ISOMORPHISMS AND STRICTLY SINGULAR OPERATORS IN MIXED TSIRELSON SPACES

DENKA KUTZAROVA, ANTONIS MANOUSSAKIS, AND ANNA PELCZAR-BARWACZ

Abstract. We study the family of isomorphisms and strictly singular operators in mixed Tsirelson spaces and their modified versions setting. We show sequential minimality of modified mixed Tsirelson spaces $T_M[(\mathcal{S}_n, \theta_n)]$ satisfying some regularity conditions and present results on existence of strictly singular non-compact operators on subspaces of mixed Tsirelson spaces defined by the families $(\mathcal{A}_n)_n$ and $(\mathcal{S}_n)_n$.

Introduction

In the celebrated paper [20] W.T. Gowers started his classification program for Banach spaces. The goal is to identify classes of Banach spaces which are

(1) hereditary, i.e. if a space belongs to a given class, then all of its closed infinite dimensional subspaces as well,
(2) inevitable, i.e. any Banach space contains an infinite dimensional subspace in one of those classes,
(3) defined in terms of richness of family of bounded operators in the space.

The famous Gowers' dichotomy brought first two classes: spaces with unconditional basis and hereditary indecomposable spaces. The further classification, described in terms of isomorphisms, concerned minimality and strict quasiminimality. A Banach space X is *minimal* if every closed infinite dimensional subspace of X contains a further subspace isomorphic to X. A Banach space X is called *quasiminimal* if any two infinite dimensional subspaces Y, Z of X contain further isomorphic subspaces. The classical spaces ℓ_p, $1 \leq p < \infty$, c_0 are minimal and the Tsirelson space $T[\mathcal{S}_1, 1/2]$ is the first known strictly quasiminimal space (i.e. without minimal subspaces), [15]. The results of W.T. Gowers lead to the question of the refinement of the classes and classification of already known Banach space. Further step in the first direction was made by the third named author, [30], who proved that a strictly quasiminimal Banach space contains a subspace with no subsymmetric sequence. An extensive refinement of list of the classes and study of exampes were made recently by V. Ferenczi and C. Rosendal [16, 17].

The mixed Tsirelson spaces $T[(\mathcal{M}_n, \theta_n)_n]$, for $\mathcal{M}_n = \mathcal{A}_n$ or $\mathcal{S}_n$, as the basic examples of spaces not containing ℓ_p or c_0, form a natural class to be studied with respect to the classification program. The first step was made by T. Schlumprecht, [5], who proved that his famous space $S = T[(\mathcal{A}_n, 1/\log_2(n+1))_n]$ is complementably minimal. The result of

Key words and phrases. quasiminimality, strictly singular operator, mixed Tsirelson space .

The research of the third author was supported by the Polish Ministry of Science and Higher Education grant N N201 421739.

Schlumprecht holds for a certain class of mixed Tsirelson spaces $T[(\mathcal{A}_{k_n}, \theta_n)_n]$ by [27]. On the other hand, the Tzafriri's space $T[(\mathcal{A}_n, c/\sqrt{n})_n]$ [34] is not minimal by [21]. However the original Tsirelson space $T[\mathcal{S}_1, 1/2]$ is not minimal [15], every its normalized block sequence is equivalent to a subsequence of the basis. We show that mixed Tsirelson spaces $T[(\mathcal{A}_n, \theta_n)_n]$, for which Tzafriri space is a prototype, are saturated with subspaces with this "blocking principle".

V. Ferenczi and C. Rosendal [16] introduced and studied a stronger notion of quasiminimality. A Banach space X with a basis is *sequentially minimal* [16], if any block subspace of X contains a block sequence (x_n) such that every block subspace of X contains a copy of a subsequence of (x_n). The related notions in mixed Tsirelson spaces defined by families $(\mathcal{S}_n)$ and their relation to existence of ℓ_1^ω-spreading models were studied in [25, 22]. In [28] it was shown that the spaces $T[(\mathcal{A}_n, \theta_n)_n]$, as well as $T[(\mathcal{S}_n, \theta_n)_n]$ satisfying the regularity condition $\theta_n/\theta^n \searrow$, where $\theta = \lim_n \theta_n^{1/n}$, are sequentially minimal. We show that the modified mixed Tsirelson spaces $T_M[(\mathcal{S}_n, \theta_n)_n]$ with the above property are also sequentially minimal.

The major tool in the study of mixed Tsirelson spaces $T[(\mathcal{S}_n, \theta_n)_n]$ are the tree-analysis of norming functionals and the special averages introduced in [7], see also [11]. The basic idea to prove quasiminimality is to produce in every subspace a sequence of appropriate special averages of rapidly increasing lengths and show these sequences span isomorphic subspaces. The major obstacle in study of modified mixed Tsirelson spaces is estimating the norms of splitting a vector into pairwise disjoint parts instead of consecutive parts as in non-modified setting. In order to overcome it, we introduced special types of averages, so-called Tsirelson averages, describing in fact local representation of the Tsirelson space $T[\mathcal{S}_1, \theta]$, with $\theta = \sup_n \theta_n^{1/n}$, in the considered space. Then we are able to control the action of a norming functional on a linear combination of Tsirelson averages by the action of a norming functional on suitable averages in the Tsirelson space $T[\mathcal{S}_1, \theta]$ and vice versa. Using those estimations we prove the sequential minimality of modified mixed Tsirelson space satisfying the regularity condition. Tsirelson averages are also the main tool for proving arbitrary distortability of $T_M[(\mathcal{S}_n, \theta_n)]$ in case $\theta_n/\theta^n \searrow 0$, the result known before in non-modified setting under the condition $\theta_n/\theta^n \to 0$, [3].

In the second part of the paper we deal with the existence of strictly singular non-compact operators in mixed Tsirelson spaces. The existence of non-trivial strictly singular operators, i.e. operators whose none restriction to an infinite dimensional subspace is an isomorphism, was also studied in context of classification program of Banach space, both in search for sufficient conditions and examples on known spaces. A space on which all the bounded operators are compact perturbations of multiple of the identity was constructed recently by S.A. Argyros and R. Haydon, [10], who solved "scalar-plus-compact". The existence of strictly singular non-compact operators was shown on Gowers-Maurey spaces and Schlumprecht space [6], as well as on a class of spaces defined by families $(\mathcal{S}_n)_n$ [19]. Th. Schlumprecht [33] studying the richness of the family of operators on a Banach space in connection with the "scalar-plus-compact" problem defined two classes of Banach spaces. Class 1 refers to a variation of a "blocking principle", while Class 2 means existence of a striclty singular non-compact operator in any subspace (see Def. 3.3). T. Schlumprecht asked if any Banach space contains a subspace with a basis which is either of Class 1 or Class 2. We show that a mixed Tsirelson space $T[(\mathcal{A}_n, \frac{c_n}{n^{1/q}})_n]$ belongs to Class 1 if $\inf_n c_n > 0$ and to Class 2 if $\lim_n c_n = 0$.

In [23] a block sequence $(x_n)_{n\in\mathbb{N}}$ generating ℓ_1-spreading model was constructed in Schlumprecht space S. This result combined with the result of I. Gasparis [19] led to the question if some biorthogonal sequence to $(x_n)_n$ generates a c_0-spreading model in S^*. We remark that this is not the case. In general, it is still unknown if any sequence in S^* generates a c_0-spreading model. Finally we show that in mixed (modified) Tsirelson spaces defined by $(\mathcal{S}_n)$ containing a block sequence generating ℓ_1^ω-spreading model there is a strictly singular non-compact operator on a subspace.

We describe now briefly the content of the paper. In the first section we recall the basic notions in the theory of mixed Tsirelon spaces and their modified versions, including the canonical representation of these spaces and the notion of a tree-analysis of a norming functional (Def. 1.8). The second section is devoted to the study of modified mixed Tsirelson spaces $T[(\mathcal{S}_n, \theta_n)_n]$ satisfying the regularity condition. We extend the notion of an averaging tree (Def. 2.2) and present the notions of averages of different types, providing also upper (Lemma 2.10) and lower (Lemma 2.14) "Tsirelon-type" estimates. We conclude the section with the result on arbitrary distortion for spaces with $\theta_n/\theta^n \searrow 0$ (Theorem 2.19) and sequential minimality (Theorem 2.20). In the last section we study the existence of non-compact strictly singular operators in mixed Tsirelson spaces $T[(\mathcal{A}_n, \theta_n)_n]$ (Theorem 3.4). We discuss the behaviour of a biorthogonal sequence to the sequence generating ℓ_1-spreading model in Schlumprecht space (Proposition 3.6) and the case of mixed Tsirelson spaces $T[(\mathcal{S}_n, \theta_n)_n]$ admitting ℓ_1^ω-spreading model (Theorem 3.8). We finish with the comments and questions concerning the Tzafriri space and richness of the set of subsymmetric sequences in a Banach space.

1. Preliminaries

We recall the basic definitions and standard notation.

By a *tree* we shall mean a non-empty partially ordered set $(\mathcal{T}, \preceq)$ for which the set $\{y \in \mathcal{T} : y \preceq x\}$ is linearly ordered and finite for each $x \in \mathcal{T}$. If $\mathcal{T}' \subseteq \mathcal{T}$ then we say that $(\mathcal{T}', \preceq)$ is a *subtree* of $(\mathcal{T}, \preceq)$. The tree $\mathcal{T}$ is called *finite* if the set $\mathcal{T}$ is finite. The *initial* nodes of $\mathcal{T}$ are the minimal elements of $\mathcal{T}$ and the *terminal* nodes are the maximal elements. A *branch* in $\mathcal{T}$ is a maximal linearly ordered set in $\mathcal{T}$. The *immediate successors* of $x \in \mathcal{T}$, denoted by $\succ (x)$, are all the nodes $y \in \mathcal{T}$ such that $x \preceq y$ but there is no $z \in \mathcal{T}$ with $x \preceq z \preceq y$. If X is a linear space, then a *tree in* X is a tree whose nodes are vectors in X.

Let X be a Banach space with a basis (e_i). The *support* of a vector $x = \sum_i x_i e_i$ is the set $\operatorname{supp} x = \{i \in \mathbb{N} : x_i \neq 0\}$, the *range* of x, denoted by $\operatorname{range}(x)$ is the minimal interval containing $\operatorname{supp} x$. Given any $x = \sum_i a_i e_i$ and finite $E \subset \mathbb{N}$ put $Ex = x_E = \sum_{i\in E} a_i e_i$. We write $x < y$ for vectors $x, y \in X$, if $\max\operatorname{supp} x < \min\operatorname{supp} y$. A *block sequence* is any sequence $(x_i) \subset X$ satisfying $x_1 < x_2 < \dots$, a *block subspace* of X - any closed subspace spanned by an infinite block sequence. A subspace spanned by a block sequence (x_n) we denote by $[x_n]$.

Notation 1.1. *Given any two vectors $x, y \in X$ we write $x \preceq y$, if $\operatorname{supp} x \subset \operatorname{supp} y$, and we say that x and y are incomparable, if $\operatorname{supp} x \cap \operatorname{supp} y = \emptyset$.*

Given a block sequence $(x_n) \subset X$ *and a functional* $f \in X^*$ *we say that* f *begins in* x_n*, if* $\operatorname{minsupp} f \in (\operatorname{maxsupp} x_{n-1}, \operatorname{maxsupp} x_n]$ *(set* $x_0 = 0$*).*

A basic sequence (x_n) $C-$*dominates* a basic sequence (y_n), $C \geq 1$, if for any scalars (a_n) we have

$$\|\sum_n a_n y_n\| \leq C \|\sum_n a_n x_n\| \,.$$

Two basic sequences (x_n) and (y_n) are C-*equivalent*, $C \geq 1$, if (x_n) $C-$dominates (y_n) and (y_n) $C-$dominates (x_n).

Definition 1.2. Let E be a Banach space with a 1-subsymmetric basis (u_n), i.e. 1-equivalent to any of its infinite subsequences. Let (x_n) be a seminormalized basic sequence in a Banach space X. We say that $(x_n)_n$ generates (u_n) as a spreading model, if for any $k \in \mathbb{N}$ and any $(a_i)_{i=1}^k \subset \mathbb{R}$ we have

$$\lim_{n_1 \to \infty} \lim_{n_2 \to \infty} \dots \lim_{n_k \to \infty} \|\sum_{i=1}^k a_i x_{n_i}\|_X = \|\sum_{i=1}^k a_i u_i\|_E \,.$$

We say that a Banach space X with a basis is ℓ_p-asymptotic, $1 \leq p \leq \infty$, if any block sequence $(x_i)_{i=1}^n$ is C-equivalent to the u.v.b. of ℓ_p^n, for some universal $C \geq 1$.

By [13] any seminormalized basic sequence admits a subsequence generating spreading model. We say that (x_n) generates ℓ_p- (resp. c_0-)spreading model, if (u_n) is equivalent to the u.v.b. of ℓ_1 (resp. c_0).

Recall that by Krivine theorem for any Banach space X with a basis there is some $1 \leq p \leq \infty$ such that ℓ_p is finitely block (almost isometrically) represented in X, i.e. for any $\varepsilon > 0$ and any $n \in \mathbb{N}$ there is a normalized block sequence $x_1 < \dots < x_n$ in X which is $(1+\varepsilon)$-equivalent to the u.v.b. of ℓ_p^n.

We work on two types of families of finite subsets of $\mathbb{N}$: $(\mathcal{A}_n)_{n \in \mathbb{N}}$ and $(\mathcal{S}_\alpha)_{\alpha < \omega_1}$. Let

$$\mathcal{A}_n = \{F \subset \mathbb{N} : \#F \leq n\}, \quad n \in \mathbb{N}\,.$$

Schreier families $(\mathcal{S}_\alpha)_{\alpha<\omega_1}$, introduced in [1], are defined by induction:

$$\mathcal{S}_0 = \{\{k\} : \ k \in \mathbb{N}\} \cup \{\emptyset\},$$
$$\mathcal{S}_{\alpha+1} = \{F_1 \cup \dots \cup F_k : \ k \leq F_1 < \dots < F_k, \ f_1, \dots, F_k \in \mathcal{S}_\alpha\}, \quad \alpha < \omega_1\,.$$

If α is a limit ordinal, choose $\alpha_n \nearrow \alpha$ and set

$$\mathcal{S}_\alpha = \{F : \ F \in \mathcal{S}_{\alpha_n} \text{ and } n \leq F \text{ for some } n \in \mathbb{N}\}\,.$$

Given a family $\mathcal{M} = \mathcal{A}_n$ or $\mathcal{S}_n$ we say that a sequence $E_1, \dots, E_k$ of subsets of $\mathbb{N}$ is

(1) $\mathcal{M}$-*admissible*, if $E_1 < \dots < E_k$ and $(\min E_i)_{i=1}^k \in \mathcal{M}$,

(2) $\mathcal{M}$-*allowable*, if $(E_i)_{i=1}^k$ are pairwise disjoint and $(\min E_i)_{i=1}^k \in \mathcal{M}$.

Let X be a Banach space with a basis. We say that a sequence $x_1 < \dots < x_n$ is $\mathcal{M}$-*admissible* (resp. *allowable*), if $(\operatorname{supp} x_i)_{i=1}^n$ is $\mathcal{M}$-admissible (resp. allowable).

Definition 1.3 (Mixed and modified mixed Tsirelson space)**.** Fix a sequence of families $(\mathcal{M}_n) = (\mathcal{A}_{k_n})$ or $(\mathcal{S}_{k_n})$ and sequence $(\theta_n) \subset (0,1)$ with $\lim_{n\to\infty} \theta_n = 0$. Let $K \subset c_{00}$ be the smallest set satisfying the following:

(1) $(\pm e_n^*)_n \subset K$,
(2) for any $f_1 < \cdots < f_k$ in K, if $(f_i)_{i=1}^k$ is $\mathcal{M}_n$-admissible for some $n \in \mathbb{N}$, then $\theta_n(f_1 + \cdots + f_k) \in K$.

