Singular limit and exact decay rate of a nonlinear elliptic equation

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Abstract

For any $n\ge 3$, $0<m\le(n-2)/n$, and constants $\eta>0$, $\beta>0$, α, satisfying $\alpha\le\beta(n-2)/m$, we prove the existence of radially symmetric solution of $\frac{n-1}{m}\Delta v^m+\alpha v+\beta x\cdot\nabla v=0$, $v>0$, in $\mathbb{R}^n$, $v(0)=\eta$, without using the phase plane method. When $0<m<(n-2)/n$, $n\ge 3$, and $\alpha=2\beta/(1-m)>0$, we prove that the radially symmetric solution v of the above elliptic equation satisfies $\lim_{|x|\to\infty}\frac{|x|^2v(x)^{1-m}}{\log|x|}=\frac{2(n-1)(n-2-nm)}{\beta(1-m)}$. In particular when $m=\frac{n-2}{n+2}$, $n\ge 3$, and $\alpha=2\beta/(1-m)>0$, the metric $g_{ij}=v^{\frac{4}{n+2}}dx^2$ is the steady soliton solution of the Yamabe flow on $\mathbb{R}^n$ and we obtain $\lim_{|x|\to\infty}\frac{|x|^2v(x)^{1-m}}{\log|x|}=\frac{(n-1)(n-2)}{\beta}$. When $0<m<(n-2)/n$, $n\ge 3$, and $2\beta/(1-m)>\max(\alpha,0)$, we prove that $\lim_{|x|\to\infty}|x|^{\alpha/\beta}v(x)=A$ for some constant $A>0$. For $\beta>0$ or $\alpha=0$, we prove that the radially symmetric solution $v^{(m)}$ of the above elliptic elliptic equation converges uniformly on every compact subset of $\mathbb{R}^n$ to the solution u of the equation $(n-1)\Delta\log u+\alpha u+\beta x\cdot\nabla u=0$, $u>0$, in $\mathbb{R}^n$, $u(0)=\eta$, as $m\to 0$.

Key words: existence of solution, nonlinear elliptic equations, singular limit, exact decay rate, Yamabe flow

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0 Introduction

Recently there is a lot of interest in the following singular diffusion equation [A], [DK], [P],

$$u_t = \frac{n-1}{m}\Delta u^m \quad \text{in } \mathbb{R}^n \times (0,T) \tag{0.1}$$

which arises in the study of many physical models. When $m > 1$, (0.1) is called the porous medium equation which models the the flow of gases through porous medium. When $m = 1$, (0.1) is the well known heat equation with diffusivity coefficient equal to $(n-1)/m$. When $0 < m < 1$, (0.1) is called the fast diffusion equation. Interested reader can read the book [DK] by P. Daskalopoulos and C.E. Kenig and the book [V1] by J.L. Vazquez for the most recent results on (0.1).

For any $n \in \mathbb{Z}^+$, $n \ge 3$, $0 < m < 1$, $\eta > 0$, suppose v is the solution of

$$\begin{cases} \frac{n-1}{m}\Delta v^m + \alpha v + \beta x \cdot \nabla v = 0, v > 0, \quad \text{in } \mathbb{R}^n \\ v(0) = \eta. \end{cases} \tag{0.2}$$

Then as observed by B.H. Gilding and L.A. Peletier [GP] and others [DS], [V1], [V2], the function

$$u_1(x,t) = t^{-\alpha} v(xt^{-\beta})$$

is a solution of (0.1) in $\mathbb{R}^n \times (0,\infty)$ if

$$\alpha = \frac{2\beta - 1}{1-m} \tag{0.3}$$

and for any $T > 0$ the function

$$u_2(x,t) = (T-t)^{\alpha} v(x(T-t)^{\beta})$$

is a solution of (0.1) in $\mathbb{R}^n \times (0,T)$ if

$$\alpha = \frac{2\beta + 1}{1-m} > 0 \tag{0.4}$$

and the function

$$u_3(x,t) = e^{-\alpha t} v(xe^{-\beta t})$$

is an eternal solution of (0.1) in $\mathbb{R}^n \times (-\infty,\infty)$ if

$$\alpha = \frac{2\beta}{1-m}. \tag{0.5}$$

On the other hand P. Daskalopoulos and N. Sesum [DS] proved that a locally conformally flat gradient Yamabe soliton with positive sectional curvature must be radially symmetric and the metric $g_{ij} = v^{\frac{4}{n+2}} dx^2$ satisfies (0.2) or

$$\frac{n-1}{m}\left((v^m)'' + \frac{n-1}{r}(v^m)'\right) + \alpha v + \beta r v' = 0, v > 0, \tag{0.6}$$

in $(0,\infty)$ and

$$\begin{cases} v(0)=\eta \\ v'(0)=0 \end{cases} \tag{0.7}$$

for some constant $\eta>0$ where dx^2 is the standard metric on $\mathbb{R}^n$ with $m=(n-2)/(n+2)$, $n\ge 3$, and

$$\alpha=\frac{2\beta+\rho_1}{1-m} \tag{0.8}$$

for some constants $\beta>0$, α, and ρ_1 where $\rho_1=0$ if g_{ij} is a Yamabe steady soliton, $\rho_1<0$ if g_{ij} is a Yamabe expander soliton, and $\rho_1>0$ if g_{ij} is a Yamabe shrinker soliton.

Since the asymptotic behaviour of the solutions of (0.1) are usually similar to either the functions u_1, u_2 or u_3, it is important to study the solutions of (0.2) in order to understand the behaviour of solutions of (0.1) and the locally conformally flat gradient Yamabe solitons. Existence and uniqueness of radially symmetric solution of (0.2) for α,β, satisfying (0.4) and

$$0<m<\frac{n-2}{n}, n\ge 3, \tag{0.9}$$

is proved by M.A. Peletier, H. Zhang [PZ] and J.R. King [K] using phase plane method (cf. Proposition 7.4 of [V1]). Existence of radially symmetric solution of (0.2) for $\alpha,\beta>0$, satisfying (0.3) and (0.9) is proved on P.22 of [DS]. A sketch of the proof of the existence of radially symmetric solution of (0.2) for $m=(n-2)/(n+2)$, $n\ge 3$, and $\alpha,\beta>0$, satisfying (0.5) is given on P.22-23 of [DS]. This existence result is also noted without proof in [GaP].

In [DS] P. Daskalopoulos and N. Sesum also proved that if $m=(n-2)/(n+2)$, $n\ge 6$ and $\alpha,\beta>0$, satisfy (0.5), then the radially symmetric solution of (0.2) satisfies

$$C_1\frac{\log|x|}{|x|^2}\le v(x)^{1-m}\le C_2\frac{\log|x|}{|x|^2} \qquad \text{as } |x|\to\infty \tag{0.10}$$

for some constants $C_2>C_1>0$.

