrak{a}'(\lambda_i) - k\mathfrak{a}(\lambda_i)/\lambda_i}{4P'(\lambda_i)} \frac{2\lambda_i^l}{P'(\lambda_i)} \delta^{(3)}(x-y) + \dots \right].$$

Therefore P_l^{ii} is mapped to

$$P_l^{ii} \mapsto (g_l^{ii}\delta^{(1)}(x-y) + \dots) + \epsilon^2 \left[\left(E_l^{ii}(\lambda) - \frac{\mathfrak{a}'(\lambda_i) - k\mathfrak{a}(\lambda_i)/\lambda_i}{4P'(\lambda_i)} \frac{2\lambda_i^l}{P'(\lambda_i)} \right) \delta^{(3)}(x-y) + \dots \right],$$

so the term in ϵ^2 in P_l^{ii} is reduced to

$$\epsilon^2 \left[\frac{(k-l)\lambda_i^{l-1}\mathfrak{a}(\lambda_i)}{2(P'(\lambda_i))^2} \delta^{(3)}(x-y) + \dots \right].$$

Now we specialize to $k = 0$, so that the exactness condition $\sum_h \frac{\partial g_k^{ii}}{\partial u^h} = 0$, $i = 0, \dots, r-1$ holds, and furthermore we assume $l = 1$ so that the central invariants are polynomials of degree r. Now observe that in the case $k = 0$, $l = 1$, the canonical coordinates $u^i = \lambda_i$, so the vector field $e = \sum_h \frac{\partial}{\partial u^h} = \sum_h \frac{\partial}{\partial \lambda_h}$. Let us observe also

that $\sum_h \frac{\partial}{\partial \lambda_h} P'(\lambda_i) = 0$. Taking into account these two facts and applying $\frac{1}{r!}\text{Lie}_e^r$ to the term ϵ^2 in $P_{l=1}^{ij}$ we obtain

$$P_{l=1}^{ij} = (g_1^{ij}\delta^{(1)}(x-y) + \dots) + \epsilon^2 \left(-\frac{a_r}{2(P'(\lambda_i))^2}\delta^{(3)}(x-y) + \dots \right),$$

that is, using Lie_e^r we can map an r-KdV-CH system associated to $\mathfrak{a}(\lambda)$ to an r-KdV-CH system associated to $\frac{1}{r!}\frac{d^r}{d\lambda^r}\mathfrak{a}(\lambda)$. $\blacksquare$

6 Deformations of homogeneous Poisson pencils of hydrodynamic type

Let us consider a deformation P_λ of a homogenous Poisson pencil of hydrodynamic type ω_λ. We will call it *homogenous* if its central invariants are homogenous functions of the *same* degree D in the canonical coordinates:

$$\sum_{k=1}^n u^k \frac{\partial c_i}{\partial u^k} = Dc_i.$$

As usual we can assume, without loss of generality, that the pencil P_λ has the form

$$P_\lambda = \omega_2 + \sum_{k=1}^\infty \epsilon^{2k} P_2^{(2k)} - \lambda\omega_1.$$

In this case we know that

$$P_2^{(2)} = d_{\omega_1}d_{\omega_2}\left(\sum_{i=1}^N \int c^i(u^i)u_x^i \ln u_x^i \, dx \right).$$

Using the properties (1.12) and (1.13) and the identity

$$\text{Lie}_E \sum_{i=1}^N \int c^i(u^i)u_x^i \ln u_x^i \, dx = (D+1)\sum_{i=1}^N \int c^i(u^i)u_x^i \ln u_x^i \, dx$$

we obtain

$$\text{Lie}_E P_2^{(2)} = [2(d-1)+D]\, P_2^{(2)} \tag{6.1}$$

Indeed we have

$$\text{Lie}_E P_2^{(2)} = \text{Lie}_E d_{\omega_1}d_{\omega_2}\left(\sum_{i=1}^N \int c^i(u^i)u_x^i \ln u_x^i \, dx \right) =$$

$$d_{\omega_1}d_{\omega_2}\text{Lie}_E\left(\sum_{i=1}^N \int c^i(u^i)u_x^i \ln u_x^i \, dx \right) + (2d-3)d_{\omega_1}d_{\omega_2}\left(\sum_{i=1}^N \int c^i(u^i)u_x^i \ln u_x^i \, dx \right) =$$

$$(D+1+2d-3)d_{\omega_1}d_{\omega_2}\left(\sum_{i=1}^N \int c^i(u^i)u_x^i \ln u_x^i \, dx \right) = [2(d-1)+D]\, P_2^{(2)}.$$