We define a norm on c_{00} by $\|x\| = \sup\{f(x) : f \in K\}$, $x \in c_{00}$. The *mixed Tsirelson space* $T[(\mathcal{M}_n, \theta_n)_n]$ is the completion of $(c_{00}, \|\cdot\|)$.

The *modified mixed Tsirelson space* $T_M[(\mathcal{M}_n, \theta_n)_n]$ is defined analogously, by replacing admissibility by allowability of the sequences.

It is standard to verify that the norm $\|\cdot\|$ is the unique norm on c_{00} satisfying the equation

$$\|x\| = \max\left\{\|x\|_\infty, \sup\left\{\theta_n \sum_{i=1}^k \|E_i x\| : \ (E_i)_{i=1}^k - \mathcal{M}_n - \text{admissible}, \ n \in \mathbb{N}\right\}\right\}.$$

It follows immediately that the u.v.b. (e_n) is 1-unconditional in the space $T[(\mathcal{M}_n, \theta_n)_n]$. It was proved in [7] that any $T[(\mathcal{S}_{k_n}, \theta_n)_n]$ is reflexive, also any $T[(\mathcal{A}_{k_n}, \theta_n)_n]$ is reflexive, provided $\theta_n > \frac{1}{k_n}$ for at least one $n \in \mathbb{N}$, [11].

Taking $\mathcal{M}_n = \mathcal{M}$ and $\theta_n = \theta$ for any n we obtain the classical Tsirelson-type space $T[\mathcal{M}, \theta]$. Recall that $T[\mathcal{A}_n, \theta] = c_0$ if $\theta \leq 1/n$ and $T[\mathcal{A}_n, \theta] = \ell_p$, if $\theta = 1/\sqrt[q]{n}$ for q satisfying $1/p + 1/q = 1$, [12, 11]. The space $T[\mathcal{S}_1, 1/2]$ is the Tsirelson space.

Schlumprecht space S is the space $T[(\mathcal{A}_n, \frac{1}{\log_2(n+1)})_n]$, Tzafriri space is $T[(\mathcal{A}_n, \frac{c}{\sqrt{n}})_n]$ for $0 < c < 1$. Modified Tsirelson-type spaces are isomorphic to their non-modified version, whereas the situation is quite different in mixed setting, [9].

We present now the canonical form of (modified) mixed Tsirelson space in both cases $\mathcal{M}_n = \mathcal{A}_{k_n}$ or $\mathcal{S}_{k_n}$, $n \in \mathbb{N}$.

Definition 1.4. [27] A mixed Tsirelson space $T[(\mathcal{A}_{k_n}, \theta_n)_{n\in\mathbb{N}}]$ is called a *$p-$space*, for $p \in [1, \infty)$, if there is a sequence $(p_N)_N \subset (1, \infty)$ such that

(1) $p_N \to p$ as $N \to \infty$, and $p_N \geq p_{N+1} > p$ for any $N \in \mathbb{N}$,
(2) $T[(\mathcal{A}_{k_n}, \theta_n)_{n=1}^N]$ is isomorphic to ℓ_{p_N} for any $N \in \mathbb{N}$.

A $p-$space $T[(\mathcal{A}_n, \theta_n)_{n\in\mathbb{N}}]$ is called *regular*, if $\theta_n \searrow 0$ and $\theta_{nm} \geq \theta_n\theta_m$ for any $n, m \in \mathbb{N}$. Recall that any $p-$space is isometric to a regular $p-$space [28].

Notation 1.5. *Let $T[(\mathcal{A}_n, \theta_n)_{n\in\mathbb{N}}]$ be a regular $p-$space. If we set $\theta_n = 1/n^{1/q_n}$ with $q_n \in (1, \infty)$, $n \in \mathbb{N}$, then $q = \lim_n q_n = \sup_n q_n \in (0, \infty]$, where $1/p + 1/q = 1$, with usual convention $1/\infty = 0$.*

In the situation as above let $c_n = \theta_n n^{1/q} \in (0, 1)$, $n \in \mathbb{N}$, if $p > 1$. To unify the notation put $c_n = \theta_n$, $n \in \mathbb{N}$, in case $p = 1$.

A space $T_M[(\mathcal{S}_n, \theta_n)_{n\in\mathbb{N}}]$ with $\theta_n \searrow 0$ and $\theta_{n+m} \geq \theta_n\theta_m$ is called a *regular* space. Notice that any modified mixed Tsirelson space is isometric to a regular modified mixed Tsirelson space (cf. [3]).

Notation 1.6. *For a regular modified mixed Tsirelson space $T_M[(\mathcal{S}_n, \theta_n)_n]$ let $\theta = \lim_n \theta_n^{1/n} = \sup_n \theta_n^{1/n} \in (0, 1]$. We shall use also the following condition:*

$$(\clubsuit) \qquad (\theta_n/\theta^n)_n \searrow \quad \textit{i.e.} \quad \theta_{n+m} \leq \theta_n \theta^m \textit{ for any } n, m \in \mathbb{N}.$$

Lemma 1.7. *The space $T_M[(\mathcal{S}_n[A_2],\theta_n)_n]$ is 3-isomorphic to $T_M[(\mathcal{S}_n,\theta_n)_n]$.*

The proof of the above follows that of Lemma 4.5, [28] with "admissible" sequences replaced by "allowable" ones.

The following notion provides a useful tool for estimating norms in Tsirelson type spaces, mixed Tsirelson spaces and their modified versions:

Definition 1.8. [The tree-analysis of a norming functional] Let $f \in K$, the norming set of $T[(\mathcal{M}_n,\theta_n)_n]$ (resp. $T_M[(\mathcal{M}_n,\theta_n)_n]$). By a *tree-analysis* of f we mean a finite family $(f_\alpha)_{\alpha\in\mathcal{T}}$ indexed by a tree $\mathcal{T}$ with a unique root $0 \in \mathcal{T}$ (the smallest element) such that the following hold

(1) $f_0 = f$ and $f_\alpha \in K$ for all $\alpha \in \mathcal{T}$,
(2) $\alpha \in T$ is maximal if and only if $f_\alpha \in (\pm e_n^*)$,
(3) for every not maximal $\alpha \in T$ there is some $n \in \mathbb{N}$ such that $(f_\beta)_{\beta\in\mathrm{succ}(\alpha)}$ is an $\mathcal{M}_n$-admissible (resp. -allowable) sequence and $f_\alpha = \theta_n(\sum_{\beta\in\mathrm{succ}(\alpha)} f_\beta)$. We call θ_n the *weight* of f_α.

For any $\alpha \in \mathcal{T}$, $\alpha > 0$, we define the tag $t(\alpha) = t(f_\alpha)$ as $t(\alpha) = \prod_{\alpha>\beta\geq 0} weight(f_\beta)$.

For any $\alpha \in \mathcal{T}$we define also inductively the order of α as follows: $\mathrm{ord}(0) = 0$ and for any $\beta \in \mathrm{succ}(\alpha)$ we put $\mathrm{ord}(\beta) = \mathrm{ord}(\alpha) + n$, where $weight(f_\alpha) = \theta_n$.

Notice that every functional $f \in K$ admits a tree-analysis, not necessarily unique.

We shall use repeatedly the following

Fact 1.9. *Let $X = T_M[(\mathcal{S}_n,\theta_n)_n]$ with (♣). Let $(f_\alpha)_{\alpha\in\mathcal{T}}$ be a norming tree of a norming functional $f \in K$ and α not a terminal node. Let $f_\alpha = \theta_{r_\alpha}\sum_{\beta\in succ(\alpha)} f_\beta$. Then for every $k \in [\mathrm{ord}(\alpha), \mathrm{ord}(\alpha) + r_\alpha]$ we get*

$$f_\alpha = \theta_{r_\alpha} \sum_{t\in A_\alpha} \sum_{s\in F_t} f_s$$

where $(f_s)_{s\in F_t}$ is $\mathcal{S}_{r_\alpha-(k-\mathrm{ord}(\alpha))}$-allowable, for any $t \in A_\alpha$, and $(g_t)_{t\in A_\alpha}$ is $\mathcal{S}_{k-\mathrm{ord}(\alpha)}$-allowable, for $g_t = \theta_{r_\alpha-(k-\mathrm{ord}(\alpha))}\sum_{s\in F_t} f_t$, $t \in A_\alpha$. In particular by (♣) we get

$$f_\alpha(x) \leq \theta^{k-\mathrm{ord}(\alpha)} \sum_{t\in A_\alpha} g_t(x).$$

Moreover using that $t(\alpha) \leq \theta_{\mathrm{ord}(\alpha)} \leq \theta^{\mathrm{ord}(\alpha)}$ we have $t(\alpha)f_\alpha(x) \leq \theta^k \sum_{t\in A_\alpha} g_t(x)$.

2. Modified mixed Tsirelson spaces defined on Schreier families

In this section we present the main results on sequential minimality and arbitrary distortability of a regular modified mixed Tsirelson spaces $T_M[(\mathcal{S}_n,\theta_n)]$ with (♣). In the first subsection we discuss the notions of averages of different types, in the next two subsections we present estimations on their norms. Since the u.v.b. in any (modified) mixed Tsirelson space and its dual is unconditional, we work in the sequel on functionals and vectors with non-negative coefficients.

2.1. **Averages.** In this part we present the notion of special averages and recall basic facts. Let X be a Banach space with a basis. We will use a version of the notion of special averages introduced in [7].

Definition 2.1. A vector $x \in X$ is called an (M, ε)-average of a block sequence $(x_i)_i \subset X$, for $M \in \mathbb{N}$ and $\varepsilon > 0$, if $x = \sum_{i \in G} a_i x_i$ for some $G \in \mathcal{S}_M$ and $(a_i)_{i \in G} \subset (0,1]$ with $\sum_{i \in G} a_i = 1$ and for any $F \in \mathcal{S}_{M-1}$ we have $\sum_{i \in F} a_i < \varepsilon$.

We use the notion of an averaging admissible tree, [3], with additional features:

Definition 2.2. We call a tree $(x_i^j)_{j=0,i=1}^{M,N^j}$ in X with weights $(N_i^j)_{j=1,i=1}^{M,N^j} \subset \mathbb{N}$ and errors $(\varepsilon_i^j)_{j=1,i=1}^{M,N^j} \subset (0,1)$, an averaging tree, if

(1) $(x_i^j)_{i \in I_j}$ is a block sequence for any j, $1 = N^M \leq \cdots \leq N^0$.
Moreover for any $j = 1, \ldots, M$ and $i = 1, \ldots, N^j$ we have the following
(2) there exists a nonempty interval $I_i^j \subset \{1, \ldots, N^{j-1}\}$ with $\# I_i^j = N_i^j$ such that $\operatorname{succ}(x_i^j) = (x_s^{j-1})_{s \in I_i^j}$,
(3) $x_i^j = 1/N_i^j \sum_{s \in I_i^j} x_s^{j-1}$,
(4) $2/\varepsilon_i^j < N_i^j \leq \operatorname{minsupp} x_i^j$,
(5) $\varepsilon_{i+1}^j < 1/(2^i \operatorname{maxsupp} x_i^j)$, $\operatorname{maxsupp} x_i^j < N_{i+1}^j$.

Remark 2.3. In the situation as above we define coefficients $(a_i^j)_{j=0,i=1}^{M,N^j} \subset (0,1]$, as satisfying $x^M = \sum_{i=1}^{N^j} a_i^j x_i^j$. It follows straightforward that for any $j = 0, \ldots, M$, $i = 1, \ldots, N^j$ we have the following

(6) $\sum_{i=1}^{N^j} a_i^j = 1$,
(7) $a_i^j = \prod_{r=j+1}^{M} \frac{1}{N_{i_r}^r}$, where $x_{i_r}^r \succeq x_i^j$ for each $M \geq r > j$,
(8) $a_i^j = \sum_{m:\ x_m^0 \preceq x_i^j} a_m^0$.

Notice that any x_i^j is a (j, ε_i^j)-average of $(x_m^0)_{x_m^0 \preceq x_i^j}$.

Proof. To show the last statement notice that by (4) for any $j, i \geq 1$ the block sequence $\operatorname{succ}(x_i^j)$ is $\mathcal{S}_1$-admissible, thus any block sequence $(x_m^0)_{x_m^0 \preceq x_i^j}$ is $\mathcal{S}_j$-admissible. To complete the proof notice that by the standard reasoning (cf for example [29], last part of the proof of Proposition 3.6) we have the following fact:
Fact Fix a block sequence $(x_m)_m$ and let $(x_i)_{i=1}^N$ be a block sequence of $(M-1, \varepsilon_i)$-averages of $(x_m)_{m \in A_i}$ such that $N > 2/\varepsilon$ and $\varepsilon_{i+1} < 1/2^i \operatorname{maxsupp} x_i$. Then $x = \frac{1}{N}(x_1 + \cdots + x_N)$ is a (M, ε)-average of $(x_m)_{m \in A_i, i=1,\ldots,N}$. □

The above Lemma together with the construction of an averaging tree presented in [3] yields the standard

Fact 2.4. *For any block sequence $(x_m)_m$ of X, any $\varepsilon > 0$ and any $M \in \mathbb{N}$ there is an (M, ε)-average x of (x_m).*

From now on we fix a regular modified mixed Tsirelson space $X = T_M[(\mathcal{S}_n, \theta_n)]$. We shall use the following facts in the sequel.

Fact 2.5. [8] *Let* $x=\sum_{i\in F}a_ix_i$ *be an* (M,ε)*-average of normalized vectors* $(x_i)_{i\in F}$, $M\in\mathbb{N}$, $\varepsilon>0$ *and* $\mathcal{E}$ *an* $\mathcal{S}_{M-1}$ *allowable family of sets. Then there is some* $G\subset F$ *such that for every* $i\in G$ *the set* $\{Ex_i : E\in\mathcal{E}, Ex_i\neq 0\}$ *is* $\mathcal{S}_1$*-allowable and*

$$\sum_{E\in\mathcal{E}}\|Ex\|\leq\sum_{E\in\mathcal{F}}\|E(\sum_{i\in G}a_ix_i)\|+2\varepsilon/\theta_M\,.$$

Fact 2.6. *Let* $x=\sum_{i\in F}a_ix_i$ *be an* (M,ε)*-average of normalized vectors* $(x_i)_{i\in F}$, $M\in\mathbb{N}$, $\varepsilon>0$ *and* f *a norming functional with a tree-analysis* $(f_\alpha)_{\alpha\in\mathcal{T}}$*. Then there is subtree* $\mathcal{T}'$ *of* $\mathcal{T}$ *such that any terminal node of* $\mathcal{T}'$ *has order at least* M *and the functional* f' *defined by the tree-analysis* $(f_\alpha)_{\alpha\in\mathcal{T}'}$ *satisfies* $f(x)\leq f'(x)+2\varepsilon$.