In this paper we will extend the result of [DS] and give a new simple rigorous proof of the existence of radially symmetric solutions of (0.2) for any $\eta>0$ and α,β,n,m, satisfying

$$0<m\le\frac{n-2}{n}, \quad n\ge 3, \tag{0.11}$$

and

$$\alpha\le\frac{\beta(n-2)}{m} \quad \text{and} \quad \beta>0 \tag{0.12}$$

without using the phase plane method. Note that if (0.11) holds, then (0.12) holds if $\beta>0$ and

$$\alpha\le\frac{2\beta}{1-m}$$

hold. For

$$\beta>0 \quad \text{or} \quad \alpha=0, \tag{0.13}$$

we prove that the radially symmetric solution $v^{(m)}$ of (0.2) converges uniformly on every compact subset of $\mathbb{R}^n$ to the solution u of the equation

$$\begin{cases} (n-1)\Delta \log u + \alpha u + \beta x \cdot \nabla u = 0, u > 0, \text{ in } \mathbb{R}^n \\ u(0) = \eta \end{cases} \tag{0.14}$$

as $m \to 0$. When α, β, m, satisfy (0.9) and

$$\alpha = 2\beta/(1-m) > 0, \tag{0.15}$$

we prove that the radially symmetric solution v of (0.2) satisfies

$$\lim_{|x|\to\infty} \frac{|x|^2 v(x)^{1-m}}{\log |x|} = \frac{2(n-1)(n-2-nm)}{\beta(1-m)}. \tag{0.16}$$

When $m = (n-2)/(n+2)$ and (0.15) hold, this result says that the locally conformally flat gradient steady Yamabe solitons $g_{ij} = v^{\frac{4}{n+2}} dx^2$, $n \geq 3$, has exact decay rate

$$\lim_{|x|\to\infty} \frac{|x|^2 v(x)^{1-m}}{\log |x|} = \frac{(n-1)(n-2)}{\beta}. \tag{0.17}$$

In Theorem 3.2 of [V1] J.L.Vazquez by using phase plane method proved that if (0.3) and (0.9) holds, then the radially symmetric solution v of (0.2) satisfies

$$\lim_{|x|\to\infty} |x|^{\alpha/\beta} v(x) = A \tag{0.18}$$

for some constant $A > 0$. In this paper we will extend this theorem and use a modification of the technique of [Hs] to give a new simple proof of the result that if (0.9) and

$$\frac{2\beta}{1-m} > \max(\alpha, 0) \tag{0.19}$$

hold and v is the radially symmetric solution of (0.2), then (0.18) for some constant $A > 0$.

The plan of the paper is as follows. In section 1 we will prove the existence of radially symmetric solutions of (0.2) when (0.11) and (0.12) hold. We will also prove the singular limit of the radially symmetric solution of (0.2) as $m \to 0$. In section 2 we will prove the exact decay rate (0.16) of the radially symmetric solution of (0.2) when (0.9) and (0.15) hold. In section 3 we will prove the decay rate (0.18) of the radially symmetric solution of (0.2) when (0.9) and (0.19) hold. We let

$$k = \frac{\beta}{\alpha} \quad \text{if } \alpha \neq 0.$$

and we will assume that (0.11) holds for the rest of the paper.

1 Existence and singular limit of solutions

In this section we will prove the existence of radially symmetric solutions of (0.2) and the singular limit of radially symmetric solutions of (0.2) as $m \to 0$.

Lemma 1.1. *Let $m, \alpha \neq 0, \beta \neq 0$, satisfy* (0.11) *and*

$$\frac{m\alpha}{\beta} \le n-2. \tag{1.1}$$

For any $R_0 > 0$ and $\eta > 0$, let v be the solution of (0.6), (0.7), *in* $(0, R_0)$. *Then*

$$v + krv'(r) > 0 \quad \text{in } [0, R_0) \tag{1.2}$$

and

$$\begin{cases} v'(r) < 0 & \text{in } (0,R_0) \quad \text{if } \alpha > 0 \\ v'(r) > 0 & \text{in } (0,R_0) \quad \text{if } \alpha < 0. \end{cases} \tag{1.3}$$

Proof: Let $h_1(r) = v(r) + krv'(r)$. By (1.1), $(n-2) \ge m/k$. Then by direct computation,

$$h_1' + \left(\frac{(n-2)-(m/k)}{r} - (1-m)\frac{v'}{v} + \frac{\beta}{n-1} r v^{1-m}\right) h_1 = \frac{(n-2)-(m/k)}{r} v \ge 0 \quad \text{in } (0,R_0). \tag{1.4}$$

Let

$$f(r) = v(r)^{m-1} exp\left(\frac{\beta}{n-1}\int_0^r \rho v(\rho)^{1-m}\, d\rho\right). \tag{1.5}$$

By (1.4),

$$\begin{aligned} & (r^{n-2-(m/k)} f(r) h_1(r))' \ge 0 && \forall 0 < r < R_0 \\ \Rightarrow \quad & r^{n-2-(m/k)} f(r) h_1(r) > 0 && \forall 0 < r < R_0 \\ \Rightarrow \quad & h_1(r) > 0 && \forall 0 \le r < R_0 \end{aligned}$$

and (1.2) follows. By (0.6), (0.7), and (1.2),

$$\begin{aligned} & \frac{n-1}{m}\frac{1}{r^{n-1}}(r^{n-1}(v^m)')' = -\alpha h_1 \begin{cases} < 0 & \text{in } (0,R_0) \quad \text{if } \alpha > 0 \\ > 0 & \text{in } (0,R_0) \quad \text{if } \alpha < 0 \end{cases} \\ \Rightarrow \quad & \begin{cases} r^{n-1}(v^m)' < 0 & \text{in } (0,R_0) \quad \text{if } \alpha > 0 \\ r^{n-1}(v^m)' > 0 & \text{in } (0,R_0) \quad \text{if } \alpha < 0 \end{cases} \end{aligned}$$

and (1.3) follows. □

Theorem 1.2. *Let $\eta > 0$ and let $\alpha, \beta \in \mathbb{R}$, m, satisfy* (0.11) *and* (0.12). *Then there exists a unique solution v of* (0.6), (0.7), *in* $(0, \infty)$. *Moreover the function*

$$w_1(r) = r^2 v(r)^{2k} \tag{1.6}$$

satisfies $w_1'(r) > 0$ for all $r > 0$.

Proof: We will use a modification of the proof of Theorem 1.3 of [Hs] to prove the theorem. If $\alpha = 0$, the constant function $v(r) \equiv \eta$ is the unique solution of (0.6), (0.7), in $(0,\infty)$ and then $w_1(r) = \eta^{2k}r^2$ satisfies $w'(r) > 0$ for any $r > 0$. Hence we may assume $\alpha \neq 0$ in the proof.

We next note that uniqueness of solution of (0.6), (0.7), in $(0,\infty)$ follows by standard O.D.E. theory. Hence we only need to prove existence of solution of (0.6), (0.7), in $(0,\infty)$. Local existence of solution of (0.6), (0.7), in a neighbourhood of the origin follows by standard O.D.E. theory.

Let $(0, R_0)$ be the maximal interval of existence of solution of (0.6), (0.7). Suppose $R_0 < \infty$. Then there exists a sequence $\{r_i\}_{i=1}^{\infty}$, $r_i \nearrow R_0$ as $i \to \infty$, such that either

$$|v'(r_i)| \to \infty \text{ as } i \to \infty \quad \text{or} \quad v(r_i) \searrow 0 \text{ as } i \to \infty \quad \text{or} \quad v(r_i) \to \infty \text{ as } i \to \infty.$$

By (0.12), (1.1) holds. Hence by Lemma 1.1,

$$w_1'(r) = 2rv^{2k} + 2kr^2v^{2k-1}v' = 2rv^{2k-1}(v + krv') > 0 \quad \forall 0 < r < R_0. \tag{1.7}$$

We now divide the proof into two cases.

<u>**Case 1**</u>: $\alpha > 0$.