The behaviour of the the flat pencil g_λ with respect to the Euler vector field has some consequences also on the form of the higher order deformations. We have the following theorem

Theorem 10. *A homogenous Poisson pencil P_λ can be reduced by a Miura transformation to a Poisson pencil Q_λ of the form*

$$Q_\lambda = \omega_2 + \sum_{k=1}^{\infty} \epsilon^{2k} Q_2^{(2k)} - \lambda\omega_1 \tag{6.2}$$

with

$$\mathrm{Lie}_E Q_2^{(2k)} = [(k+1)(d-1) + kD]Q_2^{(2k)}, \qquad k = 1, 2, \ldots \tag{6.3}$$

Proof. We can prove the theorem by induction. Suppose that

$$\mathrm{Lie}_E P_2^{(2k)} = [(k+1)(d-1) + kD]P_2^{(2k)}, \qquad k = 1, \ldots, N$$

and that

$$\mathrm{Lie}_E P_2^{(2N+2)} \neq [(N+2)(d-1) + (N+1)D]P_2^{(2N+2)}, \qquad k = 1, \ldots, N$$

then we show that there exists a Miura transformation, canonical with respect to ω_1, reducing the pencil P_λ to the form

$$Q_\lambda = \omega_2 + \sum_{k=1}^{\infty} \epsilon^{2k} Q_2^{(2k)} - \lambda\omega_1 \tag{6.4}$$

with

$$\mathrm{Lie}_E Q_2^{(2N+2)} = [(N+2)(d-1) + (N+1)D]Q_2^{(2N+2)}. \tag{6.5}$$

We need to show that

$$d_{\omega_1}\left(\mathrm{Lie}_E P_2^{(2N+2)} - [(N+2)(d-1) + (N+1)D]P_2^{(2N+2)}\right) = 0$$

$$d_{\omega_2}\left(\mathrm{Lie}_E P_2^{(2N+2)} - [(N+2)(d-1) + (N+1)D]P_2^{(2N+2)}\right) = 0$$

The first property follows immediately (1.12) and from $d_{\omega_1}P_2^{(2k)} = 0$ (which follows from the compatiblity of the pencil P_λ), while the second one can be proved using the induction hypothesis. Indeed, taking into account that

$$d_{\omega_2}P_2^{(2N+2)} = -\frac{1}{2}\sum_{k=1}^{N}\left[P_2^{(2k)}, P_2^{(2N+2-2k)}\right]$$

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we obtain

$$d_{\omega_2}\left(\mathrm{Lie}_E P_2^{(2N+2)} - [(N+2)(d-1) + (N+1)D]P_2^{(2N+2)}\right) =$$

$$\mathrm{Lie}_E d_{\omega_2} P_2^{(2N+2)} - (d-1)d_{\omega_2}P_2^{(2N+2)} - [(N+2)(d-1) + (N+1)D]d_{\omega_2}P_2^{(2N+2)} =$$

$$-\frac{1}{2}\mathrm{Lie}_E \sum_{k=1}^{N}\left[P_2^{(2k)}, P_2^{(2N+2-2k)}\right] - [(N+3)(d-1) + (N+1)D]d_{\omega_2}P_2^{(2N+2)} =$$

$$-\sum_{k=1}^{N}\left[\mathrm{Lie}_E P_2^{(2k)}, P_2^{(2N+2-2k)}\right] - [(N+3)(d-1) + (N+1)D]d_{\omega_2}P_2^{(2N+2)} =$$

$$-\sum_{k=1}^{N}[(k+1)(d-1) + kD]\left[P_2^{(2k)}, P_2^{(2N+2-2k)}\right] - [(N+3)(d-1) + (N+1)D]d_{\omega_2}P_2^{(2N+2)}$$

Since $[P_2^{(2k')}, P_2^{(2N+2-2k')}] = [P_2^{(2N+2-2k')}, P_2^{(2k')}]$, we can reorganize the sum in such a way that the terms with $k = k'$ and $k = N+1-k'$ have the *same* coefficient

$$\frac{1}{2}\left\{[(k'+1)(d-1) + k'D] + [(N+2-k')(d-1) + (N+1-k')D]\right\} =$$

$$\frac{1}{2}[(N+3)(d-1) + (N+1)D],$$

independent on k'. The result is

$$-\sum_{k=1}^{N}[(k+1)(d-1) + kD]\left[P_2^{(2k)}, P_2^{(2N+2-2k)}\right] - [(N+3)(d-1) + (N+1)D]d_{\omega_2}P_2^{(2N+2)}$$

$$[(N+3)(d-1) + (N+1)D]\left(-\frac{1}{2}\sum_{k=1}^{N}\left[P_2^{(2k)}, P_2^{(2N+2-2k)}\right] - d_{\omega_2}P_2^{(2N+2)}\right) = 0.$$