Proof. Let $\mathcal{E}$ be the collection of all terminal nodes of $\mathcal{T}$ of order smaller than M. Let $G=\{i\in F:\ \text{ some } f_\alpha \text{ begins in } x_i,\ \alpha\in\mathcal{E}\}$. Since the set $(f_\alpha)_{\alpha\in\mathcal{E}}$ is $\mathcal{S}_{M-1}$-allowable, it follows $G\setminus\{\min G\}\in\mathcal{S}_{M-1}$ and $f(\sum_{i\in G}a_ix_i)\leq a_{\min G}+\sum_{i\in G\setminus\{\min G\}}a_i\leq 2\varepsilon$. We let $\mathcal{T}'$ be the tree $\mathcal{T}$ with removed nodes from the family $\mathcal{E}$. Then $f(x)\leq f'(x)+f(\sum_{i\in G}a_ix_i)\leq f'(x)+2\varepsilon$. □

2.2. **General estimations.** We are able to control the norm of splitting a vector into allowable, not only admissible parts, by comparing it to the norm of splitting of a corresponding vector in the original Tsirelson space $T[\mathcal{S}_1,\theta]$. In this section we present the upper "Tsirelson-type" estimate for usual (M,ε)-averages.

For the rest of chapter we assume that the considered regular modified mixed Tsirelson space $X=T_M[(\mathcal{S}_n,\theta_n)_n]$ satisfies (♣). First we present a classical fact.

Lemma 2.7. *Let* $x=\sum_i a_ix_i$ *be an* (M,ε)*-average of a normalized block sequence* $(x_i)_i\subset X$, $M\in\mathbb{N}$*. Then for any* $j\in\mathbb{N}$, $j<M$ *and* $\mathcal{S}_j$*-allowable* $(E_l)_l$ *we have*

$$\sum_l\|E_lx\|\leq\theta_1^{-1}\theta^{M-j-1}\sum_l\sum_i a_i\|E_lx_i\|+4\varepsilon/\theta_M\,.$$

In particular $\|x\|\leq\theta_1^{-1}\theta^{M-1}+4\varepsilon/\theta_M$.

Proof. Take an $\mathcal{S}_j$-allowable sequence $(E_l)_l$. For any l take a norming functional f_l with $\|E_lx\|=f_l(x)$ and its tree-analysis $(f^l_\alpha)_{\alpha\in\mathcal{T}_l}$. Let $\mathcal{E}$ be the collection of all terminal nodes $\alpha\in\mathcal{T}_l$ for all l, such that $\text{ord}_{\mathcal{T}_l}(\alpha)\leq M-1-j$. Then the set $(f_\alpha)_{\alpha\in\mathcal{E}}$ is $\mathcal{S}_{M-1}$-allowable. By Fact 2.6 we can assume with error 2ε that all terminal nodes of all $\mathcal{T}_l$ have order at least $M-j$.

We will add in the tree-analysis $(f^l_\alpha)_{\alpha\in\mathcal{T}_l}$'s additional nodes $(h_t)_t$ of order $M-j-1$, by grouping some of nodes of $\mathcal{T}_l$, and by (♣) obtain the desired estimation.

For any l let $\mathcal{E}_l$ be collection of all $\alpha\in\mathcal{T}_l$ which are maximal with respect to the property $\text{ord}_{\mathcal{T}_l}(\alpha)\leq M-j-1$. Fix $\alpha\in\mathcal{E}_l$. Then by the above reduction α is not terminal, so $f^l_\alpha=\theta_{r_\alpha}\sum_{s\in\text{succ}(\alpha)}f^l_s$ for some $\mathcal{S}_{r_\alpha}$-allowable (f^l_s). By Fact 1.9 for $k=M-j-1$, there exists $\mathcal{S}_{M-j-1-\text{ord}(\alpha)}$-allowable functionals $(h_t)_{t\in A_\alpha}$ with

$$t(\alpha)f^l_\alpha(x)\leq\theta^{M-j-1}\sum_{t\in A_\alpha}h_t(x)\,.$$

It follows that $(h_t)_{t\in A_l}$ is $\mathcal{S}_{M-j-1}$-allowable, where $A_l=\cup_{\alpha\in E_l}A_\alpha$. Now we have

$$
\begin{aligned}
\|E_l x\| = f_l(x) &= \sum_{\alpha\in\mathcal{E}_l} t(\alpha) f^l_\alpha(E_l x) \\
&\leq \sum_{\alpha\in\mathcal{E}_l}\theta^{M-j-1}\sum_{t\in A_\alpha} h_t(E_l x) = \theta^{M-j-1}\sum_{t\in A_l} h_t(E_l x).
\end{aligned}
$$

Taking into account the error from erasing nodes with too small orders we obtain

$$
\sum_l \|E_l x\| \leq \theta^{M-j-1}\sum_l\sum_{t\in A_l} h_t(E_l x) + 2\varepsilon \leq \dots
$$

Notice that $(h_t)_{t\in A}$ is $\mathcal{S}_{M-1}$-allowable. By Fact 2.5 with error $2\varepsilon/\theta_M$ we assume that the family $(h_t(x_i))_{t:h_t(x_i)\neq 0}$ is $\mathcal{S}_1$-allowable for each i and thus we have:

$$
\begin{aligned}
\dots &\leq \theta^{M-j-1}\sum_l\sum_i a_i \sum_{\operatorname{minsupp} h_t\leq \operatorname{minsupp} x_i} h_t(E_l x_i) + 4\varepsilon/\theta_M \\
&\leq \theta^{M-j-1}\theta_1^{-1}\sum_l\sum_i a_i\|E_l x_i\| + 4\varepsilon/\theta_M \\
&= \theta_1^{-1}\theta^{M-j-1}\sum_l\sum_i a_i\|E_l x_i\| + 4\varepsilon/\theta_M.
\end{aligned}
$$

□

In order to deal with allowable splittings, we need the next result, stating - roughly speaking - that a restriction of an average x with an averaging tree high enough is still an average y, with a strict control on the error on the new average y - depending on the error in the averaging tree of x corresponding to $\operatorname{minsupp} y$.

Lemma 2.8. *Let (x_i^j), (N_i^j), (a_i^j), (ε_i^j) form an averaging tree for a $(M+\tilde{M},\varepsilon)$-average x, $M,\tilde{M}\in\mathbb{N}$, $\varepsilon>0$, of normalized block sequence $(x_i^0)_i$, satisfying*

(1) *for any i,j we have $N_i^j = 2^{k_i^j}$ for some k_i^j,*
(2) *for any i,j we have $\varepsilon_{i+1}^j\leq \theta_M\varepsilon/2^i \operatorname{maxsupp} x_i^j$, $\varepsilon_1^j\leq\theta_M\varepsilon/2$ for any i,j.*

Then for any $I\subset\mathbb{N}$ with $N^M_{\min I}\sum_{i\in I}a_i^M\in\mathbb{N}$ the vector $y=\sum_{i\in I}a_i^M x_i^M$ is a restriction of an $(M,\varepsilon^M_{\min I})$-average of some block sequence (y_k^0) with $\|y_k^0\|\leq 1$ and such that the following property holds:

(P) for every k,i,l either $x_i^M\preceq y_k^l$ or $x_i^M\succeq y_k^l$ or x_i^M and y_k^l are incomparable, where $(y_k^l)_{k,l}$ is the family of nodes of averaging tree of y.

Proof. Let $\varepsilon_I=\varepsilon^M_{\min I}$. We represent $y=\sum_{i\in I}a_i^M x_i^M$ as a restriction of an (M,ε_I)-average. We construct inductively on $l=M,M-1,\dots,0$ an averaging tree $(y_k^l)_{l=0,k=1}^{M,K_l}$ with weights (W_k^l) and coefficients (c_k^l), where $y_k^l=1/W_k^l\sum_{s\in J_k^l}y_s^{l-1}$ and $c_k^l=\prod_{r>l:y_k^l\preceq y^r_{k_r}}\frac{1}{W^r_{k_r}}$, such that $y_1^M=y$ and the following is satisfied

(P_0) $c_k^l y_k^l=\sum_{m\in A_k^l}a_m^0 x_m^0$, $c_k^l=\sum_{m\in A_k^l}a_m^0$ for every k and $l<M$,
(P_1) for every k,i,l either $x_i^l\preceq y_k^l$ or x_i^l is incomparable with y_k^l,
(P_2) for every i,j,k,l either $x_i^j\preceq y_k^l$ or $x_i^j\succeq y_k^l$ or x_i^j and y_k^l are incomparable,

(P_3) for every k,l we have $W^l_k=\min\{N^l_i:\ x^l_i\preceq y^l_k\}$.

We allow one difference from the original definition: $\#J^M_1=L=N^M_{\min I}\sum_{i\in I}a^M_i$, not W^M_1, to occur, otherwise $\#J^l_k=W^l_k$ for any $l<M$.

We let $y^M_1=\sum_{i\in I}a^M_i x^M_i=\sum_{m\in A}a^0_m x^0_m$, $c^M_1=1$, $A^M_1=A$ and $W^M_1=N^M_{\min I}\leq$ minsupp y. All properties (P_0)-(P_3) are obviously satisfied.

Assume we have $(y^l_k)_k$, $(W^l_k)_k$ and $(c^l_k)_k$ for some $M\geq l>2$ satisfying the above.

Fix k and consider A^l_k. Pick any $m\in A^l_k$. By (P_1) in inductive assumption we have $x^r_{i_r}\preceq y^r_{k_r}$ for any $l\leq r\leq M$, i_r,k_r with $x^0_m\preceq x^r_{i_r}$ and $x^0_m\preceq y^r_{k_r}$. Therefore $N^r_{i_r}\geq W^r_{k_r}$ for any $l\leq r\leq M$, i_r,k_r as above. By Remark 2.3 and (P_3) we have

$$a^0_m=\prod_{r=1}^{M}\frac{1}{N^r_{i_r}}\leq\prod_{r=l}^{M}\frac{1}{N^r_{i_r}}\leq\prod_{r=l}^{M}\frac{1}{W^r_{k_r}}=\frac{c^l_k}{W^l_k}\,.$$

Recall that all coefficients $a^0_m,c^l_k,1/W^l_k$ are some powers of $1/2$ and $(a^0_m)_m$ is non-increasing. Moreover for $l<M$ we have $\sum_{m\in A^l_k}a^0_m=c^l_k$, hence we can split A^l_k into W^l_k-many successive sets $(A^{l-1}_s)_{s=1}^{W^l_k}$ such that for each s we have

$$\sum_{m\in A^{l-1}_s}a^0_m=\frac{c^l_k}{W^l_k}\,.$$

In case $l=M$ we have $\sum_{m\in A^M_1}a^0_m=L/W^M_1$, hence we can split A^M_1 into L-many sets $(A^{M-1}_s)_{s=1}^{L}$ such that for each s we have

$$\sum_{m\in A^{M-1}_s}a^0_m=\frac{c^M_1}{W^M_1}=\frac{1}{W^M_1}\,.$$

We define then $(y^{l-1}_s)_s$ and $(c^{l-1}_s)_s$ by

$$\frac{c^l_k}{W^l_k}y^{l-1}_s=\sum_{m\in A^{l-1}_s}a^0_m x^0_m,\qquad c^{l-1}_s=\frac{c^l_k}{W^l_k}\,.$$

Hence obviously $y^l_k=1/W^l_k\sum_s y^{l-1}_s$. We let also $W^{l-1}_s=\min\{N^{l-1}_i:\ x^{l-1}_i\preceq y^{l-1}_s\}$ and thus we finish construction of vectors on level $l-1$ satisfying (P_0) and (P_3).

Now we verify property (P_1). Notice that by property (P_1) on level l for each k we have $\operatorname{supp}y^l_k=\cup\{\operatorname{supp}x^l_i:\ x^l_i\preceq y^l_k\}=\cup\{\operatorname{supp}x^{l-1}_s:\ x^{l-1}_s\preceq y^l_k\}$. In case $l<M$ by Remark 2.3 and (P_0) for l we have

$$\sum_{r:\ y^{l-1}_r\preceq y^l_k}c^{l-1}_r=W^l_k\frac{c^l_k}{W^l_k}=c^l_k=\sum_{m\in A^l_k}a^0_m=\sum_{s:\ x^{l-1}_s\preceq y^l_k}a^{l-1}_s\,,$$

and as in the construction each $a^{l-1}_s\leq c^l_k/W^l_k=c^{l-1}_r$. In case of $l=M$ we have

$$\sum_{r:\ y^{M-1}_r\preceq y^M_k}c^{M-1}_r=L\frac{c^M_1}{W^M_1}=\frac{L}{W^l_k}=\sum_{m\in A^M_1}a^0_m=\sum_{s:\ x^{M-1}_s\preceq y^M_k}a^{M-1}_s\,,$$

and each $a_s^{M-1} \le 1/W_1^M = c_r^{M-1}$. Since all coefficients are the powers of $1/2$ and the sequence $(a_s^{l-1})_s$ is non-increasing we can partition the set $\{s: \ x_s^{l-1} \preceq y_k^l\}$ into $\cup\{B_r : y_r^{l-1} \preceq y_k^l\}$ such that for any r we have $c_r^{l-1} = \sum_{s\in B_r} a_s^{l-1}$. Consequently for any $y_r^{l-1} \preceq y_k^l$ and $x_s^{l-1} \preceq y_k^l$ we have either $y_r^{l-1} \succeq x_s^{l-1}$ or y_r^{l-1} and x_s^{l-1} are incomparable.

The property (P$_2$) can be verified analogously by induction. If for some l,k,j we have $\operatorname{supp} y_k^l = \cup\{\operatorname{supp} x_i^j : \ x_i^j \preceq y_k^l\}$, then we show that for any $y_r^{l-1} \preceq y_k^l$ and $x_s^{j-1} \preceq y_k^l$ we have either $y_r^{l-1} \succeq x_s^{j-1}$ or y_r^{l-1} and x_s^{j-1} are incomparable. The same argument works if $\operatorname{supp} x_i^j = \cup\{\operatorname{supp} y_k^l : \ x_i^j \succeq y_k^l\}$ for some i,j,l.

Define for each $l = M,\dots,1$ and $k = 1,\dots,K_l$ the error δ_k^l. For $k=1$ let $\delta_1^l = \varepsilon_I$, for any $l = M,\dots,1$. By property (P$_1$) for any l,k there is some $i_k \ge k$ with

$$\operatorname{maxsupp} y_k^l \le \operatorname{maxsupp} x_{i_k}^l < \operatorname{minsupp} x_{i_k+1}^l \le \operatorname{minsupp} y_{k+1}^l \,.$$

Let $\delta_{k+1}^l = \varepsilon_{i_k+1}^l$ for any $k \ge 1$. We verify condition (5) of Definition 2.2. For $k=1$ and $l = M,\dots,1$ we have $W_1^l \ge N_{\min I}^M \ge 2/\varepsilon_{\min I}^M = 2/\delta_1^l$. On the other hand we have for any $l = M-1,\dots,1$ and $k = 1,\dots,K_l - 1$

$$\delta_{k+1}^l = \varepsilon_{i_k+1}^l < 1/2^{i_k} \operatorname{maxsupp} x_{i_k}^l \le 1/2^k \operatorname{maxsupp} y_k^l \,,$$

and $W_{k+1}^l \ge N_{i_k+1}^l > 2/\varepsilon_{i_k+1}^l = 2/\delta_{k+1}^l$.

Hence $(y_k^l)_{k,l}$, $(W_k^l)_{k,l}$, $(c_k^l)_{k,l}$, $(\delta_k^l)_{k,l}$ form an averaging tree and thus y is (M,ε_I)-average of $(y_k^0)_k$. Notice that

$$\|c_k^0 y_k^0\| = \|\sum_{m\in A_k^0} a_m^0 x_m^0\| \le \sum_{m\in A_k^0} a_m^0 = c_k^0 \,,$$

therefore $\|y_k^0\| \le 1$. Moreover property (P$_2$) includes property (P). □

Remark 2.9. Note that by the construction each sequence $(y_s^{l-1})_{s\in J_k^l}$ is $\mathcal{S}_1$-admissible for any k,l. Hence it readily follows that for every set F of incomparable nodes (y_k^l) the functional $\sum_{y_k^l\in F} \theta^{M-l} e^*_{\operatorname{minsupp} y_k^l}$ is a norming functional on the space $T[\mathcal{S}_1,\theta]$.