By (1.7),

$$\begin{aligned} & w_1(r) = r^2v^{2k} \geq w_1(R_0/2) > 0 \quad \forall R_0/2 \leq r < R_0 \\ \Rightarrow\quad & v(r) \geq (R_0^{-2}w_1(R_0/2))^{\frac{1}{2k}} \quad \forall R_0/2 \leq r < R_0. \end{aligned} \tag{1.8}$$

By Lemma 1.1 $v' < 0$ on $(0, R_0)$. Hence

$$0 < v(r) \leq v(0) = \eta \quad \forall 0 \leq r < R_0. \tag{1.9}$$

By (0.6), (0.7), and (1.9),

$$\frac{n-1}{m}\frac{1}{r^{n-1}}(r^{n-1}(v^m)')' = -(\alpha v + \beta rv') \quad \text{in } (0, R_0)$$

$$\Rightarrow\quad (n-1)r^{n-1}(v^m)' = -m\left(\alpha\int_0^r \rho^{n-1}v(\rho)\,d\rho + \beta\int_0^r \rho^n v'(\rho)\,d\rho\right) \quad \text{in } (0, R_0)$$

$$\Rightarrow\quad (n-1)(v^m/m)' = -\beta rv(r) + \frac{(n\beta-\alpha)}{r^{n-1}}\int_0^r \rho^{n-1}v(\rho)\,d\rho \quad \text{in } (0, R_0) \tag{1.10}$$

$$\Rightarrow\quad (n-1)v(r)^{m-1}|v'(r)| \leq \left(\beta + \frac{|n\beta-\alpha|}{n}\right)R_0v(0) \quad \text{in } (0, R_0)$$

$$\Rightarrow\quad (n-1)|v'(r)| \leq \left(\beta + \frac{|n\beta-\alpha|}{n}\right)R_0v(0)^{2-m} \quad \text{in } (0, R_0). \tag{1.11}$$

By (1.8), (1.9), (1.11), a contradiction arises. Hence no such sequence $\{r_i\}_{i=1}^{\infty}$ exists. Thus $R_0 = \infty$ and there exists a unique solution of (0.6), (0.7), in $(0,\infty)$.

<u>**Case 2**</u>: $\alpha < 0$.

By Lemma 1.1,

$$0 < v'(r) \le \frac{v}{|k|r} \quad \text{in } (0, R_0). \tag{1.12}$$

By (1.12) and an argument similar to the proof of case 2 of Theorem 1.3 of [Hs], there exists a constant $C > 0$ such that

$$0 < v'(r) \le Cv(r) \quad \forall 0 \le r < R_0.$$

Then

$$v(0) \le v(r) \le v(0)\exp(CR_0) \quad \forall 0 \le r < R_0 \tag{1.13}$$

and

$$0 < v'(r) \le Cv(0)\exp(CR_0) \quad \forall 0 \le r < R_0. \tag{1.14}$$

By (1.13) and (1.14) , a contradiction arises. Hence no such sequence $\{r_i\}_{i=1}^{\infty}$ exists. Thus $R_0 = \infty$ and there exists a unique solution of (0.6), (0.7), in $(0, \infty)$. By case 1, case 2, and (1.7) the lemma follows. □

Theorem 1.3. *Let $\eta > 0$ and m, n, α, β, satisfy* (0.11) *and* (0.13) *and let $v^{(m)}$ be the radially symmetric solution of* (0.2). *Then $v^{(m)}$ converges uniformly on every compact subset of $\mathbb{R}^n$ to the solution of* (0.14) *as $m \to 0$.*

Proof: If $\alpha = 0$, then $v^{(m)} \equiv \eta$ on $\mathbb{R}^n$ which satisfies (0.14) and we are done. Hence we may assume that $\alpha \ne 0$. Then by (0.11) and (0.13) there exists a constant $m_0' \in (0, (n-2)/n)$ such that (0.12) holds for any $0 < m \le m_0'$. Without loss of generality we may assume that $0 < m \le m_0'$ in the proof. Note that $v^{(m)}(x) = v^{(m)}(|x|)$ satisfies (0.6) and (0.7) in $(0, \infty)$. Let $\{m_i\}_{i=1}^{\infty}$ be a sequence such that $0 < m_i < m_0'$ for all $i \in \mathbb{Z}^+$ and $m_i \to 0$ as $i \to \infty$. We now divide the proof into two cases.

<u>Case 1</u>: $\alpha > 0$.

By the proof of Theorem 1.2, $v^{(m)}$ satisfies (1.9), (1.10), and (1.11) in $(0, \infty)$. Hence

$$0 < v^{(m)}(r) \le \eta \quad \forall r \ge 0, \tag{1.15}$$

$$\begin{aligned}
&(n-1)(v^{(m)m}/m)' = -\beta r v^{(m)}(r) + \frac{(\beta n - \alpha)}{r^{n-1}} \int_0^r \rho^{n-1} v^{(m)}(\rho)\, d\rho \quad \text{in } (0, \infty) \\
\Rightarrow\quad & \frac{v^{(m)}(r)^m - 1}{m} - \frac{\eta^m - 1}{m} \\
&= -\frac{\beta}{n-1} \int_0^r \rho v^{(m)}(\rho)\, d\rho + \frac{n\beta - \alpha}{n-1} \int_0^r \frac{1}{\sigma^{n-1}} \left(\int_0^\sigma \rho^{n-1} v^{(m)}(\rho)\, d\rho \right) d\sigma \quad \text{in } (0, \infty),
\end{aligned} \tag{1.16}$$

and for any $r_0 > 0$,

$$\begin{aligned}
&(n-1) \left| \frac{d}{dr} v^{(m)}(r) \right| \le \left(\beta + \frac{|n\beta - \alpha|}{n} \right) \eta^{2-m} r_0 \quad \forall 0 \le r \le r_0 \\
\Rightarrow\quad & |v^{(m)}(r_1) - v^{(m)}(r_2)| \le C_1 r_0 |r_1 - r_2| \quad \forall 0 \le r_1, r_2 \le r_0
\end{aligned} \tag{1.17}$$

where

$$C_1 = (n-1)^{-1}(2\beta + (|\alpha|/n))\max(1,\eta)^2.$$

By (1.15) and (1.17), the sequence $\{v^{(m_i)}\}_{i=1}^\infty$ is equi-Holder continuous on every compact subset of $[0,\infty)$. By the Ascoli Theorem the sequence $\{v^{(m_i)}\}_{i=1}^\infty$ has a subsequence which we may assume without loss of generality to be the sequence itself that converges uniformly on every compact subset of $[0,\infty)$ to some continuous function u as $i \to \infty$ and $u(0) = \eta$. By (1.17),

$$\begin{aligned} & |u(r) - u(0)| \le C_1 r_0 r \quad \forall 0 < r \le r_0 \\ \Rightarrow \quad & \limsup_{r\to 0}\left|\frac{u(r)-u(0)}{r}\right| \le C r_0 \quad \forall r_0 > 0 \\ \Rightarrow \quad & \lim_{r\to 0}\left|\frac{u(r)-u(0)}{r}\right| = 0 \quad \text{as } r_0 \to 0. \end{aligned}$$

Hence u is differentiable at $r = 0$ with $u'(0) = 0$. Thus

$$u(0) = \eta, \quad u'(0) = 0 \tag{1.18}$$

hold. By (1.17),

$$\begin{aligned} & v^{(m)}(r) \ge v^{(m)}(0) - (\eta/2) = \eta/2 \quad \forall 0 \le r \le \min(1, \eta/(2C_1)) \\ \Rightarrow \quad & u(r) \ge \eta/2 \quad \forall 0 \le r \le \min(1,\eta/(2C_1)) \quad \text{as } m = m_i \to \infty. \end{aligned} \tag{1.19}$$