According to the results of Liu and Zhang there exists a vector field

$$X_2^{(2N+2)} = d_{\omega_1}H_2^{(2N+2)} + d_{\omega_2}K_2^{(2N+2)}$$

such that

$$\mathrm{Lie}_E P_2^{(2N+2)} - [(N+2)(d-1) + (N+1)D]P_2^{(2N+2)} = d_{\omega_1}X^{(2N+2)} = d_{\omega_1}d_{\omega_2}K_2^{(2N+2)}.$$

The Miura transformation generated by the vector field $d_{\omega_1}\tilde{K}_2^{(2N+2)}$ reduces the pencil to the form $\omega_2 + \sum_k \epsilon^{2k}Q_2^{(2k)}$ where

$$Q_2^{(2k)} = P_2^{(2k)}, \qquad k = 1, \ldots, N$$
$$P_2^{(2N+2)} = P_2^{(2N+2)} - d_{\omega_1}d_{\omega_2}\tilde{K}_2^{(2N+2)}.$$

Moreover the "new" pencil satisfies

$$\mathrm{Lie}_E Q_2^{(2N+2)} = \mathrm{Lie}_E(P_2^{(2N+2)} - d_{\omega_1}d_{\omega_2}\tilde{K}_2^{(2N+2)}) =$$

$$d_{\omega_1}d_{\omega_2}K_2^{(2N+2)} + [(N+2)(d-1) + (N+1)D]P_2^{(2N+2)} - \mathrm{Lie}_E d_{\omega_1}d_{\omega_2}\tilde{K}_2^{(2N+2)} =$$

$$d_{\omega_1}d_{\omega_2}K_2^{(2N+2)} + [(N+2)(d-1) + (N+1)D]P_2^{(2N+2)} +$$

$$-d_{\omega_1}d_{\omega_2}\left(\mathrm{Lie}_E \tilde{K}_2^{(2N+2)} + (2d-3)\tilde{K}_2^{(2N+2)}\right).$$

If the functional $\tilde{K}_2^{(2N+2)}$ satisfies the further condition

$$\text{Lie}_E \tilde{K}_2^{(2N+2)} + (2d-3)\tilde{K}_2^{(2N+2)} - K_2^{(2N+2)} = [(N+2)(d-1) + (N+1)D]\tilde{K}_2^{(2N+2)}$$

that is

$$\text{Lie}_E \tilde{K}_2^{(2N+2)} - [N(d-1) + (N+1)D + 1]\tilde{K}_2^{(2N+2)} = K_2^{(2N+2)} \tag{6.6}$$

then

$$\text{Lie}_E Q_2^{(2N+2)} = [(N+2)(d-1) + (N+1)D]Q_2^{(2N+2)}$$

as required.

To conclude the proof we have to show that an equation of the form

$$\text{Lie}_E \tilde{K} + C\tilde{K} = K \tag{6.7}$$

always admits solutions (for $C = 0$ it has been proved in [17]). In canonical coordinates (6.7) reads

$$\sum_{i=1}^n \int_{S^1} \left[u^i \frac{\partial \tilde{k}}{\partial u^i} + \sum_{s=1}^\infty u^i_{(s)} \frac{\partial \tilde{k}}{\partial u^i_{(s)}} \right] dx = \int_{S^1} k\, dx$$

A solution can be found solving the equation

$$\sum_{i=1}^n \left[u^i \frac{\partial \tilde{k}}{\partial u^i} + \sum_{s=1}^\infty u^i_{(s)} \frac{\partial \tilde{k}}{\partial u^i_{(s)}} \right] = k \tag{6.8}$$

for the density of the functional $\tilde{K}$. The differential operator

$$\sum_{s=1}^\infty u^i_{(s)} \frac{\partial}{\partial u^i_{(s)}}$$

acts on the single monomials of $\tilde{K}$ multiplying them by the their degree as polinomials in the variables u^i_x, u^i_{xx}, ..., $u^i_{(s)}$, ... Notice that this is different from the degree they have as differential polinomials. In the last case $\deg(u^i_{(s)}) = s$ while in the former case $\deg(u^i_{(s)}) = 1$. This means that the equation (6.8) is equivalent to the quasilinear equations