The next Lemma provides a "Tsirelson-type" upper estimate for the norms of averages.

Lemma 2.10. *Let (x_i^j), (N_i^j), (a_i^j), (ε_i^j) form an averaging tree for a $(2M-3,\varepsilon)$-average x, $M>1$, $\varepsilon>0$, of normalized block sequence $(x_i^0)_i$, satisfying additionally the following conditions:*

1. *for any i,j we have $N_i^j = 2^{k_i^j}$ for some k_i^j,*
2. *for any i,j we have $\varepsilon_{i+1}^j \le \theta_M\varepsilon/2^i \operatorname{maxsupp} x_i^j$, $\varepsilon_1^j \le \theta_M\varepsilon/2$ for any i,j.*

Fix an $\mathcal{S}_{M-4}$-allowable family $\mathcal{E}$ of subsets of $\mathbb{N}$, such that the family $\{E\in\mathcal{E}: \ Ex_i^M \neq 0\}$ is $\mathcal{S}_1$-allowable for any i, and coefficients $(t_E)_{E\in\mathcal{E}} \subset [0,1]$.

Then there is a partition $(V_E)_{E\in\mathcal{E}}$ of nodes $(x_i^0)_i$, with $\operatorname{minsupp} x^0_{\min V_E} \ge \min E$, such that

$$\sum_{E\in\mathcal{E}} t_E\|Ex\| \le C\sum_{E\in\mathcal{E}} t_E\|\sum_{i\in V_E} a_i^0 e_{\operatorname{minsupp} x_i^0}\|_{T[\mathcal{S}_1,\theta]} + C\varepsilon$$

for some universal constant C depending only on θ_1 and θ.

Proof. STEP 1. Let us recall that x is an $(M-3,\varepsilon)$-average of $(x_i^M)_i$. First let $\mathcal{E}_i=\{E\in\mathcal{E}:\ E$ begins at $x_i^M\}$ and $J=\{i:\ \mathcal{E}_i\neq\emptyset\}$. As $(x_i^M)_{i\in J}$ is $\mathcal{S}_{M-4}$-admissible, we have

$$\sum_{E\in\mathcal{E}} t_E\|E\sum_{i\in J}a_i^M x_i^M\|\leq\sum_{i\in J}a_i^M\sum_{E\in\mathcal{E}}\|Ex_i^M\|\leq\theta_1^{-1}\sum_{i\in J}a_i^M\leq\theta_1^{-1}2\varepsilon.$$

For any $E\in\mathcal{E}$ let $I_E=\{i\notin J:\ Ex_i^M\neq 0\}$, $i_E=\min I_E$ and $\varepsilon_E=\varepsilon_{i_E}^M$. Compute

$$\sum_{E\in\mathcal{E}}\varepsilon_E\leq\varepsilon\theta_M\sum_{i\in J}\sum_{E\in\mathcal{E}_i}1/2^{i_E-1}\operatorname{maxsupp}x_{i_E-1}^M$$
$$\leq\varepsilon\theta_M\sum_{i\in J}\operatorname{maxsupp}x_i^M/2^i\operatorname{maxsupp}x_i^M\leq\varepsilon\theta_M.$$

STEP 2. Fix $E\in\mathcal{E}$. Let $\sum_{i\in I_E}a_i^M x_i^M=\sum_{m\in K}a_m^0 x_m^0$. Notice that each $a_m^0\leq 1/N_{i_E}^M$ and $(a_m^0)_m$ is non-increasing, therefore we can partition K into intervals $A<B$ with $\sum_{m\in A}a_m^0=L/N_{i_E}^M$ and $\sum_{m\in B}a_m^0=\delta/N_{i_E}^M$ for some $L\in\mathbb{N}$ and $0\leq\delta<1$. Hence we can erase $\sum_{m\in B}a_m^0x_m^0$ with error $\delta/N_{i_E}^M\leq 1/N_{i_E}^M\leq\varepsilon_E$.

After this reduction by Lemma 2.8 the vector $y=\sum_{i\in I_E}a_i^Mx_i^M$ is a restriction of an $(M-2,\varepsilon_E)$-average $\sum_k c_k^2y_k^2$ with $\|y_k^2\|\leq 1$ and property (P) given by a suitable averaging tree $(y_k^l)_{k,l}$ with proper weights, coefficients and errors.

We take the family $K=\{k:\operatorname{minsupp}x_i^M\in\operatorname{range}y_k^2$ for some $x_i^M\}$. Since $(x_i^M)_i$ is an $\mathcal{S}_{M-3}$-admissible family and y is an $(M-2,\varepsilon_E)$-average of (y_k^2), we can erase $\sum_{k\in K}c_k^2y_k^2$ with error $2\varepsilon_E$. For any i let

$$l_{E,i}=\min\{M\geq l\geq 0: y_k^l\succeq x_i^M\}.$$

By the above reduction and (P) we can assume that $l_{E,i}\geq 2$ for all $i\in I_E$. Let

$$K_{E,i}=\{k: y_k^2\preceq x_i^M\}\text{ for any }i\in I_E.$$

Compute by Lemma 2.7 for the $(M-2,\varepsilon_E)$-average $\sum_k c_k^2y_k^2$ and $j=0$

$$\|Ex\|=\|E\sum_k c_k^2y_k^2\|\leq\|\sum_{k\notin K}c_k^2Ey_k^2\|+2\varepsilon_E$$
$$\leq\theta_1^{-1}\theta^{M-3}\sum_{i\in I_E}\sum_{k\in K_{E,i}}c_k^2\|Ey_k^2\|+6\varepsilon_E/\theta_M$$
$$=\theta_1^{-1}\theta^{M-3}\sum_{i\in I_E}a_i^M\sum_{k\in K_{E,i}}\|\frac{c_k^2}{a_i^M}Ey_k^2\|+6\varepsilon_E/\theta_M.$$

STEP 3. Fix $i\notin J$. Put $\mathcal{F}_i=\{E\in\mathcal{E}:\ i\in I_E\}=\{E\in\mathcal{E}: Ex_i^M\neq 0\}$. For any $E\in\mathcal{F}_i$ and $k\in K_{E,i}$ let $w_k=\frac{c_k^2}{a_i^M}Ey_k^2$. For each $k\in K_{E,i}$ take the norming functional f_k with $f_k(w_k)=\|w_k\|$ and $\operatorname{supp}f_k\subset\operatorname{supp}w_k$.

We gather all the terminal nodes in the tree-analysis of f_k for all $k\in K_{E,i}$, $E\in\mathcal{F}_i$, of order smaller than $M-l_{E,i}$. By the assumption on $\mathcal{E}$ and the fact that $l_{E,i}\geq 2$ they form an $\mathcal{S}_{M-1}$-allowable family, hence as x_i^M is an (M,ε_i^M)-average, we can erase these nodes with total error $2\varepsilon_i^M$.

By Fact 1.9, adding nodes in the tree-analysis of each f_k, $k\in K_{E,i}$, on the level $M-l_{E,i}$, we get $\|w_k\|\leq \theta^{M-l_{E,i}}\sum_l f^l_k(x^M_i)$ for some $\mathcal{S}_{M-l_{E,i}}$-allowable functionals $(f^l_k)_l$. Pick E_i with $t_{E_i}\theta^{-l_{E_i,i}}=\max\{t_E\theta^{-l_{E,i}}:E\in\mathcal{E}\}$. Let $l_i=l_{E_i,i}$ and compute

$$\sum_{E\in\mathcal{F}_i}t_E\sum_{k\in K_{E,i}}\|\frac{c_k^2}{a_i^M}Ey_k^2\|\leq\sum_{E\in\mathcal{F}_i}t_E\theta^{M-l_{E,i}}\sum_{k\in K_{E,i}}\sum_l f^l_k(x^M_i)+2\varepsilon^M_i\leq\dots$$

Notice again that $(f^l_k)_{l,k\in K_{E,i},E\in\mathcal{E}}$ is an $\mathcal{S}_{M-1}$-allowable family (as before by $l_{E,i}\geq 2$ and assumption on $\mathcal{E}$). As x^M_i is an (M,ε^M_i)-average of suitable $(x^0_m)_m$, by Fact 2.5 with error $2\varepsilon^M_i/\theta_M$, we may assume that for any m the family $(\operatorname{supp}f^l_k\cap\operatorname{supp}x^0_m)_{l,k\in K_{E,i},E\in\mathcal{E}}$ is $\mathcal{S}_1$-allowable. Therefore we continue the estimation

$$\dots\leq t_{E_i}\theta^{M-l_i}\sum_{E\in\mathcal{F}_i}\sum_{k\in K_{E,i}}\sum_l f^l_k(x^M_i)+4\varepsilon^M_i/\theta_M\leq\theta_1^{-1}t_{E_i}\theta^{M-l_i}+4\varepsilon^M_i/\theta_M\,.$$

STEP 4. We define $J_E=\{i:\ E=E_i\}\subset I_E$ for any $E\in\mathcal{E}$. Notice that $(J_E)_{E\in\mathcal{E}}$ are pairwise disjoint. By STEP 1, STEP 2 and STEP 3 we have

$$\begin{aligned}\sum_{E\in\mathcal{E}}t_E\|Ex\|&\leq\sum_{E\in\mathcal{E}}t_E\|E\sum_{i\in J}a^M_ix^M_i\|+\sum_{E\in\mathcal{E}}t_E\|E\sum_{i\in I_E}a^M_ix^M_i\|\\&\leq 2\theta_1^{-1}\varepsilon+\theta_1^{-1}\theta^{M-3}\sum_{E\in\mathcal{E}}\sum_{i\in I_E}a^M_i\sum_{k\in K_{E,i}}t_E\|\frac{c_k^2}{a_i^M}Ey_k^2\|+6\sum_{E\in\mathcal{E}}\varepsilon_E/\theta_M\\&=\theta_1^{-1}\theta^{M-3}\sum_{i\notin J}a^M_i\sum_{E\in\mathcal{F}_i}\sum_{k\in K_{E,i}}t_E\|\frac{c_k^2}{a_i^M}Ey_k^2\|+(6+2\theta_1^{-1})\varepsilon\\&\leq\theta_1^{-2}\theta^{M-3}\sum_{i\notin J}a^M_it_{E_i}\theta^{M-l_i}+4\sum_i\varepsilon^M_i/\theta_M+(6+2\theta_1^{-1})\varepsilon\\&\leq\theta_1^{-2}\theta^{M-3}\sum_{E\in\mathcal{E}}t_E\sum_{i\in J_E}a^M_i\theta^{M-l_i}+(10+2\theta_1^{-1})\varepsilon\leq\dots\end{aligned}$$

Fix $E\in\mathcal{E}$. Notice that for any l the sequence $(x^M_i)_{x^M_i\preceq y^l_k,l=l_i}$ is $\mathcal{S}_1$-admissible, hence by Remark 2.9 the formula $\sum_{i\in I}\theta^{M-l_i+1}e^*_{\operatorname{minsupp}x^M_i}$ defines a norming functional in $T[\mathcal{S}_1,\theta]$. Therefore for any $E\in\mathcal{E}$ we have

$$\sum_{i\in J_E}a^M_i\theta^{M-l_i}\leq\theta^{-1}\|\sum_{i\in J_E}a^M_ie_{\operatorname{minsupp}x^M_i}\|_{T[\mathcal{S}_1,\theta]}\,,$$

and we continue the above estimation

$$\dots\leq\theta_1^{-2}\theta^{M-4}\sum_{E\in\mathcal{E}}t_E\|\sum_{i\in J_E}a^M_ie_{\operatorname{minsupp}x^M_i}\|_{T[\mathcal{S}_1,\theta]}+(10+2\theta_1^{-1})\varepsilon\leq\dots$$

Consider $z_i^M = 1/a_i^M \sum_{x_m^0 \preceq x_i^M} a_m^0 e_{\operatorname{minsupp} x_m^0}$, for $i = 1, \dots, N^M$, which are (M, ε_i^M)-averages in $T[\mathcal{S}_1, \theta]$ by Remark 2.3. As $\|z_i^M\|_{T[\mathcal{S}_1,\theta]} \geq \theta^M$ for each i, we continue

$$\begin{aligned} \dots &\leq \theta_1^{-2}\theta^{-4} \sum_{E\in\mathcal{E}} t_E \| \sum_{i\in J_E} a_i^M z_i^M \|_{T[\mathcal{S}_1,\theta]} + (10 + 2\theta_1^{-1})\varepsilon \\ &\leq \theta_1^{-2}\theta^{-4} \sum_{E\in\mathcal{E}} t_E \| \sum_{i\in J_E} \sum_{x_m^0 \preceq x_i^M} a_m^0 e_{\operatorname{minsupp} x_m^0} \|_{T[\mathcal{S}_1,\theta]} + C\varepsilon , \end{aligned}$$

which ends the proof with $C = 10 + 2\theta_1^{-2}\theta^{-4}$ and $V_E = \{m : x_m^0 \preceq x_i^M, i \in J_E\}$ for each $E \in \mathcal{E}$. $\square$

2.3. **Special types of averages.** We present the lower "Tsirelson-type" estimate in a regular modified mixed Tsirelson space X with (♣). In order to achieve this we need special types of averages. We start with Corollary 4.10 [28] recalled below

Proposition 2.11. *For any block subspace Y of X, any $M \in \mathbb{N}$ and $\varepsilon > 0$, there is an (M, ε)-average $x \in Y$ of some normalized block sequence in Y such that*

$$\theta^{M-j} D \geq \sup \left\{ \sum_i \|E_i x\| : \ \mathcal{S}_j\text{-allowable } (E_i) \right\} \geq \theta^{M-j}/D$$

for any $0 \leq j \leq M$ and some universal constant D depending only on θ_1 and θ.

Proof. We recall Lemma 4.9 [28], whose proof is valid, line after line, also in the modified case. Lemma 4.9 [28] and Lemma 2.7 yield the Proposition. $\square$

Definition 2.12. A special (M, ε)-average x, $M \in \mathbb{N}$, $\varepsilon > 0$, is any (M, ε)-average satisfying assertion of Proposition 2.11.

For the next lemma we shall need the following observation.

Fact 2.13. *Fix $M \in \mathbb{N}$. Then for any $G \in \mathcal{S}_M$ and any $z = \sum_{i\in G} a_i e_i \in T[\mathcal{S}_1, \theta]$, $(a_i)_{i\in G} \subset [0, 1]$, there is a norming functional f with a tree-analysis with height at most M, such that $\|z\|_{T[\mathcal{S}_1,\theta]} \leq 2f(z)$.*

Proof. Take a norming functional g with a tree-analysis $(g_t)_{t\in T}$ satisfying $g(z) = \|z\|_{T[\mathcal{S}_1,\theta]}$. Let I be the set of all terminal nodes of $\mathcal{T}$ with order at most M and let g_1 be the restriction of g to I and $g_2 = g - g_1$. If $g_1(z) \geq g_2(z)$ then we let $f = g_1$. Assume that $g_1(z) \leq g_2(z)$ and compute

$$g(z) \leq 2g_2(z) \leq 2\theta^{M+1} \sum_{i\in G\setminus I} a_i \leq 2\theta^M \sum_{i\in G} a_i = 2f(z) ,$$

where $f = \theta^M \sum_{i\in G} e_i^*$, which ends the proof. $\square$

The major obstacle in obtaining the lower "Tsirelson-type" estimate for norm is the fact that given an (M, ε)-average $x = \sum_{i\in F} a_i x_i$ we do not control the norm of $\sum_{i\in G} a_i x_i$, $G \subset F$, in general case. The next result provides a block sequence (x_i) whose any $\mathcal{S}_M$-admissible subsequence dominates suitable subsequence of the basis in the original Tsirelson space.