By (1.19) there exists a maximal interval $(0, R_1)$ such that $u(r) > 0$ in $(0, R_1)$. Suppose $R_1 < \infty$. Then $u(R_1) = 0$. For any $0 < \delta < R_1$, since

$$\inf_{0\le r\le R_1-\delta} u(r) := c_0 > 0,$$

there exists $i_0 \in \mathbb{Z}^+$ such that

$$v^{(m_i)}(r) \ge c_0/2 \quad \forall 0 \le r \le R_1 - \delta, i \ge i_0. \tag{1.20}$$

By (1.15), (1.20), and the mean value theorem,

$$\begin{aligned} \left|\frac{v^{(m_i)}(r)^{m_i} - 1}{m_i} - \log u(r)\right| =& |e^{\xi_i}\log v^{(m_i)} - \log u(r)| \\ \le& e^{\xi_i}|\log v^{(m_i)} - \log u(r)| + |e^{\xi_i} - 1||\log u(r)| \\ \le& e^{m_i M}|\log v^{(m_i)} - \log u(r)| + |e^{\xi_i} - 1||\log u(r)| \\ \to& 0 \quad \text{uniformly on } [0, R_1-\delta] \quad \text{as } i \to \infty \end{aligned} \tag{1.21}$$

for some ξ_i satisfying $|\xi_i| \le m_i M$ for any $i \in \mathbb{Z}^+$ where $M = \max(|\log\eta|, |\log(c_0/2)|)$. Putting $m = m_i$ in (1.16) and letting $i \to \infty$, by (1.21),

$$(n-1)\log u(r) = -\beta\int_0^r \rho u(\rho)\,d\rho + (n\beta - \alpha)\int_0^r \frac{1}{\sigma^{n-1}}\left(\int_0^r \sigma^{n-1}u(\rho)\,d\rho\right)d\sigma \quad \text{in } (0, R_1). \tag{1.22}$$

Since the right hand side of (1.22) is a differentiable function of $r \in [0, R_1)$, $u(r)$ is a differentiable function of $r \in [0, R_1)$. Differentiating (1.22) with respect to r,

$$(n-1)\frac{u'(r)}{u(r)} = -\beta r u(r) + \frac{n\beta - \alpha}{r^{n-1}} \int_0^r \rho^{n-1} u(\rho)\, d\rho \quad \text{in } (0, R_1) \tag{1.23}$$

$$\Rightarrow \quad (n-1)\left(\frac{r^{n-1}u'}{u}\right) = -\alpha \int_0^r \rho^{n-1} u(\rho)\, d\rho - \beta \int_0^r \rho^n u'(\rho)\, d\rho \quad \text{in } (0, R_1). \tag{1.24}$$

Since the right hand side of (1.24) is a differentiable function of $r \in [0, R_1)$, $u'(r)/u(r)$ is a differentiable function of $r \in [0, R_1)$. Differentiating (1.24) with respect to r,

$$(n-1)\left(\frac{u'}{u}\right)' + \frac{n-1}{r}\frac{u'}{u} = (n-1)\frac{1}{r^{n-1}}\left(\frac{r^{n-1}u'}{u}\right)' = -\alpha u - \beta r u' \quad \text{in } (0, R_1). \tag{1.25}$$

Hence u is a classical solution of (1.25) and satisfies (1.18). By Theorem 1.3 of [Hs] and a rescaling there exists a unique positive solution $\overline{u}$ of (1.25) in $[0, \infty)$ that satisfies (1.18). By uniqueness of solution,

$$u(r) \equiv \overline{u}(r) \quad \forall 0 \le r \le R_1 \quad \Rightarrow \quad u(R_1) \equiv \overline{u}(R_1) > 0$$

and contradiction arises. Hence $R_1 = \infty$ and $u(r) > 0$ for all $r \ge 0$. By the above argument the solution u satisfies (1.25) and (1.18) and $u(r) \equiv \overline{u}(r)$ is the a unique positive solution $\overline{u}$ of (1.25) in $[0, \infty)$ that satisfies (1.18). Since the sequence $\{m_i\}_{i=1}^\infty$ is arbitrary, $v^{(m)}$ converges uniformly on every compact subset of $\mathbb{R}^n$ to the solution u of (0.14) as $m \to 0$.

Case 2: $\alpha < 0$.

By the proof of Theorem 1.2, $v^{(m)}$ satisfies (1.10) and (1.12) in $(0, \infty)$. We choose $m_0 \in (0, m_0']$ such that

$$\frac{1}{2} \le (2\eta)^m \le 2 \quad \forall 0 < m \le m_0 \tag{1.26}$$

and let $r_0 = \min((8C_2\eta)^{-\frac{1}{2}}, (8C_2\eta)^{-1})$ where $C_2 = (2\beta + (|\alpha|/n))/(n-1)$. Let

$$r_m = \sup\{\delta' > 0 : v(r) \le 2\eta \quad \forall 0 \le r \le \delta'\}.$$

By (0.7), $r_m > 0$. We claim that

$$r_m \ge r_0 \quad \forall 0 < m \le m_0. \tag{1.27}$$

Suppose (1.27) does not hold. Then there exists $m' \in (0, m_0]$ such that $r_{m'} < r_0$. Then by (1.10) and (1.26),

$$\left|\frac{dv^{(m')}}{dr}(r)\right| \le \frac{1}{n-1}\left(\beta r v(r) + \frac{n\beta - \alpha}{r^{n-1}} \int_0^r \rho^{n-1} v(\rho)\, d\rho\right) v(r)^{1-m'} \quad \forall 0 \le r \le r_{m'}$$

$$\Rightarrow \quad \left|\frac{dv^{(m')}}{dr}(r)\right| \le (n-1)^{-1}(2\beta + (|\alpha|/n))(2\eta)^{2-m'} r = 8C_2\eta^2 r \quad \forall 0 \le r \le r_{m'} \tag{1.28}$$

$$\Rightarrow \quad v^{(m')}(r) \le \eta + 4C_2\eta^2 r_0^2 \le 3\eta/2 \quad \forall 0 \le r \le r_{m'}. \tag{1.29}$$

By (1.29) and continuity there exists a constant $\delta_1 > 0$ such that $v^{(m')}(r) \le 2\eta$ in $[0, r_{m'} + \delta_1]$. This contradicts the choice of $r_{m'}$. Hence no such m' exists and (1.27) holds. By (1.12) and (1.28),

$$0 \le \frac{dv^{(m)}}{dr}(r) \le 8C_2\eta^2 r_0 = \eta \quad \forall 0 \le r \le r_0, 0 < m \le m_0 \tag{1.30}$$

$$\Rightarrow \quad \eta \le v^{(m)}(r) \le 2\eta \quad \forall 0 \le r \le r_0, 0 < m \le m_0. \tag{1.31}$$

By (1.12) and (1.31),

$$v^{(m)}(r_0) \le v^{(m)}(r) \le v^{(m)}(r_0)(r/r_0)^{\frac{1}{|k|}} \quad \forall r \ge r_0, 0 < m \le m_0$$

$$\Rightarrow \quad \eta \le v^{(m)}(r) \le 2\eta(r/r_0)^{\frac{1}{|k|}} \quad \forall r \ge r_0, 0 < m \le m_0. \tag{1.32}$$