$$\sum_{i=1}^n u^i \frac{\partial \tilde{A}_j}{\partial u^i} + c_j \tilde{A}_j = A_j, \qquad \sum_{i=1}^n u^i \frac{\partial \tilde{B}_{jm}}{\partial u^i} + c_{jm} \tilde{B}_{jm} = B_{jm}, \qquad \ldots$$

for the coefficients $\tilde{A}_i, \tilde{B}_{ij}$, etc. of the homogenous differential polynomial

$$\tilde{k} = \tilde{A}_i u^i_{(N)} + \tilde{B}_{ij} u^i_x u^j_{(N-1)} + \ldots.$$

The constant c_j, c_{jm}, ... are equal to C plus the degree of the monomial containing A_j, B_{jm}, and so on. For instance $c_j = C + 1$, $c_{jm} = C + 2$, etc. It is well known (see

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for instance [9] or [25]) that equations of this form admit n functional independent solutions.

Indeed plugging-in $u^{n+1} = \tilde{A}$ and looking for solutions of

$$\sum_{i=1}^{n} u^i \frac{\partial u^{n+1}}{\partial u^i} + cu_{n+1} = A(u^1, \ldots, u^n)$$

in implicit form:

$$\phi(u^1, \ldots, u^{n+1}) = \text{cost}$$

we obtain

$$-\sum_{i=1}^{n} u^i \frac{\frac{\partial \phi}{\partial u^i}}{\frac{\partial \phi}{\partial u^{n+1}}} = A(u^1, \ldots, u^n) - cu^{n+1}$$

that is

$$\sum_{i=1}^{n} u^i \frac{\partial \phi}{\partial u^i} + (A(u^1, \ldots, u^n) - cu^{n+1}) \frac{\partial \phi}{\partial u^{n+1}}$$

which is the quasilinear equation for the first integrals of the vector field

$$\sum_{i=1}^{n} \frac{\partial}{\partial u^i} + (A(u^1, \ldots, u^n) - cu^{n+1}) \frac{\partial}{\partial u^{n+1}}.$$

■

An interesting consequence of the previous Theorem is that the tensorial quantities appearing in the ϵ-expansion of the Poisson pencil have a well-defined degree of homogeneity with respect to E. This is detailed in the following:

Corollary 11. *The tensor fields $A^{ij}_{(2)k,0}$, $k = 2, 4, 6, \ldots$ appearing in the leading terms of the ϵ-expansion of the Poisson pencil (6.2)*

$$Q^{ij}_\lambda = \omega_2^{ij} + \sum_{k \geq 1} \epsilon^{2k} \sum_{l=0}^{2k+1} A^{ij}_{(2)2k,l}(q, q_x, \ldots, q_{(l)}) \delta^{(2k-l+1)}(x-y) - \lambda \omega_1^{ij}$$

satisfy the homogeneity condition

$$\text{Lie}_E A_{(2)2k,0} = [(k+1)(d-1) + kD] A^{ij}_{(2)2k,0}, \qquad k = 1, \ldots, N. \qquad (6.9)$$

Here we identify the Euler vector field on the manifold with the corresponding vector field on the loop space.

Proof: The proof is a direct consequence of the formula

$$\text{Lie}_E Q_2^{ij} = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (6.10)$$

$$\sum_{k,s} \left(\partial_x^s E^k(u(x), \ldots) \frac{\partial Q_2^{ij}}{\partial u_{(s)}^k(x)} - \frac{\partial E^i(u(x), \ldots)}{\partial u_{(s)}^k(x)} \partial_x^s Q_2^{kj} - \frac{\partial E^j(u(y), \ldots)}{\partial u_{(s)}^k(y)} \partial_y^s Q_2^{ik} \right),$$

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Indeed, considering only the leading terms in the ϵ-expansion equation (6.3) reads

$$\left[\mathrm{Lie}_E A^{ij}_{(2)2k,0}\delta^{(2k+1)}(x-y) + \ldots\right] = [(k+1)(d-1)+kD]\left[A^{ij}_{(2)2k,0}\delta^{(2k+1)}(x-y) + \ldots\right]$$
(6.11)