Lemma 2.14. *For every block subspace Y and every $M \in \mathbb{N}$, $\delta > 0$, there exists a block sequence (x_i) of Y satisfying for any $G \in \mathcal{S}_M$ and scalars $(a_i)_{i\in G}$*

$$\|\sum_{i\in G} a_i x_i\| \geq \frac{1}{2}(1-\delta)\|\sum_{i\in G} a_i\|x_i\| e_{\mathrm{minsupp}\, x_i}\|_{T[\mathcal{S}_1,\theta]}. \tag{2.1}$$

Proof. Assume the contrary. Notice first that for any $M \in \mathbb{N}$ we have

$$(\sqrt[m]{\theta_m})^M \leq \sqrt[m]{\theta_{Mm}} \leq \sqrt[m]{\theta^{mM}},$$

thus $\lim_{m\to\infty} \sqrt[m]{\theta_{Mm}} = \theta^M$. Pick $m \in \mathbb{N}$ such that $\sqrt[m]{\theta_{Mm}} > \sqrt[m]{D^2}(1-\delta)\theta^M$ with D as in Prop. 2.11. Take a block sequence $(x_i^0)_i$ of special (Mm, ε)-averages, for some $\varepsilon > 0$.

Since (2.1) fails there is an infinite sequence $G^1_{k_1}$ of successive elements of $\mathcal{S}_M$ and coefficients $(a_i^1)_{i\in G^1_{k_1}}$ such that

$$\|\sum_{i\in G^1_{k_1}} a_i^1 x_i^0\| < \frac{1}{2}(1-\delta)\|\sum_{i\in G^1_{k_1}} a_i^1\|x_i^0\| e_{m_i^0}\|_{T[\mathcal{S}_1,\theta]},$$

where $m_i^0 = \mathrm{minsupp}\, x_i^0$ for each i. Set $x^1_{k_1} = \sum_{i\in G^1_{k_1}} a_i^1 x_i^0$, $k_1 \in \mathbb{N}$, and by Fact 2.13 take norming functionals $f^1_{k_1}$ of the space $T[\mathcal{S}_1,\theta]$ of height at most M with

$$\|\sum_{i\in G^1_{k_1}} a_i^1\|x_i^0\| e_{m_i^0}\|_{T[\mathcal{S}_1,\theta]} \leq 2 f^1_{k_1}\left(\sum_{i\in G^1_{k_1}} a_i^1\|x_i^0\| e_{m_i^0}\right).$$

Assume that we have defined $(x^{j-1}_{k_{j-1}})_{k_{j-1}}$ and $(f^{j-1}_{k_{j-1}})_{k_{j-1}}$ for some $j < m$. Then the failure of (2.1) implies the existence of a sequence $(G^j_{k_j})_k$ of successive elements of $\mathcal{S}_M$ and a sequence $(a_i^j)_{\in G^j_{k_j}}$ such that

$$\|\sum_{i\in G^j_{k_j}} a_i^j x_i^{j-1}\| < \frac{1}{2}(1-\delta)\|\sum_{i\in G^j_{k_j}} a_i^j\|x_i^{j-1}\| e_{m_i^{j-1}}\|_{T[\mathcal{S}_1,\theta]},$$

where $m_i^{j-1} = \mathrm{minsupp}\, x_i^{j-1}$. Set $x^j_{k_j} = \sum_{i\in G^j_{k_j}} a_i^j x_i^{j-1}$, for $k_j \in \mathbb{N}$, and take norming trees $f^j_{k_j}$ of the space $T[\mathcal{S}_1,\theta]$ of height at most M such that

$$\|\sum_{i\in G^j_{k_j}} a_i^j\|x_i^{j-1}\| e_{m_i^{j-1}}\|_{T[\mathcal{S}_1,\theta]} \leq 2 f^j_{k_j}\left(\sum_{i\in G^j_{k_j}} a_i^j\|x_i^{j-1}\| e_{m_i^{j-1}}\right).$$

The inductive construction ends once we get the vector x_1^m and the functional f_1^m.

Each functional $f^j_{k_j}$ is of the form $\sum_{i\in G^j_{k_j}} \theta^{l_i^j} e^*_{m_i^{j-1}}$, by construction satisfying

$$\|x^j_{k_j}\| < (1-\delta)\sum_{i\in G^j_{k_j}} \theta^{l_i^j} a_i^j \|x_i^{j-1}\|.$$

Inductively, beginning from f_1^m we produce a tree-analysis of some norming functional f on $T[\mathcal{S}_1,\theta]$ by substituting each terminal node $e^*_{m^j_k}$, $j=1,\dots,m$, by the tree-analysis of the functional f^j_k.

Put $G=\cup_{k_{m-1}\in G_1^m}\cup_{k_{m-2}\in G_{k^{m-1}_{m-1}}}\cdots\cup_{k_1\in G^2_{k_2}} G^1_{k_1}$. Let $(l_i)_{i\in G}$ be such that $f=\sum_{i\in G}\theta^{l_i}e^*_{m^0_i}$. Notice that $l_i\leq mM$ for any $i\in G$, as the height of each f^j_i does not exceed M. We compute the norm of x_1^m, which is of the form

$$x_1^m=\sum_{k_{m-1}\in G_1^m}\sum_{k_{m-2}\in G_{k^{m-1}_{m-1}}}\cdots\sum_{k_1\in G^2_{k_2}}\sum_{i\in G^1_{k_1}} a^m_{k_{m-1}}\dots a^1_i x^0_i=\sum_{i\in G} b_i x^0_i\,.$$

Since each x_i^0 is a special (mM,ε)-average, for some $\mathcal{S}_{mM-l_i}$-allowable sequence $(E_l)_{l\in L_i}$ we have $\|x_i^0\|\leq D^2\theta^{mM-l_i}\sum_{l\in L_i}\|E_l x^0_i\|$.

We have on one hand by the above construction

$$\begin{aligned}\|x_1^m\|&\leq (1-\delta)^m\sum_{i\in G}\theta^{l_i}b_i\|x_i^0\|\\&\leq (1-\delta)^m D^2\sum_{i\in G}\theta^{l_i}b_i\theta^{mM-l_i}\sum_{l\in L_i}\|E_l x_i^0\|\\&=(1-\delta)^m D^2\theta^{mM}\sum_{i\in G}b_i\sum_{l\in L_i}\|E_l x_i^0\|\,.\end{aligned}$$

Notice that $(E_l)_{l\in\cup_{i\in G}L_i}$ is $\mathcal{S}_{mM}$-allowable by the definition of f and $(l_i)_{i\in G}$, thus

$$\|x_1^m\|\geq\theta_{mM}\sum_{i\in G}b_i\sum_{l\in L_i}\|E_l x_i^0\|\,,$$

which brings $\theta_{mM}\leq(1-\delta)^m D^2\theta^{mM}$, a contradiction with the choice of m. □

Definition 2.15. A Tsirelson (M,ε)-average x, $M\in\mathbb{N}$, $\varepsilon>0$, is an (M,ε)-average $x=\sum_{i\in F}a_ix_i$ of a normalized block sequence (x_i) satisfying the assertion of the Lemma 2.14 with $\delta=1/2$.

Definition 2.16. A RIS of (special, Tsirelson) averages is any block sequence of (special, Tsirelson) $(n_k,\varepsilon/2^k)$-averages (x_k) for $\varepsilon>0$ and $(n_k)_k\subset\mathbb{N}$ satisfying

$$\theta_{l_{k+1}}\|x_k\|_{\ell_1}\leq\frac{\varepsilon}{2^{k+1}},\quad k\in\mathbb{N}\,,$$

where $l_k=\max\{l\in\mathbb{N}:\ 4l\leq n_k\}$, $k\in\mathbb{N}$.

We need the following technical lemma, mostly reformulating Lemma 7, [22]:

Fact 2.17. *Take RIS of normalized averages (x_k), for some $(n_k)\subset\mathbb{N}$ and $\varepsilon>0$, and some $x=\sum_k b_kx_k$ with $(b_k)\subset[0,1]$. Then for any norming functional f with a tree-analysis $(f_\alpha)_{\alpha\in\mathcal{T}}$ there is a subtree $\mathcal{T}'$ such that the corresponding functional f' defined by the tree-analysis $(f_\alpha)_{\alpha\in\mathcal{T}'}$ satisfies $f(x)\leq f'(x)+3\varepsilon$ and the following holds for any k*

(a) *any node α of $\mathcal{T}'$ with $f_\alpha(x_k)\neq 0$ satisfies* $\mathrm{ord}(\alpha)<n_{k+1}/4$,
(b) *any terminal node α of $\mathcal{T}'$ with $f_\alpha(x_k)\neq 0$ satisfies* $\mathrm{ord}(\alpha)\geq n_k$.

Proof. In order to prove (a) we repeat the reasoning from the proof of Lemma 7 [22]. For any k let $\mathcal{F}_k$ be the collection of all nodes in $\mathcal{T}$ which are minimal with respect to the property $\operatorname{ord}(\alpha) \geq n_{k+1}/4$ and $f_\alpha(x_k) \neq 0$. Then

$$\sum_{\alpha\in\mathcal{F}_k} t(\alpha) f_\alpha(x_k) \leq \theta_{l_{k+1}} \|x_k\|_{\ell_1} \leq \frac{\varepsilon}{2^{k+1}}\,.$$

Thus we can erase all nodes from $\mathcal{F}_k$ restricted to supports of x_k, for all k, with error $\sum_k b_k \frac{\varepsilon}{2^{k+1}} \leq \varepsilon$.

For (b) we use Fact 2.6 for erasing all terminal nodes α of $\mathcal{T}$ with $f_\alpha(x_k) \neq 0$ with error $2\varepsilon_k$, for any k. □

Lemma 2.18. *Let $x = \sum_k a_k x_k$ be an (M, ε)-average of RIS of normalized special averages (x_k), for $(n_k) \subset \{M+3, M+4, \dots\}$ and $\varepsilon > 0$, with $\varepsilon < \theta_M$.*

Then $\|x\| \leq D'\theta_M$, for some universal constant D' depending only on θ and θ_1.

Proof. Take a norming functional f with a tree-analysis $(f_\alpha)_{\alpha\in\mathcal{T}}$ such that $\|x\| = f(x)$. Using Fact 2.17 pick the subtree $\mathcal{T}'$ satisfying (a) and (b) and the corresponding functional f'.

Let $\mathcal{E}$ be collection of all $\alpha \in \mathcal{T}'$ maximal with respect to the property $\operatorname{ord}(\alpha) \leq M-1$. Notice that $\mathcal{E}$ is $\mathcal{S}_{M-1}$ - allowable.

Fix $\alpha \in \mathcal{E}$. Then α is not terminal, so $f_\alpha = \theta_{r_\alpha} \sum_{s\in\operatorname{succ}(\alpha)} f_s$. As in Fact 1.9 we partition $\operatorname{succ}(\alpha) = \bigcup_{t\in A_\alpha} F_t$ in such a way that $(f_s)_{s\in F_t}$ is $\mathcal{S}_{\operatorname{ord}(s)-(M-1)}$-allowable for every $t \in A_\alpha$ and $(g_t)_{t\in A_\alpha}$ is $\mathcal{S}_{M-1-\operatorname{ord}(\alpha)}$-allowable, where $g_t = \sum_{s\in F_t} f_s$. Let $A = \cup_{\alpha\in\mathcal{E}} A_\alpha$ and notice that $(g_t)_{t\in A}$ is $\mathcal{S}_{M-1}$-allowable. Let H denote the set of all k such that some g_t, $t \in A$, begins in x_k. Since x is an (M, ε)-average we have $\|\sum_{k\in H} a_k x_k\| \leq \sum_{k\in H} a_k \leq 2\varepsilon$.

By definition of H for any $\alpha \in \mathcal{E}$ and $k \notin H$ with $f_\alpha(x_k) \neq 0$ there is an immediate successor of α beginning before x_k. Thus by (a) we have for any $k \notin H$

(c) for any $\alpha \in \mathcal{E}$ with $f_\alpha(x_k) \neq 0$ the order of immediate successors of α is at most $n_k/4$,

(d) $\{g_t : \ t \in A, \ g_t(x_k) \neq 0\}$ restricted to $\operatorname{supp} x_k$ is $\mathcal{S}_1$ - allowable.

Fix $k \notin H$ and $t \in A$ with $g_t(x_k) \neq 0$ and let $B_t^k = \{s \in F_t : \ f_s(x_k) \neq 0\}$.

Fix $s \in B_t^k$ and take the subtree $\mathcal{T}_s$ of $\mathcal{T}'$ consisting of s (as a root) and of all successors of s in $\mathcal{T}'$. By Fact 1.9, using (b) and (c) we can add nodes in $\mathcal{T}_s$ on level $n_k - \operatorname{ord}(s)$ obtaining $(h_{s,r})_{r\in C_s}$ which is $\mathcal{S}_{n_k-\operatorname{ord}(s)}$-allowable satisfying

$$f_s(x_k) \leq \sum_{r\in C_s} \theta^{n_k-\operatorname{ord}(s)} h_{s,r}(x_k).$$

Compute for $k \notin H$ using the above and (♣)

$$f'(x_k) = \sum_{t\in A} \sum_{s\in B_t^k} t(s) f_s(x_k)$$
$$\leq \theta_M \sum_{t\in A} \sum_{s\in B_t^k} \theta^{\mathrm{ord}(s)-M} \sum_{r\in C_s} \theta^{n_k-\mathrm{ord}(s)} h_{s,r}(x_k)$$
$$\leq \theta_M \sum_{t\in A} \sum_{s\in B_t^k} \sum_{r\in C_s} \theta^{n_k-M} h_{s,r}(x_k)\,.$$

Notice that the family $\{h_{s,r}:\ r\in C_s,\ s\in B_t^k\}$ for any fixed $t\in A, k\notin H$ is $\mathcal{S}_{n_k-M+1}$-allowable. Therefore by (d) the family $\{h_{s,r}:\ r\in C_s,\ s\in B_t^k,\ t\in A\}$ for any fixed $k\notin H$ is $\mathcal{S}_{n_k-M+2}$-allowable and hence since x_k is a normalization of a (n_k,ε_k)-special average, we continue the estimation

$$\cdots \leq \theta_M \theta^{n_k-M} D^2 \theta^{-n_k+M-2} = D^2\theta^{-2}\theta_M\,.$$

We compute

$$f(x) \leq f'(x) + 3\varepsilon \leq \sum_{k\notin H} a_k f(x_k) + 5\varepsilon \leq D^2\theta^{-2}\theta_M + 5\varepsilon \leq (D^2\theta^{-2}+5)\theta_M\,,$$

which ends the proof of Lemma. □

2.4. **Main results.**

Theorem 2.19. *Let X be a regular modified mixed Tsirelson space $T_M[(\mathcal{S}_n,\theta_n)_n]$. If $\theta_n/\theta^n \searrow 0$, then X is arbitrary distortable.*

Proof. Theorem follows immediately from Proposition 2.11 and Lemma 2.18. □

Recall that a Banach space X with a basis is called sequentially minimal ([16]), if any block subspace of X contains a block sequence (x_n) such that every block subspace of X contains a copy of a subsequence of (x_n). Notice that this property implies quasiminimality of X.