By (1.12), (1.30), (1.31) and (1.32), for any $r_1 > 0$ there exists a constant $M_{r_1} > 0$ such that

$$\begin{cases} 0 \le \dfrac{dv^{(m)}}{dr}(r) \le M_{r_1} \quad \forall 0 \le r \le r_1, 0 < m \le m_0 \\ \eta \le v^{(m)}(r) \le M_{r_1} \quad \forall 0 \le r \le r_1, 0 < m \le m_0. \end{cases} \tag{1.33}$$

By (1.33) the sequence $\{m_i\}_{i=1}^\infty$ is equi-Holder continuous on every compact subset of $[0, \infty)$. By the Ascoli theorem the sequence $\{m_i\}_{i=1}^\infty$ has a subsequence which we may assume without loss of generality to be the sequence itself that converges uniformly to some continuous function u on every compact subset of $[0, \infty)$ as $i \to \infty$. By an argument similar to the proof of case 1 u is the a unique positive solution $\overline{u}$ of (1.25) in $[0, \infty)$ that satisfies (1.18). Since the sequence $\{m_i\}_{i=1}^\infty$ is arbitrary, $v^{(m)}$ converges uniformly on every compact subset of $\mathbb{R}^n$ to the solution u of (0.14) as $m \to 0$ and the theorem follows. □

2 Exact decay rate for $\alpha = \frac{2\beta}{1-m} > 0$

In this section we will prove the exact decay rate (0.16) for the radially symmetric solution v of (0.2) when (0.9) and (0.15) hold. We let

$$h(r) = v + \frac{1-m}{2} r v'(r) \quad \text{and} \quad w(r) = r^2 v(r)^{1-m}.$$

Lemma 2.1. *Let $\eta > 0$, and α, β, m, satisfies* (0.11) *and*

$$\frac{2\beta}{1-m} \ge \alpha > 0. \tag{2.1}$$

Let v be the radially symmetric solution of (0.2). *Then $h(r) > 0$ for any $r \ge 0$ and $w'(r) > 0$ for any $r > 0$.*

Proof: By direct computation,

$$
\begin{aligned}
&h'(r)+\left(\frac{n-2-mn}{(1-m)r}-(1-m)\frac{v'}{v}+\frac{\beta}{(n-1)}rv^{1-m}\right)h\\
=&\frac{n-2-mn}{1-m}\cdot\frac{v}{r}+\frac{1}{n-1}\left(\frac{2\beta}{1-m}-\alpha\right)rv^{2-m}\ge 0\quad\forall r>0.
\end{aligned}\tag{2.2}
$$

Let f be given by (1.5). Then by (2.2),

$$
(r^{\frac{n-2-mn}{1-m}}f(r)h(r))'\ge 0\quad\Rightarrow\quad h(r)>0\quad\forall r\ge 0.
$$

Hence

$$
w'(r)=2rv(r)^{-m}h(r)>0\quad\forall r>0
$$

and the lemma follows. □

Let $\eta>0$, and let m, α, β, ρ_1, satisfy (0.8) and (0.9). Suppose v is a radially symmetric solution of (0.2). Let $s=\log r$ and $v_1=w^{\frac{1}{1-m}}$. Then v_1 satisfies

$$
(v_1^m)_{ss}+\frac{n-2-(n+2)m}{1-m}(v_1^m)_s-\frac{2m(n-2-nm)}{(1-m)^2}v_1^m+\frac{m\beta}{n-1}v_{1,s}+\frac{m\rho_1}{(1-m)(n-1)}v_1=0 \tag{2.3}
$$

in $(-\infty,\infty)$ and w satisfies

$$
w_{ss}=\frac{1-2m}{1-m}\cdot\frac{w_s^2}{w}-\frac{n-2-(n+2)m}{1-m}w_s-\frac{\beta}{n-1}ww_s-\frac{\rho_1}{n-1}w^2+\frac{2(n-2-nm)}{1-m}w \tag{2.4}
$$

in $(-\infty,\infty)$ or equivalently

$$
w_{rr}+\left(1+\frac{n-2-(n+2)m}{1-m}\right)\frac{w_r}{r}-\frac{1-2m}{1-m}\cdot\frac{w_r^2}{w}+\frac{\beta}{n-1}\frac{ww_r}{r}+\frac{\rho_1}{n-1}\frac{w^2}{r^2}-\frac{2(n-2-nm)}{1-m}\frac{w}{r^2}=0 \tag{2.5}
$$

in $(0,\infty)$. When $\rho_1=0$, (2.3), (2.4), and (2.5) reduce to

$$
(v_1^m)_{ss}+\frac{n-2-(n+2)m}{1-m}(v_1^m)_s-\frac{2m(n-2-nm)}{(1-m)^2}v_1^m+\frac{m\beta}{n-1}v_{1,s}=0\quad\text{in }(-\infty,\infty), \tag{2.6}
$$

$$
w_{ss}=\frac{1-2m}{1-m}\cdot\frac{w_s^2}{w}-\frac{n-2-(n+2)m}{1-m}w_s-\frac{\beta}{n-1}ww_s+\frac{2(n-2-nm)}{1-m}w\quad\text{in }(-\infty,\infty) \tag{2.7}
$$

and

$$
w_{rr}+\left(1+\frac{n-2-(n+2)m}{1-m}\right)\frac{w_r}{r}-\frac{1-2m}{1-m}\cdot\frac{w_r^2}{w}+\frac{\beta}{n-1}\frac{ww_r}{r}-\frac{2(n-2-nm)}{1-m}\frac{w}{r^2}=0 \tag{2.8}
$$

in $(0,\infty)$.

Lemma 2.2. *Let $\eta > 0$ and let m, α, β, satisfy* (0.9) *and* (0.15). *Let v be the radially symmetric solution of* (0.2). *Then there exist constants $C_1 > 0$, $C_2 > 0$, $C_3 > 0$, such that*

$$\frac{rw_r(r)}{w(r)} \le C_1 \quad \forall r \ge 0 \tag{2.9}$$

and

$$C_2 \le rw_r(r) \le C_3 \quad \forall r \ge 1. \tag{2.10}$$

Moreover

$$w(r) \to \infty \qquad \text{as } r \to \infty. \tag{2.11}$$

Proof: Note that $v_1(-\infty) = v_{1,s}(-\infty) = 0$ and by Lemma 2.1 $v_{1,s} > 0$ on $(-\infty, \infty)$. Let

$$b_0 = \frac{n-2-(n+2)m}{1-m} \quad \text{and} \quad b_1 = \frac{2m(n-2-nm)}{(1-m)^2}. \tag{2.12}$$

If $b_0 \ge 0$, then by (2.6),

$$(v_1^m)_{ss} - b_1 v_1^m \le 0 \quad \Rightarrow \quad (v_1^m)_s \le b_1 v_1^m \quad \Rightarrow \quad \frac{rw_r(r)}{w(r)} \le \frac{(1-m)b_1}{m} \quad \forall r \ge 0$$

and (2.9) follows.