■

As an example, we check the homogeneity predictions of the previous Theorem in the case of non trivial bi-Hamiltonian deformations of the Poisson pencil $\omega_\lambda = \omega_2 + \lambda\omega_1 = u\delta'(x-y) + \frac{1}{2}u_x\delta(x-y) + \lambda\delta'(x-y)$, up to order ϵ^6. We will need the following auxiliary formula

$$\frac{\partial}{\partial u_{(s)}}\partial_x^l = \sum_{t=0}^{l}\binom{l}{t}\partial_x^t\frac{\partial}{\partial u_{(s-l+t)}},$$
(6.12)

and the following simple

Lemma 12. *The following identity holds*

$$\sum_{s=0}^{l}(\partial_x^s u)\frac{\partial}{\partial u_{(s)}}(\partial_x^l u^D) = D\partial_x^l u^D.$$
(6.13)

Proof: By (6.12) we can write the left hand side of (6.13) as

$$\sum_{s=0}^{l}(\partial_x^s u)\frac{\partial}{\partial u_{(s)}}(\partial_x^l u^D) = \sum_{s=0}^{l}(\partial_x^s u)\sum_{t=0}^{l}\binom{l}{t}\partial_x^t\left(\frac{\partial}{\partial u_{(s-l+t)}}u^D\right) =$$

$$= D\sum_{s=0}^{l}(\partial_x^s u)\binom{l}{l-s}\partial_x^{l-s}u^{D-1} = D\partial_x^l u^D.$$

■

The non trivial bi-Hamiltonian deformations of ω_λ up the ϵ^6 have been computed in [3] (extending previous results of [23]) providing the following result:

Theorem 13. *Up to Miura transformations, the deformations of the pencil $\omega_\lambda = u\delta^{(1)}(x-y) + \frac{1}{2}u_{(1)}\delta(x-y) - \lambda\delta^{(1)}(x-y)$ can be reduced to the following form:*

$$Q_\lambda = \omega_\lambda - \epsilon^2\left\{\partial_x^2\left(c_2\delta^{(1)}(x-y)\right) + c_2\delta^{(3)}(x-y) + (\partial_x c_2)\delta^{(2)}(x-y)\right\}$$
$$-\epsilon^4\left\{\partial_x^4\left(c_4\delta^{(1)}(x-y)\right) + c_4\delta^{(5)}(x-y) + (\partial_x c_4)\delta^{(4)}(x-y)\right\}$$
$$-\epsilon^6\left\{\partial_x^6\left(c_6\delta^{(1)}(x-y)\right) + c_6\delta^{(7)}(x-y) + (\partial_x c_6)\delta^{(6)}(x-y)\right\}$$
$$+\epsilon^6\left\{h\delta^{(3)}(x-y) + (\partial_x h)\delta^{(2)}(x-y) + \partial_x^2\left(h\delta^{(1)}(x-y)\right)\right\}$$
$$+\epsilon^6\left\{\partial_x^3\left((\partial_x^2 g)\delta^{(2)}(x-y)\right) + \partial_x\left((\partial_x^3 g)\delta^{(3)}(x-y)\right) + (\partial_x^2 g)\delta^{(5)}(x-y) + (\partial_x^3 g)\delta^{(4)}(x-y)\right\},$$
(6.14)

where $c_2(u)$ is the central invariant and c_4 and c_6 are related to c_2 via the following equations:

$$c_4 = -\frac{\partial}{\partial u}(c_2)^2, \tag{6.15}$$

$$c_6 = -\frac{1}{2}\frac{\partial}{\partial u}\left(c_2^2\,\frac{\partial c_2}{\partial u}\right), \tag{6.16}$$

while g is given by

$$g = \frac{1}{2}\int\left\{\frac{3}{2}c_2^2\,\frac{\partial^3 c_2}{\partial u^3} + \left(\frac{\partial c_2}{\partial u}\right)^3 + \frac{19}{3}c_2\,\frac{\partial^2 c_2}{\partial u^2}\frac{\partial c_2}{\partial u}\right\}\,du \tag{6.17}$$

and $h := h_1 + h_2 + h_3 + h_4$ and the h_i's have the following expression in terms of the central invariant $c_2(u)$:

$$h_1 = u_{xx}^2\left(\frac{97}{60}c_2\left(\frac{\partial^2 c_2}{\partial u^2}\right)^2 + \frac{8}{3}\left(\frac{\partial c_2}{\partial u}\right)^2\frac{\partial^2 c_2}{\partial u^2} + \frac{21}{40}c_2^2\frac{\partial^4 c_2}{\partial u^4} + \frac{49}{15}c_2\left(\frac{\partial^3 c_2}{\partial u^3}\right)\frac{\partial c_2}{\partial u}\right) \tag{6.18}$$