Theorem 2.20. *Let X be a regular modified mixed Tsirelson space $T_M[(\mathcal{S}_n,\theta_n)_n]$. If $\theta_n/\theta^n \searrow$, then X is sequentially minimal.*

The theorem follows immediately from the following result:

Lemma 2.21. *Let $(x_k)_k$, $(y_k)_k$ be RIS of Tsirelson $(2M_k-3,\varepsilon_k)$-averages, $M_k>4$, $\varepsilon<(6C)^{-1}$, with C as in Lemma 2.10, such that*

1. *x_k has an averaging tree $(x_{k,i}^j)_{i,j}$, $(N_{k,i}^j)_{i,j}$, $(\varepsilon_{k,i}^j)_{i,j}$, $(a_{k,i}^j)_{i,j}$, y_k has an averaging tree $(y_{k,i}^j)_{i,j}$, $(N_{k,i}^j)_{i,j}$, $(\varepsilon_{k,i}^j)_{i,j}$, $(a_{k,i}^j)_{i,j}$, both satisfying conditions (1) and (2) of Lemma 2.10 for any k,*
2. *$\mathrm{minsupp}\, x_{k,i}^0 = \mathrm{minsupp}\, y_{k,i}^0$ and $\|x_{k,i}^0\| = \|y_{k,i}^0\| = 1$ for any k,i,*
3. *$\varepsilon_k \leq \theta_{2M_k-3}\theta^{2M_k-3}\varepsilon/2^{k+2}$ for any k.*

Then $(x_k/\|x_k\|)_k$ and $(y_k/\|y_k\|)_k$ are equivalent.

Notice first that Lemma above yields Theorem 2.20, as given a block sequence (w_n) in X and a block subspace Y of $[w_n]$ and $k \in \mathbb{N}$, we can choose block sequences $(u_i) \subset [w_n]$ and $(v_i) \subset Y$ satisfying the assertion of Lemma 2.14 for $2M_k - 3$. Passing to a subsequences if necessary and using a small perturbations we obtain block sequences (u_i') and (v_i') of the form $u_i' = u_i + \delta_i e_{m_i}$, $v_i' = v_i + \delta_i e_{m_i}$, for some $(m_i) \subset \mathbb{N}$ with $m_i = \operatorname{minsupp} u_i' = \operatorname{minsupp} v_i'$ for each i and small $(\delta_i) \subset (0,1)$, which are equivalent to (u_i) and (v_i) respectively and satisfy still the assertion of Lemma 2.14 for $2M_k - 3$. Then construct on these sequences two Tsirelson $(2M_k - 3, \varepsilon_k)$-averages with averaging trees as in Lemma 2.21 with equal systems of weights, errors and coefficients, obtaining x_k and y_k.

Now we proceed to the proof of Lemma 2.21.

Proof. Notice first that by Lemmas 2.7 and 2.14 we have estimation

$$\theta^{2M_k-3}/4 \leq \|x_k\| \leq 5\theta_1^{-2}\theta^{2M_k-3}, \qquad k \in \mathbb{N},$$

and the same estimation for $\|y_k\|$, $k \in \mathbb{N}$.

We show first that $(y_k/\|y_k\|)_k$ dominates $(x_k/\|x_k\|)_k$. Let $x = \sum_k d_k x_k/\|x_k\|$ be of norm 1, with $(d_k) \subset [0,1]$, and take its norming functional f with a tree-analysis $(f_\alpha)_{\alpha \in \mathcal{T}}$. Let $y = \sum_k d_k y_k/\|y_k\|$. By Fact 2.17 we can assume with error ε that $\operatorname{ord}(\alpha) < M_{k+1}/4 \leq M_{k+1} - 4$ for any $\alpha \in \mathcal{T}$ with $f_\alpha(x_k) \neq 0$. For any $k > 1$ let

$$\mathcal{E}_k = \{\alpha \in \mathcal{T} : \ f_\alpha \text{ begins at } x_k \text{ and has a sibling beginning before } x_k\}.$$

By our reduction $\operatorname{ord}(\alpha) < M_k - 4$ for any $\alpha \in \mathcal{E}_k$, $k \geq 2$. We replace in the tree-analysis of f each functional f_α, $\alpha \in \mathcal{E}_k$, by two functionals $g_\alpha = f_\alpha|_{\operatorname{supp} x_k}$ and $k_\alpha = f_\alpha - g_\alpha$, obtaining a tree-analysis of a functional g on the space $X_2 = T[(\mathcal{S}_n[\mathcal{A}_2], \theta_n)_n]$, which by Lemma 1.7 is 3-isomorphic to X.

Notice that $(g_\alpha)_{\alpha \in \mathcal{E}_k, k \geq 2}$ have pairwise disjoint supports and $(\bigcup_{\alpha \in \mathcal{E}_k} \operatorname{supp} g_\alpha) \cap \operatorname{supp} x_k = \operatorname{supp} f \cap \operatorname{supp} x_k$, hence $f|_{\operatorname{supp} x_k} = \sum_{\alpha \in \mathcal{E}_k} t(\alpha) g_\alpha$. For each $k \geq 2$ consider the set $J_k = \{i :$ some g_α begins at $x_{k,i}^{M_k}\}$. Notice that by our reduction $(g_\alpha)_{\alpha \in \mathcal{E}_k}$ is $\mathcal{S}_{M_k-4}$-allowable, thus $(x_{k,i}^{M_k})_{i \in J_k}$ is $\mathcal{S}_{M_k-4}$-admissible and recall that x_k is an $(M_k - 3, \varepsilon_k)$-average of $(x_{k,i}^{M_k})$. Let g_α', $\alpha \in \mathcal{E}_k$, be the restriction of g_α to $\cup_{i \notin J_k} \operatorname{supp} x_{k,i}^{M_k}$. Then we have the following estimation

$$\begin{aligned} f(x) &= \frac{d_1}{\|x_1\|} f(x_1) + \sum_{k \geq 2} \frac{d_k}{\|x_k\|} f(x_k) \\ &\leq \frac{d_1}{\|x_1\|} f(x_1) + \sum_{k \geq 2} \frac{d_k}{\|x_k\|} \sum_{\alpha \in \mathcal{E}_k} t(\alpha) g_\alpha'(x_k) + \sum_k \frac{d_k}{\|x_k\|} \sum_{i \in J_k} a_{k,i}^{M_k} \|x_{k,i}^{M_k}\| \\ &\leq \frac{d_1}{\|x_1\|} f(x_1) + \sum_{k \geq 2} \frac{d_k}{\|x_k\|} \sum_{\alpha \in \mathcal{E}_k} t(\alpha) g_\alpha'(x_k) + 4 \sum_k \varepsilon_k \theta^{-2M_k+3} \\ &\leq \frac{d_1}{\|x_1\|} f(x_1) + \sum_{k \geq 2} \frac{d_k}{\|x_k\|} \sum_{\alpha \in \mathcal{E}_k} t(\alpha) g_\alpha'(x_k) + \varepsilon. \end{aligned}$$

Fix $k \geq 2$. Notice that by definition the set $\{g_\alpha' : \ g_\alpha'(x_{k,i}^{M_k}) \neq 0\}$ restricted to the support of $x_{k,i}^M$ is $\mathcal{S}_1$-allowable for any i. Therefore by Lemma 2.10 we pick suitable partition $(V_\alpha)_{\alpha \in \mathcal{E}_k}$

of nodes $(x^0_{k,i})_i$ with $\operatorname{minsupp} x^0_{k,\min V_\alpha} \geq \operatorname{minsupp} g'_\alpha$ for each $\alpha \in \mathcal{E}_k$ and applying Lemma 2.14 we have

$$\begin{aligned}
\sum_{\alpha\in\mathcal{E}_k} t(\alpha) g'_\alpha(x_k) &\leq C \sum_{\alpha\in\mathcal{E}_k} t(\alpha) \| \sum_{i\in V_\alpha} a^0_{k,i} e_{\operatorname{minsupp} x^0_{k,i}} \|_{T[\mathcal{S}_1,\theta]} + C\varepsilon_k \\
&\leq C \sum_{\alpha\in\mathcal{E}_k} t(\alpha) \| \sum_{i\in V_\alpha} a^0_{k,i} e_{\operatorname{minsupp} y^0_{k,i}} \|_{T[\mathcal{S}_1,\theta]} + C\varepsilon_k \\
&\leq 2C \sum_{\alpha\in\mathcal{E}_k} t(\alpha) \| \sum_{i\in V_\alpha} a^0_{k,i} y^0_{k,i} \| + C\varepsilon_k \\
&\leq 2C \sum_{\alpha\in\mathcal{E}_k} t(\alpha) h_\alpha(y_k) + C\varepsilon_k \,,
\end{aligned}$$

where h_α is a norming functional on X with $h_\alpha(y_k) = \|\sum_{i\in V_\alpha} a^0_{k,i} y^0_{k,i}\|$ and $\operatorname{minsupp} h_\alpha \geq \operatorname{minsupp} x^0_{k,\min V_\alpha} \geq \operatorname{minsupp} g'_\alpha$ for each $\alpha \in \mathcal{E}_k$.

We modify the tree-analysis of g, replacing each node g_α, $\alpha \in \mathcal{E}_k$, $k \geq 2$, by the functional h_α. As $\operatorname{minsupp} h_\alpha \geq \operatorname{minsupp} g_\alpha$ for each α, we obtain a tree-analysis of some norming functional h on X_2. We compute, by Lemma 1.7 and above estimations including the estimation on the norms of $(x_k)_k$ and $(y_k)_k$,

$$\begin{aligned}
1 = f(x) &\leq d_1 + \sum_{k\geq 2} \frac{d_k}{\|x_k\|} \sum_{\alpha\in\mathcal{E}_k} t(\alpha) g_\alpha(x_k) + \varepsilon \\
&\leq d_1 + 40C\theta^{-2} \sum_{k\geq 2} \frac{d_k}{\|y_k\|} \sum_{\alpha\in\mathcal{E}_k} t(\alpha) h_\alpha(y_k) + 4C \sum_{k\geq 2} \frac{\varepsilon_k}{\theta_{2M_k-3}} + \varepsilon \\
&\leq d_1 + 40C\theta^{-2} h(\sum_{k\geq 2} \frac{d_k}{\|y_k\|} y_k) + 3C\varepsilon \\
&\leq 121C\theta^{-2}\|y\| + 1/2 \,,
\end{aligned}$$

which means that $(y_k/\|y_k\|)_k$ dominates $(x_k/\|x_k\|)_k$. Since the conditions are symmetric, the opposite domination follows analogously. □

3. Strictly singular non-compact operators

3.1. **Spaces defined by families $(\mathcal{A}_n)_n$.** As in mixed Tsirelson spaces defined by Schreier families the crucial tool will be formed by ℓ_p-averages.

Definition 3.1. A vector $x \in X$ is called a $C - \ell_r-$*average* of length m, for $r \in [1,\infty]$, $m \in \mathbb{N}$ and $C \geq 1$ if $x = \sum_{i=1}^m x_i / \|\sum_{i=1}^m x_i\|$ for some normalized block sequence $(x_n)_{n=1}^m$ which is C-equivalent to the unit vector basis of ℓ_r^m.

Definition 3.2. [33] Let X be a Banach space with a basis (e_n). Then X is in

1. Class 1, if every normalized block sequence in X has a subsequence equivalent to some subsequence of (e_n).
2. Class 2, if each block sequence has further normalized block sequences (x_n) and (y_n) such that the map $x_n \mapsto y_n$ extends to a bounded strictly singular operator between $[x_n]$ and $[y_n]$.

T. Schlumprecht asked if any Banach space contains a subspace with a basis which is either of Class 1 or Class 2 and gave some sufficient condition (Thm. 1.1 [33]) for the existence of strictly singular non-compact operator in the space.

Theorem 3.3. [33] *Let (x_n) and (y_n) be two normalized basic sequences generating spreading models (u_n) and (v_n) respectively. Assume that (u_n) is not equivalent to the u.v.b. of c_0 and (u_n) strongly dominates (v_n), i.e.*

$$\|\sum_{i=1}^{\infty} a_i v_i\| \leq \max_{n\in\mathbb{N}} \delta_n \max_{\#F\leq n} \|\sum_{i\in F} a_i u_i\|$$

for some sequence (δ_n) with $\delta_n \searrow 0$, $n\to\infty$. Then the map $x_n \mapsto y_n$ extends to a bounded strictly singular operator between $[x_n]$ and $[y_n]$.

Theorem 3.4. *Let $X = T[(\mathcal{A}_n, \frac{c_n}{n^{1/q}})_n]$ be a regular $p-$space, with $p\in[1,\infty)$. Then*

(1) *if $\inf_n c_n > 0$, then X is saturated with subspaces of Class 1.*
(2) *if $c_n \to 0$, $n\to\infty$, then X is in Class 2.*

Proof. PART (1). We show that any block subspace of X contains a normalized block sequence $(u_s)_s$ with the following "blocking principle": any normalized block sequence $(y_j)_j$ is equivalent to any $(u_{k_j})_j$, with $y_j < u_{k_{j+1}}$ and $u_{k_j} < y_{j+1}$. It follows that the subspace $[(u_s)]$ is sequentially minimal..

By Prop. 2.10 [28] any block subspace of X contains an ℓ_p-asymptotic subspace of X. Let W be such ℓ_p-asymptotic subspace, spanned by a normalized block sequence $(w_k)_k$. Let C be the asymptotic constant of W, i.e. any normalized block sequence $(z_i)_{i=1}^n$ with $z_1 > n$ in W is C-equivalent to the u.v.b. of ℓ_p^n.

For any block subspace Y of X spanned by normalized block sequence (y_n) let $\|\sum_n a_n y_n\|_{Y,\infty} = \sup_{n\in\mathbb{N}} |a_n|$.

Fix two strictly increasing sequences of integers $(m_n)_n \subset \mathbb{N}$ and $(N_j)_j \subset \mathbb{N}$ and take normalized block sequences $(v_n)_n$ of $(w_k)_k$ and $(u_j)_j$ of $(v_n)_n$ such that

(1) $v_n > m_n$ in W for any n,
(2) for any $y \in [(v_i)_{i>n}]$ we have $\|y\|_{W,\infty} < 1/(8m_n^5)$, for any n,
(3) $u_j > N_j$ in $V = [(v_n)_n]$ for any j,
(4) for any $y\in[(u_i)_{i>j}]$ we have $\|y\|_{V,\infty} < 1/(8N_j^5)$, for any j,
(5) $\sqrt[p]{N_j} \geq C2^{j+7}$ for any j
(6) $N_j\theta_{m_n} < 1/2^{n+5}$ for any $n\geq j$ (in particular $m_n \geq N_j$ for any $n\geq j$)
(7) $\theta_{m_n}\sum_{i<n} \#\operatorname{supp} v_i < 1/2^{n+5}$ for any n

Notice that every vector $y\in[(v_i)_{i>n}]$ is an $2C-\ell_p$-average of length m_n of some normalized block sequence $(y_i)_{i=1}^{m_n}$ of $(w_k)_k$. Indeed, by Claim 3.8 [28] and condition (2) split y into $(Fy_i)_{i=1}^{m_n}$ with almost equal norm and obtaining by condition (1) and ℓ_p-asymptoticity of W that y is a suitable average. The same holds in V: every vector $y\in[(u_i)_{i>j}]$ is an $2C-\ell_p$-average of length N_j of some normalized block sequence $(y_i)_{i=1}^{N_j}$ (block with respect to $(v_n)_n$).