If $b_0 < 0$, by (2.6),

$$(v_1^m)_{ss} + b_0 (v_1^m)_s - b_1 v_1^m \le 0. \tag{2.13}$$

Let $p = (v_1^m)_s / v_1^m$. Then by (2.13),

$$p_s = \frac{(v_1^m)_{ss}}{v_1^m} - \frac{(v_1^m)_s^2}{v_1^{2m}} \le |b_0| p + b_1 - p^2 = -(p - (|b_0|/2))^2 + b_1 + (b_0^2/4). \tag{2.14}$$

Let

$$b_2 = \max\left(\frac{3m}{1-m}, \sqrt{b_1 + b_0^2} + |b_0|\right).$$

We claim that

$$p(s) \le b_2 \quad \forall s \in \mathbb{R}. \tag{2.15}$$

Suppose (2.15) does not hold. Then there exists $s_0 \in \mathbb{R}$ such that $p(s_0) > b_2$. Since

$$p = m\frac{v_{1,s}}{v_1} = \frac{m}{1-m}\frac{w_s}{w} = \frac{m}{1-m}\frac{rw_r}{w} = \frac{2m}{1-m}\left(1 + \frac{1-m}{2} \cdot \frac{rv_r(r)}{v(r)}\right), \tag{2.16}$$

$p(s = -\infty) = 2m/(1-m)$. Let $s_1 = \inf\{s' < s_0 : p(s) > b_2 \quad \forall s' \le s \le s_0\}$. Then $-\infty < s_1 < s_0$, $p(s) > b_2$ for any $s \in (s_1, s_0)$, and $p(s_1) = b_2$. By (2.14), $p_s(s) < 0$ for any $s \in (s_1, s_0)$. Hence $p(s_0) \le p(s_1) = b_2$. Thus contradiction arises and (2.15) follows. Then by (2.15) and (2.16), (2.9) holds with $C_1 = b_2/m$.

Let

$$a_1 = \frac{2(n-2-nm)}{1-m}, \quad a_2 = \frac{\beta}{(n-1)a_1}, \quad \text{ and } a_3 = a_1^{-1}\max(|b_0|, |1-2m|/(|1-m|w(1))).$$

Since $w_s > 0$ for any $s \in \mathbb{R}$, $w(s) \ge w(1)$ for any $s \ge 1$. Then by (2.7),

$$w_{ss} \ge a_1((1-a_2w_s)w - a_3(w_s + w_s^2)) \quad \forall s \ge 1. \tag{2.17}$$

Suppose $w_s \le C_2' := \min(1, (2a_2)^{-1}, w(1)/(8a_3))$ for all $s \ge 1$. Then by (2.17),

$$w_{ss} \ge a_1 w(1)/4 > 0 \quad \forall s \ge 1. \tag{2.18}$$

Hence $w_s \to \infty$ as $s \to \infty$ and contradiction arises. Thus there exists $s_1 > 1$ such that $w_s(s_1) > C_2'$. Suppose there exists $s_2 > s_1$ such that $w_s(s_2) < C_2'$. Let $s_3 = \inf\{s' < s_2 : w_s(s) < C_2' \quad \forall s' \le s \le s_2\}$. Then $s_1 < s_3 < s_2$ and $w_s(s_3) = C_2'$. Then by the above argument (2.18) holds in (s_3, s_2). Hence $w_s(s_2) > w_s(s_3) = C_2'$ and contradiction arises. Thus $w_s(s) \ge C_2'$ for any $s \ge s_1$. Since $w_s(s) > 0$ for all $s \in \mathbb{R}$, the left hand side of (2.10) holds with $C_2 = \min(C_2', \min_{[0,s_1]} w_s(s)) > 0$ and (2.11) holds.

Let

$$a_4 = \frac{\beta}{3(n-1)} \quad \text{ and } \quad a_5 = \frac{\beta(1-m)}{3(n-1)}.$$

By (2.7) and (2.9),

$$w_{ss} \le \begin{cases} (|b_0| - a_4 w)w_s + a_1 w(1-(a_2/3)w_s) + (1-m)^{-1}w_s[(1-2m)C_1 - a_5 w] & \text{if } 0 < m < 1/2 \\ (|b_0| - a_4 w)w_s + a_1 w(1-(a_2/3)w_s) & \text{if } 1/2 \le m < (n-2)/n. \end{cases} \tag{2.19}$$

By (2.11) there exists a constant $s_0 > 0$ such that

$$w > \max((1-2m)C_1/a_5, |b_0|/a_4) \quad \forall s \ge s_0. \tag{2.20}$$

By (2.19) and (2.20),

$$w_{ss} \le a_1 w(1-(a_2/3)w_s) \quad \forall s \ge s_0. \tag{2.21}$$

We claim that there exists a constant $s_1' > s_0$ such that

$$w_s \le C_3' := \max(5/a_2, 2w_s(s_0)) \quad \forall s \ge s_1'. \tag{2.22}$$

Suppose (2.22) does not hold. Then there exists a constant $s_2' > s_0$ such that

$$w_s(s_2') > C_3'.$$

Let $s_3' = \inf\{s_0 \le t_0 < s_2' : w_s(s) > C_3' \quad \forall t_0 \le s \le s_2'\}$. Then $s_0 < s_3' < s_2'$, $w_s > C_3'$ for any $s_3' < s < s_2'$, and $w_s(s_3') = C_3'$. Then by (2.21) $w_{ss} < 0$ in (s_3', s_2'). Hence $w_s(s_2') \le w_s(s_3') = C_3'$ and contradiction arises. Thus no such constant s_2' exists and there exists a constant $s_1' > s_0$ such that (2.22) holds. Then the right hand side of (2.10) holds with $C_3 = \max(C_3', \max_{[0,s_1']} w_s(s)) > 0$. □

Theorem 2.3. *Let $\eta > 0$ and let m, α, β, satisfy* (0.9) *and* (0.15). *Let v be the radially symmetric solution of* (0.2). *Then* (0.16) *holds.*

Proof: Let $q(r) = rw_r(r)$,

$$a_0 = \frac{2(n-2-nm)(n-1)}{(1-m)\beta}, \quad q_1 = q - a_0,$$

and let b_0 be given by (2.12). By (2.8),

$$(r^{b_0}q(r)w(r)^{\frac{2m-1}{1-m}})' = \frac{\beta}{n-1}\cdot\frac{w^{\frac{m}{1-m}}}{r^{1-b_0}}(a_0 - q(r)) \quad \forall r > 0 \tag{2.23}$$

$$\Rightarrow \quad q_r + \frac{b_0}{r}q + \frac{\beta}{n-1}\frac{w}{r}(q - a_0) = \frac{1-2m}{1-m}\cdot\frac{q^2}{rw} \quad \forall r > 0$$

$$\Rightarrow \quad q_{1,r} + \frac{b_0}{r}q_1 + \frac{\beta}{n-1}\frac{w}{r}q_1 = \frac{1-2m}{1-m}\cdot\frac{q^2}{rw} - \frac{b_0 a_0}{r} \quad \forall r > 0. \tag{2.24}$$

Since

$$r^{b_0}q(r)w(r)^{\frac{2m-1}{1-m}} = r^{b_0}\cdot(r^2v(r)^{1-m})^{\frac{2m-1}{1-m}}\cdot 2r^2v(r)^{-m}h(r) = 2r^{\frac{n-2-nm}{1-m}}v(r)^m\left(1 + \frac{1-m}{2}\cdot\frac{rv_r(r)}{v(r)}\right)$$

$$\Rightarrow \quad \lim_{r\to 0} r^{b_0}q(r)w(r)^{\frac{2m-1}{1-m}} = 0,$$

integrating (2.23) over $(0, r)$,

$$r^{b_0}q(r)w(r)^{\frac{2m-1}{1-m}} = \frac{\beta}{n-1}\int_0^r \rho^{b_0-1}w(\rho)^{\frac{m}{1-m}}(a_0 - q(\rho))\,d\rho \tag{2.25}$$

$$\Rightarrow \quad q(r) = \frac{\beta}{n-1}\cdot\frac{\int_0^r \rho^{b_0-1}w(\rho)^{\frac{m}{1-m}}(a_0 - q(\rho))\,d\rho}{r^{b_0}w(r)^{\frac{2m-1}{1-m}}} \tag{2.26}$$