$$h_2 = u_x^4\left(\begin{array}{c}\frac{254}{3}\left(\frac{\partial c_2}{\partial u}\right)^2\frac{\partial^4 c_2}{\partial u^4} + \frac{17}{5}\left(c_2\right)^2\frac{\partial^6 c_2}{\partial u^6} + \frac{176}{3}c_2\left(\frac{\partial^3 c_2}{\partial u^3}\right)^2 \\ +\frac{4018}{45}c_2\left(\frac{\partial^4 c_2}{\partial u^4}\right)\frac{\partial^2 c_2}{\partial u^2} + \frac{1684}{45}c_2\frac{\partial^5 c_2}{\partial u^5}\frac{\partial c_2}{\partial u} + \frac{14512}{45}\left(\frac{\partial c_2}{\partial u}\right)\left(\frac{\partial^2 c_2}{\partial u^2}\right)\frac{\partial^3 c_2}{\partial u^3}\end{array}\right) \tag{6.19}$$

$$h_3 = u_{xxx}u_x\left(\frac{3}{10}c_2^2\frac{\partial^4 c_2}{\partial u^4} + \frac{2}{3}\left(\frac{\partial c_2}{\partial u}\right)^2\frac{\partial^2 c_2}{\partial u^2} + \frac{1}{15}c_2\left(\frac{\partial^2 c_2}{\partial u^2}\right)^2 + \frac{28}{15}c_2\left(\frac{\partial^3 c_2}{\partial u^3}\right)\frac{\partial c_2}{\partial u}\right) \tag{6.20}$$

$$h_4 = u_{xx}u_x^2\left(\begin{array}{c}\frac{139}{10}\left(\frac{\partial c_2}{\partial u}\right)\left(\frac{\partial^2 c_2}{\partial u^2}\right)^2 + \frac{178}{15}\left(\frac{\partial c_2}{\partial u}\right)^2\frac{\partial^3 c_2}{\partial u^3} + \frac{21}{20}c_2^2\frac{\partial^5 c_2}{\partial u^5} \\ +\frac{259}{30}c_2\left(\frac{\partial^4 c_2}{\partial u^4}\right)\frac{\partial c_2}{\partial u} + 13\,c_2\left(\frac{\partial^2 c_2}{\partial u^2}\right)\frac{\partial^3 c_2}{\partial u^3}\end{array}\right). \tag{6.21}$$

Observe that the scalar pencil ω_λ is automatically homogeneous with Euler vector field $E = u\frac{\partial}{\partial u}$. We want to check that the equation (6.3) holds explicitly in this case for $k = 1, 2, 3$ with arbitrary D and with $d = 0$. In particular, formula (6.3) gives in this specific case $\mathrm{Lie}_E Q_2^{(2k)} = (kD - k - 1)Q_2^{(2k)}$. On the other hand, since by formula (6.10)

$$\mathrm{Lie}_E Q_2^{(2k)} = \sum_{s\geq 0}(\partial_x^s u)\frac{\partial Q_2^{(2k)}}{\partial u_{(s)}} - 2Q_2^{(2k)},$$

to check directly (6.3) is equivalent to show

$$\sum_{s\geq 0}(\partial_x^s u)\frac{\partial Q_2^{(2k)}}{\partial u_{(s)}} = (kD - k + 1)Q_2^{(2k)}, \tag{6.22}$$

where D is the degree of the central invariant c_2 as a function of u. Now for $c_2 := u^D$ we have $Q_2^{(2)} = \partial_x^2 \left(u^D \delta^{(1)}(x-y) \right) + u^D \delta^{(3)}(x-y) + (\partial_x u^D)\delta^{(2)}(x-y)$ and it is immediate to show that $\sum_{s \geq 0}(\partial_x^s u)\frac{\partial Q_2^{(2)}}{\partial u_{(s)}} = DQ_2^{(2)}$ using the identity (6.13). This confirms equation (6.22) in the case $k = 1$.

For the case $k = 2$, we have $Q_2^{(4)} = \partial_x^4 \left(c_4 \delta^{(1)}(x-y) \right) + c_4 \delta^{(5)}(x-y) + (\partial_x c_4)\delta^{(4)}(x-y)$. If $c_2 = u^D$, then $c_4 = -2Du^{2D-1}$ one gets

$$Q_2^{(4)} = \partial_x^4 \left(-2Du^{2D-1}\delta^{(1)}(x-y) \right) - 2Du^{2D-1}\delta^{(5)}(x-y) + (\partial_x(-2Du^{2D-1}))\delta^{(4)}(x-y).$$

Again applying (6.13) we see that $\sum_{s \geq 0}(\partial_x^s u)\frac{\partial Q_2^{(4)}}{\partial u_{(s)}} = (2D-1)Q_2^{(4)}$, which is exactly (6.22) with $k = 2$.