We show that under such conditions we can prove the above Theorem repeating the proof of Theorem 3.1 [28]. We consider any normalized block sequence (y_j) of (u_j) and as (z_j) we take (u_{k_j}) with $y_j < u_{k_{j+1}}$ and $u_{k_j} < y_{j+1}$. By the above observation $y_j =$

$(y_1^j + \cdots + y_{N_j}^j)/\|y_1^j + \cdots + y_{N_j}^j\|$ and $u_{k_j} = (u_1^j + \cdots + u_{N_j}^j)/\|u_1^j + \cdots + u_{N_j}^j\|$, where $(y_j^i)_{i=1}^{N_j}$ and $(u_j^i)_{i=1}^{N_j}$ are normalized block sequences with respect to $(v_j)_j$. Notice that (N_j) are big enough by condition (5). We again use the above observation obtaining that each y_j^i and v_j^i is an ℓ_p-average of a block sequence, of $(w_k)_k$, of suitable length with parameters satisfying the assertion of a version of Lemma 3.2 [28] for C-averages instead of 2-averages (by conditions (6) and (7)). Therefore repeating the proof of Theorem 3.1 [28] we obtain uniform equivalence of (y_j) and (u_{k_j}) and hence "blocking principle" stated above.

PART (2). Fix a block subspace Y of X. By Theorem 2.9 [28] p is in Krivine set of Y. Take finite normalized block sequences $(y_i)_i$ such that for some $(m_i)_i \subset \mathbb{N}$

(1) each y_i is $2-\ell_p-$averages of length $N_i \geq (2m_i)^p$,
(2) $\theta_{m_i} \sum_{j<n} \# \operatorname{supp} y_j \leq 1/2^{i+5}$ for any i,
(3) $2^{i+5}\theta_{m_i} \to 0$, $i \to \infty$.

Passing to a subsequence we can assume that (y_i) generates a spreading model (v_i).

Lemma 3.5. *The spreading model (v_i) is strongly dominated by the u.v.b. of ℓ_p.*

Proof. Take $k \in \mathbb{N}$ and $(a_i)_{i=1}^N \in c_{00}$ with $\|(a_i)\|_\infty \leq 1/k^2$ and $\|(a_i)\|_{\ell_p} = 1$. Choose M by (3) in definition of (y_i) with $N\theta_{m_{i+M}} \leq 1/2^{i+M+5}$ for any i and $1/2^M \leq 1/k$. We have $\|\sum_{i=1}^N a_i v_i\| \leq 2\|\sum_{i=1+M}^{N+M} \tilde{a}_i y_i\|$, where $\tilde{a}_{i+M} = a_i$, $i = 1, \ldots, N$.

Take a norming functional f with a tree-analysis $(f_t)_{t\in\mathcal{T}}$ and $\operatorname{supp} f \subset \operatorname{supp} y$, where $y = \sum_{i=1+M}^{N+M} \tilde{a}_i y_i$. By Lemma 2.5 [28] up to multiplying by 36 we can assume that for any f_t and y_i we have either $\operatorname{supp} f_t \subset y_i$, $\operatorname{supp} f_t \supset \operatorname{supp} y_i \cap \operatorname{supp} f$ or $\operatorname{supp} f_t \cap \operatorname{supp} y_i = \emptyset$. We say that f_t covers y_i, if t is maximal in $\mathcal{T}$ with $\operatorname{supp} f_t \supset \operatorname{supp} y_i \cap \operatorname{supp} f$.

Let $A = \{t \in T : f_t \text{ covers some } y_i\}$. Given any $t \in A$ let $I_t = \{i = 1+M, \ldots, N+M : f_t$ covers $y_i\}$. Let θ_{m_t} be the weight of f_t. If $m_t > m_i$ for some $i \in I_t$ let i_t be the maximal element of I_t with this property. Otherwise let $i_t = 0$.

For any $i \in I_t$ let $J_i = \{s \in \operatorname{succ}(t) : \operatorname{supp} f_s \subset \operatorname{supp} y_i\}$. By Lemma 2.8 [28] we have $\sum_{s\in J_i} f_s(y_i) \leq 8(\# J_i)^{1/q}$ for each $i \in I_t, i > i_t$.

First let $L_t = \{i \notin I_t : \operatorname{supp} y_i \cap \operatorname{supp} f \subset \operatorname{supp} f_t\}$. Notice that for any $i \in L_t$ there is some f_{t_i} - successor of f_t so that $\operatorname{supp} y_i \cap \operatorname{supp} f \subset \operatorname{supp} f_{t_i}$. Hence

$$f_t(\sum_{i\in L_t} \tilde{a}_i y_i) \leq \theta_{m_{i_t}}(\sum_{i\in L_t} f_{t_i}(\tilde{a}_i y_i)) \leq N\theta_{m_{i_t}} \leq 1/2^{i_t+2}\,.$$

Thus $f(\sum_{t\in A, i\in L_t} y_i) \leq 1/2^M$ and we erase this part for all t with error $\leq 1/k$. Notice that by condition (2) in choice of (y_i) we have

$$f_t(\sum_{i\in I_t, i<i_t} y_i) \leq \theta_{m_{i_t}} \sum_{i<i_t} \# \operatorname{supp} y_i \leq 1/2^{i_t+2}\,,$$

so we can again erase this part for all t with error $1/k$.

Let g be the restriction of f to $\cup_{t\in A} \operatorname{supp} y_{i_t}$ and $h = f - g$. First we consider $g(y) = \sum_{t\in A} t(f_t)\tilde{a}_{i_t} f_t(y_{i_t})$. Let $B = \{t \in A : \operatorname{ord}(f_t) \leq k\}$, hence $\# B \leq k$. Then $\sum_{t\in B} \tilde{a}_{i_t} f_t(y_{i_t}) \leq \# B/k^2 \leq 1/k$, hence we can erase this part with error $1/k$. Notice that $\sum_{t\in A\setminus B} \frac{1}{\operatorname{ord}(f_t)^{1/q}} e^*_{i_t}$

is a norming functional on ℓ_p, hence

$$\sum_{t\in A\setminus B}\tilde{a}_{i_t}t(f_t)f_t(y_{i_t})\leq\sum_{t\in A\setminus B}\tilde{a}_{i_t}\frac{c_{\mathrm{ord}(f_t)}}{(\mathrm{ord}(f_t))^{1/q}}\leq\max_{n\geq k}c_n\|(\tilde{a}_{i_t})_{t\in A\setminus B}\|_{\ell_p}\leq\max_{n\geq k}c_n\,.$$

We consider $h(y)=\sum_{t\in A}\sum_{i\in I_t,i>i_t}\tilde{a}_i\sum_{s\in J_i}t(f_s)f_s(y_i)$. Let $D=\{s\in J_i,i\in I_t,i>i_t,t\in A:\ \mathrm{ord}(f_s)\leq k\}$. Then

$$\sum_{t\in A}\sum_{i\in I_t,i>i_t}\sum_{s\in J_i\cap D}\tilde{a}_if_s(y_i)\leq\#D/k^2\leq 1/k\,,$$

and we again erase this part with error $1/k$. For any $i\in I_t,i>i_t$ for some $t\in A$ we let $r_i=\mathrm{ord}(f_t)m_t$ and compute, using Hölder inequality,

$$\begin{aligned}\sum_{t\in A}\sum_{i\in I_t,i>i_t}\sum_{s\in J_i\setminus D}\tilde{a}_it(f_s)f_s(y_i)\leq&\sum_{t\in A}\sum_{i\in I_t,i>i_t}\tilde{a}_i8(\#J_i)^{1/q}\theta_{r_i}\\ &\leq 8\max_{n\geq k}c_n\sum_{t\in A}\sum_{i\in I_t,i>i_t}\tilde{a}_i\frac{(\#J_i)^{1/q}}{r_i^{1/q}}\\ &\leq 8\max_{n\geq k}c_n\|(\tilde{a}_i)_{i\in I_t,i>i_t,t\in A}\|_{\ell_p}\leq 8\max_{n\geq k}c_n\,.\end{aligned}$$

We put all the estimates together obtaining

$$f(y)\leq 36(9\max_{n\geq k}c_n+4/k)\,.$$

Therefore we proved that $\Delta_\varepsilon=\sup\left\{\|\sum_{i\in\mathbb{N}}a_iv_i\|:\ \sup_{i\in\mathbb{N}}|a_i|\leq\varepsilon,\ \|(a_i)_{i\in\mathbb{N}}\|_{\ell_p}=1\right\}$ converges to zero, as $\varepsilon\to 0$. By Lemma 2.4 [33] there are some $(\delta_n)_n\subset(0,\infty)$ with $\delta_n\searrow 0$ such that for any $(a_i)_i\in c_{00}$

$$\|\sum_i a_iv_i\|\leq\max_{n\in\mathbb{N}}\delta_n\max_{\#F\leq n}\|(a_i)_{i\in F}\|_{\ell_p}\,,$$

which ends the proof of Lemma. □

We continue the proof of Theorem 3.4. By the proof of Thm 2.9 [28], p is in the Krivine set of Y in Lemberg sense [24], i.e. for any n there is a normalized block sequence $(x_i^{(n)})_i\subset Y$ generating spreading model $(u_i^{(n)})_i$ such that $(u^{(n)})_{i=1}^n$ is 1-equivalent to the u.v.b. of ℓ_p^n.

Pick $(m_n)_n$ such that $\delta_{m_n}\leq 1/4^n$. Apply Prop. 3.2 [4] to constants $C_n=2^n$, $n\in\mathbb{N}$ and normalized block sequences $(x_i^{(m_n)})_i$ generating spreading models $(u_i^{(m_n)})_i$. We obtain thus a seminormalized block sequence (x_i) generating spreading model $(u_i)_i$ which C_n dominates

$(u_i^{(m_n)})_i$ for any $n \in \mathbb{N}$. By Lemma 3.5 we obtain

$$\begin{aligned}\|\sum_i a_i v_i\| &\le \max_{n\in\mathbb{N}} \delta_n \max_{\#F\le n} \|(a_i)_{i\in F}\|_{\ell_p} \\ &\le \max_{n\in\mathbb{N}} \delta_{m_n} \max_{\#F\le m_{n+1}} \|(a_i)_{i\in F}\|_{\ell_p} \\ &\le \max_{n\in\mathbb{N}} 1/4^n \max_{\#F\le m_{n+1}} \|\sum_{i\in F} a_i u_i^{(m_{n+1})}\| \\ &\le \max_{n\in\mathbb{N}} C_{n+1}/4^n \max_{\#F\le m_{n+1}} \|\sum_{i\in F} a_i u_i\| \\ &\le \max_{n\in\mathbb{N}} 2/2^n \max_{\#F\le m_{n+1}} \|\sum_{i\in F} a_i u_i\|\,.\end{aligned}$$

Notice that (u_i) is not equivalent to c_0, thus by Theorem 3.3 we finish the proof. □

In [19] the construction of non-compact strictly singular operators was based on c_0-spreading model of higher order in the dual space. However this method does not follow straightforward in case of $p-$spaces, as the observation below shows. We consider the Schlumprecht space $S = T[(\mathcal{A}_n, \frac{1}{\log_2(n+1)})_n]$ introduced in [32]. In [23] it was shown that S contains a block sequence generating ℓ_1-spreading model.

Proposition 3.6. *Consider the sequence (y_k) generating ℓ_1-spreading model constructed in [23], $y_k = \sum_{m=1}^k v_{k,m}$, $k \in \mathbb{N}$. Take any block sequence $(y_k^*) \subset S^*$ so that $y_k^*(y_l) = \delta_{l,k}$. Then the sequence (y_k^*) does not generate c_0-spreading model.*

Proof. We can assume that $\operatorname{supp} y_k^* = \operatorname{supp} y_k$, $k \in \mathbb{N}$. Consider two cases:

CASE 1. There is $m_0 \in \mathbb{N}$, $\delta > 0$ and an infinite $K \subset \mathbb{N}$ with $|y_k^*(\sum_{m=1}^{m_0} v_{k,m})| \ge \delta$ for any $k \in K$.

Let z_k^* be the restriction of y_k^* to the support of $\sum_{m=1}^{m_0} v_{k,m}$, $k \in K$. Then $(z_k^*)_{k\in K}$ is a seminormalized block sequence in S^*, majorized by $(y_k^*)_{k\in K}$. Since by the form of $(v_{m,k})$ the length of $\operatorname{supp}(\sum_{m=1}^{m_0} v_{k,m})$ is constant, we can pick some subsequence $(z_k^*)_{k\in L}$ of $(z_k^*)_{k\in K}$ consisting of, up to controllable error, equally distributed vectors. As the u.v.b. in S is subsymmetric, the same holds for $(z_k^*)_{k\in L}$, thus $(z_k^*)_{k\in L}$ is equivalent to spreading model generated by itself. It follows that (y_k^*) cannot generate c_0-spreading model.

CASE 2. If the first case does not hold, pick increasing $(N_j) \subset \mathbb{N}$ so that

$$\left| y_{N_j}^* \left(\sum_{m=1}^{N_{j-1}} v_{N_j,m} \right) \right| \le 1/2^j\,.$$

Consider the norm of vectors $z_j^* = y_{N_1}^* + \dots + y_{N_j}^*$. Put

$$x_{N_1} = y_{N_1}, \quad x_{N_j} = \sum_{m=N_{j-1}+1}^{N_j} v_{N_j,m}, \quad j > 1\,.$$

By the choice of (N_j) we have $y_{N_j}^*(x_{N_j}) \ge 1 - 1/2^j$.

We estimate the norm of $x_j = x_{N_1} + \cdots + x_{N_j}$. We can assume at the beginning that (N_j) was chosen to increase fast enough so that (x_{N_j}) is D-equivalent to the unit basis of S (see Remark 5, Lemma 2 [23]). Therefore $\|x_j\| \leq Dj/f(j)$.

By the choice of (N_j) and definition of x_{N_j} we have $z_j^*(x_j) \geq j-1$. Hence

$$\|z_j^*\| \geq z_j^*(x_j)/\|x_j\| \geq f(j)(j-1)/Dj \geq f(j)/2D\,.$$

Notice that the same scheme works if we replace $N_1, \ldots, N_j$ by any $N_{n_1}, \ldots, N_{n_j}$ in definition of z_j, hence no subsequence of (y_k^*) can produce a c_0-spreading model. □

3.2. **Spaces defined by families $(\mathcal{S}_n)_n$.** Regarding the existence of strictly singular operators from subspaces of mixed Tsirelson spaces we prove the following result, which is in "localization" of Schlumprecht result in mixed Tsirelson spaces. First recall the definition of a higher order ℓ_1-spreading models.

Definition 3.7. We say that a normalized basic sequence $(x_n)_{n\in\mathbb{N}}$ in a Banach space generates an $C-\ell_1^\alpha$-spreading model, $\alpha < \omega_1$, $C \geq 1$, if for any $F \in \mathcal{S}_\alpha$ the sequence $(x_n)_{n\in F}$ is $C-$equivalent to the u.v.b. of $\ell_1^{\#F}$. In case of $\alpha = 1$ we obtain the classical ℓ_1-spreading model.

We recall that $[M]$, $M \subset \mathbb{N}$, denotes the family of all infnite subsequences of M, $[M]^<$ - the family of all finite subsequences of M.

Theorem 3.8. *Let $X = T[(\mathcal{S}_n, \theta_n)_n]$ or $T_M[(\mathcal{S}_n, \theta_n)_n]$ be a regular (modified) mixed Tsirelson space. If X contains a block sequence (y_n) generating ℓ_1^ω-spreading model then there are a subspace $Y \subset [(y_n)]$ and a strictly singular operator $T : Y \to X$.*

We recall that in [25] it was proved that if a regular sequence (θ_n) satisfies $\lim_m \limsup_n \frac{\theta_{m+n}}{\theta_n} > 0$ then the mixed Tsirelson space $X = T[(\mathcal{S}_n, \theta_n)_n]$ is subsequentially minimal if and only if any block subspace of X admits an ℓ_1^ω-spreading model, if and only if any block subspace of X has Bourgain ℓ_1-index greater than ω^ω. These conditions hold in particular if $\sup \theta_n^{1/n} = 1$ [27]. In [22] analogs of these results were studied in the partly modified setting.