Let $\{r_i\}_{i=1}^\infty$ be a sequence of positive numbers such that $r_i \to \infty$ as $i \to \infty$. By Lemma 2.2 there exist constants $C_1 > 0$, $C_2 > 0$, $C_3 > 0$, such that (2.9) and (2.10) holds. Then by (2.10) the sequence $\{r_i\}_{i=1}^\infty$ has a subsequence which we may assume without loss of generality to be the sequence itself such that $q(r_i) \to q_\infty$ as $i \to \infty$ for some constant q_∞ satisfying

$$C_2 \le q_\infty \le C_3. \tag{2.27}$$

Suppose $q_\infty \ne a_0$. Let

$$f_1(r) = \exp\left(\frac{\beta}{n-1}\int_0^r \rho^{-1}w(\rho)\,d\rho\right).$$

We now divide the proof into three cases.
Case 1: $(n-2)/(n+2) < m = 1/2 < (n-2)/n$.
Then $b_0 < 0$. Since $m = 1/2$, by (2.24),

$$(r^{b_0}f_1(r)q_1(r))' \le 0 \quad \Rightarrow \quad q_1(r) \le 0 \quad \Rightarrow \quad 0 \le q(r) \le a_0 \quad \forall r > 0. \tag{2.28}$$

Hence $q_\infty < a_0$. Then by (2.26) and (2.28),

$$q(r_i) = \frac{\beta}{n-1} r_i^{|b_0|} \int_0^{r_i} \rho^{b_0-1} w(\rho)^{\frac{m}{1-m}} (a_0 - q(\rho))\, d\rho \to \infty \quad \text{as } i \to \infty$$

and contradiction arises. Hence $q_\infty = a_0$.
<u>Case 2</u>: $(n-2)/(n+2) < m < (n-2)/n$ and $m \neq 1/2$.
Since $b_0 < 0$, by Lemma 2.2, (2.25), (2.26), (2.27), and the l'Hosiptal rule,

$$\begin{aligned}
q_\infty =& |\lim_{i\to\infty} q(r_i)| = \frac{\beta}{n-1} \left| \lim_{i\to\infty} \frac{r_i^{|b_0|} \int_0^{r_i} \rho^{b_0-1} w(\rho)^{\frac{m}{1-m}} (a_0 - q(\rho))\, d\rho}{w(r_i)^{\frac{2m-1}{1-m}}} \right| \\
=& \frac{\beta}{n-1} \left| \lim_{i\to\infty} \frac{|b_0| r_i^{|b_0|-1} \int_0^{r_i} \rho^{b_0-1} w(\rho)^{\frac{m}{1-m}} (a_0 - q(\rho))\, d\rho + r_i^{-1} w(r_i)^{\frac{m}{1-m}} (a_0 - q(r_i))}{\frac{2m-1}{1-m} w(r_i)^{\frac{3m-2}{1-m}} w_r(r_i)} \right| \\
=& \frac{\beta(1-m)}{(n-1)|2m-1|} q_\infty^{-1} \left| \lim_{i\to\infty} [|b_0|(n-1)\beta^{-1} q_\infty w(r_i) + w(r_i)^2 (a_0 - q_\infty)] \right| \\
=& \infty.
\end{aligned}$$

Hence contraction arises. Thus $q_\infty = a_0$.
<u>Case 3</u>: $0 < m \le (n-2)/(n+2)$.
Then $b_0 \ge 0$. By (2.24),

$$\begin{aligned}
& r^{b_0} f_1(r) q_1(r) = f_1(1) q_1(1) - a_0 b_0 \int_1^r \rho^{b_0-1} f_1(\rho)\, d\rho + \frac{1-2m}{1-m} \int_1^r \frac{\rho^{b_0-1} q(\rho)^2 f_1(\rho)}{w(\rho)}\, d\rho \quad \forall r \ge 1 \\
\Rightarrow \quad & q_1(r) = \frac{f_1(1) q_1(1) - a_0 b_0 \int_1^r \rho^{b_0-1} f_1(\rho)\, d\rho + \frac{1-2m}{1-m} \int_1^r \frac{\rho^{b_0-1} q(\rho)^2 f_1(\rho)}{w_1(\rho)}\, d\rho}{r^{b_0} f_1(r)} \quad \forall r \ge 1. \qquad (2.29)
\end{aligned}$$

By (2.9), (2.11), and the l'Hosiptal rule,

$$\liminf_{r\to\infty} \frac{f_1(r)}{w(r)} = \frac{\beta}{n-1} \liminf_{r\to\infty} \frac{r^{-1} w(r) f_1(r)}{w'(r)} \ge \frac{\beta}{(n-1)C_1} \liminf_{r\to\infty} f_1(r) = \infty.$$

Thus there exists a constant $R_1 > 1$ such that

$$\frac{f_1(r)}{w(r)} \ge 1 \quad \forall r \ge R_1. \qquad (2.30)$$

By (2.10) and (2.30),

$$\int_1^r \frac{\rho^{b_0-1} q(\rho)^2 f_1(\rho)}{w(\rho)}\, d\rho \ge C \int_{R_1}^r \rho^{b_0-1}\, d\rho \to \infty \quad \text{as } r \to \infty. \qquad (2.31)$$

On the other hand

$$\lim_{r\to\infty} \frac{\int_1^r \rho^{b_0-1} f_1(\rho)\, d\rho}{r^{b_0} f_1(r)} = \lim_{r\to\infty} \frac{r^{b_0-1} f_1(r)}{b_0 r^{b_0-1} f_1(r) + \beta(n-1)^{-1} r^{b_0-1} w(r) f_1(r)} = \lim_{r\to\infty} \frac{1}{b_0 + \beta(n-1)^{-1} w(r)} = 0. \qquad (2.32)$$

By (2.27), (2.29), (2.31), (2.32) and the l'Hosiptal rule,

$$\begin{aligned}\lim_{i\to\infty} q_1(r_i) =&\lim_{i\to\infty}\frac{f_1(1)q_1(1) - a_0b_0\int_1^r \rho^{b_0-1}f_1(\rho)\,d\rho + \frac{1-2m}{1-m}\int_1^{r_i}\frac{\rho^{b_0-1}q(\rho)^2f_1(\rho)}{w(\rho)}\,d\rho}{r_i^{b_0}f_1(r_i)}\\ =&\frac{1-2m}{1-m}\lim_{i\to\infty}\frac{r_i^{b_0-1}q(r_i)^2w(r_i)^{-1}f_1(r_i)}{b_0r_i^{b_0-1}f_1(r_i)+\beta(n-1)^{-1}r_i^{b_0-1}w(r_i)f_1(r_i)}\\ =&\frac{1-2m}{1-m}\lim_{i\to\infty}\frac{q(r_i)^2w(r_i)^{-2}}{b_0w(r_i)^{-1}+\beta(n-1)^{-1}}\\ =&0\end{aligned}$$

Hence $q_\infty = a_0$.