To check the homogeneity at ϵ^6 is slightly more delicate. If $c_2 = u^D$, then $c_6 = -\frac{1}{2}D(3D-1)u^{3D-2}$, and consequently for the component of $Q_2^{(6)}$ given by $Q_{2,1}^{(6)} := \partial_x^6 \left(c_6 \delta^{(1)}(x-y) \right) + c_6 \delta^{(7)}(x-y) + (\partial_x c_6)\delta^{(6)}(x-y)$ we have

$$\sum_{s \geq 0}(\partial_x^s u)\frac{\partial Q_{2,1}^{(6)}}{\partial u_{(s)}} = (3D-2)Q_{2,1}^{(6)},$$

which is (6.22) for $k = 3$. Moreover, for the choice $c_2 = u^D$, the function g appearing in (6.17) is equal to $au^{3D-2} + b$, where a is a suitable constant depending on D while b is a constant of integration. If we call $Q_{2,2}^{(6)} := \partial_x^3 \left((\partial_x^2 g)\delta^{(2)}(x-y) \right) + \partial_x \left((\partial_x^3 g)\delta^{(3)}(x-y) \right) + (\partial_x^2 g)\delta^{(5)}(x-y) + (\partial_x^3 g)\delta^{(4)}(x-y)$, again using (6.13) we find

$$\sum_{s \geq 0}(\partial_x^s u)\frac{\partial Q_{2,2}^{(6)}}{\partial u_{(s)}} = (3D-2)Q_{2,2}^{(6)}.$$

Finally we need to check the homogeneity of $Q_{2,3}^{(6)} := h\delta^{(3)}(x-y) + (\partial_x h)\delta^{(2)}(x-y) + \partial_x^2 \left(h\delta^{(1)}(x-y) \right)$. Using the expressions for h written above we find $h = \alpha u_{xx}^2 u^{3D-4} + \beta u_x^4 u^{3D-6} + \gamma u_{xxx} u_x u^{3D-4} + \rho u_{xx} u_x^2 u^{3D-5}$, where $\alpha, \beta, \gamma, \rho$ are constants depending on D. This boils down to prove that

$$\sum_{s \geq 0}(\partial_x^s u)\frac{\partial h}{\partial u_{(s)}} = (3D-2)h,$$

which is immediate. Thus we have verified explicitly formula (6.3) up to ϵ^6 for the non trivial bi-Hamiltonian deformations of the scalar pencil ω_λ.

7   Conclusions

In this paper we have studied some general properties of deformations of exact or homogeneous Poisson pencils of hydrodynamic type. In particular, we focused our

attention on those characteristics of the fully deformed pencils that are inherited from properties of the dispersionelss limit (like exactness and homogeneity) coupled with suitable conditions about the central invariants.

In the case of exact Poisson pencil we proved that their deformations can be reduced to a suitable normal form via a Miura transformation. It turned out that each term of the deformation is annihilated by a sufficiently high power of the operator Lie$_e$. As a byproduct of this result we showed that there exists a map between Poisson pencils with polynomial central invariants and Poisson pencil with constant central invariants having the same dispersionless limit. It would be interesting to extend this map to a wider class of central invariants and to provide different maps performing the inverse task, namely starting from a pencil with constant central invariants and providing as an output pencils with polynomial central invariants (not constants). In principle, it is much more difficult to construct this last class of maps.

Using similar ideas we also constructed normal forms of deformations of homogeneous Poisson pencil of hydrodynamic type having homogenous central invariants. In this case it turned out that each term of the deformation is a homogeneous bivector of specific degree.

As future investigations, it would be interesting to determine other characteristics of the fully deformed pencil that are controlled by the dispersionless limit. In particular, one might ask if the requirement of the fulfillment of Virasoro constraints can be interpreted as a suitable property of dispersionless pencil.

References

[1] M. Antonowicz, A.P. Fordy, *Coupled KdV equations with multi-Hamiltonian structures*, Physica D **28** (1987) 345–357.

[2] M. Antonowicz, A.P. Fordy, *Coupled Harry Dym equations with multi-Hamiltonian structures*, J. Phys. A **21** (1988) 269–275.