To prove the theorem we first define an index measuring the best constant of the ℓ_1^α-spreading models generated by subsequences of a given sequence. Let $\vec{x} := (x_n)_{n\in\mathbb{N}}$ be a normalized block sequence. We set

$$\delta_\alpha(\vec{x}) = \sup\{\delta > 0 : \exists M \in [\mathbb{N}] \text{ such that } (x_n)_{n\in M} \text{ generates } \delta - \ell_1^\alpha \text{ spr. model}\}\,.$$

The following properties of $\delta_\alpha(\vec{x})$ follows readily from the definition.

a) $\delta_\alpha((x_n)_{n\in\mathbb{N}}) = \delta_\alpha((x_n)_{n\geq n_0})$ for all $n_0 \in \mathbb{N}$.
b) $\delta((x_n)_{n\in M}) \leq \delta_\alpha((x_n)_{n\in\mathbb{N}})$ for all $M \in [\mathbb{N}]$.
c) $(\delta_\alpha(\vec{x}))_{\alpha<\omega_1}$ is non-increasing family.

By standard arguments we may stabilize $\delta_\alpha(\vec{x})$. Namely passing to a subsequence we may assume that $\delta_\alpha((x_n)_{n\in\mathbb{N}}) = \delta_\alpha((x_n)_{n\in M})$ for every $M \in [\mathbb{N}]$.

By Bourgain's $\ell_1 - index$ it follows that $\delta_\alpha((x_n)_{n\in\mathbb{N}}) > 0$ countable many $\alpha's$, enumerate them as $(\alpha_n)_n$. In particular for an asymptotic ℓ_1 space it follows that $\delta_n(\vec{x}) > 0$ for all $n \in \mathbb{N}$.

Inductively we choose $M_1 \supset M_2 \supset \dots$ infinite subsets of $\mathbb{N}$ such that

$$\delta_{\alpha_n}((x_n)_{n\in M_n}) = \delta_{\alpha_n}(x_n)_{n\in L} \;\forall L \in [M_n]\,.$$

We define the family

$$\mathcal{F}_{2\delta_{\alpha_n}(\vec{x})} = \{A \in [\mathbb{N}]^{<} : \exists x^* \in B_{X^*} \text{ with } x^*(x_i) > 2\delta_{\alpha_n}((x_n)_{n\in\mathbb{N}}) \text{ for all } i \in A\}.$$

By I. Gasparis theorem [18] there exists $N \in [M_n]$ such that

$$\text{either } \mathcal{S}_{\alpha_n} \cap [N] \subset \mathcal{F}_{2\delta_{\alpha_n}} \text{ or } \mathcal{F}_{2\delta_{\alpha_n}} \cap [N] \subset \mathcal{S}_{\alpha_n}.$$

In the first case by 1-unconditionality of the basis it follows that $(x_n)_{n\in N}$ and hence $(x_k)_{k\in M_n}$ contains a subsequence which generates $2\delta_{\alpha_n} - \ell_1^{\alpha_n}$-spreading model, a contradiction. So additionally we may assume that there exists $M_n \in [M_{n-1}]$ with

$$\mathcal{F}_{2\delta_{\alpha_n}}(M_n) \subset \mathcal{S}_{\alpha_n}, \tag{3.1}$$

$$\mathcal{S}_{\alpha_{n-1}} \cap \{m_n, m_n+1, \dots\} \subset \mathcal{S}_{\alpha_n}\,. \tag{3.2}$$

Let $M = (m_i)_i$ be a diagonal set. Passing to a subsequence we may assume that $\sum_n n\delta_{\alpha_n} < 0.25$. Let $\|\sum_i a_i x_{m_i}\| = 1$ and let $x^* \in B_{X^*}$ such that $\sum_i a_i x^*(x_{m_i}) = 1$. By the unconditionality we may assume that $x^*(x_{m_i}) \geq 0$ for every i. Let $2\delta_{\alpha_0} = 1$ and

$$F_k = \{i : x^*(x_{m_i}) \in (2\delta_{\alpha_k}, 2\delta_{\alpha_{k-1}}]\}$$

and $F_k^1 = F_k \cap \{1, \dots, k-1\}$, $F_k^2 = F_k \cap \{k, k+1, \dots\}$.

From (3.1),(3.2) we get $F_k^2 \in \mathcal{S}_{\alpha_k} \cap \{k, k+1, \dots\} = \mathcal{G}_k$. It follows

$$\begin{aligned}
\|\sum_i a_i x_{m_i}\| &= \sum_i a_i x^*(x_{m_i}) = \sum_{k=1}^{\infty} \sum_{i\in F_k} a_i x^*(x_{m_i}) \\
&= \sum_{k=1}^{\infty} \left(\sum_{i\in F_k^1} a_i x^*(x_{m_i}) + \sum_{i\in F_k^2} a_i x^*(x_{m_i}) \right) \\
&\leq \sum_{k=2}^{\infty} 2\delta_{\alpha_{k-1}}(k-1) \max_i |a_i| + \sum_{k=1}^{\infty} 2\delta_{\alpha_{k-1}} \sum_{i\in F_k^2} |a_i| \\
&\leq 0.5 \|\sum_i a_i x_{m_i}\| + \sum_{k=1}^{\infty} 2\delta_{\alpha_{k-1}} \sup_{F\in\mathcal{G}_k} \sum_{i\in F} |a_i|\,,
\end{aligned}$$

and therefore $\|\sum_i a_i x_{m_i}\| \leq 4 \sum_{k=1}^{\infty} \delta_{\alpha_{k-1}} \sup_{F\in\mathcal{G}_k} \sum_{i\in F} |a_i|$.

So we have the following

$$\|\sum_i a_i x_{m_i}\| \leq 4 \sum_{k=1}^{\infty} \delta_{\alpha_{k-1}} \sup_{F\in\mathcal{G}_k} \sum_{i\in F} |a_i| \text{ for all } (a_i)_i, \tag{3.3}$$

where $\mathcal{G}_k = \mathcal{S}_{\alpha_k} \cap \{k, k+1, \dots\}$.

Proof of the Theorem 3.8. Let $\vec{e} = (e_n)_{n\in\mathbb{N}}$ be the basis of X. Using that for every $j \in \mathbb{N}$ and every $\sum_{i\in F} a_i e_i$ special convex combination of the basis it holds

$$\theta_n \leq \|\sum_{i\in F} a_i e_i\| \leq 2\theta_n \,,$$

see [7, 9]. It follows readily that $\delta_n(\vec{e}) \in [\theta_n, 2\theta_n]$ and $\delta_\omega = 0$.

Since the space X contains a a block sequence $(y_n)_{n\in\mathbb{N}}$ generating ℓ_1^ω-spreading model it follows that

$$\|\sum_i a_i y_i\| \geq c \sum_{i\in F} |a_i| \ \ \forall n \in \mathbb{N}, F \in \mathcal{S}_n \cap \{n, n+1, \dots\}.$$

By the previous reasoning we pick a $M = (m_i) \in [\mathbb{N}]$ and a sequence $\alpha_k \nearrow \omega$ such that $\sum_k k\delta_{\alpha_k} < \infty$ and (3.3) holds. Setting $M = \sum_k \theta_{\alpha_{k-1}}$ we have

$$\begin{aligned} \|\sum_i a_i e_{m_i}\| &\leq 8 \sum_k \theta_{\alpha_{k-1}} \sup_{F\in\mathcal{G}_k} \sum_{i\in F} |a_i| \\ &\leq \frac{8M}{c} \sup_k c \sup_{F\in\mathcal{G}_k} \sum_{i\in F} |a_i| \\ &\leq \frac{8M}{c} \|\sum_i a_i y_i\| \,. \end{aligned}$$

It follows that the operator extending the mapping $y_n \to x_{m_n}$ factors through a c_0-saturated space and hence is strictly singular. □

3.3. **Remarks and questions.** As a corollary to Theorem 3.4, part (1), we obtain that the (non-modified) Tzafriri space Y has an asymptotic ℓ_2 subspace Z which satisfies a blocking principle in the sense of [14]. The only known spaces with a blocking principle so far were similar to T, T^* and their variations. The two major ingredients used in [14] for proving the minimality of T^* are the blocking principle and the saturation with ℓ_∞^n's. It is shown in [21] that Tzafriri space Y contains uniformly ℓ_∞^n's. It is not known whether Y is uniformly saturated with ℓ_∞^n's. In the opposite direction, we do not know if Z contains a convexified Tsirelson space $T^{(2)}$ (which is equivalent to its modified version).

In 1977 Altshuler [2] (cf. e.g. [26]) constructed a Banach space with a symmetric basis which contains no ℓ_p or c_0, and all its symmetric basic sequences are equivalent. In 1981 C. Read [31] constructed a space with, up to equivalence, precisely two symmetric bases. More precisely, Read proved that any symmetric basic sequence in his space CR is equivalent either to the u.v.b. of ℓ_1 or to one of the two symmetric bases of CR. A careful look at the papers of Altshuler and Read shows that their proofs work similarly for the more general case of all subsymmetric basic sequences. This observation leads to the following questions:

Question 1. Does there exist a space in which all subsymmetric basic sequences are equivalent to one basis, and that basis is not symmetric?

We remark that Altshuler's space has a natural subsymmetric version but we do not know if it satisfies the above property.

Question 2. Does there exist a space with exactly two subsymmetric bases, which are not symmetric?

References

[1] D. Alspach, S.A. Argyros, *Complexity of weakly null sequences*, Diss. Math. 321, 1–44 (1992).

[2] Z. Altshuler, *A Banach space with a symmetric basis which contains no ℓ_p or c_0, and all its symmetric basic sequences are equivalent*, Composition Math. 35 (1977), 189–195.

[3] G. Androulakis, E. Odell, *Distorting mixed Tsirelson spaces*, Israel J. Math. 109 (1999), 125–149.

[4] G. Androulakis, E. Odell, T. Schlumprecht, N. Tomczak-Jaegermann, *On the structure of the spreading models of a Banach space*, Canad. J. Math., 57 (2005), no. 4, 673–707.

[5] G. Androulakis, T. Schlumprecht, *The Banach space S is complementably minimal and subsequentially prime*, Studia Math. 156 (2003), no. 3, 227–242.

[6] G. Androulakis, T. Schlumprecht, *Strictly singular, non-compact operators exist on the space of Gowers and Maurey*, J. London Math. Soc. (2) 64 (2001) 655–674.

[7] S.A. Argyros, I. Deliyanni, *Examples of asymptotic ℓ_1 Banach Spaces*, Trans. Amer. Math. Soc. 349 (1997), 973–995.

[8] S.A. Argyros, I. Deliyanni, A. Manoussakis, *Distortion and spreading models in modified mixed Tsirelson spaces*, Studia Math. 157 (3) (2003), 199–236.

[9] S.A. Argyros, I. Deliyanni, D. Kutzarova, A. Manoussakis, *Modified mixed Tsirelson spaces*, J. Funct. Anal. 159 (1998), 43–109.

[10] S.A. Argyros, R. Haydon, *A hereditarily indecomposable $\mathcal{L}_\infty$ -space that solves the scalar-plus-compact problem*, Acta Math. 206 (2011), no. 1, 1–54.

[11] S.A. Argyros, S. Todorcevic, *Ramsey methods in analysis.* Advanced Courses in Mathematics, CRM Barcelona, Birkhäuser Verlag, Basel, 2005.

[12] S. Bellenot, *Tsirelson superspaces and ℓ_p*, J. Funct. Anal. 69 (1986), no. 2, 207–228.

[13] A. Brunel, L.Sucheston, *On B-convex Banach spaces*, Math. Systems Theory, 7 (1974), no. 4, 294–299.

[14] P. Casazza, W.B. Johnson, L. Tzafriri, *On Tsirelson's space*, Israel J. Math. 47 (1984), 81–98.

[15] P. Casazza, E. Odell, *Tsirelson's space and minimal subspaces*, Longhorn Notes, University of Texas, Texas Functional Analysis Seminar (1982-3), 61–76

[16] V. Ferenczi, C. Rosenthal, *Banach spaces without minimal subspaces*, J. Funct. Anal. 257 (2009), 149–193.

[17] V. Ferenczi, C. Rosenthal, *Banach spaces without minimal subspaces - Examples (Part II)*, to appear in Annales de l'Institut Fourier.

[18] I. Gasparis, *A dichotomy theorem for subsets of the power set of the natural numbers*, Proc. Amer. Math. Soc. 129 (2001), no. 3, 759–764

[19] I. Gasparis, *Strictly singular non-compact operators on hereditarily indecomposable Banach spaces*, Proc. Amer. Math. Soc. 131 (2002), 1181–1189.

[20] W.T. Gowers, *An infinite Ramsey theorem and some Banach space dichotomies*, Annals of Mathematics 156 92002), 797-833

[21] M. Junge, D. Kutzarova,, E. Odell *On asymptotically symmetrical Banach spaces*, Studia Math. 173, no. 3, 203–231 (2006).

[22] D. Kutzarova, D. Leung, A. Manoussakis, W.-K. Tang, *Minimality properties of Tsirelson type spaces*, Studia Math. 187 (2008), 233–263.

[23] D. Kutzarova, P. K. Lin, *Remarks about Schlumprecht space*, Proc. Amer. Math. Soc. 128 (2000), 2059–2068.

[24] H. Lemberg, *Sur un théorème de J.-L. Krivine sur la finie représentation de ℓ_p dans un espace de Banach*, C. R. Acad. Sci. Paris Sér. I Math. 292 (1981), no. 14, 669–670.

[25] D. Leung, W.-K. Tang, *ℓ_1-spreading models in subspaces of mixed Tsirelson spaces*, Studia Math. 172, no. 1, 47–128 (2006).

[26] J. Lindenstrauss, L. Tzafriri, *Classical Banach Spaces, I, Sequence Spaces*, 1977, Springer.

[27] A. Manoussakis, *On the structure of a certain class of mixed Tsirelson spaces*, Positivity, 5 (2001), no. 3, 193–238.

[28] A. Manoussakis, A.M. Pelczar, *Quasiminimality of mixed Tsirelson spaces*, Math. Nachr. DOI 10.1002/mana.200910845

[29] E. Odell, N. Tomczak-Jaegermann, R. Wagner, *Proximity to ℓ_1 and Distortion in Asymptotic ℓ_1 Spaces*, J. Funct. Anal. 150 (1997), 101–145.

[30] A. Pelczar, *Subsymmetric sequences and minimal spaces*, Proc. Amer. Math. Soc. 131 (3) (2003) 765–771.

[31] C.J. Read, *A Banach space with, up to equivalence, precisely two symmetric bases*, Israel J. Math. 40 (1981), 33–53.

[32] T. Schlumprecht, *An arbitrary distortable Banach space*, Israel J. Math. 76 (1991), 81–95.

[33] T. Schlumprecht, *How many operators exist on a Banach space?* Trends in Banach spaces and operator theory (Memphis, TN, 2001), 295–333, Contemp. Math. 321, Amer. Math. Soc., Providence, RI, 2003.

[34] L. Tzafriri, *On the type and cotype of Banach spaces*, Israel J. Math. 32 (1979), 3–38.

Institute of Mathematics, Bulgarian Academy of Sciences, current address: Department of Mathematics, University of Illinois at Urbana-Champaign
E-mail address: denka@math.uiuc.edu

Department of Sciences, Technical University of Crete, 73100 Chania (Crete), Greece
E-mail address: amanousakis@isc.tuc.gr

Institute of Mathematics, Jagiellonian University, Łojasiewicza 6, 30-348 Kraków, Poland
E-mail address: anna.pelczar@im.uj.edu.pl