By case 1, case 2, and case 3, $q(r_i) \to a_0$ as $i \to \infty$. Since the sequence $\{r_i\}_{i=1}^{\infty}$ is arbitrary, $q(r) \to a_0$ as $r \to \infty$. □

Corollary 2.4. *The metric* $g_{ij} = v^{\frac{4}{n+2}}dx^2$, $n \geq 3$, *of a locally conformally flat gradient steady Yamabe soliton where* v *satisfies* (0.2) *has the exact decay rate* (0.17).

By Theorem 1.3, Theorem 2.3, and the result of [Hs] we have the following result.

Corollary 2.5. *Let* $\beta > 0$, $\eta > 0$, *and* $n \geq 3$. *For any* $0 < m < (n-2)/n$, *let* $\alpha_m = 2\beta/(1-m)$ *and let* $v^{(m)}$ *be the radially solution of* (0.2) *with* $\alpha = \alpha_m$. *Then*

$$\lim_{|x|\to\infty}\lim_{m\to 0}\frac{|x|^2v^{(m)}(x)^{1-m}}{\log|x|} = \lim_{m\to 0}\lim_{|x|\to\infty}\frac{|x|^2v^{(m)}(x)^{1-m}}{\log|x|} = \frac{2(n-1)(n-2)}{\beta}.$$

3 Decay rate for $\frac{2\beta}{1-m} > \max(\alpha, 0)$

In this section we will use a modification of the technique of [Hs] to prove the decay rate (0.18) of the radially symmetric solution of (0.2) when (0.9) and (0.19) hold.

Theorem 3.1. *Let* $\eta > 0$ *and let* m, n, α, β, *satisfies* (0.9) *and* (0.19). *Let* v *be the solution of* (0.2). *Then* (0.18) *holds for some constant* $A > 0$.

Proof: Let $q(r) = r^{\alpha/\beta}v(r)$. Then by Lemma 1.1,

$$q'(r) = (\alpha/\beta)r^{\frac{\alpha}{\beta}-1}(v(r) + krv'(r)) > 0 \quad \forall r > 0. \tag{3.1}$$

By direct computation,

$$\left(\frac{q'}{q}\right)' + \frac{n-1-(2m\alpha/\beta)}{r}\cdot\frac{q'}{q} + m\left(\frac{q'}{q}\right)^2 + \frac{\beta r^{1-\frac{\alpha}{\beta}(1-m)}q'}{(n-1)q^m} = \frac{\alpha}{\beta}\cdot\frac{n-2-(m/k)}{r^2}. \tag{3.2}$$

Let

$$f_2(r) = \exp\left(\frac{\beta}{n-1}\int_1^r \rho^{1-\frac{\alpha}{\beta}(1-m)} q(\rho)^{1-m}\, d\rho\right).$$

Then $f_2'(r) = (n-1)^{-1}\beta r^{1-\frac{\alpha}{\beta}(1-m)} q(r)^{1-m} f_2(r)$ and

$$f_2(r) \ge \exp\left(\frac{\beta q(1)^{1-m}}{n-1}\int_1^r \rho^{1-\frac{\alpha}{\beta}(1-m)}\, d\rho\right) = \exp\left(c_0(r^{2-\frac{\alpha}{\beta}(1-m)}-1)\right) \to \infty \quad \text{as } r\to\infty. \tag{3.3}$$

where

$$c_0 = \frac{\beta q(1)^{1-m}}{(n-1)(2-\frac{\alpha}{\beta}(1-m))}.$$

Let $c_1 = q(1)^{m-1}q'(1)f_2(1)$ and $c_2 = (\alpha/\beta)(n-2-(m/k))$. Multiplying (3.2) by $r^{n-1-(2m\alpha/\beta)}q(r)^m f_2(r)$ and integrating over $(1,r)$,

$$r^{n-1-(2m\alpha/\beta)}q(r)^m f_2(r)\cdot\frac{q'(r)}{q(r)} = c_1 + c_2\int_1^r \rho^{n-3-(2m\alpha/\beta)}q(\rho)^m f_2(\rho)\, d\rho \quad \forall r>1. \tag{3.4}$$

By (3.3), (3.4), and the l'Hosiptal rule,

$$\begin{aligned}\limsup_{r\to\infty} r^p\frac{q'(r)}{q(r)} \le& \limsup_{r\to\infty}\frac{c_1 + c_2\int_1^r \rho^{n-3-(m\alpha/\beta)}q(\rho)^m f_2(\rho)\, d\rho}{r^{n-p-1-(2m\alpha/\beta)}q(r)^m f_2(r)}\\ \le& c_2\limsup_{r\to\infty}\frac{r^{n-3-(2m\alpha/\beta)}q(r)^m f_2(r)}{F(r)} \quad \forall p>0 \end{aligned} \tag{3.5}$$

where

$$\begin{aligned}F(r) =&(n-p-1-(2m\alpha/\beta))r^{n-p-2-(2m\alpha/\beta)}q(r)^m f_2(r) + mr^{n-p-1-(2m\alpha/\beta)}q(r)^{m-1}q'(r)f_2(r)\\ &+ r^{n-p-1-(2m\alpha/\beta)}q(r)^m f_2'(r)\\ \ge&(n-p-1-(2m\alpha/\beta))r^{n-p-2-(2m\alpha/\beta)}q(r)^m f_2(r) + r^{n-p-1-(2m\alpha/\beta)}q(r)^m f_2'(r)\\ =&(n-p-1-(2m\alpha/\beta))r^{n-p-2-(2m\alpha/\beta)}q(r)^m f_2(r) + (n-1)^{-1}\beta r^{n-p-(\alpha/\beta)(1+m)}q(r)f_2(r) \end{aligned} \tag{3.6}$$

By (3.1), (3.5) and (3.6),

$$\begin{aligned}0 \le& \limsup_{r\to\infty} r^p\frac{q'(r)}{q(r)}\\ \le& c_2\limsup_{r\to\infty}\frac{r^{n-3-(2m\alpha/\beta)}q(r)^m f_2(r)}{(n-p-1-\frac{2m\alpha}{\beta})r^{n-p-2-(2m\alpha/\beta)}q(r)^m f_2(r) + \beta(n-1)^{-1}r^{n-p-(\alpha/\beta)(1+m)}q(r)f_2(r)}\\ \le& c_2\limsup_{r\to\infty}\frac{1}{(n-p-1-\frac{2m\alpha}{\beta})r^{1-p} + \beta(n-1)^{-1}r^{3-p-(\alpha/\beta)(1-m)}q(r)^{1-m}}\\ =&0 \qquad \forall 1<p<3-(\alpha/\beta)(1-m).\end{aligned}$$

Hence

$$\lim_{r\to\infty} r^p \frac{q'(r)}{q(r)} = 0 \quad \forall 1 < p < 3 - (\alpha/\beta)(1-m). \tag{3.7}$$

Let $p_0 = 2 - (\alpha/2\beta)(1-m)$. By (0.19), $1 < p_0 < 3 - (\alpha/\beta)(1-m)$. By (3.7),

$$|\log q(r) - \log q(1)| \le \int_1^r |(\log q)'(\rho)|\, d\rho \le C \int_1^r \rho^{-p_0}\, d\rho \le C_3 \quad \forall r \ge 1. \tag{3.8}$$

for some constant $C_3 > 0$. Hence

$$e^{-C_3} q(1) \le q(r) \le e^{C_3} q(1) \quad \forall r \ge 1. \tag{3.9}$$

By (3.1) and (3.9), $q(r)$ increases to some constant $A \in \mathbb{R}$ as $r \to \infty$ and the theorem follows.
□

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