[3] A. Arsie, P. Lorenzoni, *On bi-Bihamiltonian deformations of exact pencils of hydrodynamic type*, J. Phys. A: Math. Theor. **44** (2011)

[4] A. Buryak, H. Posthuma, and S. Shadrin, *A polynomial bracket for Dubrovin-Zhang hierarchies*, arXiv1009.5351.

[5] A. Buryak, H. Posthuma, and S. Shadrin, *On deformations of quasi-Miura transformations and the Dubrovin-Zhang bracket*, arXiv:1104.2722.

[6] R. Camassa and D. Holm, *An integrable shallow water equation with peaked solitons* Phys. Rev. Lett. **81** 1661-4 (1993).

[7] M. Chen, S.-Q. Liu, Y. Zhang, *A two-component generalization of the Camassa-Holm equation and its solutions*, Lett. Math. Phys. **75** (2006) 1–15.

[8] Chen, Ming; Liu, Si-Qi; Zhang, Youjin, *Hamiltonian structures and their reciprocal transformations for the r-KdV-CH hierarchy*, J. Geom. Phys. **59** (2009), no. 9, 1227–1243.

[9] R. Courant, D. Hilbert, *Methods of mathematical physics. Vol. II. Partial differential equations.* Reprint of the 1962 original. Wiley Classics Library. A Wiley-Interscience Publication. John Wiley and Sons, Inc., New York, 1989.

[10] L. Degiovanni, F.Magri, V. Sciacca, *On deformation of Poisson manifolds of hydrodynamic type*, Comm. Math. Phys. **253**(1), 1–24 (2005).

[11] B. Dubrovin, *Flat pencils of metrics and Frobenius manifolds*, In: Proceedings of 1997 Taniguchi Symposium, Integrable Systems and Algebraic Geometry, Editors M.-H. Saito, Y.Shimizu and K.Ueno, 47-72. World Scientific, 1998.

[12] B. Dubrovin, S.Q. Liu, Y. Zhang, *Hamiltonian peturbations of hyperbolic systems of conservation laws I. Quasi-triviality of bi-Hamiltonian perturbations*, Comm. Pure Appl. Math. **59**(4), 559–615 (2006).

[13] B. Dubrovin, S.Q. Liu, Y. Zhang, *Frobenius Manifolds and Central Invariants for the Drinfeld - Sokolov Bihamiltonian Structures*, Advances in Mathematics **219** (3), 780–837 (2008).

[14] B. Dubrovin, S.P. Novikov, *On Poisson brackets of hydrodynamic type*, Soviet Math. Dokl. **279:2** (1984) 294–297.

[15] B. Dubrovin, Y. Zhang, *Normal forms of integrable PDEs, Frobenius manifolds and Gromov-Witten invariants*, math.DG/0108160.

[16] G. Falqui, *On a Camassa-Holm type equation with two dependent variables.* J. Phys. A **39** (2006), no. 2, 327–342.

[17] G.Falqui, P. Lorenzoni, *Exact Poisson pencils, τ-structures and topological hierarchies*, arXiv:1106.1546.

[18] E.V. Ferapontov, *Compatible Poisson brackets of hydrodynamic type*, J. Phys. A **34** (2001) 2377–2388.

[19] I. M. Gelfand and I Zakharevich, *On the local geometry of a Bihamiltonian structure*, in: Gelfand Seminar 1990/92, Birkhauser 1993.

[20] E. Getzler, *A Darboux theorem for Hamiltonian operators in the formal calculus of variations*, Duke Math. J. **111** (2002) 535–560.

[21] A. Lichnerowicz, *Les variétés de Poisson et leurs algèbres de Lie associeés*, J. Diff. Geom. **12** (1977) 253–300.

[22] S.Q. Liu, Y. Zhang, *Deformations of semisimple bihamiltonian structures of hydrodynamic type*, J. Geom. Phys. **54**(4), 427–453 (2005).

[23] P. Lorenzoni, *Deformations of bihamiltonian structures of hydrodynamic type*, J. Geom. Phys. **44** (2002) 331–375.

[24] L. Martinez Alonso, *Schrödinger spectral problems with energy-dependent potentials as sources of nonlinear Hamiltonian evolution equations*, J. Math. Phys. **21** (1980) 2342–2349.

[25] V.I. Smirnov, *A course of higher mathematics. Vol. IV* [Integral equations and partial differential equations]. Translated by D. E. Brown; translation edited by I. N. Sneddon Pergamon Press, Oxford-New York; Addison-Wesley Publishing Co., Inc., Reading, Mass.-London 1964.